A
References

1. F. Brauer & C. Castillo-Chavez, 2001. *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, New York. (Concentrates on populations, disease dynamics, and resource management, for advanced undergraduates.)

2. W. E. Boyce & R. G. DiPrima, 1997. *Elementary Differential Equations*, 6th ed., John Wiley and Sons, New York. (One of many editions of a popular, elementary text.)

3. S. L. Campbell & R. Haberman, 2008. *Introduction to Differential Equations with Dynamical Systems*, Princeton University Press, Princeton, NJ. (A recent, traditional treatment with many examples and exercises.)

4. R. V. Churchill, 1972. *Operational Mathematics*, 3rd ed., McGraw-Hill, New York. (This well-known book contains many applications and an extensive table of transforms.)

5. L. Edelstein–Keshet, 2005. *Mathematical Models in Biology*, SIAM (Society of Applied and Industrial Mathematics), Philadelphia (reprint of the 1988 edition published by Random House). (The classic text, with an outstanding introduction to differential equations.)

6. D. J. Higham & N. J. Higham, 2005. *MATLAB Guide*, 2nd ed., SIAM, Philadelphia. (An excellent resource for MATLAB® applications of all types.).

7. M. W. Hirsch, S. Smale, & R. L. Devaney, 2004. *Differential Equations, Dynamical Systems, & An Introduction to Chaos*, Elsevier, New York. (A readable, intermediate text with an attractive format.)

8. D. Hughes-Hallet, et al. 2005. *Calculus: Single Variable*, 4th ed, John Wiley, New York. (Chapter 11 of this widely used calculus text is a good introduction to simple ideas in differential equations.)

9. W. Kelley and A. Peterson, 2010. *The Theory of Differential Equations: Classical and Qualitative*, 2nd ed., Universitext, Springer, New York. (An intermediate level text covering the theory of differential equations.)

10. J. H. Kwak & S. Hong, 2004. *Linear Algebra*, 2nd ed., Birkhauser, Boston. (A readable and thorough introduction to matrices and linear algebra.)

11. J. D. Logan, 2004. *Applied Partial Differential Equations*, 2nd ed., Springer-Verlag, New York. (A very brief treatment of boundary value problems written at an elementary level.)

12. J. D. Logan, 2006. *Applied Mathematics*, 3rd ed., Wiley-Interscience, New York. (An introduction to dimensional analysis and scaling, as well as to advanced techniques in differential equations, including regular and singular perturbation methods and bifurcation theory.)

13. J. D. Logan & W. R. Wolesensky, 2009. *Mathematical Methods in Biology*, John Wiley & Sons, New York. (A treatment of differential equations models in the biological sciences, including stochastic effects.)

14. C. Neuhauser 2009. *Calculus for Biology and Medicine*, 2nd ed., Pearson Education, Upper Saddle River, NJ. (A biology-motivated calculus text with chapters on matrices and differential equations.)

15. J. Polking, 2004. *pplane7* and *dfield7*, http://www.rice.edu/~polking. (Outstanding, downloadable MATLAB m-files for graphical solutions of DEs.)

16. L. E. Shampine, I. Gladwell, & S. Thompson, 2003. *Solving ODEs with MATLAB*, Cambridge University Press, Cambridge, UK. (One of several available texts that focuses on solving DEs using MATLAB®. It has a good discussion of the origins of the algorithms.)

17. S. H. Strogatz, 1994. *Nonlinear Dynamics and Chaos*, Addison-Wesley, Reading, MA. (A noted treatment of nonlinear dynamics.)

18. P. Waltman, 1986. *A Second Course in Elementary Differential Equations*, Academic, New York. (A classic, easily accessible, text on intermediate DEs.)

B
Computer Algebra Systems

There is great diversity in differential equations courses with regard to technology use, and there is equal diversity regarding the choice of technology. MATLAB®, Maple, and *Mathematica* are common computer environments used at many colleges and universities. MATLAB®, in particular, has become an important tool in scientific computation; Maple and *Mathematica* are computer algebra systems that are used for symbolic computation. There is also an add-on symbolic toolbox for the professional version of MATLAB®; the student edition includes the toolbox. In this appendix we present a list of useful commands in Maple and MATLAB®. The presentation is only for reference and to present some standard templates for tasks commonly faced in differential equations. It is not meant to be an introduction or tutorial to these environments, but only a statement of the syntax of a few basic commands. The reader should realize that these systems are updated regularly, so there is danger that the commands will become obsolete quickly as new versions appear.

Advanced scientific calculators also perform symbolic computation. Manuals that accompany these calculators give specific instructions that are not repeated here.

B.1 Maple

Maple has single automatic commands that perform most of the calculations and graphics used in differential equations. There are excellent Maple application manuals available, but everything required can be found in the help menu in the program itself. A good strategy is to find what you want in the help menu, copy and paste it into your Maple worksheet, and then modify it to conform to your own problem. Listed below are some useful commands for plotting solutions to differential equations, and for other calculations. The output of these commands is not shown; we suggest the reader type these commands in a worksheet and observe the results. There are packages that must be loaded before making some calculations: `with(plots):` `with(DEtools):` and `with(linalg):` In Maple, a colon suppresses output, and a semicolon presents output.

Define a function $f(t,u) = t^2 - 3u$:

```
f:=(t,u) → t^2-3*u;
```

Draw the slope field for the DE $u' = \sin(t-u)$:

```
DEplot(diff(u(t),t)=sin(t-u(t)),u(t),t=-5..5,u=-5..5);
```

Plot a solution satisfying $u(0) = -0.25$ superimposed upon the slope field:

```
DEplot(diff(u(t),t)=sin(t-u(t)),u(t),t=-5..5,
u=-5..5,[[u(0)=-.25]]);
```

Find the general solution of a differential equation $u' = f(t,u)$ symbolically:

```
dsolve(diff(u(t),t)=f(t,u(t)),u(t));
```

Solve an initial value problem symbolically:

```
dsolve({diff(u(t),t) = f(t,u(t)), u(a)=b}, u(t));
```

Plot the solution to: $u'' + \sin u = 0$, $u(0) = 0.5$, $u'(0) = 0.25$.

```
DEplot(diff(u(t),t$2)+sin(u(t)),u(t),t=0..10,
[[u(0)=.5,D(u)(0)=.25]],stepsize=0.05);
```

Euler's method for the IVP $u' = \sin(t-u)$, $u(0) = -0.25$:

```
f:=(t,u) → sin(t-u):
t0:=0: u0:=-0.25: Tfinal:=3:
n:=10: h:=evalf((Tfinal-t0)/n):
t:=t0: u=u0:
for i from 1 to n do
u:=u+h*f(t,u):
t:=t+h:
print(t,u);
od:
```

Set up a matrix and calculate the eigenvalues, eigenvectors, and inverse:

```
with(linalg):
A:=array([[2,2,2],[2,0,-2],[1,-1,1]]);
eigenvectors(A);
eigenvalues(A);
inverse(A);
```

Solve a linear algebraic system:

$A\mathbf{x} = \mathbf{b}$:

```
b:=matrix(3,1,[0,2,3]);
x:=linsolve(A,b);
```

Solve a linear system of DEs with two equations:

```
eq1:=diff(x(t),t)=-y(t):
eq2:=diff(y(t),t)=-x(t)+2*y(t):
dsolve({eq1,eq2},{x(t),y(t)});
dsolve({eq1,eq2,x(0)=2,y(0)=1},{x(t),y(t)});
```

A fundamental matrix associated with the linear system $\mathbf{x}' = A\mathbf{x}$:

```
Phi:=exponential(A,t);
```

Plot a phase diagram in two dimensions:

```
with(DEtools):
eq1:=diff(x(t),t)=y(t):
eq2:=2*diff(y(t),t)=-x(t)+y(t)-y(t)^3:
DEplot([eq1,eq2],[x,y],t=-10..10,x=-5..5,y=-5..5,
{[x(0)=-4,y(0)=-4],[x(0)=-2,y(0)=-2] },
arrows=line, stepsize=0.02);
```

Plot time series:

```
DEplot([eq1,eq2],[x,y],t=0..10,
{[x(0)=1,y(0)=2] },scene=[t,x],arrows=none,stepsize=0.01);
```

Laplace transforms:

```
with(inttrans):
u:=t*sin(t):
U:=laplace(u,t,s):
U:=simplify(expand(U));
u:=invlaplace(U,s,t):
```

Display several plots on same axes:

```
with(plots):
p1:=plot(sin(t), t=0..6): p2:=plot(cos(2*t), t=0..6):
display(p1,p2);
```

Plot a family of curves:

```
eqn:=c*exp(-0.5*t):
curves:={seq(eqn,c=-5..5)}:
plot(curves, t=0..4, y=-6..6);
```

Solve a nonlinear algebraic system: `fsolve({2*x-x*y=0,-y+3*x*y=0},{x,y},{x=0.1..5,y=0..4});`

Find an antiderivative and definite integral:

```
int(1/(t*(2-t)),t); int(1/(t*(2-t)),t=1..1.5);
```

B.2 MATLAB®

There are many references on MATLAB® applications in science and engineering. Among the best is Higham & Higham (2005). The MATLAB® files *dfield7.m* and *pplane7.m*, developed by J. Polking (2004), are two excellent programs for solving and graphing solutions to differential equations. These programs can be downloaded from his website (see references). In the table we list several common MATLAB® commands. We do not include commands from the symbolic toolbox. The package's "help" file contains a very complete reference with samples of all the commands.

An m-file for Euler's Method. For scientific computation we often write several lines of code to perform a certain task. In MATLAB®, such a code, or program, is written and stored in an *m-file*. The m-file below is a program of the Euler method for solving a pair of DEs, namely, the predator–prey system

$$x' = x - 2 * x^2 - xy, \quad y' = -2y + 6xy,$$

subject to initial conditions $x(0) = 1$, $y(0) = 0.1$. The m-file *euler.m* plots the time series solution on the interval $[0, 15]$.

```
function euler
x=1; y=0.1; xhistory=x; yhistory=y; T=15; N=200; h=T/N;
for n=1:N
u=f(x,y); v=g(x,y);
x=x+h*u; y=y+h*v;
xhistory=[xhistory,x]; yhistory=[yhistory,y];
end
t=0:h:T;
plot(t,xhistory,'-',t,yhistory,'--')
xlabel('time'), ylabel('prey (solid),predator (dashed)')
function U=f(x,y)
U=x-2*x.*x-x.*y;
function V=g(x,y)
V=-2*y+6*x.*y;
```

Direction Fields. The `quiver` command plots a vector field in MATLAB®.

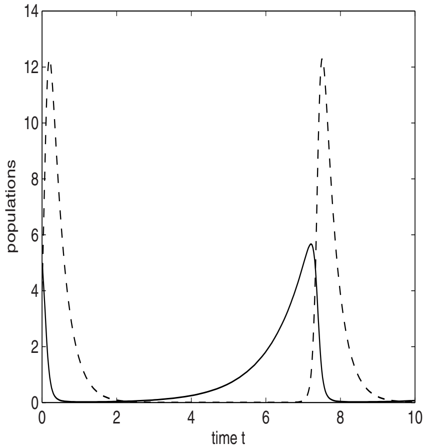

Figure B.1 Predator (dashed) and prey (solid) populations.

Consider the system

$$x' = x(8 - 4x - y), \quad y' = y(3 - 3x - y).$$

To plot the vector field on $0 < x < 3$, $0 < y < 4$ we use:

```
[x,y] = meshgrid(0:0.3:3, 0:0.4:4];
dx = x.*(8-4*x-y);  dy = y.*(3-3*x-y);
quiver(x,y,dx,dy)
```

Using the DE Packages. MATLAB® has several differential equations routines that numerically compute the solution to an initial value problem. To use these routines we define the DEs and calling routine in an m-file. The files below use the package `ode45`, which is a Runge–Kutta type solver with an adaptive step size. Consider the initial value problem

$$u' = 2u(1 - 0.3u) + \cos 4t, \quad 0 < t < 3, \quad u(0) = 0.1.$$

```
function diffeq
trange = [0  3]; ic=0.1;
[t,u] = ode45(@uprime,trange,ic);
plot(t,u,'*--')
```

We define the differential equation as follows:

```
function uprime = f(t,u)
uprime = 2*u.*(1-0.3*u)+cos(4*t);
```

Solving a System of DEs. As for a single equation, we write an m-file that calls the system of DEs. Consider the Lotka–Volterra model

$$x' = x - xy, \quad y' = -3y + 3xy,$$

with initial conditions $x(0) = 5$, $y(0) = 4$. Figure B.1 shows the time series plots.

```
function lotkatimeseries
tspan=[0 10]; ics=[5;4];
[T,X]=ode45(@lotka,tspan,ics);
plot(T,X)
xlabel('time t'), ylabel('populations')
function deriv=lotka(t,z)
deriv=[z(1)-z(1).*z(2); -3*z(2)+3*z(1).*z(2)];
```

Phase Diagrams. To produce phase plane plots we simply plot z(1) versus z(2). In the following example we draw two orbits. The calling portion of the m-file is:

```
function lotkaphase
tspan=[0 10]; ICa=[5;4]; ICb=[4;3];
[ta,ya]=ode45(@lotka,tspan, ICa);
[tb,yb]=ode45(@lotka,tspan, ICb);
plot(ya(:,1),ya(:,2), yb(:,1),yb(:,2))
```

Symbolic Solution. This script solves the logistic equation symbolically and plots the solution.

```
y=dsolve('Dy=r*y*(1-(1/K)*y)','y(0)=y0');
y=vectorize(y);
r=0.5; K=150; y0=15; t=0:.05:20; y=eval(y);
plot(t,y), ylim([0 K+10]), title('Logistic Growth')
xlabel('time (years)'), ylabel('Population')
```

To solve a system:

```
[x,y] = dsolve('Dx=r*x+4*y, Dy =4*x-3*y', 'x(0) = a, y(0) = b');
x=vectorize(x), y=vectorize(y);
a=1; b=3; r=1; t=1:.01:2;                                    1
x=eval(x); y=eval(y);
plot(t,x,t,y)
```

The command `vectorize` in the preceding scripts turns a symbolic solution into a vector solution that MATLAB® can evaluate and plot.

