Appendix
Notes and References

Preface

Harold Scott MacDonald Coxeter (1907–2003): English geometer who spent most of his life in Canada. An excellent biography is Siobhan Roberts' *King of Infinite Space: Donald Coxeter, the Man Who Saved Geometry*, Walker & Co. New York 2006.

Foundations: the Synthetic Approach

2. John Playfair (1748–1819): Scottish mathematician and scientist. Professor of Natural Philosophy at the University of Edinburgh.
   János Bolyai (1802–1860) and Nikolai Lobachevsky (1792–1856) discovered non-Euclidean geometry independently and almost simultaneously. Bolyai's work was published in 1832 as an appendix in a mathematical textbook written by his father. The English translation of Lobachevsky's book is *Geometrical Researches on the Theory of Parallels*, Open Court Publishing Co., Illinois 1914.
3. Donato Bramante (1444–1514): Italian architect. Bramante's most astonishing work is St. Peter's Basilica in Rome. He was also a painter, expert in the construction of correct perspective images, but left no written record of his methods. Leon Battista Alberti (1404–1472) was an architect, poet, linguist, philosopher—and mathematician. His *Della Pittura* (1435) contains the earliest systematic exposition of the methods of perspective drawing. English translation by JR Spencer: *On Painting*, Yale University Press 1956.
4. Albrecht Dürer (1471–1528): German painter and printmaker. 'A Man Drawing a Lute' is a woodcut published in the 1525 edition of Dürer's *Underweysung der Messung*.
5. Girard Desargues (1591–1661) was an engineer, architect, musician and mathematician. His *Brouillon project d'une atteinte aux evenements des rencontres du cone avec un plan* (1639) marks the beginning of projective methods in geometry. Its importance was appreciated at the time by eminent mathematicians such as Pascal and Leibnitz, but thereafter was largely forgotten till the 19th century. Desargues' works were collected and published as *L'oeuvre mathématique de Desargues*, Paris, 1951 (René Taton, ed.).

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DOI 10.1007/978-1-4471-4631-5, © Springer-Verlag London 2013

11. Pappus of Alexandria (4th century): one of the last of the great Greek mathematicians. He wrote eight books, most of which have survived.
18. Some important configurations and their symmetries are lucidly discussed in Coxeter, H S M, Self-dual configurations and regular graphs, *Amer. Math. Soc. Bulletin*, **56**, 413 (1950). It was reprinted in Coxeter (1968).
18. Gino Fano (1871–1952): Italian mathematician.
26. René Descartes (1596–1650): major French philosopher and mathematician.

The Analytic Approach

38. Julius Plücker (1801–1868): German mathematician and physicist who made important contributions to analytic geometry, and early investigations of cathode rays that eventually led to the discovery of the electron.
39. Hermann Grassmann (1809–1877): German mathematician and linguist. His mathematical work was far ahead of its time and its fundamental importance was not much appreciated during his lifetime.
    Grassmann's methods were originally published in *Die lineare Ausdehnungslehre: ein neuer Zweig der Mathematik*, Wiegand, Leipzig (1844). It is discussed briefly in Coolidge 1940. English translation: Kannenberg, L, *A New Branch of Mathematics*, Open Court, Chicago 1995.

