Appendix A
The gamma function and related functions

The gamma function plays a central role in the representation of the special functions. This function and other functions derived from the gamma function are collected in this appendix together with some of their basic properties.

A.1 The gamma function $\Gamma(z)$

The gamma function is not a special function in the sense that it is a solution to a second order ordinary differential equation of the kind we have analyzed in this book, i.e., a differential equation of Fuchsian type with a finite number of regular singular points. That this is the case can be deduced from the fact that the gamma function has infinitely many poles — a property that no solution to a second order ordinary differential equation with a finite number of regular singular points can have. Instead, the gamma function has to be defined by other means. In fact, it satisfies a difference equation rather than a differential equation, see below in (A.2).

There are several ways of defining the gamma function $\Gamma(z)$. We prefer to define it by an integral representation, due to Euler, [18], viz.,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1} \, dt, \quad \text{Re} \, z > 0$$  \hfill (A.1)

An explicit value is

$$\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1$$

We easily see by integration by parts that

$$\Gamma(z+1) = z\Gamma(z), \quad \text{Re} \, z > -1, \, \text{and} \, z \neq 0$$  \hfill (A.2)

Evaluated at the non-negative integers, $\Gamma(n+1)$ coincides with the factorials, i.e.,

$$\Gamma(n+1) = n! \quad n \in \mathbb{N}$$
The gamma function and related functions

which is easy to verify by induction over \( n \). This result shows that the gamma function is a generalization of the factorials from the non-negative integers to the complex plane.

The gamma function has simple poles at \( z = 0, -1, -2, -3, \ldots \), which is seen by repeated use of (A.2), see Figure A.1,

\[
\Gamma(z) = \frac{\Gamma(1+z)}{z} = \ldots = \frac{\Gamma(n+1+z)}{(n+z)(n-1+z)\ldots z} = \frac{\Gamma(n+1+z)}{(n+z)(z,n)}
\]

where we introduced the Appell symbol, \((\alpha, n)\), defined in Section A.3.

The residue of \( \Gamma(z) \) at the poles \( z = 0, -1, -2, -3, \ldots \) is determined by

\[
\text{Res}_{z=-n} \Gamma(z) = \lim_{z \to -n} (z+n)\Gamma(z) = \frac{\Gamma(1)}{(-n,n)} = \frac{(-1)^n}{n!}, \quad n \in \mathbb{N} \quad (A.3)
\]

since \((-n,n) = (-n)(-n+1)\ldots(-1) = (-1)^n n! \) and \( \Gamma(1) = 1 \).

An alternative definition of the gamma function is to use the integral representation, see Problem A.2,
The gamma function $\Gamma(z)$

$$\Gamma(z) = \frac{i}{2\sin \pi z} \int_C e^{-t}(-t)^{z-1} dt, \quad \forall z \in \mathbb{C}$$

where the contour $C$ is depicted in Figure A.2.

The reciprocal of the gamma function $1/\Gamma(z)$ is an entire function with zeros at $z = 0, -1, -2, -3, \ldots$. The Weierstrass factorization theorem states that every entire function can be written as an infinite product [9]. The gamma function has the product formula

$$\frac{1}{\Gamma(z)} = ze^{\gamma} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \quad \forall z \in \mathbb{C}$$

where $\gamma$ is the Euler–Mascheroni\(^1\) constant defined in (A.18) on page 176.

We also frequently use

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad (A.4)$$

which is proved in Problem A.1. If we evaluate this identity at $z = 1/2$, we get

$$\Gamma(1/2) = \sqrt{\pi}$$

Additional expression for products of gamma functions are, see also Problem A.5:

$$\Gamma(z)\Gamma(z+1/2) = \frac{\sqrt{\pi}}{2^{2z-1}} \Gamma(2z) \quad (A.5)$$

and

$$\Gamma(z)\Gamma(z+1/3)\Gamma(z+2/3) = \frac{2\pi}{3^{3z-1/2}} \Gamma(3z)$$

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\(^1\) Lorenzo Mascheroni (1750–1800), Italian mathematician.
A.2 Estimates of the gamma function

The gamma function for large arguments is determined by Stirling’s\(^2\) formula

\[
\ln \Gamma(z) = \ln \sqrt{2\pi} + \left( z - \frac{1}{2} \right) \ln z - z + J(z), \quad |\arg(z)| < \pi \quad (A.6)
\]

where Binet’s\(^3\) function \(J(z)\) is an analytic and bounded function in any open sector \(\Gamma_\delta = \{ z \in \mathbb{C} : |\arg(z)| < \pi - \delta \}\), where \(\delta > 0\). Moreover, \(|J(z)| \leq C(\delta)/|z|\) as \(z \to \infty\) in \(\Gamma_\delta\). The branch of the logarithm is real for real arguments. An explicit representation of Binet’s function is

\[
J(z) = \frac{1}{\pi} \int_0^\infty \frac{z}{x^2 + 1} \ln \frac{1}{1 - e^{-2\pi x}} \, dx, \quad \text{Re} z > 0
\]

From this expression of Binet’s function, it is possible to analytically continue the function \(J(z)\) to \(\Gamma_\delta\). A proof of these results is given in, e.g., [11, Sec. 8.5].

The growth rate of the gamma function is used frequently in the main text — in particular in the proofs related to the integral representations of Barnes in Chapter 5. It is convenient to summarize the results in a series of lemmas. We start with a useful estimate of the growth properties of the sine function in the complex plane.

**Lemma A.1.** For each \(\varepsilon \in (0, 1/2)\), there are positive constants, \(c_\varepsilon\) and \(C_\varepsilon\), depending only on \(\varepsilon\), but not on \(z\), such that the sine function satisfies

\[
c_\varepsilon \leq |\sin \pi z| e^{-\pi |\text{Im} z|} \leq C_\varepsilon, \quad \text{in } \{ z \in \mathbb{C} : |z + n| > \varepsilon, \forall n \in \mathbb{Z} \}
\]

**Proof.** The modulus of the sine function in the complex plane is, \(x, y \in \mathbb{R}\)

\[
|\sin \pi (x + iy)|^2 = \sinh^2 \pi y + \sin^2 \pi x = \frac{e^{2\pi |y|}}{4} \left\{ 1 + e^{-4\pi |y|} - 2e^{-2\pi |y|} \cos 2\pi x \right\}
\]

For all \(z = x + iy \in \mathbb{C}\), we have the estimate

\[
e^{-\pi |y|} |\sin \pi (x + iy)| = \frac{1}{2} \sqrt{1 + e^{-4\pi |y|} - 2e^{-2\pi |y|} \cos 2\pi x} \leq \frac{1}{2} \left( 1 + e^{-2\pi |y|} \right) \leq 1
\]

Thus, we have proved that

\[
|\sin \pi (x + iy)| \leq e^{\pi |y|}, \quad \forall z \in \mathbb{C}
\]

This is the right part of the inequality of the lemma. In fact, the constant \(C_\varepsilon = 1\) is independent of the value of \(\varepsilon\).

To prove the left inequality, it suffices, due to the periodicity of the sine function, i.e., \(|\sin \pi (z \pm 1)| = |\sin \pi z|\), to prove the lemma in the strip \(\text{Re} z = x \in [0, 1/2]\). We

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\(^2\) James Stirling (1692–1770), Scottish mathematician.  
\(^3\) Jacques Binet (1786–1856), French mathematician.
define for each $\varepsilon \in (0, 1/2)$ and for all complex numbers $z = x + iy$ in the punctuated strip $M = \{ z \in \mathbb{C} : |z| > \varepsilon, \text{Re} z \in [0, 1/2]\}$

$$f(x + iy) = 1 + e^{-4\pi|y|} - 2e^{-2\pi|y|}\cos 2\pi x, \text{ in } M$$

Our aim is to bound this function from below in $M$. Write $f_y(x) = f(x + iy)$, and we have

$$\frac{d}{dx}f_y(x) = 4\pi e^{-2\pi|y|}\sin 2\pi x \geq 0, \text{ in } M$$

Therefore, for each fixed $y$, $f_y(x)$ is a non-decreasing function as a function of $x$ in $M$. We divide the set $M$ in two parts, $I$ and $II$, see Figure A.3. In the region $I$, where $|y| \leq \varepsilon$ and $x \in [\sqrt{\varepsilon^2 - y^2}, 1/2]$, we have for $\varepsilon \in (0, 1/2)$

$$1 + e^{-4\pi|y|} - 2e^{-2\pi|y|}\cos \left(2\pi \sqrt{\varepsilon^2 - y^2}\right) = f_y(\sqrt{\varepsilon^2 - y^2}) \leq f_y(x)$$

The function on the left-hand side is bounded from below when $|y| \in [0, \varepsilon]$. To see this, rewrite the left-hand side using the notation

$$a(y) = 2\pi|y|, \quad b(y) = 2\pi\sqrt{\varepsilon^2 - y^2}, \quad c(x) = \sqrt{4\pi^2\varepsilon^2 - x^2}$$

as

$$f_y(\sqrt{\varepsilon^2 - y^2}) = 2e^{-a(y)} \left( \int_0^{a(y)} \sinh x \, dx + \int_0^{b(y)} \sin x \, dx \right)$$

$$= 2e^{-a(y)} \left( \int_0^{a(y)} \sinh x \, dx + \int_{a(y)}^{2\pi\varepsilon} x \frac{\sin c(x)}{c(x)} \, dx \right)$$

Since

$$\sinh x \geq x \geq x \frac{\sin c(x)}{c(x)} = \frac{d}{dx} \cos c(x), \quad x \geq 0$$
we get \(|y| \in [0, \varepsilon]\)

\[
\begin{align*}
f_y(\sqrt{\varepsilon^2 - y^2}) & \geq 2e^{-a(y)} \int_0^{2\pi \varepsilon} x \frac{\sin c(x)}{c(x)} \, dx \\
& \geq 2e^{-2\pi \varepsilon} \int_0^{2\pi \varepsilon} \frac{d}{dx} \cos c(x) \, dx = 2e^{-2\pi \varepsilon} (1 - \cos 2\pi \varepsilon)
\end{align*}
\]

Thus, in region \(I\), for each \(\varepsilon \in (0, 1/2)\), we have

\[
2e^{-2\pi \varepsilon} (1 - \cos 2\pi \varepsilon) = \left(2e^{-\pi \varepsilon} \sin \pi \varepsilon\right)^2 \leq f_y(x)
\]

In region \(II\), \(|y| \geq \varepsilon\), and we have

\[
(1 - e^{-2\pi \varepsilon})^2 \leq \left(1 - e^{-2\pi |y|}\right)^2 = 1 + e^{-4\pi |y|} - 2e^{-2\pi |y|} = f_y(0) \leq f_y(x)
\]