To plot a function defined by an integral:

```
clear all
f=inline('exp(-t)./t','t');
for n=0:20
t(n+1)=1+n/10;
u(n+1)=2+(quad(f,1,t(n+1))).∧2;
end
plot(t,u,xlabel('t'),ylabel('u(t)'))
```

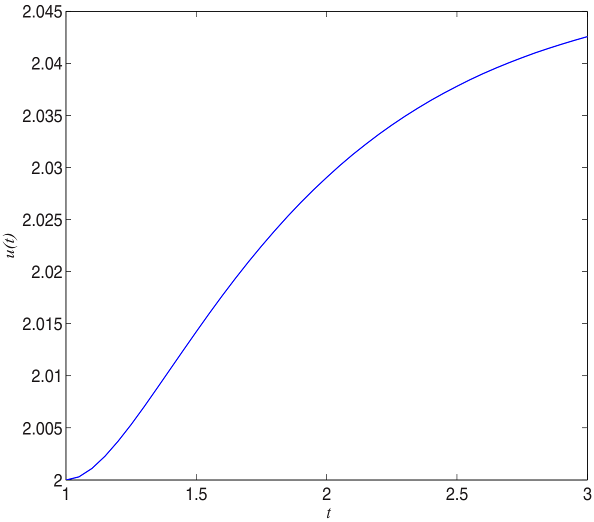

Figure B.2 Plot of $u(t) = 2 + \left(\int_1^t e^{-s}/s\, ds\right)^2$.

The following table contains several useful MATLAB® commands.

MATLAB® Command	Instruction
>>	command line prompt
;	semicolon suppresses output
clc	clear the command screen
Ctrl+C	stop a program
help *topic*	help on MATLAB *topic*
a = 4, A = 5	assigns 4 to a and 5 to A
clear a b	clears the assignments for a and b
clear all	clears all the variable assignments
x=[0,3,6,9,12,15,18]	row vector (list) assignment
x=0:3:18	defines the same vector as above
x=linspace(0,18,7)	defines the same vector as above
x'	transpose of x
+, -, *, /, ∧	operations with numbers
sqrt(a)	square root of a
exp(a), log(a)	e^a and $\ln a$
pi	the number π
.*, ./, .∧	operations on vectors of same length (with dot)
t=0:0.01:5, x=cos(t), plot(t,x)	plots $\cos t$ on $0 \le t \le 5$
xlabel('time'), ylabel('state')	labels horizontal and vertical axes
title('Title of Plot')	titles the plot
xlim([a b]), ylim([c d])	sets plot range on x and y axes
hold on, hold off	does not plot immediately; releases hold on
for n=1:N,...,end	syntax for a "for-end" loop from 1 to N
bar(x)	plots a bar graph of a vector x
plot(x)	plots a line graph of a vector x
A=[1 2; 3 4]	defines a matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
x=A\b	solves Ax=b, where b=[α;β] is a column vector
inv(A)	the inverse matrix
A'	transpose of a matrix
det(A)	determinant of A
[V,D]=eig(A)	computes eigenvalues and eigenvectors of A
q=quad(fun,a,b,tol);	Approximates $\int_a^b \text{fun}(t)dt$, tol = error tolerance
function fun=f(t), fun=t.∧ 2	defines $f(x) = t^2$ in an m-file

C
Practice Test Questions

Below are some sample questions on which students can assess their skills and review for exams.

Practice Exercises Chapters 1–2

1. Find the function $u = u(t)$ that solves the initial value problem $u' = (1+t^2)/t, \quad u(1) = 0$.

2. A particle of mass 1 moves in one dimension with *acceleration* given by $3 - v(t)$, where $v = v(t)$ is its velocity. If its initial velocity is $v = 1$, when, if ever, is the velocity equal to two?

3. Find $y'(t)$ if
$$y(t) = t^2 \int_1^t \frac{1}{r} e^{-r} \, dr.$$

4. Consider the autonomous equation
$$\frac{du}{dt} = -(u-2)(u-4)^2.$$
Find the equilibrium solutions, sketch the phase line, and indicate the type of stability of the equilibrium solutions.

5. Consider the initial value problem
$$u' = t^2 - u, \quad u(-2) = 0.$$
Use your calculator to draw the graph of the solution on the interval $-2 \le t \le 2$. Reproduce the graph on your answer sheet.

6. For the initial value problem in Problem 5, use the Euler method with stepsize $h = 0.25$ to estimate $u(-1)$.

7. For the differential equation in Problem 5, plot in the tu-plane the locus of points where the slope field has value -1.

8. At noon the forensics expert measured the temperature of a corpse and it was 85 degrees F. Two hours later it was 74 degrees. If the ambient temperature of the air was 68 degrees, use Newton's law of cooling to estimate the time of death. (Set up and solve the problem).

9. Consider the differential equation

$$\frac{du}{dt} = (t^2 + 1)u - t.$$

   a) In the tu plane sketch the graph of the of the set of points where the slope field is zero.

   b) Consider the initial value problem consisting of the differential equation (1) and the initial condition $u(1) = 3$. State precisely why you are guaranteed that the IVP has a unique solution in some small open interval containing $t = 1$.

10. Find two different solutions of the differential equation

$$t^2 u'' - 6u = 0$$

having the form $u(t) = t^m$. (That is, determine value(s) of m for which t^m is a solution.)

11. Find an explicit analytic formula for the solution to the initial value problem

$$\frac{du}{dt} = 2te^{-t^2}, \quad u(0) = 1.$$

12. Find the explicit solution to the initial value problem

$$tu\frac{du}{dt} - (2t^2 + 1) = 0, \quad u(1) = 4.$$

13. Solve the initial value problem

$$\frac{du}{dt} + \frac{2}{t}u = 3, \qquad u(1) = 5.$$

14. A roasting chicken at room temperature (70 deg) is put in a 325 deg oven to cook. The heat loss coefficient for chicken meat is 0.4 per hour. Set up an initial value problem for the temperature $T(t)$ of the chicken at time t. Set up only but do not solve.

15. Consider a population model governed by the autonomous equation

$$p' = \sqrt{2}\, p - \frac{4p^2}{1+p^2}.$$

   a) Sketch a graph of the growth rate p' versus the population p, and sketch the phase line.

   b) Find the equilibrium populations and determine their stability.

16. You are driving your truck, which has mass m, down the freeway at a constant speed of V_0 when you apply the brakes hard, exerting a constant stopping force of $-F_0$. How long does it take you to stop? (You *must* set up an initial value problem and solve it.)

17. An RC circuit with no emf has an initial charge of q_0 on the capacitor. The resistance is $R = 1$ and the capacitance is $C = 1/2$. Set up an initial value problem for the charge on the capacitor and solve to find $q = q(t)$.

18. An autonomous differential equation is given by

$$\frac{du}{dt} = (u^2 - 36)(a - u)^3,$$

where a is a fixed constant with $b > 12$.

   a) Find all equilibrium solutions and draw the phase line diagram. (Label all axes with "arrows" appropriately placed on the phase line.)

   b) Draw a rough graph of the solution curve $u = u(t)$ when the initial condition is $u(0) = 8$.

Practice Exercises Chapters 3–4

1. Find the general solution to the equation $u'' + 3u' - 10u = 0$.

2. A mass of 2 kg is hung on a spring with stiffness (spring constant) $k = 3$ N/m. After the system comes to equilibrium, the mass is pulled downward 0.25 m and then given an initial velocity of 1 m/sec. What is the amplitude of the resulting oscillation?

3. Find the general solution to the linear differential equation

$$u'' - \frac{1}{t}u' + \frac{2}{t^2}u = 0.$$

4. A particle of mass $m = 2$ moves on a u-axis under the influence of a force $F(u) = -au$, where a is a positive constant. Write down the differential equation that governs the motion of the particle and then write down the expression for conservation of energy.

5. Find the general solution $x = x(t)$ of the damped spring–mass equation

$$2x'' + x' + \frac{3}{32}x = 0.$$

6. In the previous problem, suppose there is a forcing term of magnitude $g(t) = 5t\cos 5t$. What is the form that the particular solution $x_p(t)$ takes? (Do not find the constants.)

7. The solution of a second-order, linear, homogeneous DE is $u(t) = 5+2e^{-10t}$. What is the equation?

8. A conservative mechanical system is governed by Newton's second law of motion (mass $\times$ acceleration = force):

$$2\frac{d^2x}{dt^2} = -xe^{-x^2}.$$

Find the potential energy $V(x)$ of this system for which $V(0) = 0$. Then write down the conservation of energy expression if $x(0) = 0$ and $x'(0) = 1$.

9. Using a graphing calculator, sketch the solution $u = u(t)$ of the initial value problem

$$u'' + u' - 3\cos 2t = 0, \quad u(0) = 1, \quad u'(0) = 0$$

on the interval $0 < t < 6$.

10. Consider the IVP

$$u' = \sqrt{1 + t + u}, \qquad u(1) = 7.$$

Use the modified Euler (predictor–corrector) method to approximate the value of $u(1.1)$. You may use your calculator, but show your work. Go out to 4 decimal places.

11. Transform the following nonlinear Bernoulli equation

$$u' + tu = \frac{1}{t^2u}$$

into a linear equation using a transformation of the dependent variable. DO NOT solve the linear equation.

12. An RCL circuit with no emf is governed by the circuit equation

$$Lq'' + Rq + \frac{1}{C}q = 0,$$

where $q = q(t)$ is the charge on the capacitor.

a) If the resistance is $R = 8$, shade the region in CL parameter space, or the CL plane (C is the horizontal axis, and L is the vertical) where the solution can be described as "oscillatory decay."

b) What is the decay rate?

c) If $R = 0$, what is the natural frequency of oscillation of the circuit? What is its period?

13. Find the Laplace transform of $u(t) = e^{-3t}h_2(t)$ using the integral definition of Laplace transform.

14. Find the inverse transform of

$$U(s) = \frac{1}{(s-5)^3}.$$

15. Use the convolution integral to solve the initial value problem

$$u'' + 6u = f(t), \quad u(0) = u'(0) = 0.$$

(Write down the correct integral.)

16. Solve the initial value problem

$$u' + 2u = \delta_a(t), \quad u(0) = 1,$$

where $\delta_a(t)$ is a unit impulse at some fixed time $t = a > 0$. Sketch a generic plot of the solution for $t \geq 0$.

Practice Exercises Chapters 4–6

1. Consider the system

$$x' = xy, \quad y' = 2y.$$

Find a relation between x and y that must hold on the orbits in the phase plane.

2. Consider the system

$$x' = 2y - x, \quad y' = xy + 2x^2.$$

Find the equilibrium solutions. Find the nullclines and indicate the nullclines and equilibrium solutions on a phase diagram. Draw several interesting orbits.

3. Consider the two-dimensional linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \mathbf{x}.$$

a) Find the eigenvalues and corresponding eigenvectors and identify the type of equilibrium at the origin.

b) Write down the general solution.

c) Draw a rough phase plane diagram, being sure to indicate the directions of the orbits.

4. Find the equation of the orbits in the xy plane for the system $x' = 4y, \quad y' = 2x - 2$.

5. For the following system, for which values of the constant b is the origin an unstable spiral?

$$x' = x - (b+1)y$$
$$y' = -x + y.$$

6. Consider the nonlinear system

$$x' = x(1 - xy),$$
$$y' = 1 - x^2 + xy.$$

a) Find all the equilibrium solutions.

b) In the xy plane plot the x and y nullclines.

7. Find a solution representing a linear orbit of the three-dimensional system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{x}.$$

8. Classify the equilibrium as to type and stability for the system

$$x' = x + 13y, \quad y' = -2x - y.$$

9. A two-dimensional system $\mathbf{x}x' = A\mathbf{x}$ has eigenpairs

$$-2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

a) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find a formula for $y(t)$ (where $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

b) Sketch a rough, but accurate, phase diagram.