Linear Figures

58. Herbert William Richmond (1868–1948): English geometer.
    Richmond H W, On the figure of six points in a space of four dimensions, *Quart. J. Math.* **31**, 125 (1900)
    Julius Plücker (1801–1868): German geometer who introduced the idea of using six homogeneous coordinates to investigate the properties of linear figures in 3-space. Jacob Steiner (1796–1863): German geometer. A cubic surface investigated by him carries his name (see Fig. 6.1).
    Luigi Cremona (1830–1903): Italian geometer who contributed much to our understanding of algebraic curves and surfaces.
60. James Joseph Sylvester (1814–1897): major English mathematician. Founder of the American Journal of Mathematics.
65. Ludwig Schläfli (1814–1895): Swiss mathematician. One of the first geometers to explore the geometry of more than three dimensions.
68. Beniamino Segre (1903–1977): Italian geometer who made important contributions in analytic geometry and combinatorics.
    Coxeter H S M, The polytope 2_{21}, whose twenty-seven vertices correspond to the lines on the general cubic surface, *Amer. J. Math.*, **62**, 561 (1946).
72. Henry Frederick Baker (1866–1956): English mathematician. Coxeter was one of his students.
    Baker's configuration: Coxeter has employed the duads and synthemes in quite different contexts, showing how they are related to a remarkable configurations in the finite geometries PG(3, 3) and PG(5, 3): Coxeter H S M, The chords of the non-ruled quadric in PG(3, 3). *Canad. J. Math.* **10**, 484, 1958; Twelve

points in PG(5, 3) with 95040 self-transformations, *Proc. Roy. Soc. A*, **247**, 287 (1958); reprinted in Coxeter 1968.

Quadratic Figures

79. Blaise Pascal (1623–1662): French mathematician and religious philosopher. He was sixteen years old when he wrote *Essai pour les coniques* (1639), a significant work on projective geometry containing the *hexagramum mysticum* theorem.
95. Charles Julien Brianchon (1783–1864): French mathematician and chemist.
95. The 60 Pascal lines: the discovery of all these unexpected incidences is due to many eminent 19th century mathematicians—Arthur Cayley (1821–1895), George Salmon (1819–1904) and Penyngton Kirkman (1806–1905) in Britain, Julius Plücker (1801–1868) and Jacob Steiner (1796–1863) in Germany, Luigi Cremona (1830–1903) and Giuseppe Veronese (1854–1917) in Italy...
107. Felix Klein (1849–1925) proposed that all the different kinds of geometry could be classified according to the groups of transformations that preserve their structure, and emphasized the fundamental importance in geometry of the group-theoretical concept of *symmetry*. These ideas were presented in Erlangen in 1879 under the title *Vergleichende Betrachtungen über neuere geometrische Forschungen*—which became known as 'Klein's Erlangen program'. Thus a projective space is characterized by its collineation group and the various metric geometries (including the newly-discovered 'non-Euclidean' geometries) correspond to the subgroups of collineations that leave their absolute quadric invariant.
108. William Kingdon Clifford (1845–1879): English mathematician and philosopher, now best known for 'Clifford algebras'—a generalization of Hamilton's quaternions.

Cubic Figures

119. Carl Gustav Jacobi (1804–1851): German mathematician who made very many fundamental contributions.
129. The various degenerate cases of the configuration of the 27 lines lead to a classification scheme for cubic surfaces: Schläfli, L, On the distribution of surfaces of the third order into species, in reference to the presence or absence of singularities and the reality of their lines, *Phil. Trans. Roy. Soc*. **63**, 193 (1863).
131. Rudolph Clebsch (1833–1972): German mathematician who made important contributions in algebraic geometry.
   In the 19th and early 20th centuries, models demonstrating intricate geometrical concepts were sculpted in wire, wood or plaster and displayed in mathematics departments of universities. Fischer (1986) is a fine collection of photographs of these models. Many of these displays were subsequently sadly neglected—fashions in mathematics, as elsewhere, come and go. It is to be hoped that the recent advent of 'stereolithography' will lead to a revival of this delightful aspect of geometrical studies.

Quartic Figures

133. Basset (1910) is a classic on the subject of singularities of surfaces. A much more recent book on this topic is Lane, E P, *Projective Differential Geometry of Curves and Surfaces*, Porter Press 2007. Jessop (1916) deals specifically with quartic surfaces.
138. Ernst Eduard Kummer (1810–1893): German mathematician who made major contributions to number theory and hypergeometric functions.