To summarize, for each \(\varepsilon \in (0, 1/2)\) and for all \(z = x + iy \in M\), we have the estimate

\[
4c_\varepsilon^2 \leq f_y(x) = 4e^{-2\pi |y|} |\sin \pi (x + iy)|^2
\]

where

\[
c_\varepsilon = \min \{e^{-\pi \varepsilon} \sin \pi \varepsilon, e^{-\pi \varepsilon} \sin \pi \varepsilon\} = e^{-\pi \varepsilon} \sin \pi \varepsilon
\]

Thus, we have proved that there exists a constant \(c_\varepsilon\), such that

\[
c_\varepsilon e^{\pi |\text{Im} z|} \leq |\sin \pi z|, \quad z \in M
\]

which by periodicity holds for all \(\{z \in \mathbb{C} : |z + n| > \varepsilon, \forall n \in \mathbb{Z}\}\), and the lemma is proved. \(\square\)

The argument of the sine function can also be shifted leading to the following corollary:

**Corollary A.1.** For each \(\varepsilon \in (0, 1/2)\) and \(\alpha \in \mathbb{C}\), there are positive constants, \(c_{\varepsilon, \alpha}\) and \(C_{\varepsilon, \alpha}\), depending only on \(\varepsilon\) and \(\alpha\), but not on \(z\), such that the sine function satisfies

\[
c_{\varepsilon, \alpha} \leq |\sin \pi (z + \alpha)| e^{-\pi |\text{Im} z|} \leq C_{\varepsilon, \alpha}, \text{ in } \{z \in \mathbb{C} : |z + \alpha + n| > \varepsilon, \forall n \in \mathbb{Z}\}
\]

**Proof.** Lemma A.1 applied with \(z + \alpha\) reads

\[
c_\varepsilon \leq |\sin \pi (z + \alpha)| e^{-\pi |\text{Im}(z + \alpha)|} \leq C_\varepsilon, \text{ in } \{z \in \mathbb{C} : |z + \alpha + n| > \varepsilon, \forall n \in \mathbb{Z}\}
\]

The lemma follows from the triangle inequality

\[
|\text{Im} z| - |\text{Im} \alpha| \leq |\text{Im}(z + \alpha)| \leq |\text{Im} z| + |\text{Im} \alpha|
\]

which leads to

\[
c_{\varepsilon, \alpha} = e^{-\pi |\text{Im} \alpha|} c_\varepsilon
\]
A.2 Estimates of the gamma function

Fig. A.4 The contour $C$ in the complex plane that connects the two points $z$ and $z + \alpha$, assuming $\text{Im} \alpha > 0$. The shaded area denotes the region between the two cuts.

and

$$C_{\varepsilon, \alpha} = e^{\pi |\text{Im} \alpha|} C_{\varepsilon}$$

and the proof is completed. □

We have also need for the following useful result:

**Lemma A.2.** For arbitrary $\alpha, \gamma \in \mathbb{C}$, the real part of the function

$$f(z) = (z + \gamma) (\ln(z + \alpha) - \ln z)$$

is a bounded function in $\Gamma_{\varepsilon, \alpha} = \{ z \in \mathbb{C} : |z + \alpha| \geq \varepsilon, |z| \geq \varepsilon \}$, where $\varepsilon > 0$, and where the logarithms belong to the principal branch.$^4$ Explicitly, there exist constants $c_{\varepsilon, \alpha, \gamma}$ and $C_{\varepsilon, \alpha, \gamma}$ that depend only on $\varepsilon$, $\alpha$, and $\gamma$, but not on $z$, such that

$$c_{\varepsilon, \alpha, \gamma} \leq \text{Re} f(z) \leq C_{\varepsilon, \alpha, \gamma}, \text{ in } \Gamma_{\varepsilon, \alpha} \quad (A.7)$$

**Proof.** We start with the identity

$$\ln(z + \alpha) - \ln z = \int_{C} \frac{dr}{t}, \quad z \in \Gamma_{\varepsilon, \alpha}$$

where $C$ is a contour in the complex $t$-plane that connects the two points $z$ and $z + \alpha$ without crossing the two branch cuts, $|\text{arg}(z)| = \pi$ and $|\text{arg}(z + \alpha)| = \pi$, see Figure A.4.

If the two points, $z$ and $z + \alpha$, lie on the same side of the two cuts, i.e., both $z$ and $z + \alpha$ are located outside the shaded area in Figure A.4, we can connect the points with a straight line, and the real part of the function $f(z)$ can be estimated for large

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$^4$ The principal branch of the logarithm is defined as

$$\ln z = \ln |z| + i \text{arg}(z), \quad \text{arg}(z) \in (-\pi, \pi]$$
arguments as\(^5\)

\[
|\text{Re} f(z)| = \left| \text{Re} \int_0^1 \frac{(z + \gamma) \alpha \, dt}{z + t \alpha} \right| \leq |\alpha| \int_0^1 \frac{|1 + \gamma/z|}{|1 + t \alpha/z|} \, dt \leq |\alpha| \left(1 + O\left(|z|^{-1}\right)\right)
\]

which is bounded in \(\Gamma_{\varepsilon, \alpha}\).

However, if the two points, \(z\) and \(z + \alpha\), are located on opposite sides of a branch cut, i.e., either \(z\) or \(z + \alpha\) is located in the shaded area in Figure A.4, we have to compensate with a circle around the branch point, see Figure A.5, in order to end the contour \(C\) on the principal branch. The real part of the function \(f(z)\) then is

\[
\text{Re} f(z) = \text{Re} \left\{ (z + \gamma) \left( \int_0^1 \frac{\alpha \, dt}{z + t \alpha} \pm 2\pi i \right) \right\}, \quad z \in \Gamma_{\varepsilon, \alpha}
\]

where the upper or lower sign in the last term depends on whether \(\text{Im} \alpha > 0\) or \(\text{Im} \alpha < 0\), respectively. The real part of the extra term, however, is always bounded, since \(\text{Im} z\) (in contrast to \(\text{Re} z\)) is bounded in this case, and the lemma is proved. \(\square\)

The next lemma estimates the growth properties of the gamma function from both above and below.

**Lemma A.3.** For each \(\varepsilon \in (0, 1/2)\), there are positive constants, \(c_\varepsilon\) and \(C_\varepsilon\), depending only on \(\varepsilon\), but not on \(z\), such that the gamma function satisfies\(^6\)

\[
c_\varepsilon \leq |\Gamma(z)| e^{-\text{Re}(z-\frac{1}{2}) \ln |z| + \text{Im} z \text{arg}(z) + \text{Re} z} \leq C_\varepsilon, \text{ in } \{z \in \mathbb{C} : |z + n| > \varepsilon, \forall n \in \mathbb{N}\}
\]

**Proof.** We first prove the statement in the lemma in the right-hand \(z\)-plane, or more precisely, for all \(z \in \{z \in \mathbb{C} : \text{Re} z \geq 0 \text{ and } |z| > \varepsilon\}\). For all such \(z\), Stirling’s formula, (A.6), reads

\[
\ln \Gamma(z) = \ln \sqrt{2\pi} + \left(z - \frac{1}{2}\right) \ln z - z + O\left(1/|z|\right)
\]

\(^5\) In fact, not only the real part of the function is bounded, but also the modulus of the function itself.

\(^6\) Note that \(\text{Im} z \text{arg}(z)\) is continuous across the negative real axis.
A.2 Estimates of the gamma function

The real part of this expression is

$$\Re \ln \Gamma(z) = \ln \sqrt{2\pi} + \Re \left( z - \frac{1}{2} \right) \ln |z| - \Im \arg(z) - \Re z + O(1/|z|)$$

and, therefore, for each $\epsilon > 0$, there exist constants $C_\epsilon > c_\epsilon > 0$, depending only on $\epsilon$, but not on $z$, such that

$$c_\epsilon \leq |\Gamma(z)| e^{\Re \left(z - \frac{1}{2} \right) \ln |z| + \Im \arg(z) + \Re z} \leq C_\epsilon$$

for all $z \in \{ z \in \mathbb{C} : \Re z \geq 0 \text{ and } |z| > \epsilon \}$. This proves the estimate in the right-hand complex $z$-plane. In fact, this estimate holds for all $z$ such that $|z| > \epsilon$ and $|\arg(z)| < \pi - \delta$, where $\delta > 0$. However, we also want to estimate the gamma function on the negative real axis away from the singular points at the non-positive integers.

We now focus on an estimate in the left-hand $z$-plane, more precisely, for an $\epsilon \in (0, 1/2)$, and for $z \in M_\epsilon = \{ z \in \mathbb{C} : |z + n| > \epsilon, \forall n \in \mathbb{N} \} \cap \{ z \in \mathbb{C} : \Re z < 0 \}$. We start by utilizing (A.4), i.e.,

$$\Gamma(z) = \pi \frac{e^{\Re \left( \frac{1}{2} - z \right) \ln |1 - z| - \Im \arg(1 - z)} - \Re(1 - z)}{\sin \pi z} \quad (A.8)$$

Note that $\Re(1 - z) > 1$ when $z \in M_\epsilon$, and therefore, by the result above, there exists constants $C_1 > c_1 > 0$, depending only on $\epsilon$, but not on $z$, such that

$$c_1 \leq \frac{1}{|\Gamma(1 - z)|} e^{\Re \left( \frac{1}{2} - z \right) \ln |1 - z| + \Im \arg(1 - z) - \Re(1 - z)} \leq C_1, \quad z \in M_\epsilon$$

Using Lemma A.1 we can estimate (A.8) in $M_\epsilon$. There exist constants $C_2 > c_2 > 0$, depending only on $\epsilon$, but not on $z$, such that

$$c_2 \leq |\Gamma(z)| e^{\Re \left( \frac{1}{2} - z \right) \ln |1 - z| + \Im \arg(1 - z) + |\Im z| \pi + \Re z} \leq C_2, \text{ in } M_\epsilon$$

As a consequence of Lemma A.2, see (A.7), there exist constants $C_3$ and $c_3$, depending only on $\epsilon$, but not on $z$, such that

$$c_3 \leq \Re \left( \frac{1}{2} - z \right) (\ln |1 - z| - \ln |z|) - \Im(z) (\arg(1 - z) - \arg(-z)) \leq C_3$$

and we get

$$c_\epsilon \leq |\Gamma(z)| e^{\Re \left(z - \frac{1}{2} \right) \ln |z| + \Im \arg(z) - \Re z + |\Im z| \pi + \Re z} \leq C_\epsilon$$

or

$$c_\epsilon \leq |\Gamma(z)| e^{\Re \left(z - \frac{1}{2} \right) \ln |z| + \Im \arg(z) + \Re z} \leq C_\epsilon$$

since
The combination of the results in the right- and the left-hand $z$-planes then proves the lemma. □

It is also possible to bound the shifted gamma function.