10. Consider the IVP

$$x' = -2x + 2y$$
$$y' = 2x - 5y,$$
$$x(0) = 3, \quad y(0) = -3.$$

   a) Use your calculator's graphical DE solver to plot the solution for $t > 0$ in the xy-phase plane.

   b) Using your plot in (a), sketch $y(t)$ versus t for $t > 0$.

11. Consider

$$x' = 5x - y, \quad y' = -4x - py.$$

For which values of p is the origin a saddle point?

12. In the xy phase plane, plot the orbit

$$x(t) = 2e^{-t},$$
$$y(t) = -e^{-2t}, \quad -\infty < t < \infty.$$

13. For the the system

$$x' = -2x + 4y,$$
$$y' = -5x + 2y,$$

sketch a few of the orbits in the phase plane.

14. The general solution of a linear system is

$$x(t) = c_1e^{-7t} + c_2e^{-2t},$$
$$y(t) = -c_1e^{-7t} + \frac{1}{4}c_2e^{-2t}.$$

State the type and stability of the equilibrium $(0, 0)$, and then draw the linear orbits. Draw on your diagram a few other orbits, indicating exactly their behavior as they enter the origin.

Practice Final Examination 1

1. Find the general solution of the DE $u'' = u' + \frac{1}{2}u$.
2. Find a particular solution to the DE $u'' + 8u' + 16u = t^2$.

3. Find the (implicit) solution of the DE

$$u' = \frac{1+t}{3tu^2+t}$$

that passes through the point $(1, 1)$.

4. Consider the autonomous system $u' = -u(u-2)^2$. Determine all equilibria and their stability. Draw a rough time series plot (u versus t) of the solution that satisfies the initial condition $x(0) = 1$.

5. Consider the nonlinear system

$$x' = 4x - 2x^2 - xy, \quad y' = y - y^2 - 2xy.$$

Find all the equilibrium points and determine the type and stability of the equilibrium point $(2, 0)$.

6. An RC circuit has $R = 1$, $C = 2$. Initially the voltage drop across the capacitor is 2 volts. For $t > 0$ the applied voltage (emf) in the circuit is $b(t)$ volts. Write down an IVP for the *voltage* across the capacitor and find a formula for it.

7. Solve the IVP

$$u' + 3u = \delta_2(t) + h_4(t), \quad u(0) = 1.$$

8. Use eigenvalue methods to find the general solution of the linear system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \mathbf{x}.$$

9. In a recent TV episode of *Miami: CSI*, Horatio took the temperature of a murder victim at the crime scene at 3:20 A.M. and found that it was 85.7 degrees F. At 3:50 A.M. the victim's temperature dropped to 84.8 degrees. If the temperature during the night was 55 degrees, at what time was the murder committed? Note: Body temperature is 98.6 degrees; work in hours.

10. Consider the model $u' = \lambda^2 u - u^3$, where λ is a parameter. Draw the bifurcation diagram (equilibria solutions versus the parameter) and determine analytically the stability (stable or unstable) of the branch in the first quadrant.

11. Consider the IVP $u'' = \sqrt{u+t}$, $u(0) = 3$, $u'(0) = 1$. Pick step size $h = 0.1$ and use the modified Euler method to find an approximation to $u(0.1)$.

12. A particle of mass $m = 1$ moves on the x-axis under the influence of a potential $V(x) = x^2(1-x)$.

a) Write down Newton's second law, which governs the motion of the particle.

b) In the phase plane, find the equilibrium solutions. If one of the equilibria is a center, find the type and stability of all the other equilibria.

c) Draw the phase diagram.

Practice Final Examination 2

1. Classify the type and stability of the equilibrium of the system

$$x' = -2x + y,$$
$$y' = -2x.$$

In a phase plane, draw in the nullclines (as dashed lines) and indicate which is which. Then, noting the direction field along the x axis, sketch in a couple of sample orbits.

2. A mass of $m = 1$ gm is subjected to a positive force proportional to the square root of the velocity; the initial velocity is 3 cm/sec. Find the velocity as a function of time and sketch a time series plot for $t \geq 0$.

3. Find two independent solutions of the differential equation

$$\frac{d^2y}{dt^2} + \frac{4}{t}\frac{dy}{dt} + \frac{2}{t^2}y = 0$$

of the form $y = t^\lambda$, where λ is to be determined.

4. Consider a damped spring–mass system where $x = x(t)$ is the displacement of the mass from equilibrium. Let m, c, and k denote the mass, damping constant, and spring constant, respectively.

a) If there is no damping and there is a external forcing function of magnitude $3\cos 5t$, what is the relationship between the mass m and spring constant k for which pure resonance occurs?

b) If $c = 2$ and $k = 0.1$ and there is no external forcing, what values of the mass m will lead to damped oscillations?

5. Consider the initial value problem

$$u' = 0.5u\left(1 - \frac{u}{t+10}\right), \quad u(5) = 3.$$

Use the Euler algorithm (method) to approximate the solution at $t = 5.1$.

6. Consider the autonomous equation

$$\frac{dp}{dt} = (p - h)(p^2 - 2p), \quad h > 0.$$

Clearly, $p = h$ is an equilibrium. Use an analytic equilibrium criterion, or whatever, to determine the values of h for which the equilibrium is unstable.

7. Solve the initial value problem using Laplace transforms:

$$u' + 2u = e^{-t} h_3(t), \quad u(0) = 0.$$

8. Find the general solution of the fourth-order differential equation

$$u'''' + 4u'' = 0.$$

9. Find the particular solution of

$$u'' + u = 7 + 6e^t.$$

10. Find the solution of the initial value problem

$$y' - \frac{2}{t+1} y = (t+1), \ y(0) = 3.$$

11. Find the inverse transformation of

$$U(s) = \frac{s}{(s^2 - 10)(s - 5)}$$

using convolution. Write down the appropriate convolution integral, but do not calculate it.

12. Lizards, like other reptiles, are cold-blooded. A small lizard, whose body temperature is 50 deg, comes out from under a rock into an environment with temperature 70 deg. Furthermore, through solar radiation the sun heats its body at the rate of $q(t) = 1$ deg per minute. The heat loss/gain coefficient of the lizard is h, given in per minute. Very carefully think about the model and answer the following questions.

   a) Set up an initial value problem whose solution would give the body temperature $T(t)$ of the lizard for all times $t \geq 0$. (Be sure to explain what you are doing. Of course, your model will contain the parameter h.)

   b) Find the general solution of the differential equation in part (a) using any correct method. You must show your work.

   c) From the general solution, or otherwise, determine the value of h if the long time equilibrium temperature of the lizard is 90 deg. Show your reasoning and work.

D

Solutions and Hints to Selected Exercises

This appendix contains hints and partial solutions to most of the even-numbered problems. Plots are not included, but enough information is often given to construct the required graph.

CHAPTER 1

Section 1.1

2. Both $u(t) = 1/t$ and $u(t) = 1/(t-2)$ are solutions.

4. Substitute into the differential equation and equate like coefficients.

6. Substitute into the differential equation and obtain the quadratic $m(m-1) - 6 = 0$, giving $m = -2, 3$. Therefore t^{-2} and t^3 are solutions.

8. The solution to $u' = -ku$ is $u(t) = u_0 e^{-kt}$. If $u(t) = 0.5u_0 e^{-kt}$, then $k = (\ln 2)/t_{1/2}$ is the relation between k and the half-life $t_{1/2}$. If $t_{1/2} = 5730$, then $k = 0.000121$ per year. If $u(t) = 0.2u_0$, then the solution gives $0.2 = e^{-kt}$, then $t = -(\ln 0.2)/k = 13,301$ years,

10. If $\ln T = -at + b$, then $T' = -aT$, which is Newton's law of cooling with environment temperature zero and heat loss coefficient a. From the given data, $\ln 8 = -2a + b$ and $\ln 22 = -0 \cdot a + b = b$. Then $b = \ln 22$ and $a = (\ln 22 - \ln 8)/2$. When $T = 2$, then $t = (\ln 2 - b)/a$.

12. We want to find T_e. We are given $T_0 = 46$. Then, from Newton's law of cooling, $T(t) = (46 - T_e)e^{-ht} + T_e$. Therefore, $39 = (46 - T_e)e^{-10h} + T_e$ and $33 = (46 - T_e)e^{-20h} + T_e$. These two equations determine h and T_e. For example, solve each equation for T_e and equate to obtain a single equation for h, which then can be solved using a "solver" routine on a calculator.

14. Let 1:00 P.M. correspond to $t = 0$. Substituting the initial and environmental temperatures, Newton's law of cooling has solution $T(t) = 58e^{-ht} + 10$. At 1:00 P.M., or $t = 9$, we have $57 = 58e^{-9h} + 10$. Solving for h gives $h = 0.023$. Then, at $t = 17$, we have $T(17) = 58e^{-17(.023)} + 10 \approx 49$ degrees.

16. (b) Setting $T' = 0$ we get $q - k(T - T_e) = 0$ or $T = T_e + q/k$ as the limiting temperature. (c)–(d) Setting $u = q - k(T - T_e)$, we get $u' = -kT'$. Substituting into the differential equation yields an equation for u, namely, $-(mc/k)u' = u$, or $u' = -(k/mc)u$, which is the decay equation. The solution is $u(t) = u(0)\exp(-kt/mc)$, where $u(0) = q - k(T(0) - T_e)$. Now write the solution in terms of T using $T(t) = (q + kT_e - u(t))/k$.

Section 1.2

2. We have $u' = C$, and so $tu' - u + f(u') = tC - (Ct + f(C)) + f(C) = 0$.

4. Here $f(t, u) = (t^2 + 1)u - t$ and $\partial f/\partial u = t^2 + 1$ is continuous for all t and u in the plane.

6. Here $f(t, u) = \ln(t^2 + u^2)$ is continuous for all $(t, u) \neq (0, 0)$. So a solution exists in a small interval for all initial conditions $(t_0, u_0) \neq (0, 0)$. For uniqueness, we need $\partial f/\partial u = 2u/(t^2 + u^2)$ continuous. Again, $(t_0, u_0) \neq (0, 0)$.

8. We have $u' = p(t)u + q(t)$. If u_1 and u_2 are two solutions, then $u_1' = p(t)u_1 + q(t)$, $u_2' = p(t)u_2 + q(t)$. But $(u_1 + u_2)' \neq p(t)(u_1 + u_2)u + q(t)$. So, the sum of solutions is not a solution. Is a constant times a solution again a solution? No, because $cu' = c(p(t)u + q(t)) \neq p(t)(cu) + q(t)$. If $q(t) = 0$, both these statements are true. If u_1 is a solution to $u' = p(t)u$ and u_2 is a solution to $u' = p(t)u + q(t)$, then $(u_1 + u_2)' = p(t)u_1 + p(t)u_2 + q(t) = p(t)(u_1 + u_2) + p(t)$.

10. By the hint,

$$\frac{d}{dt}((u')^2 - u^2) = 2u'u'' - 2uu' = 2u(u'' - u) = 0.$$

Therefore $(u')^2 - u^2$ must be constant. The curves $(u')^2 - u^2 = C$ plot as a family of hyperbolas in the uu' plane; that is, for each $C \neq 0$ we obtain an opposing pair of hyperbolas. When $C = 0$ we get the two straight lines $u' = u$ and $u' = -u$.

12. Note that

$$u(t) = \begin{cases} at^2 + 1, & t < 0, \\ bt^2 + 1, & t > 0, \end{cases}$$

is continuous at $t = 0$ for any constants a and b (the one-sided limits are equal). The derivative is

$$u'(t) = \begin{cases} 2at, & t < 0, \\ 2bt, & t > 0, \end{cases}$$

Therefore u' is continuous at $t = 0$. It is easy to check that $u(t)$ satisfies the differential equation and $u(0) = 1$. The right side of the differential equation $f(t, u)$ is not continuous at $t = 0$, and neither is its u-derivative. But this does not mean a solution does not exist. The theorem states that if f and f_u are continuous, there is a solution. Here we have the converse; if there is a solution that does not mean f and f_u are continuous.

Section 1.2.1

2. The isoclines are $u^2 + t^2 = C$, $C > 0$, which are circles. So, the slope is the same on each circle.

4. The isoclines are $t - u^2 = C$, which are parabolas opening to the right. The slope field is positive when $t - u^2 > 0$, which is the region to the right of the parabola $t = u^2$. In the region to the left, the slope field is negative.

Section 1.3

2. We have

$$u(t) = \int \frac{t+1}{\sqrt{t}} dt = \int (t^{1/2} + t^{-1/2}) dt = (2/3)t^{3/2} + 2t^{1/2} + C.$$

Next, $u(1) = 4$ gives $2/3 + 2 + C = 4$, or $C = 4/3$.

4. We have $u(t) = \int te^{-2t}dt + C$. The integral can be done using integration by parts. Let $w = t$ and $dv = e^{-2t}$; then $dw = dt$ and $v = -\frac{1}{2}e^{-2t}$. Then

$$u(t) = \int te^{-2t}dt + C = -\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2t}dt + C = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C.$$

6. Here,

$$u(t) = \int \frac{\cos\sqrt{t}}{\sqrt{t}} dt = 2\int \cos w dw = 2\sin w + C = 2\sin\sqrt{t} + C.$$

We made the substitution $w = \sqrt{t}$, $dw = 1/2\sqrt{t}$.