Finite Geometries

148. Évariste Galois (1811–1832): French mathematician. He developed the foundations of the modern approach to group theory (he is the initiator of the use of the word *groupe* in this context), made significant contributions to number theory and originated the idea of finite fields—now known as Galois fields. He got himself killed in a duel before he was twenty-one.
149. Ferdinand Georg Frobenius (1849–1917): German mathematician known for important work in group theory, number theory and differential equations.
156. John Horton Conway (born 1937): English mathematician.
Conway, J H, Three lectures on the exceptional groups, in *Finite Simple Groups* (Powell & Higman, eds.), Academic Press 1971; reprinted in Conway, J H and Sloane, N J A, *Sphere Packings, Lattices and Groups*, Springer 1988.
163. Émile Léonard Mathieu (1835–1890): French mathematician.
The five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24} were the first to be discovered of the 'sporadic' finite simple groups.
164. Coxeter H S M, A symmetrical arrangement of eleven hemi-icosahedra, *Annals of Discrete Mathematics* **20**, 103 (1984).
165. Coxeter H S M, Twelve points on PG(5, 3) with 95040 self-transformations, *Proc. Roy. Soc. Lond.* A **247**, 279 (1958); Reprinted in Coxeter 1968.
168. Marcel Golay (1902–1989): Swiss mathematician and information theorist.
The authoritative source for coding theory and finite groups is Conway & Sloane 1988.
The configuration of 24 points in PG(11, 2) (essentially the Golay code interpreted geometrically) was discovered and investigated by John Arthur Todd (1908–1994): A note on the linear fractional groups, *J. Lond. Math. Soc.* **7**, 195 (1932); On representations of the Mathieu groups as collineation groups, *J. Lond. Math. Soc.* **34**, 406 (1959); On representations of the Mathieu group M_{24} as a collineation group, *Ann. di Math. Pure ed Appl.* **71**, 199 (1966).

A few other papers related to finite geometry that may be of interest:

Conway, J H, The miracle octad generator, *Proc. Summer School, University College Galway* (M P J Curran, ed.) Academic Press 1977.
Curtis, R T, A new combinatorial approach to M_{24}, *Mat. Proc. Cambridge Phil. Soc.* **79**, 25 (1976)
Golay, M, Notes on digital coding, *Proc. IRE*, **37**, 657 (1949).

Lord, E A, Geometry of the Mathieu groups and Golay codes, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **98**, 153 (1988)
Mason, D R, On the construction of the Steiner system S(5, 8, 24), *J. Algebra* **47**, 77 (1977)
Paige, L J, A note on the Mathieu groups, *Can. J. Math.* **9**, 15 (1956)

Bibliography

In addition to the works mentioned in the Appendix, here are some books that you may find useful for supplementary reading if you want to delve deeper. This is a highly personal selection—simply a few books I found particularly stimulating while learning about this fascinating subject. They are historic treasures; some of them have been recently reprinted and a few of them can be downloaded free from the internet.

Baker, H F, *Principles of Geometry* (6 Vols.), Cambridge University Press, Cambridge, 1940.

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Coxeter, H S M, *Non-Euclidean Geometry* (4th edn.), University of Toronto Press, Toronto, 1961b.

Coxeter, H S M, *Twelve Geometric Essays*, Southern Illinois University Press, Illinois, 1968.

Cremona, L, *Elements of Projective Geometry*, Oxford University Press, Oxford, 1885 (tr. C Leudesdorf)

Conway, J H & Sloane, N J A, *Sphere Packings, Lattices and Groups*, Springer, Berlin, 1988.

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Henderson, A, *The Twenty-Seven Lines upon the Cubic Surface*, Cambridge University Press, Cambridge, 1911.

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Salmon, G, *A Treatise on the Analytic Geometry of Three Dimensions* (4th edn.), Hodge, Figgis & Co, Dublin (1882).

Segre, B, *The Non-Singular Cubic Surfaces*, Oxford University Press, Oxford, 1942.

Semple, J G & Kneebone, G T, *Algebraic Projective Geometry* (2nd edn.), Oxford University Press, Oxford (1952).

Todd, J A, *Projective and Analytic Geometry*, Pitman, London, 1947.

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