**Corollary A.2.** For each $\alpha \in \mathbb{C}$ and each $\varepsilon \in (0, 1/2)$, there are positive constants, $c_{\varepsilon, \alpha}$ and $C_{\varepsilon, \alpha}$, depending only on $\varepsilon$ and $\alpha$, but not on $z$, such that the gamma function satisfies

$$c_{\varepsilon, \alpha} \leq |\Gamma(z + \alpha)| e^{-\text{Re}(\frac{z+\alpha-1}{2}) \ln|z| + \text{Im} \arg(z) + \text{Re}z} \leq C_{\varepsilon, \alpha}, \text{ in } M_{\varepsilon, \alpha}$$

where $M_{\varepsilon, \alpha} = \{ z \in \mathbb{C} : |z + \alpha + n| > \varepsilon, |z| > \varepsilon, \forall n \in \mathbb{N} \}$ and $|\arg(z)| \leq \pi$.

**Proof.** From Lemma A.3 we have

$$c_{\varepsilon} \leq |\Gamma(z + \alpha)| e^{-\text{Re}(\frac{z+\alpha-1}{2}) \ln|z+\alpha| + \text{Im}(z+\alpha) \arg(z + \alpha) + \text{Re}(z + \alpha)} \leq C_{\varepsilon}, \quad z \in M_{\varepsilon, \alpha}$$

Lemma A.2, see (A.7), implies that there exist constants $c_1$ and $C_1$, depending only on $\varepsilon$ and $\alpha$, but not on $z$, such that

$$c_1 \leq \text{Re} \left( z + \alpha - \frac{1}{2} \right) (\ln |z + \alpha| - \ln |z|) - \text{Im} (z + \alpha) \left( \arg(z + \alpha) - \arg(z) \right) \leq C_1$$

$$\text{Re} \left\{ (z + \alpha - \frac{1}{2}) (\ln(z + \alpha) - \ln z) \right\}$$

We get

$$c_{\varepsilon, \alpha} \leq |\Gamma(z + \alpha)| e^{-\text{Re}(\frac{z+\alpha-1}{2}) \ln|z| + \text{Im} \arg(z) + \text{Re}z} \leq C_{\varepsilon, \alpha}, \quad z \in M_{\varepsilon, \alpha}$$

and the corollary is proved. □

The quotient between two gamma functions are frequently used in the text. The next lemma summarizes the growth properties.

**Lemma A.4.** Define for each $\varepsilon \in (0, 1/2)$ and $\alpha \in \mathbb{C}$ the set, see Figure A.6,

$$C_{\varepsilon, \alpha} = \{ z \in \mathbb{C} : |z + \alpha + n| > \varepsilon \text{ and } |z + n| > \varepsilon, \forall n \in \mathbb{N} \}$$

Then for any $\alpha \in \mathbb{C}$, and all $z \in C_{\varepsilon, \alpha}$

$$c_{\varepsilon, \alpha} e^{\text{Re} \alpha \ln |z|} \leq \left| \frac{\Gamma(\alpha + z)}{\Gamma(z)} \right| \leq C_{\varepsilon, \alpha} e^{\text{Re} \alpha \ln |z|}$$

where $C_{\varepsilon, \alpha} > c_{\varepsilon, \alpha} > 0$ are constants that depend on $\varepsilon$ and $\alpha$, but not on $z$. 

A.3 The Appell symbol

Related to the gamma function is the rising factorial, also known as the Appell symbol or Pochhammer symbol, \((\alpha, n)\), defined for non-negative integers \(n\) as [4]

\[
(\alpha, n) = \prod_{\nu=0}^{n-1} (\nu + \alpha) = \alpha(\alpha + 1)(\alpha + 2)\ldots(\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \quad , \quad (\alpha, 0) = 1
\]

As a consequence of this definition, we easily derive the following simple recursion relations:

\[
\begin{align*}
(\alpha, n) &= \alpha(\alpha + 1, n - 1) \\
(\alpha, n)(\alpha + n) &= (\alpha, n + 1) = \alpha(\alpha + 1, n)
\end{align*}
\]

(A.9)
The Appell symbol for negative integers $n$ can be defined by repeated use of (A.4). We have for non-integer values of $\alpha$

$$
\Gamma(\alpha-n) = \frac{1}{\Gamma(1-\alpha+n)} \frac{\pi}{\sin(\pi(\alpha-n))} = \frac{(-1)^n \pi}{\Gamma(1-\alpha+n) \sin(\pi(\alpha-n))}
$$

(A.10)

from which we define

$$
(\alpha,-n) = \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha,n)}
$$

(A.11)

Similarly,

$$
(-n,n) = (-n)(-n+1)(-n+2)\ldots(-1) = (-1)^n n!
$$

(A.12)

In the main text the growth rate of the Appell symbol is often analyzed. The following lemma is then useful:

**Lemma A.5.** For any $\alpha \in \mathbb{C}$, the Appell symbol satisfies

$$
\frac{\Gamma(\alpha,n)}{n!} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} (1 + O(1/n))
$$

**Proof.** We investigate the quotient

$$
\ln \frac{\Gamma(\alpha,n)}{\Gamma(\alpha)n!n^{\alpha-1}} = \ln \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!n^{\alpha-1}} = \ln \Gamma(\alpha+n) - \ln \Gamma(n+1) - (\alpha - 1) \ln n
$$

as $n \to \infty$. Stirling’s formula, (A.6), implies for integer values of $n$

$$
\ln \frac{\Gamma(\alpha,n)}{\Gamma(\alpha)n!n^{\alpha-1}} = \left(\alpha + n - \frac{1}{2}\right) \ln(\alpha+n) - \left(n + \frac{1}{2}\right) \ln(n+1)
$$

$$
-(\alpha - 1)(1 + \ln n) + O(1/n)
$$

$$
= \left(\alpha + n - \frac{1}{2}\right) \ln \frac{\alpha+n}{n+1} - (\alpha - 1) \left(1 - \ln \frac{n+1}{n}\right) + O(1/n)
$$

$$
= \left(\alpha + n + 1 - \frac{3}{2}\right) \ln \left(1 + \frac{n+1}{n}\right) + O(1/n)
$$

since $\ln(1+z) = z + O(z^2)$. Therefore,

$$
\ln \frac{\Gamma(\alpha,n)}{\Gamma(\alpha)n!n^{\alpha-1}} = -\ln \Gamma(\alpha) + O(1/n)
$$

or

$$
\frac{\Gamma(\alpha,n)}{n!n^{\alpha-1}} = \frac{1}{\Gamma(\alpha)} (1 + O(1/n))
$$
A.5 Binomial coefficient

since

\[ e^{O(1/n)} = 1 + O(1/n) \]

and the lemma is proved. \[\square\]

A.4 Psi (digamma) function

The logarithmic derivative of the gamma function is often used, and it is called the psi (digamma) function.

\[ \psi(z) = \frac{d}{dz} \ln \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \]

A relation similar to the relation (A.4) also holds for the psi function. We have, see Problem A.4,

\[ \psi(1-z) = \psi(z) + \pi \cot \pi z \quad (A.13) \]

The \( \psi \) function has closed form expressions for the half-integer values, i.e.,

\[ \psi(n + 1/2) = -\gamma - 2\ln 2 + 2 \left( 1 + \frac{1}{3} + \ldots + \frac{1}{2n-1} \right), \quad n \in \mathbb{Z}_+ \]

where \( \gamma \) is the Euler–Mascheroni constant, which is defined in (A.18). For negative half integers we use (A.13), and get

\[ \psi(-n + 1/2) = \psi(n + 1/2), \quad n \in \mathbb{Z}_+ \]

A.5 Binomial coefficient

Related to the gamma function is the binomial coefficient. For non-negative integers \( n \) and \( k \), the binomial coefficient is defined as

\[ \binom{n}{k} = \frac{n!}{(n-k)!k!} \]

which we extend to all integer values \( k \) by

\[ \binom{n}{k} = 0 \quad \text{if } k < 0 \text{ or } k > n \]

The binomial coefficient for a non-integer value \( \alpha \) can be expressed in the gamma function, viz.,
\[ \left( \frac{\alpha}{n} \right) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 - n)n!} \]  
\[ (A.14) \]

and, using (A.4) and \( z \Gamma(z) = \Gamma(z + 1) \),

\[
\left( -\frac{\alpha}{n} \right) = \frac{\Gamma(1 - \alpha)}{\Gamma(1 - \alpha - n)n!} = \frac{\pi}{\sin \pi \alpha} \frac{1}{\Gamma(\alpha)\Gamma(1 - \alpha - n)n!}
= (-1)^{n-1} \frac{\pi}{\sin \pi \alpha} \frac{1}{\Gamma(\alpha)(-\alpha - 1) \cdots (n - 1 + \alpha)}
= (-1)^{n-1} \frac{\pi}{\sin \pi \alpha} \frac{(1 + \alpha)(2 + \alpha) \cdots (n - 1 + \alpha)}{\Gamma(\alpha)\Gamma(-\alpha)n!}
= (-1)^{n-1} \frac{\pi}{\sin \pi \alpha} \frac{(1 + \alpha)(2 + \alpha) \cdots (n - 1 + \alpha)\Gamma(1 + \alpha)}{\Gamma(\alpha)n!}
= (-1)^{n-1} \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)n!} = (-1)^n \frac{\beta(\alpha, n)}{n!}
\]
\[ (A.15) \]

**A.6 The beta function \( B(x, y) \)**

The beta function \( B(x, y) \) is defined as

\[ B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad \text{Re} \, x, \text{Re} \, y > 0 \]  
\[ (A.16) \]

It can be proved that this integral is a quotient of gamma functions. We have, see Problem A.3,

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \]  
\[ (A.17) \]

**A.7 Euler–Mascheroni constant**

The Euler–Mascheroni constant \( \gamma \) is defined as

\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \right) = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \approx 0.5772156649 \ldots \]  
\[ (A.18) \]

This definition is equivalent to

\[ \gamma = \sum_{k=1}^{\infty} \left\{ \frac{1}{k} - \ln \left( 1 + \frac{1}{k} \right) \right\} \]  
\[ (A.19) \]
In fact, the partial sum of the expression can easily be rewritten as

\[
\sum_{k=1}^{n} \left\{ \frac{1}{k} - \ln \left( 1 + \frac{1}{k} \right) \right\} = \sum_{k=1}^{n} \frac{1}{k} - \ln \left( \prod_{k=1}^{n} \frac{k+1}{k} \right) = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \ln n + \ln \frac{n}{n+1}
\]

from which the equivalence follows.