8. Letting $u = ye^{3t}$ gives $u' = 3ye^{3t} + y'e^{3t}$. Substituting into the DE and simplifying yields an equation for y, namely, $y' = e^{-4t}$. Integrating, $y = -(1/4)e^{-4t} + C$. Therefore,

$$u(t) = -\frac{1}{4}e^{-t} + Ce^{3t}.$$

10.

$$\frac{d}{dt}\mathrm{erf}(\sin t) = \mathrm{erf}'(\sin t)\cos t = \frac{2}{\sqrt{\pi}}e^{-\sin^2 t}\cos t.$$

12. Write the integral equation as

$$u(t) + e^{-pt}\int_0^t e^{ps}u(s)ds = A.$$

Take the derivative, using the product rule on the second term; use the fundamental theorem of calculus on the integral. Then,

$$u'(t) + u(t) - pe^{-pt}\int_0^t e^{ps}u(s)ds = 0.$$

Using the integral equation, we get

$$u' + (1+p)u + Ap = 0.$$

14. Integrate both sides of the differential equation from 0 to t and use the fundamental theorem of calculus to compute the left side. We get

$$\int_0^t u'(s)ds = u(t) - u(0) = \int_0^t (5su(s) + 1)ds,$$

with $u(0) = 0$.

Section 1.4

2. Substitute the given expression into the equation and equate the coefficients of like terms to get $\lambda = -c/2m$ and $\omega = \sqrt{4mk - c^2}/2m$. The amplitude A is arbitrary.

4. Taking the derivative of the conservation law gives

$$\frac{d}{dt}\left[\frac{1}{2}l(\theta')^2 + g(1-\cos\theta)\right] = 0,$$

Use the chain rule to get

$$\frac{d}{dt}\left((\theta')^2\right) = 2\theta'\theta'',$$

and

$$\frac{d}{dt}\cos\theta = -(\sin\theta)\theta'.$$

Then simplify to get the equation of motion.

6. For small θ, the graphs of θ and $\sin\theta$ are nearly the same. And, θ is the first-term approximation of $\sin\theta$ in its Taylor expansion. (a) By substitution into the differential equation, we find $\omega = \sqrt{g/l}$.

   (b) From the last part, small displacements satisfy $\theta(t) = A\cos\sqrt{g/l}t$. Setting $\theta(t) = 0$ gives $\cos\sqrt{g/l}t = 0$, or $\sqrt{g/l}\,t = \pi/2$. Here, $l = 20$ and $g = 9.8$. Then, $t = 2.2$ sec. Note that the displacement does not depend on mass.

Section 1.5

2. (b) Separating variables, $e^{2u}du = dt$. Integrating,

$$\frac{1}{2}e^{2u} = t + C$$

   Therefore,

$$u(t) = \frac{1}{2}\ln|2t + C|.$$

   Evaluating at $t = 0$ and using the initial condition gives $C = e^2$.

4. Separate variables and integrate to get $x(t) = 1/(C - t^2)$. The initial condition gives $C = 1$, so $x(t) = 1/(1 - t^2)$. The maximum interval of existence is $-1 < t < 1$.

6. We have

$$\frac{1}{u(4+u)} = \frac{a}{u} + \frac{b}{4+u} = \frac{4a + (a+b)u}{u(4+u)}.$$

   Therefore, equating both sides, $a = \frac{1}{4}$ and $b = -\frac{1}{4}$. The differential equation becomes, therefore, upon separating variables and integrating,

$$\int \frac{1}{u(4+u)}du = \frac{1}{4}\int \left\{\frac{1}{u} - \frac{1}{4+u}\right\} = t + C.$$

   Then,

$$\frac{1}{4}\ln\left(\frac{u}{u+4}\right) = t + C.$$

   Then,

$$\frac{u}{u+4} = e^{4t+C},$$

   and you can solve for u.

8. Separate variables to get

$$\frac{\ln u}{u}du = (4 + 2t)dt.$$

Integrating (in the left integral make the substitution $w = \ln u$) to get

$$\frac{1}{2}(\ln u)^2 = 4t + t^2 + C.$$

Now $u(0) = e$ gives $C = 1/2$. Hence,

$$\ln u(t) = \sqrt{8t + 2t^2 + 1}, \quad u(t) = \exp(\sqrt{8t + 2t^2 + 1}).$$

The solution exists as long as $8t + 2t^2 + 1 > 0$, which is valid for $t \geq (-8 + \sqrt{56})/4$.

10. Separating variables and integrating gives the general solution

$$u(t) = 1 + (t^2 + C)^3.$$

Clearly, no value of C gives $u(t) = 1$.

12. Integrate both sides of the allometric equation to get

$$\ln |u_1| = \ln |u_2|^a + \ln C,$$

where we have written the arbitrary constant as $\ln C$. Now, exponentiate to get the stated result.

14. Integrate both sides to get, using the fundamental theorem of calculus,

$$ue^{2t} = -e^{-t} + C, \quad u(t) = -e^{-3t} + Ce^{-2t}.$$

The initial condition $u(0) = 3$ gives $C = 4$.

16. The equation is $u'/u = -at$. Integrating and solving for u gives

$$u(t) = Ce^{-at^2/2} = 100e^{-(0.2)t^2/2},$$

which is easily plotted (a bell-shaped type curve). The maximum can be found by setting $u'(t) = 0$.

18. If u is the thickness, then $u' = a/u$, $u(0) = 0.05$. Separate variables to get $udu = adt$. Integrating and solving for u gives $u(t) = \sqrt{2at + C}$. Use the initial condition to determine $C = 0.0025$. Then use $u(4) = 0.075$ to get a. This gives the formula for the thickness at any time t, in particular, $t = 10$.

20. (a) Separate variables to get $du/u = p(t)dt$. Integrate to get

$$\ln |u| = \int_0^t p(s)ds + C_1,$$

or

$$u = Ce^{\int_0^t p(s)ds}.$$

(b) Solve the problem separately on each subinterval, and require equality (continuity) at $t = 1$.

Section 1.6

2. (a) Setting $(1-x)\left(1-e^{-2x}\right) = 0$ gives $x = 1$, $x = 0$. There are two equilibria. (c) Setting $3u/(1+u^2) = 0$ gives the quadratic equation $u^2 - 3u + 1 = 0$, which has two roots $u = 3/2 \pm \sqrt{5}2$.

4. Settting $N' = f(N) = rN\left(1-(N/K)^\theta\right) = 0$ gives equilibria $N = 0$ and $N = K$. To check stability, we find

$$f_N(N) = rN\left(-\frac{\theta}{K}\left(\frac{N}{K}\right)^{\theta-1}\right) + r\left(1-\left(\frac{N}{K}\right)^{\theta}\right).$$

Therefore $f_N(0) = r > 0$ and $f_N(K) = -r\theta < 0$. Thus $N = 0$ is unstable and $N = K$ is stable.

8. Let L be the length and $m = \rho L^3$ be the mass, where ρ is the density. Then the rate of change of mass m is

$$(\rho L^3)' = \alpha L^2 - \beta L^3 \text{ or } 3\rho L^2 L' = \alpha L^2 - \beta L^3.$$

Dividing by $3\rho L^2$,

$$L' = a - bL, \quad a = \frac{\alpha}{3\rho}, \quad b = \frac{\beta}{3\rho}.$$

The equilibrium, or limiting length, is $L_\infty = a/b$. If $L(0) = 0$, then $L(t)$ increases and approaches L_∞, as a phase line would show. It is clearly stable. To solve, separate variables to get

$$\frac{dL}{bL-a} = -dt, \quad \text{or} \quad \frac{1}{b}\ln|bL-a| = -t + C.$$

Solving for L,

$$L(t) = \frac{a}{b}\left(1+e^{-bt}\right).$$

This is a good model for growth, and many plants and animals follow this pattern.

12. Setting $R' = f(R) = -rR\ln(R/k) = 0$ gives $R = k$. Notice that the equation is not defined at $R = 0$, so $R = 0$ is not technically an equilibrium. (However, $R' \to 0$ as $R \to 0$.) To check stability, note that $f_R(k) = -a < 0$, and therefore $R = k$ is stable. To solve, we separate variables and integrate to get

$$\int \frac{dR}{R\ln(R/k)} = -at + C.$$

Using the substitution $w = \ln(R/k)$, $dw = (1/R)dR$, we get

$$\int \frac{dw}{w} = -at + C \quad \text{or} \quad w = Ce^{-at}.$$

Then,

$$R(t) = k \exp\left(Ce^{-at}\right).$$

14. We have $I' = aSI$ or $I' = aI(N - I)$. This is basically the same form as the logistic equation. The equilibria are $I = 0$ and $I = N$, the entire population; $I = N$ is stable, so everyone eventually gets the disease. $I = 0$ is unstable. The number of infectives increases gradually up to the limit $I = N$.

16. Separate variables and write the equation as

$$\frac{dv}{1 - b^2v^2} = gdt, \quad b^2 = \frac{a}{mg}.$$

The denominator on the left factors into $(1 - bv)(1 + bv)$; therefore we perform a partial fraction expansion and find

$$\frac{1}{(1 - bv)(1 + bv)} = \frac{1/2}{1 - bv} - \frac{1/2}{1 + bv}.$$

Now we have

$$\frac{1}{2}\int\left(\frac{dv}{1 - bv} - \frac{dv}{1 + bv}\right) = gt + C.$$

Carrying out the integrations on the left, we find

$$\ln|(1 - bv)(1 + bv)| = -2bgt + C.$$

Applying the condition $v(0) = 0$, we get $C = 0$. Then

$$|1 - b^2v^2| = e^{-2bgt},$$

from which the solution can be found.

Section 1.7

2. (b) We have $u' = f(u) = u^3(3 - u) = 0$ when $u = 0$ and $u = 3$; these are the equilibria. A plot of $f(u)$ versus u instantly leads to the phase line and the issue of stability. To analytically check stability, we have $f_u(u) = -u^3 + 3u^2(3 - u)$, so $f_u(0) = 0$, which must be checked further, and $f_u(3) = -27 < 0$, so $u = 3$ is stable. Regarding $u = 0$, note that $f_u(u) > 0$ for u in a small neighborhood of $u = 0$, $u \neq 0$, so $u = 0$ is unstable.

   (f) Setting $u' = f(u) = -(1+u)(u^2-4) = 0$ we get equilibria $u = -1, -2, 2$. Now, $f_u(u) = -2u(1 + u) - (u^2 - 4)$. Then $f_u(-1) > 0$, and $u = -1$ is unstable; $f_u(-2) < 0$, so $u = -2$ is stable; $f_u(2) < 0$, so $u = 2$ is stable.

4. Clearly, $x = 0$ is the only equilibrium, and $f(x) = x/(x^2 + 1) > 0$ if $x > 0$, and $f(x) < 0$ if $x < 0$. Therefore, $x = 0$ is unstable. (Or, you could use the instability condition $f_x(0) > 0$.)

8. For $u' = u^3 - u + h$, we find equilibria graphically by setting $h = u - u^3 = u(1 - u^2)$ and plotting h versus u. The bifurcation diagram is found by rotating the graph to obtain the plot of u versus h. Note that $f_u(u) = 3u^2 - 1$; the stability of each segment of the bifurcation diagram may be found using the stability conditions.

10. Because $x' = f(x) = ax^2 - 1$, the equilibria are given by $x = \pm 1/\sqrt{a}$. There are only equilibria when $a > 0$. To check stability, $f_x(x) = 2ax$, so $f_x(1/\sqrt{a}) = 2a/\sqrt{a} > 0$; thus the upper branch is unstable. Similarly, $f_x(-1/\sqrt{a}) = -2a/\sqrt{a} < 0$, so the lower branch is stable.

12. We can write $N' = f(N) = (h+1)[(h-1)N+1]$. Assume $h \neq 0$; otherwise, every constant solution is an equilibrium. We have equilibria $N = 1/(1-h)$, which plots on a bifurcation diagram (N versus h) as two hyperbolas with vertical asymptote $h = 1$. Note that $f_N(N) = h^2 - 1$. Then, if $h > 1$ or $h < -1$, we have $f_N(1/(1-h)) > 0$ and we have stability; if $0 < h < 1$, then $f_N(1/(1-h)) < 0$, which gives stability.

Section 1.8.1

2. The equation is $100C' = (0.0002)(0.5) - 0.5C$, with $C(0) = 0$. The equilibrium is found by setting $C' = 0$, or $C = 0.0002$. It is stable. We can rewrite the DE as $C' = 10^{-6} - 0.005C$. By separating variables, we find the general solution
$$C(t) = Ae^{-0.005t} + 0.0002.$$
We have $C(0) = 0$, so $A = -0.0002$.