From the definition of \( \gamma \) in equation (A.18) and Lemma B.5 on page 193, with \( f(x) = 1/x \) and \( m = 1 \), we get (sharp inequalities in this explicit example)

\[
\frac{1}{n} < \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} \frac{dx}{x} = \sum_{k=1}^{n} \frac{1}{k} - \ln n < 1
\]

and in the limit as \( n \to \infty \), this gives \( 0 < \gamma < 1 \) (the limits are not reached in the limit process). A more sharp estimate of \( \gamma \) is obtained in Problem A.6.

### Problems

**A.1.** \( ^{\dagger} \) Prove equation (A.4), i.e.,

\[
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}
\]

**A.2.** \( ^{\dagger} \) Starting from (A.1), prove

\[
\Gamma(z) = \frac{i}{2\sin \pi z} \int_{C} e^{-t}(-t)^{-z-1} dt, \quad \forall z \in \mathbb{C}
\]

and

\[
\frac{1}{\Gamma(z)} = \frac{i}{2\pi} \int_{C} e^{-t}(-t)^{-z} dt, \quad \forall z \in \mathbb{C}
\]

**A.3.** \( ^{\dagger} \) Prove the relation between the beta function and the product of gamma functions in (A.17), i.e.,

\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

**A.4.** Show (A.13), i.e.,

\[
\psi(1-z) = \psi(z) + \pi \cot \pi z
\]

**A.5.** Show (A.5), i.e.,

\[
\Gamma(z)\Gamma(z+1/2) = \frac{\sqrt{\pi}}{2^{2z-1}} \Gamma(2z)
\]
A.6. Show that the Euler–Mascheroni constant \( \gamma \) satisfies

\[
0.998 \gamma \approx 0.575804 \approx \frac{3}{2} - \frac{4}{3} \ln 2 < \gamma < \frac{8}{3} - 2 \ln 2 - \frac{7}{12} \zeta(3) \approx 0.579172 \approx 1.003 \gamma
\]

where \( \zeta(3) \) is the Riemann zeta function, see Section B.4.1 on page 194. Compare this estimate with the exact value \( \gamma \approx 0.5772156649 \ldots \).

**Hint:** Use the inequality

\[
1 + \frac{1}{12} x^2 + \frac{1}{x + 1/4} < \left( \frac{x + 1}{2} \right) \ln \left( 1 + \frac{1}{x} \right) < 1 + \frac{1}{12} x^2 + \frac{1}{x}, \quad x > 0
\]
Appendix B
Difference equations

The asymptotic behavior of solutions to difference equations or recursion relations is the subject of this appendix. For simplicity, we restrict ourselves to second order recursion relations, which is the situation met in this textbook, and we examine the asymptotic behavior of their solutions for large index values. Some of the results are presented without proofs, and in these cases we give references to the relevant literature.

B.1 Second order recursion relations

The second order recursion relation of interest is

\[
\begin{align*}
    a_{n+1} &= A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{Z}_+ \\
    a_1 &= A_0 a_0
\end{align*}
\]  

(B.1)

or, if we define \( B_0 = 0 \),

\[
a_{n+1} = A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{N}
\]

The following lemma shows that the coefficients \( a_n \) can be written in terms of a determinant times the first coefficient \( a_0 \).

**Lemma B.1.** Let the coefficients \( \{a_n\}_{n=0}^{\infty} \) for given \( a_0 \), be generated by

\[
a_{n+1} = A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{Z}_+
\]

and initialized by

\[
a_1 = A_0 a_0
\]

Denote \( D_n = a_n / a_0 \). Then \( D_{n+1} \) is expressed as the determinant
Difference equations

\[ D_{n+1} = \begin{vmatrix} A_0 -1 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\ B_1 & A_1 -1 & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & B_2 & A_2 -1 & 0 & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \ldots & \ldots & \ldots & \ldots & A_{n-1} -1 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & B_n & A_n \end{vmatrix} \]

and, thus, the determinants satisfy the same recursion relation as the coefficients \( a_n \), i.e.,

\[ D_{n+1} = A_n D_n + B_n D_{n-1}, \quad n \in \mathbb{Z}_+ \]

**Proof.** First let \( n = 0 \). The relation is then

\[ A_0 = \frac{A_0 a_0}{a_0} = \frac{a_1}{a_0} = D_1 \]

which proves the statement is consistent for \( n = 0 \).

We prove this lemma by induction over \( n \in \mathbb{Z}_+ \). The relation for \( n = 1 \) is

\[ \begin{vmatrix} A_0 -1 \\ B_1 & A_1 \end{vmatrix} = A_0 A_1 + B_1 = \frac{A_0 a_1 a_0 + B_1 a_0}{a_0} = \frac{a_1 a_1 + B_1 a_0}{a_0} = \frac{a_2}{a_0} = D_2 \]

which proves the induction statement for \( n = 1 \). Assume the lemma is true for \( k = 1, 2, \ldots, n \) and prove it for \( n + 1 \). We then get by expanding the determinant along the last column and using the induction assumption

\[ \begin{vmatrix} A_0 -1 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\ B_1 & A_1 -1 & 0 & 0 & \ldots & \ldots & \ldots \\ 0 & B_2 & A_2 -1 & 0 & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ \ldots & \ldots & \ldots & \ldots & A_{n-1} -1 & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & B_n & A_n \end{vmatrix} = A_n D_n + B_n D_{n-1} \]

We then have
and the lemma is proved. □

This following lemma shows that the original recursion relation in (B.1), under certain conditions, can be transformed into a form where the first coefficient \( A_n = 1 \).

**Lemma B.2.** Let the coefficients \( \{a_n\}_{n=0}^{\infty} \), for given \( a_0 \), be generated by

\[
a_{n+1} = A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{Z}_+
\]

and initialized by

\[
a_1 = A_0 a_0
\]

If \( A_k \neq 0, k \in \mathbb{N} \), then the sequence \( \{a_n\}_{n=0}^{\infty} \), for given \( a_0 \), can be found from the recursion relation

\[
b_{n+1} = b_n + C_n b_{n-1}, \quad n \in \mathbb{Z}_+
\]

and initialized by the same value as the original recursion relation, i.e.,

\[
b_0 = a_0
\]

where

\[
C_n = \frac{B_n}{A_n A_{n-1}}, \quad n \in \mathbb{Z}_+
\]

and

\[
a_n = A_{n-1} \cdots A_1 A_0 b_n, \quad n \in \mathbb{Z}_+
\]

**Proof.** The statement of the lemma is easily seen if we insert \( a_n = \alpha_n b_n \), assuming \( \alpha_0 = 1 \), i.e.,

\[
\alpha_{n+1} b_{n+1} = A_n \alpha_n b_n + B_n \alpha_{n-1} b_{n-1}, \quad n \in \mathbb{Z}_+
\]

and determine \( \alpha_n \) such that

\[
\alpha_{n+1} = A_n \alpha_n \quad \implies \quad \alpha_{n+1} = A_{n-1} \cdots A_1 A_0, \quad n \in \mathbb{N}
\]

The \( a_n \) coefficient then is

\[
a_n = A_{n-1} \cdots A_1 A_0 b_n, \quad n \in \mathbb{Z}_+
\]

We get after division of \( \alpha_{n+1} \) (note that \( \alpha_{n+1} \neq 0 \))

\[
b_{n+1} = b_n + \frac{B_n \alpha_{n-1}}{\alpha_{n+1}} b_{n-1} = b_n + \frac{B_n}{A_n A_{n-1}} b_{n-1}, \quad n \in \mathbb{Z}_+
\]
B Difference equations

B.2 Poincaré–Perron theory

The convergence of the quotient $a_{n+1}/a_n$ as $n \to \infty$ is crucial in the development of the theory presented in this book, especially when finding the radius of convergence for power series. This result is often referred to as the Poincaré–Perron theory. A comprehensive treatment of this problem is found in Ref. 6 and originates from pioneer works by Poincaré and Perron. In this section, we give an overview of the main results of this theory.

We immediately see that if the coefficients $A_n$ and $B_n$ in (B.1) satisfy

$$\begin{align*}
\lim_{n \to \infty} A_n &= A \\
\lim_{n \to \infty} B_n &= B
\end{align*}$$

and if the limit $\lim_{n \to \infty} a_{n+1}/a_n = \lambda$ exists, then $\lambda$ must satisfy (divide (B.1) by $a_{n-1}$ and take the limit $n \to \infty$)

$$\frac{a_{n+1}}{a_n} = \frac{a_{n+1}}{a_{n-1}} = A_n \frac{a_n}{a_{n-1}} + B_n \implies \lambda^2 = A\lambda + B$$

The following theorem is instrumental for the existence of the limit (proof omitted, see also [6, Sec. 8.5]):

**Theorem B.1 (Poincaré, Perron).** Let \( \{x_n\}_{n=0}^{\infty} \) be a sequence generated by the recursion relation

$$\begin{align*}
x_{n+1} &= A_n x_n + B_n x_{n-1}, \quad n \in \mathbb{Z}_+ \\
x_1 &= A_0 x_0
\end{align*}$$

where the coefficients have well-defined limits as $n \to \infty$, i.e.,

$$\begin{align*}
\lim_{n \to \infty} A_n &= A \\
\lim_{n \to \infty} B_n &= B
\end{align*}$$

and, moreover, that all $B_n \neq 0$ for all $n \in \mathbb{Z}_+$. The roots to the characteristic equation

$$\lambda^2 = A\lambda + B$$

are denoted $\lambda = \lambda_1, \lambda_2$, i.e.,

$$\lambda_{1,2} = A \pm (A^2 + 4B)^{1/2}$$

\[\text{1}\] Henri Poincaré (1854–1912), French mathematician and theoretical physicist.

\[\text{2}\] Oskar Perron (1880–1975), German mathematician.
We assume these roots have different moduli, say $|\lambda_1| < |\lambda_2|$. Then there exists a fundamental set of solutions, \( \{a_n\}_{n=0}^\infty \) and \( \{b_n\}_{n=0}^\infty \), such that
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda_2, \quad \text{and} \quad \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lambda_1
\]

As a consequence of the Poincaré–Perron theorem, Theorem B.1, we have

**Theorem B.2.** Let \( \{x_n\}_{n=0}^\infty \) be a sequence generated by the recursion relation
\[
\begin{cases}
x_{n+1} = A_n x_n + B_n x_{n-1}, & n \in \mathbb{Z}_+ \\
x_1 = A_0 x_0
\end{cases}
\] (B.2)

and
\[
\begin{cases}
\lim_{n \to \infty} A_n = A \\
\lim_{n \to \infty} B_n = B
\end{cases}
\]
Assume the roots, \( \lambda = \lambda_1, \lambda_2 \), of the characteristic equation
\[
\lambda^2 = A\lambda + B
\] (B.3)
i.e.,
\[
\lambda_{1,2} = \frac{A \pm (A^2 + 4B)^{1/2}}{2}
\]

have different moduli, say $|\lambda_1| < |\lambda_2|$. Then the limit $\lim_{n \to \infty} x_{n+1}/x_n$ always exists, and it is
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lambda_2
\]
except when the solution is a multiple of \( \{b_n\}_{n=0}^\infty \), given in Theorem B.1. Then the limit is
\[
\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lambda_1
\]

**Proof.** As a consequence of Theorem B.1, there exists a fundamental set of solutions, \( \{a_n\}_{n=0}^\infty \) and \( \{b_n\}_{n=0}^\infty \), such that
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lambda_2, \quad \text{and} \quad \lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lambda_1
\]

Let \( \mu_1 \) and \( \mu_2 \) be real numbers, such that $|\lambda_1| < \mu_1 < \mu_2 < |\lambda_2|$. Then there exists an integer \( N \), such that
\[
\left| \frac{a_{n+1}}{a_n} \right| \geq \mu_2, \quad \text{and} \quad \left| \frac{b_{n+1}}{b_n} \right| \leq \mu_1, \quad n \geq N
\]

3 The existence of a fundamental set implies that every solution of the recursion relation can be found as a linear combination of the elements in this set.
which imply

\[ |a_n| = \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \cdots \left| \frac{a_{N+1}}{a_N} \right| |a_N| \geq \mu_2^{n-N} |a_N|, \quad n \geq N \]

and

\[ |b_n| = \left| \frac{b_n}{b_{n-1}} \right| \left| \frac{b_{n-1}}{b_{n-2}} \right| \cdots \left| \frac{b_{N+1}}{b_N} \right| |b_N| \leq \mu_1^{n-N} |b_N|, \quad n \geq N \]

The quotient \( b_n/a_n \) then converges to zero, i.e.,

\[ \lim_{n \to \infty} \left| \frac{b_n}{a_n} \right| \leq \lim_{n \to \infty} \left( \frac{\mu_1}{\mu_2} \right)^{n-N} \left| \frac{b_N}{a_N} \right| = 0 \]

We say that the solution \( \{b_n\}_{n=0}^{\infty} \) is a minimal solution (sequence). We also have

\[ \lim_{n \to \infty} \left| \frac{b_{n+1}}{a_n} \right| \leq \lim_{n \to \infty} \left( \frac{\mu_1}{\mu_2} \right)^{n-N} \mu_1 \left| \frac{b_N}{a_N} \right| = 0 \]

The general solution to (B.2) is \( x_n = \alpha a_n + \beta b_n \) for some constants \( \alpha \) and \( \beta \). Then

\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lim_{n \to \infty} \frac{\alpha a_{n+1} + \beta b_{n+1}}{\alpha a_n + \beta b_n} = \lim_{n \to \infty} \frac{\alpha a_{n+1}/a_n + \beta b_{n+1}/a_n}{\alpha + \beta b_n/a_n} = \lambda_2 \]

This is the general limit value except when \( \alpha = 0 \) and \( x_n = \beta b_n \). Then the limit is

\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lambda_1 \]

and the theorem is proved. \( \square \)

The proof of the following theorem is presented on page 402 in Ref. 6, and it provides conditions on the coefficients \( A_n \) and \( B_n \) for the solution to be minimal.

**Theorem B.3 (Pincherle\(^4\)).** Let \( \{x_n\}_{n=0}^{\infty} \) be a sequence generated by the recursion relation

\[
\begin{cases}
  x_{n+1} = A_n x_n + B_n x_{n-1}, & n \in \mathbb{Z}_+ \\
  x_1 = A_0 x_0
\end{cases}
\]

where the coefficients have well-defined limits as \( n \to \infty \), i.e.,

\[
\begin{cases}
  \lim_{n \to \infty} A_n = A \\
  \lim_{n \to \infty} B_n = B
\end{cases}
\]

Then the continued fraction

\[ \text{continued fraction} \]

\(^4\) Salvatore Pincherle (1853–1936), Italian mathematician.
B.3 Asymptotic behavior of recursion relations

The asymptotic behavior of the sequence generated by a special type of recursion relation, (B.1), for large values of \( n \) is addressed in this section.\(^5\) The results are not so general as the results in Section B.2, but they suffice for our needs, the analysis is self-contained, and it uses only standard analysis arguments.

The start of the analysis is motivated by the following simple example.

**Example B.1.** If \( A_n = 1 + \alpha \) and \( B_n = -\alpha, n \in \mathbb{Z}_+ \), where \( \alpha \) is independent of \( n \), the sequences \( a_n = a_0 (A_0 = 1) \) and \( a_n = \alpha^n a_0 (A_0 = \alpha) \) are solutions to the recursion relation (B.1). If \( |\alpha| < 1 \), the solutions converge to \( a_0 \) and 0, respectively. \( \blacksquare \)

For \( a_n = a_0 \) to be a solution to the recursion relation (B.1), it suffices to require \( A_n + B_n = 1, n \in \mathbb{Z}_+ \) and \( A_0 = 1 \). With this observation in mind, we anticipate that the size of \( A_n + B_n - 1 \) as \( n \to \infty \) is essential for the convergence. That this really is the case is shown in this section.

We prefer to collect the results in two lemmas, a theorem, and a corollary. We start by proving the lemmas.

**Lemma B.3.** Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence generated by the recursion relation

\[
\begin{align*}
A_0 + \frac{B_1}{A_1} & \quad A_1 + \frac{B_2}{A_2} = \frac{1}{A_1 + \frac{B_2}{A_2}} \\
A_2 + \frac{B_3}{A_3} & \quad A_3 + \ldots
\end{align*}
\]

or in a compact notation

\[
A_0 + \frac{\Phi}{n=1} B_n = \frac{A_n}{A_{n-1}}A_{n-1}
\]

converges if and only if the sequence \( \{x_n\}_{n=0}^{\infty} \) (\( x_0 \neq 0 \)) is a minimal solution (sequence), i.e.,

\[
\lim_{n \to \infty} \frac{x_n}{a_n} = 0
\]

where the sequence \( a_n \) is given in Theorem B.1. Moreover, in the case of convergence

\[
\frac{B_1}{A_1} = \frac{B_2}{A_2} = \ldots = -\frac{x_1}{x_0} = -A_0
\]

(B.4)

---

\(^5\) The idea behind the approach presented here is due to Anders Melin.
where the coefficients $A_n$ and $B_n$ satisfy

\[
\begin{aligned}
\lim_{n \to \infty} A_n &= 1 + \alpha \\
\lim_{n \to \infty} B_n &= -\alpha
\end{aligned}
\]

and $|\alpha| < 1$. Moreover, let $\lambda > 1$ be a real number, such that

\[ R_n = A_n + B_n - 1 = O(n^{-\lambda}) \]

Then, for all initial values $a_0$, the sequence \( \{a_n\}_{n=0}^{\infty} \) converges to a limit $d$, and

\[ a_n = d + O(n^{-\lambda + 1}), \quad n \to \infty \]

**Proof.** Rewrite the recursion relation as

\[ a_{n+1} - a_n = R_n a_n - B_n (a_n - a_{n-1}), \quad n \in \mathbb{Z}_+ \quad (B.5) \]

and we start by proving that the sequence \( \{a_n\}_{n=0}^{\infty} \) is bounded.

Let $N$ be a positive integer such that

\[ |B_n| \leq c = \frac{1 + |\alpha|}{2}, \quad n \geq N \]

Notice that $0 < c < 1$ with the assumptions made in the lemma. We also have

\[ |R_n| \leq \frac{C}{n^\lambda}, \quad n \geq N \]

for some constant $C$. Define

\[ \varepsilon_n = |a_{n+1} - a_n|, \quad n \in \mathbb{N} \]

Then from (B.5)

\[ \varepsilon_n \leq \frac{C}{n^\lambda} |a_n| + c \varepsilon_{n-1}, \quad n \geq N \quad (B.6) \]

and

\[ \sum_{k=N}^{n} \varepsilon_k \leq \sum_{k=N}^{n} \frac{C}{k^\lambda} |a_k| + c \sum_{k=N}^{n} \varepsilon_k + c \varepsilon_{N-1}, \quad n \geq N \]

From this expression we conclude that

\[ \sum_{k=N}^{n} \varepsilon_k \leq C_1 \sum_{k=N}^{n} k^{-\lambda} |a_k| + C_2, \quad n \geq N \quad (B.7) \]

where
\[
\begin{aligned}
C_1 &= \frac{C}{1-c} > 0 \\
C_2 &= \frac{cE_{n-1}}{1-c} > 0
\end{aligned}
\]

Introduce the notation
\[
\hat{a}_n = \max_{0 \leq k \leq n} |a_k|, \quad n \in \mathbb{N}
\]
and use the inequality
\[
\hat{a}_{n+1} = \max_{0 \leq k \leq n} \{|a_k|, |a_{n+1}|\} \leq \varepsilon_n + \hat{a}_n
\]

Use (B.7), and we obtain
\[
\hat{a}_{n+1} \leq \hat{a}_N + \sum_{k=0}^{n-N} \varepsilon_{n-k} \leq \hat{a}_N + \sum_{k=N}^{n} \varepsilon_k, \quad n \geq N
\]

where \(C_3 = C_2 + \hat{a}_N\). Choose the integer \(N\) large enough so that
\[
C_1 \sum_{k=N}^{\infty} k^{-\lambda} \leq \frac{1}{2}
\]
and we get
\[
\hat{a}_{n+1} \leq \frac{1}{2} \hat{a}_n + C_3, \quad n \geq N
\]

By the use of the result in footnote 6, we get
\[
\hat{a}_{n+1} \leq \frac{1}{2^{n-N+1}} \hat{a}_N + C_3 \sum_{k=0}^{n-N} 2^{-k}, \quad n \geq N
\]

In this section we make frequent use of inequalities of the type
\[
x_{n+1} \leq ax_n + b_n, \quad n \geq N
\]

By induction over \(m\), we get
\[
x_{n+1} \leq a^{m+1} x_{n-m} + \sum_{k=0}^{m} a^k b_{n-k}, \quad 0 \leq m \leq n - N
\]
\[
x_{n+1} \leq a^{n-N+1} x_N + \sum_{k=0}^{n-N} a^k b_{n-k}, \quad n \geq N
\]
or
\[
\hat{a}_{n+1} \leq \hat{a}_N + 2C_3, \quad n \geq N
\]
which proves that the sequence \( \{a_n\}_{n=0}^{\infty} \) is bounded, and the first part of the proof is completed.