4. The initial value problem is $1000C' = -2C$, $C(0) = 5/1000 = 0.005$. This is the decay equation with solution
$$C(t) = 0.005e^{-0.002t}.$$

6. The equilibrium $C^* = (-q + \sqrt{q^2 + 4kqVC_{\text{in}}})/2kV$ is stable.

8. The initial value problem is $C' = -rC$, $C(0) = C_0$. The solution is $C(t) = C_0e^{-rt}$. Therefore, $C_0 = 0.9C_0e^{-rT}$. Therefore, the residence time is $T = -(\ln 0.9)/r$.

10. (b) Set the equations equal to zero and solve for S and P. (c) With values from part (b), maximize aVP_e.

Section 1.8.2

2. The equation is $Rq' + (1/C)q = E$. Write this in separated form as
$$\frac{Rdq}{q - CE} = -\frac{1}{C}dt.$$

Integrating,

$$R \ln |q - CE| = -\frac{1}{C}t + K_1.$$

Exponentiate to get

$$|q - CE| = Ke^{-t/RC}, \quad K = K_1/R.$$

Thus

$$q = CE + Ke^{-t/RC}.$$

Using $q(0) = q_0$ gives $K = q_0 - CE$, and hence the solution to the initial value problem.

4. $LCV_c'' + RCV_c' + V_c = E(t)$.

6. Substitute $q = A\cos\omega t$ into $Lq'' + (1/C)q = 0$ to get $\omega = 1/\sqrt{LC}$, A arbitrary.

CHAPTER 2

Section 2.1

2. The integrating factor is e^t. Multiplying by this the equation becomes $(e^t u)' = e^{2t}$. Integrating gives

$$e^t u = \frac{1}{2}e^{2t} + C \text{ or } u(t) = \frac{1}{2}e^t + Ce^{-t}.$$

4. The integrating factor is e^{t^2}. Multiplying the equation by this factor gives $(ue^{t^2})' = 1$. Integrating,

$$ue^{t^2} = t + C \quad \text{or} \quad u(t) = te^{-t^2} + Ce^{-t^2}.$$

6. For example, in Exercise 4 the homogeneous solution is $u_h(t) = Ce^{-t^2}$ and the particular solution is $u_p(t) = te^{-t^2}$.

8. The integrating factor is

$$e^{\int(-1/t)dt} = e^{-\ln t} = \frac{1}{t}.$$

Multiplying by $1/t$ gives $(R/t)' = e^{-t}$, or $R/t = e^{-t} + C$. Thus $R(t) = te^{-t} + Ct$. The limit as $t \to 0$ is zero.

10. The general solution is

$$V(t) = \left(3\int te^t dt + C\right)e^{-t}.$$

The integral can be carried out using integration by parts.

12. The integrating factor is $\exp(-t^2)$. Therefore, $(u\exp(-t^2))' = \exp(-t^2)$. Integrating gives

$$ue^{-t^2} = \int_0^t e^{-s^2} ds + C = \frac{\sqrt{\pi}}{2}\text{erf}(t) + C.$$

Multiplying by $\exp(t^2)$ gives $u(t)$.

14. The integrating factor is e^{-pt}. The general solution is

$$u(t) = e^{pt}\int_{t_0}^t q(s)e^{-ps} ds + u_0 e^{pt}.$$

16. The quantities S, M, and A are in dollars, and a and r are in units of "per month". Setting $S' = 0$ in the equation gives

$$-aS + rA\frac{M-S}{M} = 0 \text{ or } S = \frac{rA}{a + rA/M}.$$

18. The initial value problem simplifies to

$$T' + 3T = 27 + 30\cos 2\pi t.$$

The integrating factor is $\exp 3t$ and we obtain, after multiplying by $\exp 3t$ and integrating,

$$T(t) = 9 + 3e^{-3t}\int e^{3t}\cos(2\pi t)\, dt + Ce^{-3t}.$$

The integral can be done using integration by parts, or using software.

20. We break up the differential equation over two intervals:

$$S' = -bS + rA, \quad 0 < t < T, \quad b \equiv a + \frac{rA}{M},$$

and

$$S' = -aS, \quad t > T.$$

The initial condition $S(0) = S_0$ applies to the first equation; the initial condition for the second equation is the value $S(T)$ obtained from solving the first equation. The solution to the first equation is

$$S(t) = \left(S_0 - \frac{rA}{b}\right)e^{-bt} + \frac{rA}{b}, \quad 0 \le t \le T.$$

and therefore

$$S(T) = \left(S_0 - \frac{rA}{b}\right)e^{-bT} + \frac{rA}{b}.$$

The solution to the equation in $t > T$ is $S(t) = Ce^{-at}$. So, $S(T) = Ce^{-aT}$. Therefore,

$$S(t) = S(T)e^{-a(t-T)}, \quad t \ge T,$$

where $S(T)$ is given above.

22. The DE for $S(t)$ is

$$S' = -\frac{E+I}{P}S + I.$$

(a) The long-time solution is the equilibrium $S_e = PI/(E+I)$. (b) The equation is first order and linear, so the solution is

$$S(t) = \left(S_0 - \frac{PI}{E+I}\right)e^{-(E+I)t/P} + \frac{PI}{E+I}.$$

(c) Compare the equilibria for two different values of E, one for the large island and one for the small island.

24. Letting $y = u^{1-n}$ we have $y' = (1-n)u^{-n}u'$. So, the DE becomes

$$\frac{u^n}{1-n}y' = a(t)u + g(t)u^n.$$

Multiplying $(1-n)u^{-n}$ gives the stated result.

26. The logistic equation is

$$u' = ru - \frac{r}{K}u^2,$$

which is a Bernoulli equation. Make the transformation $y = u^{1-2} = 1/u$. So, $y' = (-1/u^2)u'$. The DE becomes

$$y' = -ry + \frac{r}{K},$$

having solution

$$y = Ce^{-rt} + \frac{1}{K}.$$

Therefore, $u(t) = 1/(Ce^{-rt} + 1/K)$. Use $u(0) = u_0$ to obtain C. Finally,

$$u(t) = \frac{Ku_0}{(K-u_0)e^{-rt} + u_0}.$$

28. The integrating factor is $e^{P(t)}$ where

$$P(t) = \int_0^t e^{-s}/s\,ds.$$

Multiplying the DE by $e^{P(t)}$ and integrating gives

$$u(t) = e^{-P(t)}\int_0^t se^{P(s)}ds + Ce^{-P(t)}.$$

Using $u(0) = 1$ gives, because $P(0) = 0$, the arbitrary constant $C = 1$.

30. The larva equation, linear and first order, has solution

$$L(t) = Ce^{-(\mu_0+\mu)t} + \frac{\lambda}{\mu_0 + \mu}, \quad C = -\frac{\lambda}{\mu_0 + \mu}.$$

Substituting into the M equation gives

$$M' + \delta M = \frac{\mu\lambda}{\mu_0 + \mu}\left(1 - e^{-(\mu_0+\mu)t}\right).$$

This is first order and linear with integrating factor $\exp(\delta t)$, and it can be solved by the standard method.

Section 2.2.1

2. The Picard iteration scheme is $u_{n+1}(t) = 1 + \int_0^t (s - u_n(s))ds$, $u_0(t) = 1$. We get $u_1(t) = 1 - t + t^2/2 + \cdots$, and so on.

Section 2.2.3

2. Separating variables gives $du/u = \cos t\, dt$. Integrating and applying the initial condition gives the exact solution $u(t) = e^{\sin(t)}$. The Euler method gives

$$u_{n+1} = u_n + h(u_n + \cos(nh)), \quad n = 0, 1, 2, \ldots.$$

with $u_0 = 1$.

Step Size h	exact	0.4	0.2	0.1	0.05
$u(20)$	2.4917	0.3203	0.9387	1.5386	1.9595
Error	0	2.1714	1.5530	0.9531	0.5322

4. The solution is $u(t) = u_0e^{-rt}$, and the Euler algorithm is $u_{n+1} = (1-hr)u_n$, having solution $u_n = (1-hr)^n u_0$. If $1-hr < 0$ then we will get oscillations from the Euler method. To prevent that, take $h > 1/r$.

10. Add

$$u(t+h) = u(t) + u'(t)h + \frac{1}{2}u''(t)h^2 + C_1h^3$$

and

$$u(t-h) = u(t) - u'(t)h + \frac{1}{2}u''(t)h^2 + C_2h^3$$

to get

$$u(t+h) + u(t-h) = 2u(t) + u''(t) + Ch^3,$$

or

$$u''(t) = \frac{u(t+h) - u(t) + u(t-h)}{h^2} + Ch^3.$$

12. Integrating both sides of the differential equation gives, as in the text,

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} f(t, u(t))dt.$$

CHAPTER 3

Section 3.1

2. The potential energy and conservation of energy law are (with $y = x'$)

$$V(x) = -\int -x^2 dx = \frac{1}{3}x^3 \text{ or } \frac{1}{2}y^2 + \frac{1}{3}x^3 = E.$$

Setting $x(0) = 1$ and $y(0) = 0$ gives $E = 1/3$. Then

$$y = \pm\sqrt{\frac{2}{3}}\sqrt{1 - x^3}.$$

4. From Exercise 2, replacing y by dx/dt and separating variables,

$$\frac{dx}{\sqrt{1 - x^3}} = \pm\sqrt{\frac{2}{3}}\, dt.$$

Integrating from $x = 1$ to x and $t = 0$ to t,

$$\int_0^x \frac{dz}{\sqrt{1 - z^3}} dz = -\sqrt{\frac{2}{3}} t,$$

because the velocity is negative. This gives x implicitly as a function of t.

6. Solving the conservation law

$$\frac{1}{2}my^2 + V(x) = E$$

for y, replacing y by dx/dt, and then separating variables gives

$$\pm\sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}} dx = t + C.$$

8. (a) If $y = x'$, then $y' = -(2/t)y$. Separating variables and integrating gives $y = C/t^2$. Then

$$\frac{dx}{dt} = \frac{C}{t^2} \Rightarrow x = \frac{C_1}{t} + C_2.$$

(b) Using $y = x'$ and $x'' = y\, dy/dx$ we get

$$y\frac{dy}{dx} = xy.$$

Therefore $y = 0$ or $dy/dx = x$, giving $y = (1/2)x^2 + C$. Now, replace y by dx/dt, separate variables and integrate to get $x = x(t)$.

(e) Setting $y = x'$ the equation becomes $ty' + y = 4t$, which is a first-order linear equation. Solve to get y=2t+C/t. Thus, $x = t^2 + c\ln t + C_2$.

10. We have $F(x) = -dV/dx = -2(x+1)(x-2)(2x-1)$. (b) The conservation law is $y^2 + (x+1)^2(x-2)^2 = E$, or $y = \pm\sqrt{E - (x+1)^2(x-2)^2}$. One easily sketches these curves for different values of E. (c) When $y > 0$ we have $x' > 0$ and x is increasing in time; when $y < 0$ we have $x' < 0$ and x is decreasing in time. (d) When $x = 0$ and $y = 3$ we get $E = 13$. The maximum x-value occurs when $y = 0$, or $13 - (x+1)^2(x-2)^2 = 0$.

Section 3.2

2. (a) The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$, giving $\lambda = 2, 2$. Therefore

$$u(t) = ae^{2t} + bte^{2t}.$$

The initial conditions give $a = 1$ and $b = -2$.

(e) The characteristic equation is $\lambda^2 - 2\lambda = 0$, giving $\lambda = 0, 2$. Therefore

$$u(t) = a + be^{2t}.$$

The initial conditions give $a = 0$ and $b = 1$.

4. The characteristic equation is $\lambda^2 + (1/8)\lambda + 1 = 0$, giving

$$\lambda = \frac{1}{2}\left(-\frac{1}{16} \pm i\sqrt{\frac{255}{256}}\right).$$

Thus,

$$u(t) = e^{-t/16}\left(A\cos\sqrt{\frac{255}{256}}t + B\sin\sqrt{\frac{255}{256}}t\right).$$

6. The characteristic equation is $L\lambda^2 + \lambda + 1 = 0$, giving

$$\lambda = \frac{1}{2L}\left(-1 \pm \sqrt{1-4L}\right).$$

Therefore, if $L \leq 1/4$, the eigenvalues are negative and real, giving decay; if $L > \frac{1}{4}$, the eigenvalues are complex with negative real parts, representing a decaying oscillation.

8. If $\lambda = 4, -6$, then the characteristic equation factors into $(\lambda-4)(\lambda+6) = 0$. So, the differential equation is $u'' + 2u' - 24u = 0$.

10. If $\lambda = \pm 4i$, then $\lambda^2 + 16 = 0$, giving the differential equation $u'' + 16u = 0$.

12. $u(0) = 3$ and $u'(0) = -2$.

Section 3.3.1

2. (a) The characteristic polynomial for the homogeneous equation is $\lambda^2+7=0$, giving $\lambda = \pm\sqrt{7}$. The two independent solutions are $\cos\sqrt{7}t$ and $\sin\sqrt{7}t$. Therefore, the particular solution has the form $u_p(t) = (a+bt)e^{3t}$. Calculating $u_p''(t)$ and substituting into the differential equation gives equations $16a + 6b = 0,\ 16b = 0$. Thus $b = 1/16$ and $a = -3/128$.