We are now able to use the boundedness of the sequence \( \{a_n\}_{n=0}^{\infty} \) to prove that this sequence is a Cauchy sequence of complex numbers, and therefore has a limit. Use (B.6), and we get for a suitable constant \( C \)
\[
\varepsilon_n \leq c\varepsilon_{n-1} + \frac{C}{n^\lambda}, \quad n \geq N
\]
Use the result in footnote 6 to get
\[
\varepsilon_n \leq c^{n-N+1}\varepsilon_{N-1} + C \sum_{k=N}^{n} \frac{c^{n-k}}{k^\lambda}, \quad n \geq N \tag{B.8}
\]
However, the sequence
\[
s_n = \sum_{k=1}^{n} \frac{c^k}{k^\lambda}
\]
is bounded, see Lemma B.8 on page 197, which implies that
\[
\sum_{k=1}^{n} \frac{c^{n-k}}{k^\lambda} \leq \frac{C'}{n^\lambda}
\]
for a suitable constant \( C' \). Therefore, \( \varepsilon_n \) in (B.8) is estimated as
\[
\varepsilon_n \leq c^{n-N+1}\varepsilon_{N-1} + \frac{C''}{n^\lambda}, \quad n \geq N
\]
which can be arbitrary small for all \( n \geq N' \), provided \( N' > N \) is large enough.

Moreover, the sequence \( \{a_n\}_{n=0}^{\infty} \) is a Cauchy sequence, since we have from the results above that\(^7\)
\[
|a_n - a_m| \leq \sum_{k=m}^{n-1} \varepsilon_k \leq \frac{c^{m-N+1} - c^{n-N+1}}{1 - c} \varepsilon_{N-1} + C'' \sum_{k=m}^{n-1} \frac{1}{k^\lambda}
\]
\[
\leq \frac{c^{m-N+1} - c^{n-N+1}}{1 - c} \varepsilon_{N-1} + \frac{\lambda}{\lambda - 1} \frac{C''}{m^{\lambda-1}}, \quad n > m \geq N' > N
\]
\(^7\) For example, use the estimate (B.9) \( n > m \geq 1 \) and \( \lambda > 1 \)
\[
\sum_{k=m}^{n-1} \frac{1}{k^\lambda} \leq \frac{1}{m^\lambda} + \int_{m}^{n-1} \frac{dx}{x^\lambda} = \frac{1}{m^\lambda} + \frac{1}{\lambda - 1} \left( \frac{1}{m^{\lambda-1}} - \frac{1}{(n-1)^{\lambda-1}} \right)
\]
\[
\leq \left( 1 + \frac{1}{\lambda - 1} \right) \frac{1}{m^{\lambda-1}} \leq \frac{\lambda}{\lambda - 1} \frac{1}{m^{\lambda-1}}
\]
B.3 Asymptotic behavior of recursion relations

which can be made arbitrarily small for \( n > m \geq N' \), provided \( N' \) is chosen sufficiently large. If the limit of the sequence \( \{a_n\}_{n=0}^{\infty} \) is denoted \( d \), we also have from above that

\[
|a_n - d| \leq \sum_{k=n}^{\infty} \varepsilon_k \leq \frac{c^{n-N+1}}{1-c} \varepsilon_{N-1} + \frac{\lambda}{\lambda - 1} \frac{C''}{n^{\lambda-1}}, \quad n \geq N'
\]

Therefore, \( a_n = d + O(n^{-\lambda + 1}) \), and the lemma is proved. \( \square \)

The next lemma shows how the convergence of specific combination of the coefficients in the recursion relation can be improved.

**Lemma B.4.** If \( \{a_n\}_{n=0}^{\infty} \) is a sequence generated by the recursion relation

\[
\begin{align*}
\{a_{n+1} = A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{Z}_+ \\
a_1 = A_0 a_0
\end{align*}
\]

where the coefficients \( A_n \) and \( B_n \) satisfy

\[
\begin{align*}
A_n &= \beta_0 + \frac{\beta_1}{n} + \frac{\beta_2}{n^2} + O(1/n^3) \\
B_n &= \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + O(1/n^3)
\end{align*}
\]

as \( n \to \infty \) where it is assumed that \( \beta_0 + 2\gamma_0 \neq 0 \). Then the sequence \( \{a'_n\}_{n=0}^{\infty} \) defined by

\[
a'_n = a_n \prod_{k=1}^{n} \left( 1 + \frac{c_1}{k} + \frac{c_2}{k^2} \right), \quad n \in \mathbb{Z}_+, \quad a'_0 = a_0
\]

where

\[
c_1 = -\frac{\beta_1 + \gamma_1}{\beta_0 + 2\gamma_0}, \quad c_2 = \frac{c_1(\beta_0 - 2\gamma_1 - \beta_1 + \gamma_0 - c_1\gamma_0) - \gamma_2 - \beta_2}{\beta_0 + 2\gamma_0}
\]

satisfies the recursion relation

\[
\begin{align*}
a'_{n+1} = A'_n a'_n + B'_n a'_{n-1}, \quad n \in \mathbb{Z}_+ \\
a'_1 = A'_0 a'_0
\end{align*}
\]

where \( A'_0 = A_0 \) and

\[
\begin{align*}
A'_n &= A_n \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \quad n \in \mathbb{Z}_+ \\
B'_n &= B_n \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} \right)
\end{align*}
\]

Moreover, the coefficients \( A'_n \) and \( B'_n \) satisfy
\[ A'_n + B'_n - \beta_0 - \gamma_0 = O(n^{-3}) \]

**Proof.** The idea behind the proof is to replace the sequence \( \{a_n\}_{n=0}^{\infty} \) by the sequence \( \{a'_n\}_{n=0}^{\infty} \) defined by

\[
a'_n = a_n \prod_{k=1}^{n} \left( 1 + \frac{c_1}{k} + \frac{c_2}{k^2} \right), \quad n \in \mathbb{Z}_+, \quad a'_0 = a_0
\]

where \( c_1 \) and \( c_2 \) are to be determined, such that the claims of the lemma are satisfied.

The new sequence \( \{a'_n\}_{n=0}^{\infty} \) satisfies

\[
\begin{cases}
    a'_{n+1} = A'_n a'_n + B'_n a'_{n-1}, & n \in \mathbb{Z}_+ \\
    a'_1 = A'_0 a'_0
\end{cases}
\]

where \( A'_0 = A_0 \) and

\[
\begin{align*}
    A'_n &= A_n \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \\
    B'_n &= B_n \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} \right)
\end{align*}
\]

If we insert the expansions of the coefficients \( A_n \) and \( B_n \), we get

\[
\begin{align*}
    A'_n &= \left( \beta_0 + \frac{\beta_1}{n} + \frac{\beta_2}{n^2} + O(1/n^3) \right) \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \\
    &= \beta_0 + \frac{\beta_1 + \beta_0 c_1}{n} + \frac{\beta_1 c_1 + \beta_2 + \beta_0 c_2 - \beta_0 c_1}{n^2} + O(1/n^3) \\
    B'_n &= \left( \gamma_0 + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + O(1/n^3) \right) \left( 1 + \frac{c_1}{n+1} + \frac{c_2}{(n+1)^2} \right) \left( 1 + \frac{c_1}{n} + \frac{c_2}{n^2} \right) \\
    &= \gamma_0 + \frac{\gamma_1 + 2 \gamma_0 c_1}{n} + \frac{\gamma_2 + \gamma_0 (2c_2 + c_1^2 - c_1) + 2 \gamma_1 c_1}{n^2} + O(1/n^3)
\end{align*}
\]

We also denote

\[ h_n = A'_n + B'_n - \beta_0 - \gamma_0 \]

and our aim is to choose \( c_1 \) and \( c_2 \) such that \( h_n = O(n^{-3}) \).

\[
h_n = \frac{\beta_1 + \beta_0 c_1 + \gamma_1 + 2 \gamma_0 c_1}{n} + \frac{\beta_1 c_1 + \beta_2 + \beta_0 c_2 - \beta_0 c_1 + \gamma_2 + \gamma_0 (2c_2 + c_1^2 - c_1) + 2 \gamma_1 c_1}{n^2} + O(1/n^3)
\]

and we see that the conditions in the lemma are proved, if we choose \( c_1 \) and \( c_2 \) as

\[
\beta_1 + c_1 (\beta_0 + 2 \gamma_0) + \gamma_1 = 0 \quad \Rightarrow \quad c_1 = -\frac{\beta_1 + \gamma_1}{\beta_0 + 2 \gamma_0}
\]
and
\[
\beta_1 c_1 + \beta_2 + \beta_0 c_2 - \beta_0 c_1 + \gamma_2 + \gamma_0 (2c_2 + c_1^2 - c_1) + 2\gamma_1 c_1 = 0
\]
which we simplify to
\[
c_2 = \frac{c_1 (\beta_0 - 2\gamma_1 - \beta_1 + \gamma_0 - c_1 \gamma_0) - \gamma_2 - \beta_2}{\beta_0 + 2\gamma_0}
\]
\[\square\]

The main theorem of the section that proves convergence of sequences can now be formulated.

**Theorem B.4.** Let \(\{a_n\}_{n=0}^{\infty}\) be a sequence generated by the recursion relation

\[
\begin{align*}
a_{n+1} &= A_n a_n + B_n a_{n-1}, \quad n \in \mathbb{Z}_+ \\
a_1 &= A_0 a_0
\end{align*}
\]

where the coefficients \(A_n\) and \(B_n\) satisfy

\[
\begin{align*}
A_n &= 1 + \alpha + \frac{\beta_1}{n} + \frac{\beta_2}{n^2} + O(1/n^3) \quad \text{as } n \to \infty \\
B_n &= -\alpha + \frac{\gamma_1}{n} + \frac{\gamma_2}{n^2} + O(1/n^3)
\end{align*}
\]

and \(|\alpha| < 1\). Then, provided the sequence \(\{a_n\}_{n=0}^{\infty}\) does not converge to zero, the sequence for large \(n\) behaves as

\[
a_n = C n^{(\beta_1 + \gamma_1)/(1 - \alpha)} \left(1 + \frac{c}{n} + O(1/n^2)\right), \quad n \to \infty
\]

for some constants \(C\) and \(c\).

**Proof.** We prove the theorem by applying Lemmas B.3 and B.4. With the notation and the results of these lemmas, there exists a sequence \(\{a'_n\}_{n=0}^{\infty}\) defined by (notice that \(1 + \alpha + 2(-\alpha) = 1 - \alpha \neq 0\))

\[
a'_n = a_n \prod_{k=1}^{n} \left(1 + \frac{c_1}{k} + \frac{c_2}{k^2}\right), \quad a'_0 = a_0
\]

\[
c_1 = -\frac{\beta_1 + \gamma_1}{1 - \alpha}, \quad c_2 = \frac{c_1 (1 - 2\gamma_1 - \beta_1 + c_1 \alpha) - \gamma_2 - \beta_2}{1 - \alpha}
\]

which is converging to a limit \(d \neq 0\), such that (\(\lambda = 3\))

\[
da'_n - d = O(n^{-2}), \quad \text{as } n \to \infty
\]

Note that required asymptotic behavior needed in Lemma B.4 is one order higher, here \(\lambda = 3\), than the result of Lemma B.3.

We write the original sequence as
\[ a_n = a'_n e^{q_n} = d e^{q_n} \left( 1 + O(n^{-2}) \right) \]

where
\[ q_n = - \sum_{k=1}^{n} \ln \left( 1 + \frac{c_1}{k} + \frac{c_2}{k^2} \right) \]

We proceed by finding an asymptotic expansion of \( q_n \) valid for large values of \( n \). Start with the Taylor expansion
\[ \ln \left( 1 + c_1 x + c_2 x^2 \right) = F(x) = \sum_{m=1}^{\infty} \frac{F^{(m)}(0)}{m!} x^m, \quad 0 \leq x \leq 1 \]

where \( F'(0) = c_1 \). We also need the asymptotic expansions of the following sums:
\[ \sum_{k=1}^{n} \frac{1}{k} = \ln n + c_{1,0} + \frac{c_{1,1}}{n} + O(n^{-2}) \]

and
\[ \sum_{k=1}^{n} \frac{1}{k^m} = c_{m,0} + \frac{c_{m,m-1}}{n^{m-1}} + O(n^{-m}) = c_{m,0} + \frac{c_{m,m-1}}{n^{m-1}} + O(n^{-2}), \quad m \geq 2 \]

We also define \( c_{m,j} = 0 \) for \( j = 1, 2, \ldots, m - 2, m \geq 3 \). Specific values of the constants are, see Lemmas B.6 and B.7

\[
\begin{align*}
\left\{ \begin{array}{l}
c_{1,0} = \gamma \\
c_{1,1} = \frac{1}{2}
\end{array} \right. \\
\left\{ \begin{array}{l}
c_{m,0} = \zeta(m) \\
c_{m,m-1} = \frac{1}{1-m}
\end{array} \right. \\
m \geq 2
\end{align*}
\]

where \( \gamma \) is the Euler–Mascheroni constant defined in (A.18) on page 176, and \( \zeta(m) = \sum_{n=1}^{\infty} n^{-m} \) is the Riemann zeta function. We get
\[
\sum_{k=1}^{n} \ln \left( 1 + \frac{c_1}{k} + \frac{c_2}{k^2} \right) = \sum_{k=1}^{n} \sum_{m=1}^{\infty} \frac{F^{(m)}(0)}{m!} k^{-m} = \sum_{m=1}^{\infty} \frac{F^{(m)}(0)}{m!} \sum_{k=1}^{n} k^{-m}
\]

\[
= F'(0) \ln n + \sum_{m=1}^{\infty} \frac{F^{(m)}(0)}{m!} \left( c_{m,0} + \frac{c_{m,1}}{n} + O(n^{-2}) \right)
\]

\[
= c_1 \ln n + C_0 + \frac{C_1}{n} + O(n^{-2})
\]

where \( O(n^{-2}) \) has the meaning \( |O(n^{-2})| \leq Cn^{-2} \), with \( C \) independent of \( m \). Moreover, the constants \( C_0 \) and \( C_1 \) are
B.4 Estimates of some sequences and series

\[
\begin{align*}
C_0 &= \sum_{m=1}^{\infty} \frac{F(m)(0)}{m!} c_{m,0} \\
C_1 &= \sum_{m=1}^{\infty} \frac{F(m)(0)}{m!} c_{m,1} = \sum_{m=1}^{2} \frac{F(m)(0)}{m!} c_{m,1}
\end{align*}
\]

and consequently

\[q_n = -c_1 \ln n - C_0 - C_1 n + O(n^{-2})\]

The sequence of interest then is

\[a_n = d e^{-c_1 \ln n - C_0 - C_1 n + O(n^{-2})} (1 + O(n^{-2})) = C n^{-c_1} \left(1 + \frac{c}{n} + O(n^{-2})\right)\]

for some constants \(C\) and \(c\), which concludes the proof. \(\square\)

**Corollary B.1.** With the assumptions in Theorem B.4, we have that

\[
\frac{a_{n+1}}{a_n} = 1 + \frac{\beta_1 + \gamma_1}{n(1-\alpha)} + O(1/n^2), \quad \text{as } n \to \infty
\]

**Proof.** Use the result of Theorem B.4, i.e.,

\[a_n = C n^{(\beta_1+\gamma_1)/(1-\alpha)} \left(1 + \frac{c}{n} + O(1/n^2)\right), \quad n \to \infty\]

which implies that

\[
\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n}\right)^{(\beta_1+\gamma_1)/(1-\alpha)} \frac{1 + \frac{c}{n+1} + O(1/n^2)}{1 + \frac{c}{n} + O(1/n^2)} = 1 + \frac{\beta_1 + \gamma_1}{n(1-\alpha)} + O(1/n^2)
\]

and the corollary follows. \(\square\)

Notice that the result of Corollary B.1 is consistent with the result of Theorem B.1 since \(\lambda_{1,2} = 1, \alpha\).

### B.4 Estimates of some sequences and series

For convenience we here collect a series of lemmas on estimates of sequences and series that are used above. We start by stating a general lemma that relates a series to the corresponding integral.

**Lemma B.5.** Let the real-valued function \(f(x)\) be non-increasing in the interval \([m,n], m,n \in \mathbb{N}\). Then

\[
f(n) \leq \sum_{k=m}^{n} f(k) - \int_{m}^{n} f(x) \, dx \leq f(m), \quad n \geq m \quad (B.9)
\]
Proof. This lemma is easily proved by estimating the Riemann sum of the integral, see Figure B.1. We have for integers \( m < n \)

\[
\sum_{k=m+1}^{n} f(k) \leq \int_{m}^{n} f(x) \, dx \leq \sum_{k=m}^{n-1} f(k)
\]

which is identical to the statement in the lemma. \( \square \)

\section*{B.4.1 Riemann zeta function}

\textbf{Lemma B.6.} Let \( \lambda > 1 \) be a real number. Then, the partial sum

\[
s_n = \sum_{k=1}^{n} \frac{1}{k^\lambda}, \quad n \in \mathbb{Z}_+
\]

has the following asymptotic expansion:

\[
s_n = s + \frac{n^{1-\lambda}}{1-\lambda} + O(n^{-\lambda})
\]

where \( s = \zeta(\lambda) \) is the Riemann zeta function.

\textit{Proof.} The sequence \( s_n \) is increasing, since \( s_{n+1} - s_n = (n+1)^{-\lambda} \geq 0 \). Moreover, the sequence \( s_n \) is bounded from above, which is proved using (B.9) with \( f(x) = x^{-\lambda} \) and \( m = 1 \).

\[
0 \leq s_n \leq \int_{1}^{n} x^{-\lambda} \, dx + 1 = \frac{1}{\lambda - 1} \left( 1 - n^{-\lambda+1} \right) + 1 < \frac{1}{\lambda - 1} + 1 = \frac{\lambda}{\lambda - 1}
\]

The sequence \( s_n \) therefore has a limit \( s \).

The remaining part of the lemma is proved using the notation
B.4 Estimates of some sequences and series

\( q_n = n^{\lambda-1} (s - s_n) = n^{\lambda-1} \sum_{k=n+1}^{\infty} \frac{1}{k^{\lambda}} \geq 0 \)

Use (B.9) with \( f(x) = n^{\lambda-1}x^{-\lambda}, \) \( m = n + 1, \) and the upper limit approaching infinity. We get as \( f(x) \to 0 \) as \( x \to \infty, \)

\[
0 \leq q_n - n^{\lambda-1} \int_{n+1}^{\infty} x^{-\lambda} \, dx \leq n^{\lambda-1}(n+1)^{-\lambda}
\]
or by evaluating the integral

\[
0 \leq q_n - \frac{1}{\lambda - 1} \left( 1 + \frac{1}{n} \right)^{1-\lambda} \leq \frac{1}{n} \left( 1 + \frac{1}{n} \right)^{-\lambda}
\]

which proves that

\[
q_n = \frac{1}{\lambda - 1} + O(n^{-1}), \quad \text{as } n \to \infty
\]

and

\[
s_n = s - q_n n^{1-\lambda} = s + \frac{n^{1-\lambda}}{1 - \lambda} + O(n^{-\lambda})
\]

and the lemma is proved. \( \square \)

B.4.2 The sum \( \sum_{k=1}^{n} k^{-1} \)

The following lemma shows the asymptotic behavior of the series \( \sum_{k=1}^{n} k^{-1}, \) which also is used above:

**Lemma B.7.** Define the partial sums

\[
s_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{Z}_+
\]

Then the sequence \( s_n \) has the asymptotic expansion

\[
s_n = \ln n + \gamma + \frac{1}{2n} + O(n^{-2})
\]

where the Euler–Mascheroni constant \( \gamma \) is

\[
\gamma = \lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \frac{1}{k} - \ln n \right\}
\]

A more detailed treatment of the Euler–Mascheroni constant is presented in Appendix A on page 176.
Proof. Introduce the notation
\[ q_n = \sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n} \frac{dx}{x} = s_n - \ln n \]

The sequence \( q_n \) is a decreasing sequence since
\[ q_n - q_{n-1} = \frac{1}{n} - \int_{n-1}^{n} \frac{dx}{x} = \frac{1}{n} \ln \left( 1 - \frac{1}{n} \right) \leq 0 \]
since \( \ln(1+x) \leq x \) for \( x \geq -1 \). Utilize the inequality (B.9) with \( f(x) = 1/x \) and \( m = 1 \),
\[ 0 < \frac{1}{n} \leq q_n \leq 1, \quad n \in \mathbb{Z}_+ \quad \text{(B.10)} \]
which shows that the decreasing sequence \( q_n \) is bounded from below, and therefore the limit \( q_n \to \gamma \) as \( n \to \infty \) exists, i.e.,
\[ \gamma = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} - \ln n \right\} = \sum_{k=1}^{\infty} \left\{ \frac{1}{k} - \ln \left( 1 + \frac{1}{k} \right) \right\} \]
where we in the last equality have used (A.19) on page 176.