   (f) We have $u' + u = 4e^{-t}$. The homogeneous equation is $u' + u = 0$, so $u_h(t) = Ce^{-t}$. A guess for the particular solution is $u_p = Ae^{-t}$, but that duplicates the homogenous solution. Therefore, $u_p = Ate^{-t}$. Taking u_p' and substituting u_p and u_p' into the differential equation gives $A = 4$. Therefore

$$u(t) = Ce^{-t} + 4te^{-t}.$$

4. The characteristic equation is $L\lambda^2 - 3\lambda + 40 = 0$ with roots $\lambda = 8, -5$. The homogeneous solution is therefore $u_h = c_1e^{8t} + c_2e^{-5t}$. A particular solution has the form $u_p = Ae^{-t}$. Substituting into the DE gives $A = 2$. The general solution is

$$u(t) = c_1e^{8t} + c_2e^{-5t} + 2e^{-t}.$$

   The initial conditions give $c_1 = -8/13$, $c_2 = -18/13$.

8. The homogeneous solution is $u_h(t) = c_1\cos\sqrt{2/5}t + c_2\sin\sqrt{2/5}t$. The particular solution is $u_p = 5$. Then,

$$u(t) = c_1\cos\sqrt{2/5}t + c_2\sin\sqrt{2/5}t + 5.$$

   The initial conditions give $c_1 = 10$, $c_2 = 4\sqrt{5/2}$.

10. The initial value problem is

$$q'' + 8q' + 25q = 55, \quad q(0) = 5,\ q'(0) = 0.$$

   The eigenvalues are $\lambda = -4\pm 3i$, giving $q_h(t) = e^{-4t}(c_1\cos\sqrt{3}t + c_2\sin\sqrt{3}t)$. The particular solution is $u_p = 11/5$. Therefore, $q(t) = q_h(t) + q_p(t)$. Setting $q(0) = 5$ gives $c_1 = 14/5$; setting $q'(0) = 0$ gives $c_2 = 56/(5\sqrt{3})$.

Section 3.3.2

4. The equation is

$$Lq'' = \frac{1}{C}q = V_0\sin\beta t.$$

   The homogeneous solutions are $\cos\sqrt{1/LC}\,t$ and $\sin\sqrt{1/LC}\,t$. Resonance occurs when $\sqrt{1/LC} = \beta$, or $L = 1/C\beta$.

6. The characteristic equation is $\lambda^2+0.01\lambda+4=0$ with roots $\lambda=-1/200\pm i\beta$, where $\beta=1.9999$. Thus the homogeneous solution is

$$u_h(t)=\exp(t/200)(c_1\cos\beta t+c_2\sin\beta t)$$

. The particular solution up has the form $u_p=a\cos 2t+b\sin 2t$. Substituting into the differential equation gives $a=0$ and $b=50$. Therefore the general solution is

$$u(t)=e^{t/200}(c_1\cos\beta t+c_2\sin\beta t)+50\cos 2t.$$

Applying the initial conditions gives $c_1=-50$ and $c_2=1/4\beta=0.125$.

Section 3.4

2. $\beta=1$.

4. Let $u=\sum_{k=0}a_kt^k$ and substitute into the differential equation to get

$$u(t)=\sum_{k=2}k(k-1)a_kt^{k-2}+\sum_{k=0}a_kt^k.$$

Replacing $k-2$ by k in the first sum gives

$$u(t)=\sum_{k=0}(k+2)(k+1)a_{k+2}t^k+\sum_{k=0}a_kt^k.$$

Setting the coefficients equal to zero gives the recursion relation

$$a_{k+2}=\frac{1}{(k+2)(k+1)}a_k,\quad k=0,1,2,...$$

Computing all the coefficients recursively in terms of a_0 and a_1 gives

$$a_{2n}=\frac{1}{(2n)!}a_0,\quad a_{2n+1}=\frac{1}{(2n+1)!}a_1.$$

Thus,

$$a_0\sum_{n=0}\frac{1}{(2n)!}t^{2n}+a_1\sum_{n=0}\frac{1}{((2n+1)!}t^{2n+1}=a_0\cos t+a_1\sin t.$$

6. Let $u=\sum_{k=0}a_kt^k$ and substitute into the differential equation to get

$$u(t)=\sum_{k=2}k(k-1)a_kt^{k-2}+\sum_{k=2}k(k-1)a_kt^k+\sum_{k=0}a_kt^k=0.$$

Shifting indices in the first series gives

$$u(t)=\sum_{k=0}(k+2)(k+1)a_{k+2}t^k+\sum_{k=2}k(k-1)a_kt^k+\sum_{k=0}a_kt^k=0.$$

The recursion is

$$(k+2)(k+1)a_{k+2} + (k(k-1)+1)a_k = 0.$$

Calculating the first few coefficients in terms of a_0 and a_1 gives

$$u(t) = a_0\left(1 - \frac{1}{2}t^2 - \frac{3}{4!}t^4 + \cdots\right) + a_1\left(1 - \frac{1}{3!}t^3 - \frac{3}{5!!}t^5 + \cdots\right).$$

8. Set $n = 0$ in the equation to get $u'' - 2tu' = 0$, which obviously has solution $u(t) = H_0(t) = a \cdot 1$. Setting $n = 1$ in the equation gives $u'' - 2tu' + 2u = 0$. Try a linear solution $u = a + bt$ and substitute to get $a = 0$, b arbitrary. So $u(t) = H_1(t) = bt$. When $n = 2$, the equation is $u'' - 2tu' + 4u = 0$; try $u = at^2 + bt + c$, and substitute to get $b = 0$ and $a = c$. Thus, $u(t) = H_2(t) = a(t^2 + 1)$. Continue this process.

10. Let $u = tv$. Then $u' = tv' + v$ and $u'' = tv'' + 2v'$. Therefore, the equation for v reduces to $v'' - v' = 0$, having one solution $v = e^t$. Therefore, another solution is given by $u = te^t$.

12. The first part is straightforward. Next, solve the z equation. Separating variables gives
$$\frac{dz}{z} = \frac{-2y' - py}{y}dt = -2\frac{y'}{y}dt - pdt.$$
Integrate both sides to get
$$\ln z = -2\ln y - \int p\,dt + C, \quad \text{or} \quad z = C\frac{-\int p\,dt}{y^2}.$$

14. Take the derivative of the Wronskian expression $W = u_1u_2' - u_1'u_2$ and use the fact that u_1 and u_2 are solutions to the differential equation to show $W' = -p(t)W$. Solving gives $W(t) = W(0)\exp(-\int p(t)dt)$, which is always of one sign.

16. The given Riccati equation can be transformed into the Cauchy–Euler equation $u' - (3/t)u' = 0$.

18. (a) $tp(t) = t \cdot t^{-1} = 1$, and $t^2q(t) = t^2(1-k^2)/t^2) = t^2 - k^2$, which are both power series about $t = 0$.

Section 3.5

2. $u(x) = -(1/6)x^3 + (1/240)x^4 + (100/3)x$. The rate that heat leaves the right end is $-Ku'(20)$ per unit area.

4. There are no nontrivial solutions when $\lambda \le 0$. There are nontrivial solutions $u_n(x) = \sin n\pi x$ when $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$.

6. Integrate the steady-state heat equation from 0 to L and use the fundamental theorem of calculus. This expression states: the rate that heat flows in at $x = 0$ minus the rate it flows out at $x = L$ equals the net rate that heat is generated in the bar.

8. $\lambda = -1 - n^2\pi^2$, $n = 1, 2, \ldots$.

10. Hint: This is a Cauchy–Euler equation. Consider three cases where the values of λ give characteristic roots that are real and unequal, real and equal, and complex.

Section 3.6

2. The characteristic equation is $\lambda^4 + \lambda^2 - 4\lambda - 4 = 0$. It is easy to guess a root $\lambda = -1$, so $\lambda + 1$ is a factor. Dividing out this factor, we find the remaining factor is $\lambda^2 - 4$. So, $\lambda = -1, 2, -2$. Therefore, $u(t) = ae^{-t} + be2t + ce^{-2t}$.

4. We have $u''' + 2u'' - 5u' - u = 0$. Letting $u' = v$, $v' = u'' = w$, we get $w' = -u + 5v - 2w$. In summary, the system is

$$u' = v, \quad v' = w, \quad w' = -u + 5v - 2w.$$

Section 3.7

2. $u(t) = (\frac{1}{2} - \sin t)^{-1}$, $-7\pi/6 < t < \pi/6$.

4. Let $u = \sum_{k=0} a_k t^k$ and substitute into the differential equation to get, after shifting the indices,

$$u(t) = a_2 + \sum_{k=0} (k+3)(k+2)a_{k+3}t^{k+1} + \sum_{k=0} (k+1)a_{k+1}t^{k+1} + \sum_{k=0} a_k t^{k+1}.$$

Then $a_2 = 0$ and the recursion is

$$(k+3)(k+2)a_{k+3} = -(k+1)a_{k+1} - a_k, \quad k = 0, 1, 2, \ldots.$$

Additional coefficients can be calculated recursively in terms of a_0 and a_1.

6. $r(t) = -kt + r_0$.

8. The characteristic polynomial is $(\lambda - 2)(\lambda + 1) = 0$, and the homogeneous solution set is $u_1 = e^{-t}$, $u_2 = e^{2t}$. The Wronskian is $W(t) = 3e^t$. Therefore, a particular solution is

$$u_p(t) = -\frac{1}{3}e^{-t} \int e^t \cosh t \, dt + \frac{1}{3}e^{2t} \int e^{-2t} \cosh t \, dt.$$

These integrals may be easily calculated by replacing $\cosh t = (e^t + e^{-t})/2$.

10. Let $u = \ln u$, $y' = u'/u$. Then the equation simplifies to $y' = 4t - 2/t$. Integrating, $y(t) = 2t^2 - 2\ln t + C$. Therefore, $u(t) = e^{y(t)}$.

12. $u(t) = t - 3t\ln t + 2t^2$.

CHAPTER 4

Section 4.1

2. Write

$$U(s) = \int_0^t \sin(at)e^{-st}dt$$

and integrate by parts twice. Problem 4 gives an easier method.

4. We know $(\sin at)'' = -a^2 \sin at$. Therefore,

$$\begin{aligned}\mathcal{L}(\sin at) &= -\frac{1}{a^2}\mathcal{L}((\sin at)'') \\ &= -\frac{1}{a^2}[(s^2\mathcal{L}(\sin at) - a\sin 0 - a\cos 0] \\ &= -\frac{1}{a^2}[(s^2\mathcal{L}(\sin at) - a].\end{aligned}$$

Solving for $\mathcal{L}(\sin at)$ gives

$$\mathcal{L}(\sin at) = \frac{a}{s^2 + a^2}.$$

6. We have

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1}, \quad \mathcal{L}(\sin(t - \pi/2)) = \mathcal{L}(-\cos t) = \frac{-s}{s^2+1},$$

and $\mathcal{L}(h_{\pi/2}(t)\sin(t - \pi/2)) = e^{-pis/2}(1/(s^2+1))$.

8. Use, for example,

$$\cosh t = (e^t + e^{-t})/2.$$

So, $\mathcal{L}(\cosh t) = (1/2)(\frac{1}{s-1} + \frac{1}{s+1})$.

10. We have

$$\mathcal{L}(e^{-3} + 4\sin kt) = \frac{1}{s+3} + \frac{4k}{s^2+k^2}, \quad \mathcal{L}(e^{-3t}\sin 2t) = \frac{2}{(s+3)^2 + k^2}.$$

12. By definition,

$$\mathcal{L}(u(at)) = \int_0^\infty u(at)e^{-st}dt = \int_0^\infty u(r)e^{-(s/a)r}d(r/a) = \frac{1}{a}U(s/a).$$

14. The function $\exp(t^2)$ grows too fast as t gets large, and so is not of exponential order; the integral diverges.

16. We have

$$\mathcal{L}(f(t)) = \mathcal{L}\left(\sum_{n=0}^{\infty}(-1)^n h_n(t)\right) = \sum_{n=0}^{\infty}(-1)^n e^{-ns} = \frac{1}{1+e^{-s}}.$$

18. Taking the derivative of formula for the Laplace transform,

$$U'(s) = \frac{d}{dt}\int_0^{\infty} u(t)e^{-st}dt = \int_0^{\infty} u(t)(-t)e^{-st}dt = -tU(s) = -\mathcal{L}(tu(t))$$

Take the inverse transform to get the other formula.

20. Use induction.

22. $\mathcal{L}(t^2 h_1(t)) = e^{-s}\mathcal{L}((t+1)^2) = e^{-s}\mathcal{L}(t^2+2t+1)$.

Section 4.2

2. (a) $\mathcal{L}(e^{-6t}t^4) = 4!/(s+6)^5$.

4. (a) Taking the transform of the differential equation and solving for $U(s)$ gives

$$U(s) = \frac{1}{s+5} + \frac{1}{s(s+5)}e^{-2s}.$$

(h) Taking the transform of the differential equation and solving for $U(s)$ gives

$$U(s) = \frac{3}{(s^2+9)^2},$$

giving

$$u(t) = \frac{1}{18}\sin(3t) - \frac{1}{6}t\cos(3t).$$

6. Taking the transform of each equation, we get $sX(x) = a + 2X(s) - Y(s)$ and $sY(s) = X(s)$. Then,

$$sX(s) = a + 2X(s) - \frac{1}{s}X(s), \quad \text{or} \quad X(s) = \frac{as}{(s-1)^2}.$$

By partial fractions,

$$\frac{as}{(s-1)^2} = \frac{a}{s-1} + \frac{a}{(s-1)^2}.$$

The first term on the right inverts to ae^t and the second term on the right inverts to ate^t. Thus,

$$x(t) = ae^t + ate^t, \quad y(t) = ate^t.$$

Section 4.3

2. We have

$$t \star t^2 = \int_0^t (t-\tau)\tau^2 d\tau = t\int_0^t (\tau^2 - \tau^3) = t(t^3/3 - t^4)4.$$

4. $\mathcal{L}(1 \cdot e^t) = 1/(s-1)$, but $\mathcal{L}(1) \cdot \mathcal{L}(e^t) = s/(s-1)$.

6. $(u \star v)(t) = \int_0^t u(t-\tau)v(\tau)d\tau = -\int_t^0 u(r)v(t-r))dr = (v \star u)(t)$, where we made the substitution $r = t - \tau$, $dr = -d\tau$.

10. Taking the transform, $s^2U(s) - sU(s) = F(s)$. Therefore,

$$U(s) = \frac{1}{s(s-1)}F(s).$$

But,

$$\mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right) = -1 + e^{-t}.$$

Thus,

$$u(t) = (-1 + e^{-t}) * f(t) = \int_0^t (-1 + e^{t-\tau})f(\tau)d\tau.$$

12. Taking the transform, while using convolution on the integral, gives $U(s) = F(s) + K(s)U(s)$, which yields $U(s) = F(s)/(1 - K(s))$. Here, $K(s)$ is the transform of $k(t)$.

14. Taking the transform,

$$F(s) = \frac{1}{\sqrt{\pi}}U(s)\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}}U(s)\frac{\Gamma(1/2)}{s^{1/2}}.$$

Then $U(s) = F(s)s^{1/2}$, and

$$u(t) = f(t) * \mathcal{L}^{1-}(s^{1/2})(t) = f(t) * t^{-3/2}\frac{1}{\Gamma(-1/2)}.$$

Section 4.4

2. We have

$$\mathcal{L}(t^2h_3(t)) = e^{-3s}\mathcal{L}((t+3)^2) = e^{-3s}\mathcal{L}(t^2 + 6t + 9)$$
$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

4. $U(s) = (1/s)(3 - e^{-2s} + 4e^{-\pi s} - 6e^{-7s})$.

6. Solving for $U(s)$, we get

$$U(s) = \frac{s}{(s^2+4)^2} - \frac{s}{(s^2+4)^2}e^{-2\pi s}.$$

Using the table,

$$u(t) = \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin(2(t-2\pi))h_{2\pi}(t).$$

Note $\sin(2(t-2\pi)) = \sin(4t)$.

8. The differential equation is

$$q'' + q = t + (9-t)h_9(t).$$

Taking transforms and using the zero initial conditions, we get

$$U(s) = \frac{1}{s^2+1}\left(\frac{1}{s^2} - \frac{1}{s^2}e^{-9s}\right).$$

By convolution,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = t * \sin t = \int_0^t \tau\sin(t-\tau)d\tau.$$

Similarly, by the switching theorem,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}e^{-9s}\right) = \int_0^{t-9} \tau\sin(t-9-\tau)d\tau.$$

10. Taking the transform of the differential equation and solving for $U(s)$ gives

$$U(s) = \frac{s}{s^2+\pi^2} + \frac{\pi^2}{s(s^2+\pi^2)} - \frac{\pi^2}{s(s^2+\pi^2)}e^{-s}.$$

The first term inverts to $\cos\pi t$, and the second term inverts to $1-\cos\pi t$ (by convolution), and the third term inverts to $1-\cos(\pi(t-1))h_1(t)$.

12. We have $f(t) = 1 - 2h_a(t) + 2h_{2a}(t) - 2h_{3a}(t) - \cdots$. Therefore,

$$\begin{aligned}\mathcal{L}(f(t)) &= \frac{1}{s}\left(2 - 2e^{-as} + 2e^{-2as} - 2e^{-3as} + \cdots\right) - \frac{1}{s}\\ &= \frac{2}{s}\sum_{n=0}(-1)^n e^{-ans} - \frac{1}{s}\\ &= \frac{2}{s}\frac{1}{1+e^{-as}} - \frac{1}{s}.\end{aligned}$$

Use the fact that that $\tanh x = \sinh x/\cosh x$; then

$$\frac{1}{s}\tanh\left(\frac{as}{2}\right) = \frac{1}{s}\frac{1-e^{-as}}{1+e^{-as}}.$$

Section 4.5

2. Solving for the transform,

$$U(s) = \frac{1}{s+3} + \frac{1}{s+3}e^{-s} + \frac{1}{s(s+3)}e^{-4s}.$$

Therefore,

$$u(t) = e^{-3t} + e^{-3(t-1)}h_1(t) + \frac{1}{3}\left(1 - e^{-3(t-4)}\right)h_4(t).$$

4. Solving for the transform

$$U(s) = \frac{1}{s^2+1}e^{-2s}.$$

Therefore

$$u(t) = \sin(t-2)h_2(t).$$

6. Solving for the transform

$$U(s) = \frac{1}{s^2+4}e^{-2s} - \frac{1}{s^2+4}e^{-5s}.$$

Therefore

$$u(t) = \frac{1}{2}\sin(2(t-2))h_2(t) - \frac{1}{2}\sin(2(t-5))h_5(t).$$

8. The transformed equation is

$$U(s) = \frac{1}{s^2+1}\left(1 + e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s} + e^{-4\pi s} + \cdots\right) = \frac{1}{s^2+1}\frac{1}{1-e^{s\pi}}.$$

Therefore, from the table of transforms,

$$u(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\frac{1}{1-e^{s\pi}}\right) = \sum_0^\infty \sin(t-n\pi)h_{n\pi}(t).$$

Note $\sin(t-n\pi) = (-1)^n \sin t$.

CHAPTER 5

Section 5.1

2. We have $x(t) = 2\exp(t)$, $y(t) = -3\exp(t)$, and $x'(t) = 2\exp(t)$, $y'(t) = -3\exp(t)$. Substituting into the differential equation shows we have a solution. Also, dividing, $y/x = -3/2$, so the orbit lies on the straight line with slope $-3/2$, in the fourth quadrant. Also, $x(0) = 2$, $y(0) = -3$ and $x(t), y(t) \to 0$ as $t \to -\infty$, $x(t), y(t) \to \infty$ as $t \to +\infty$. The tangent vector along the orbit is $(x'(t), y'(t)) = (2\exp(t), -3\exp(t))$.

4. Taking the derivative of the second equation and substituting from the first gives $y''+7y'+6y=0$. The characteristic equation has roots, or eigenvalues, -1 and -6. Therefore, $y(t) = c_1e^{-t} + c_2e^{-6t}$, and thus $x(t) = \int y(t)dt = -c_1e^{-t} - (1/6)c_2e^{-6t}$. The initial conditions give $c_1 = -24/5$, $c_2 = 24/5$. As $t \to -\infty$, $(x(t), y(t)) \to (0, 0)$. Because e^{-t} dominates as e^{-6t} for large t, the orbit enters the origin tangent to the line

$$\frac{y}{x} = \frac{c_1e^{-t}}{-c_1e^{-t}} = -1.$$

Section 5.3

2. (a) The right sides of the DEs are proportional, so there are infinitely many equilibria consisting of the entire line $y = -3x$. (b) Dividing the two equations we get $dy/dx = -\frac{1}{2}$, or parallel lines, $y = -\frac{1}{2}x + C$, which the orbits in terms of x and y. In terms of time, we note $x' > 0$ when $y > -3x$, and $x' < 0$ when $y < -3x$; therefore, the orbits are going to the right as $t \to +\infty$ along the parallel lines to the right of the line of equilibria, and to the left on the other side of the line of equilibria. As $t \to -\infty$ the orbits approach the equilibria line.

4. The equations are $x' = -bx + ay$, $y' = r + bx - (a + c)y$. Setting both to zero, we find a single equilibrium at $x = ar/bc$, $y = r/c$. The x nullcline, where the vector field is vertical is the straight line $y = y = bx/a$, and the y nullcline, where the vector field is horizontal, is the straight line $y = bx/(a + c) + r/(a + c)$; note that this line has a smaller slope than the former. Finding the directions in the four regions bounded by the nullclines, we see that all orbits approach the equilibrium as $t \to +\infty$. It has the appearance of a nodal structure.

6. Assume the differential equations are

$$x' = ax + by, \quad y' = cx + dy.$$

Substituting the solution $x = e^{-t}$, $y = 2e^{-t}$ into the DEs gives

$$-1 = a + 2b, \quad -2 = c + 2d.$$

Substituting the solution $x = e^{-4t}$, $y = -e^{-4t}$ into the DEs gives

$$-4 = a - b, \quad 4 = c - d.$$

So, we have four equations for a, b, c, and d. Solving gives $a = -3$, $b = 1$, $c = 2$, $d = -2$.

8. The straight-line orbits are

$$x(t) = c_1 e^{-2t}, \quad y(t) = -3c_1 e^{-2t},$$

and

$$x(t) = c_2 e^{4t}, \quad y(t) = c_2 e^{4t},$$

These are along straight lines $y = -3x$ and $y = x$, respectively. The eigenvalues are real of opposite sign, so the origin is a saddle; the negative eigenvalue -2 corresponds to the separatrix $y = -3x$, and the rays enter the origin; the positive eigenvalue 4 corresponds to the separatrix $y = x$ and come out of the origin. To draw the saddle structure, note that as $t \to +\infty$, the terms with the positive eigenvalue 4 dominate, and the orbits approach the line $y = x$.

10. We check when the eigenvalues are complex with positive real part. The coefficient matrix is

$$A = \begin{pmatrix} a & a \\ -1 & 6 \end{pmatrix}.$$

The trace is $\operatorname{tr} A = a + 6$ and the $\det A = 7a$. So, $a > -6$ and $a > 0$; so, $a > 0$. The discriminant is $a^2 - 16a + 36$, and we require $a^2 - 16a + 36 < 0$ to have complex roots. The roots of this quadratic are $a = 8 \pm \sqrt{28}$, which are both positive. Because the parabola is concave up, we require $8 - \sqrt{28} < a < 8 + \sqrt{28}$.

12. (b) The coefficient matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -12 & -7 \end{pmatrix}.$$

We have $\operatorname{tr} A = -7$ and $\det A = 12$. The eigenvalues are $\lambda = -3, -4$, and thus $(0, 0)$ is a stable node.

(e) The coefficient matrix is

$$A = \begin{pmatrix} 2 & 5 \\ 0 & -2 \end{pmatrix}.$$

We have $\operatorname{tr} A = 0$ and $\det A = -2$. Therefore, $(0, 0)$ is a saddle.

(j) The coefficient matrix is

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}.$$

We have $\operatorname{tr} A = \alpha + \gamma > 0$ and $\det A = \alpha\gamma > 0$. Now, the discriminant is

$$(\alpha + \gamma)^2 - 4\alpha\gamma = (\alpha - \gamma)^2 > 0.$$

Therefore the eigenvalues are real and $(0, 0)$ is an unstable node.

14. The equilibrium is

$$L_0 = \frac{\lambda}{\mu + \mu_0}, \ M_0 = \frac{\mu}{\delta} L_0 = \frac{\mu\lambda}{\delta(\mu + \mu_0)}.$$

The L nullcline is the vertical line $L = \lambda/(\mu + \mu_0)$ and the straight line $M = (\mu/\delta)L$. Sketching the direction field shows that (L_0, M_0) is a stable node.

Section 5.4

2. The equilibria are $(1, 0)$ and $(-1, 0)$; $x' = 0$ on the y axis and $y' = 0$ on the parabola $y = 1 - x^2$. A sketch of the vector field easily reveals that $(-1, 0)$ is a saddle point. The point $(1, 0)$ has a circular rotation to it and could be a spiral or center.

4. The equilibria are $(0, 0)$ and $(1/2, 1)$. The x nullclines are $x = 0$, $y = 1$, and the y nullclines are $y = 0$, $x = 1/2$. A sketch of the vector field reveals $(0, 0)$ is a stable node and $(1/2, 1)$ is a saddle point.

6. The equilibrium $(1, -1)$ clearly shows a saddle structure, and $(-1, -1)$ appears to be an unstable spiral.

8. $x' = 0$ when $\sin y = 0$, or $y = \pm n\pi$; and, $y' = 0$ when $x = 0$. Therefore, there are infinitely many isolated equilibrium along the y axis, at $(0, \pm n\pi)$, $n = 0, 1, 2, ...$

10. (a) We have $x' = (x + y)(x - y)$, and $y' = x - y$. Clearly $x' = y' = 0$ on the line $y = x$. So, there is a continuum of equilibria. Dividing the equations, we get

$$\frac{dx}{dy} - x = y,$$

which is a first-order linear equation for $x = x(y)$. An integrating factor is e^{-y}. Multiplying by the factor and integrating gives

$$x = e^y \int ye^{-y} dy + Ce^y = -(1 + y) + Ce^y.$$

(c) At $t = 0$, setting $y = 0$, $x = 1/4$ gives $C = 5/4$, and the orbit is $x = -(1 + y) + (5/4)e^y$. The orbit begins at $(1/4, 0)$ and increases into the positive xy plane as $t \to \infty$.

Section 5.5

4. Apply the SIR model. We have $N = 500$ and $I(0) = 25$. Then, $S(0) = 475$. It takes 4 days to recover, so the recovery rate is $r = \frac{1}{4} = 0.25$. The average time to get the infection is $1/aN = 2$ days, so $a = 0.001$. From the

equations, the number that escape infection is 93 and the maximum number of infected at any one time is $I_m = 77$ individuals, and the maximum occurs when $S_m = r/a = 250$.

6. The equations are $x' = rx - axy - h$, $y' = -my + bxy$. The equilibrium is

$$\left(\frac{m}{b}, \frac{r}{a} - \frac{bh}{am}\right).$$

The nonzero equilibrium for the Lotka–Volterra model ($h = 0$) is $(m/b, r/a)$. Therefore, harvesting the prey lowers the predator equilibrium!

8. We have $x' = rx - axy$, $y' = -my + bxy - M$. The equilibrium is

$$\left(\frac{m}{b} + \frac{Ma}{br}, \frac{r}{a}\right).$$

So, migration of the predator increases the prey equilibrium.

10. The equations are $S' = -aSI - vS$, $I' = aSI - rI$. Note that $S' < 0$, so S is always decreasing; $S' = 0$ only along $S = 0$. Note that $I' = 0$ along $S = r/a$ and $I = 0$. The origin $(0, 0)$ is the only equilibrium in the first quadrant. In this case, a sketch of the nullclines and vector field shows that $S(t)$, $I(t) \to (0, 0)$ as $t \to \infty$. There are no susceptibles that escape the disease.

12. Write $x' = x(1 - x - ay)$, $y' = y(c - cy - bx)$. The x nullclines are $x = 0$ and $y = (1 - x)/a$, and the y nullclines are $y = 0$ and $y = 1 - (b/c)x$. The equilibria are $(0, 1)$, $(1, 0)$, and $(0, 0)$. Note, by the conditions on the constants, the nonzero nullclines do not intersect each other. It is straightforward to sketch the vector field; clearly, $(1, 0)$ is a saddle, and $(0, 1)$ is a stable node. The origin is an unstable node. Note that the growth rate of the y species is greater than its death rate, so the y species dominates, as may be expected.

Section 5.6

2. For the system $x' = f(x, y)$, $y' = g(x, y)$ the modified Euler method may be outlined as follows. Let $t_n = t_0 + nh$ and x_n and y_n denote the approximations of $x(t_n)$ and $y(t_n)$, where h is the step size. Let $x(t_0)$ and $y(t_0)$ be given; then, the predictor is the Euler formula,

$$\tilde{x}_{n+1} = x_n + hf(x_n, y_n), \quad \tilde{y}_{n+1} = y_n + hg(x_n, y_n).$$

The corrector is

$$x_{n+1} = x_n + 0.5h[f(x_n, y_n) + f(\tilde{x}_{n+1}, \tilde{y}_{n+1})]$$
$$y_{n+1} = y_n + 0.5h[g(x_n, y_n) + g(\tilde{x}_{n+1}, \tilde{y}_{n+1})].$$

CHAPTER 6

Section 6.2

1. $\det(A - \lambda I) = \lambda^2 - 5\lambda - 2$.

2. $x = 3/2$, $y = 1/6$.

4. $\det(A - \lambda I) = \lambda^2 - 5\lambda - 2 = 0$, so $\lambda = \frac{5}{2} \pm \frac{1}{2}\sqrt{33}$.

5. $\det A = 0$ so A^{-1} does not exist.

6. If $m = -5/3$ then there are infinitely many solutions, and if $m \neq -5/3$, no solution exists.

7. $m = 1$ makes the determinant zero.

8. Use expansion by minors.

10. $\det(A) = -2$, so A is invertible and nonsingular.

11. $\mathbf{x} = a(2, 1, 2)^{\mathrm{T}}$, where a is any real number.

12. Set $c_1(2, -3)^{\mathrm{T}} + c_2(-4, 8)^{\mathrm{T}} = (0, 0)^{\mathrm{T}}$ to get $2c_1 - 4c_2 = 0$ and $-3c_1 + 8c_2 = 0$. This gives $c_1 = c_2 = 0$.

13. Pick $t = 0$ and $t = \pi$.

14. Set a linear combination of the vectors equation to the zero vector and find coefficients c_1, c_2, c_3.

16. $\mathbf{r}_1(t)$ plots as an ellipse; $\mathbf{r}_2(t)$ plots as the straight line $y = 3x$; $\mathbf{r}_2(t)$ plots as a curve approaching the origin along the direction $(1, 1)^{\mathrm{T}}$. Choose $t = 0$ to get $c_1 = c_3 = 0$, and then choose $t = 1$ to get $c_2 = 0$.

Section 6.3

1. For A the eigenpairs are 3, $(1, 1)^{\mathrm{T}}$ and 1, $(2, 1)^{\mathrm{T}}$. For B the eigenpairs are 0, $(3, -2)^{\mathrm{T}}$ and -8, $(1, 2)^{\mathrm{T}}$. For C the eigenpairs are $\pm 2i$, $(4, 1 \mp i)^{\mathrm{T}}$.

2. $\mathbf{x} = c_1(1, 5)^{\mathrm{T}} e^{2t} + c_2(2, -4)^{\mathrm{T}} e^{-3t}$. The origin has saddle point structure.

3. The origin is a stable node.

4. (a) $\mathbf{x} = c_1(-1, 1)^T e^{-t} + c_2(2, 3)^T e^{4t}$ (saddle), (c) $\mathbf{x} = c_1(-2, 3)^T e^{-t} + c_2(1, 2)^T e^{6t}$ (saddle), (d) $\mathbf{x} = c_1(3.1)^T e^{-4t} + c_2(-1, 2)^T e^{-11t}$ (stable node), (f) $x(t) = c_1 e^t(\cos 2t - \sin 2t) + c_2 e^t(\cos 2t + \sin 2t)$, $y(t) = 2c_1 e^t \cos 2t + 2c_2 e^t \sin 2t$ (unstable spiral), (h) $x(t) = 3c_1 \cos 3t + 3c_2 \sin 3t$, $y(t) = -c_1 \sin 3t + c_2 \cos 3t$ (center).

6. (a) Equilibria consist of the entire line $x - 2y = 0$. (b) The eigenvalues are 0 and 5; there is a linear orbit associated with 5, but not 0.

7. The eigenvalues are $\lambda = 2 \pm \sqrt{a+1}$; $a = -1$ (unstable node), $a < -1$ (unstable spiral), $a > -1$ (saddle).

9. The eigenvalues are never purely imaginary, so cycles are impossible.

Section 6.4

2. The equations are $V_1 x' = (q + r)c - qx - rx$, $V_2 y' = qx - qy$. The steady state is $x = y = c$. When fresh water enters the system, $V_1 x' = -qx - rx$, $V_2 y' = qx - qy$. The eigenvalues are both negative ($-q$ and $-q - r$), and therefore the solution decays to zero. The origin is a stable node.

5. A fundamental matrix is

$$\Phi(t) = \begin{pmatrix} 2e^{-4t} & -e^{-11t} \\ 3e^{-4t} & 2e^{-11t} \end{pmatrix}.$$

The particular solution is $\mathbf{x}_p = -(\frac{9}{42}, \frac{1}{21})^{\mathrm{T}} e^{-t}$.

6. $\det A = r_2 r_3 > 0$ and $\mathrm{tr}(A) = r_1 - r_2 - r_3 < 0$. So the origin is asymptotically stable and both x and y approach zero. The eigenvalues are $\lambda = \frac{1}{2}(\mathrm{tr}(A) \pm \frac{1}{2}\sqrt{\mathrm{tr}(A)2 - 4\det A})$.

7. In the equations in Problem 6, add D to the right side of the first (x') equation. Over a long time the system will approach the equilibrium solution: $x_e = D/(r_1 + r_2 + r_1 r_3/r_2)$, $y_e = (r_1/r_2)x_e$.

Section 6.5

1. The eigenpairs of A are 2, $(1, 0, 0)^{\mathrm{T}}$; 6, $(6, 8, 0)^{\mathrm{T}}$; -1, $(1, -1, 7/2)^{\mathrm{T}}$. The eigenpairs of C are 2, $(1, 0, 1)^{\mathrm{T}}$; 0, $(-1, 0, 1)^{\mathrm{T}}$; 1, $(1, 1, 0)^{\mathrm{T}}$.

2(a). $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t}$.

2(b). $\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} \cos 0.2t \\ \sin 0.2t \\ -\cos 0.2t - \sin 0.2t \end{pmatrix} + c_3 \begin{pmatrix} -\sin 0.2t \\ \cos 0.2t \\ -\cos 0.2t + \sin 0.2t \end{pmatrix}$.

2(d). $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t}$.

4. The eigenvalues are $\lambda = 2, \rho \pm 1$.

CHAPTER 7

Section 7.1

1. $y = C(e^x - 1)$.

2. $y^2 - x^2 - 4x = C$.

3. Equilibria are $(0,0)$ (a saddle structure) and $(2,4)$ (stable node) and null-clines: $y = x^2$ and $y = 2x$.

4. $a < 0$ (no equilibria); $a = 0$ (origin is equilibrium); $a > 0$ (the equilibria are $(-\sqrt{a}/2, 0)$ and $(\sqrt{a}/2, 0)$, a stable node and a saddle).

6. $(-1,0)$ (stable spiral); $(1,0)$ (saddle).

8. $(2,4)$ (saddle); $(0,0)$ (stable node). The Jacobian matrix at the origin has a zero eigenvalue.

10. $\mathrm{tr}(A) < 0, \quad \det A > 0$. Thus the equilibrium is asymptotically stable.

12. The force is $F = -1 + x^2$, and the system is $x' = y, \quad y' = -1 + x^2$. The equilibrium $(1,0)$ is a saddle and $(-1,0)$ is a center. The latter is determined by noting that the orbits are $\frac{1}{2}y^2 + x - \frac{1}{3}x^3 = E$.

13. (a) $\frac{dH}{dt} = H_x x' + H_y y' = H_x H_y + H_y(-H_x) = 0$. (c) The Jacobian matrix at an equilibrium has zero trace. (e)

$$H = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3}.$$

14. $(0,0)$ is a center.

15. (c) The eigenvalues of the Jacobian matrix are never complex.

16. $(0,0)$, $(0, \frac{1}{2})$, and $(K, 0)$ are always equilibria. If $K \geq 1$ or $K \leq \frac{1}{2}$ then no other positive equilibria occur. If $\frac{1}{2} < K \leq 1$ then there is an additional positive equilibrium.

17. $a = 1/8$ (one equilibrium); $a > 1/8$, (no equilibria); $0 < a < 1/8$ (two equilibria).

19. The characteristic equation is $\lambda^2 = f'(x_0)$. The equilibrium is a saddle if $f'(x_0) > 0$.

Section 7.2

2. There are no equilibrium, and therefore no cycles.

3. $f_x + g_y > 0$ for all x, y, and therefore there are no cycles (by Dulac's criterion).

4. $(1, 0)$ is always a saddle, and $(0, 0)$ is unstable node if $c > 2$ and an unstable spiral if $c < 2$.

6. $(0, 0)$ is a saddle, $(\pm 1, 0)$ are stable spirals.

7. The equilibria are $H = 0$, $P = \phi/a$ and
$$H = \frac{\varepsilon\phi}{c} - \frac{a}{b}, \quad P = \frac{c}{\varepsilon b}.$$

8. In polar coordinates, $r' = r(a - r^2)$, $\theta' = 1$. For $a \le 0$ the origin is a stable spiral. For $a > 0$ the origin is an unstable spiral with the appearance of a limit cycle at $r = \sqrt{a}$.

9. The characteristic equation is $\lambda^2 + k\lambda + V''(x_0) = 0$ and has roots $\lambda = \frac{1}{2}(-k \pm \sqrt{k^2 - 4V''(x_0)})$. These roots are never purely imaginary unless $k = 0$.

10. Use Dulac's criterion.

11. Equilibria at $(0, 0)$, $(1, 1,)$, and $(4, 4)$.

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