To proceed, rewrite
\[ \sum_{k=1}^{n} \left\{ \frac{1}{k} - \ln \left( 1 + \frac{1}{k} \right) \right\} = \sum_{k=1}^{n} \frac{1}{k} - \ln \left( \prod_{k=1}^{n} \frac{k+1}{k} \right) = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \]
\[ = \sum_{k=1}^{n} \frac{1}{k} - \ln n + \ln \frac{n}{n+1} = q_n - \ln \left( 1 + \frac{1}{n} \right) \]
and we get
\[ \gamma - q_n + \ln \left( 1 + \frac{1}{n} \right) = \sum_{k=n+1}^{\infty} \left\{ \frac{1}{k} - \ln \left( 1 + \frac{1}{k} \right) \right\} \]
Now use the inequality (B.9) with
\[ f(x) = n \left( \frac{1}{x} - \ln \left( 1 + \frac{1}{x} \right) \right) \geq 0, \quad x > 1 \]
Letting the upper limit in (B.9) approach infinity, \( m = n + 1 \), and using \( f(x) \to 0 \) as \( x \to \infty \) for each \( n \), we get
\[ 0 \leq n \left\{ \gamma - q_n + \ln \left( 1 + \frac{1}{n} \right) - \int_{n+1}^{\infty} \left\{ \frac{1}{x} - \ln \left( 1 + \frac{1}{x} \right) \right\} dx \right\} \]
\[ \leq -\frac{n}{n+1} - n \ln \left( 1 + \frac{1}{n+1} \right) = O(1/n) \]
since \( \ln(1 + x) = x - x^2/2 + O(x^3) \) as \( x \to 0 \). We conclude that
\[
n(\gamma - q_n) + 1 = n \int_{n+1}^{\infty} \left\{ \frac{1}{x} - \ln \left( 1 + \frac{1}{x} \right) \right\} \, dx + O(n^{-1})
\]
The integral can be evaluated as
\[
n \int_{n+1}^{\infty} \left\{ \frac{1}{x} - \ln \left( 1 + \frac{1}{x} \right) \right\} \, dx = n \int_{0}^{1/(n+1)} \frac{x - \ln(1 + x)}{x^2} \, dx
\]
\[
= n \int_{0}^{1/(n+1)} \frac{1}{2} \, dx + O(n^{-1}) = \frac{1}{2} + O(n^{-1})
\]
We finally get
\[
q_n = s_n - \ln n = \gamma + \frac{1}{2n} + O(n^{-2})
\]
and
\[
s_n = \ln n + \gamma + \frac{1}{2n} + O(n^{-2})
\]
and the lemma is proved. \( \square \)

**B.4.3 Convergence of a sequence**

**Lemma B.8.** Let \( c \in (0, 1) \) and \( \lambda \in \mathbb{C} \). Then the sequence
\[
s_n = \sum_{k=1}^{n} c^{n-k} \left( \frac{n}{k} \right)^{\lambda}, \quad n \in \mathbb{Z}_{+}
\]
is convergent with limit \( s = 1/(1 - c) \).

**Proof.** The sequence \( \{s_n\}_{n=1}^{\infty} \) satisfies
\[
s_{n+1} = 1 + \frac{c(n+1)^{\lambda}}{n^{\lambda}} \sum_{k=1}^{n} c^{n-k} \left( \frac{n}{k} \right)^{\lambda} = 1 + c \left( 1 + \frac{1}{n} \right)^{\lambda} s_n = 1 + b_n s_n, \quad n \in \mathbb{Z}_{+}
\]
where
\[
b_n = c \left( 1 + \frac{1}{n} \right)^{\lambda} \to c, \quad n \to \infty
\]
If the sequence \( \{s_n\}_{n=1}^{\infty} \) has a limit \( s \), then it must satisfy
\[
s = 1 + cs \quad \Rightarrow \quad s = \frac{1}{1 - c} > 1
\]
Therefore, write the sequence \( \{s_n\}_{n=1}^{\infty} \) as
\[ s_n = s + t_n, \quad n \in \mathbb{Z}_+ \]

and the lemma follows, if we can prove that the sequence \( \{t_n\}_{n=1}^{\infty} \) converges to zero.

The sequence \( \{t_n\}_{n=1}^{\infty} \) satisfies

\[ t_{n+1} = \frac{b_n - c}{1 - c} + b_n t_n = c \frac{(1 + \frac{1}{n})^\lambda - 1}{1 - c} + b_n t_n, \quad n \in \mathbb{Z}_+ \]

which implies that

\[ |t_{n+1}| \leq \frac{C}{n} + |b_n| |t_n|, \quad n \in \mathbb{Z}_+ \]

for some constant \( C > 0 \) since

\[ \lim_{x \to 0} \left( \frac{(1 + x)^\lambda - 1}{x} \right) = \lambda \]

To proceed, introduce the constant \( d = (1 + c)/2 \in (c, 1) \), and let \( N \) be an integer such that

\[ |b_n| < d, \quad n \geq N \]

which is possible since \( b_n \to c \) as \( n \to \infty \). Then

\[ |t_{n+1}| \leq \frac{C}{n} + d |t_n|, \quad n \geq N \quad (B.11) \]

Moreover, introduce

\[ r_n = |t_n| - \frac{A}{n}, \quad n \geq N \]

where the positive constant \( A \) is chosen as

\[ A = \frac{C(N + 1)}{N(1 - d) - d} \]

which is positive if \( N > d/(1 - d) \). Then from (B.11)

\[ r_{n+1} \leq \frac{C}{n} - \frac{A}{n + 1} + \frac{Ad}{n} + dr_n, \quad n \geq N \]

The chosen value of \( A \) implies that

\[ \frac{C}{n} - \frac{A}{n + 1} + \frac{Ad}{n} \leq \frac{C + dA - A_{n+1}^n}{n} \leq \frac{C + dA - A_N^N}{n} = 0, \quad n \geq N \]

and we obtain

\[ r_{n+1} \leq dr_n, \quad n \geq N \]

Iteration of this expression gives
\[ r_{n+1} \leq d^{n-N+1} r_N, \quad n \geq N \]

which proves that \( r_n \to 0 \), and consequently \( |t_n| \to 0 \), as \( n \to \infty \), and the lemma is proved. \( \square \)
Appendix C
Partial fractions

A **rational function**, \( r(z) \), is a quotient between two polynomials, \( p(z) \) and \( q(z) \).

\[
 r(z) = \frac{p(z)}{q(z)}
\]

The rational function is defined in a domain in the complex \( z \)-plane, if we exclude the isolated zeros of \( q(z) \), which we denote by

\[
 z_i, \quad i = 1, 2, 3, \ldots, k
\]

By polynomial division, it is always possible to write \( r(z) \) as

\[
 r(z) = s_0(z) + \frac{p_0(z)}{q(z)}
\]

where \( s_0(z) \) is a polynomial, and where the degree of the polynomial \( p_0(z) \) is less than the degree of the polynomial \( q(z) \). The rational function \( p_0(z)/q(z) \) is analytic at infinity.

If \( m_i \) is the multiplicity of \( z_i \), then \( q(z) = (z - z_i)^m q_i(z) \), where \( q_i(z) \) is a polynomial and \( q_i(z_i) \neq 0 \). Let the principal parts of \( r(z) \) be

\[
 s_i(z) = \sum_{j=1}^{m_i} \frac{a_{i,j}}{(z - z_i)^j}, \quad i = 1, 2, 3, \ldots, k
\]

The **partial fraction decomposition** of \( r(z) \) then is

\[
 r(z) = \sum_{i=0}^{k} s_i(z)
\]
Appendix D
Circles and ellipses in the complex plane

The circle and the ellipse occur frequently in the analysis in this book. In particular, we use circles and ellipses in the treatment of the convergence properties of the series solution of Heun’s equation in Chapter 8. Their equations in the complex plane are reviewed in this appendix.

D.1 Equation of the circle

The equation for the circle, centered at \( z = z_0 \) and radius \( r \), in the complex \( z \)-plane is

\[
|z - z_0| = r
\]

or equivalently

\[
(z - z_0)(z^* - z_0^*) = r^2
\]

where the star, \( ^* \), denotes complex conjugate of the complex number. Denote \( a = -z_0 \) and \( b = |z_0|^2 - r^2 \in \mathbb{R} \). The equation of the circle then is

\[
zz^* + a^*z + az^* + b = 0, \quad b \text{ real}
\]

Every circle in the complex plane has this form.

The analysis above implies that the equation

\[
zz^* + \alpha z + \beta z^* + \gamma = 0 \quad (D.1)
\]

represents a circle if and only if

\[
\alpha = \beta^* \quad \text{and} \quad \alpha \beta - \gamma \text{ non-negative real number} \quad (D.2)
\]

If these conditions are fulfilled, the circle has its center at \( z = -\beta \) and the radius is

\[
r = \sqrt{|\alpha|^2 - \gamma}.
\]
D.1.1 Harmonic circles

Let $k$ be a real, positive number, and find the complex numbers $z$ that satisfy

$$\frac{|z-a|}{|z-b|} = k \quad (D.3)$$

where $a$ and $b$ are distinct complex numbers, i.e., $a \neq b$. The problem has an equivalent formulation:

$$|z-a| = k|z-b|$$

and we observe that the problem is to find the complex numbers $z$, whose distances to the points $a$ and $b$ have a constant quotient $k$.

If $k = 1$, the solution is the straight line perpendicular to, and passing through the midpoint of the line connecting $a$ and $b$.

We now show that the solution is a circle if $k \neq 1$. The equation is equivalent to

$$(z-a)(z^*-a^*) = k^2(z-b)(z^*-b^*)$$

or

$$(1-k^2)zz^* + (k^2b^*-a^*)z + (k^2b-a)z^* + |a|^2 - k^2|b|^2 = 0$$

This is the equation of a circle, see (D.1), provided the conditions in (D.2) are fulfilled. The first one in (D.2) is apparently satisfied, and the second is also satisfied, since

$$\frac{|k^2b-a|^2}{(1-k^2)^2} - \frac{|a|^2 - k^2|b|^2}{1-k^2} = \frac{|k^2b-a|^2 - (1-k^2)(|a|^2 - k^2|b|^2)}{(1-k^2)^2}$$

$$= \frac{k^2(|a|^2 + |b|^2 - 2\text{Re}ab^*)}{(1-k^2)^2} = \frac{k^2|a-b|^2}{(1-k^2)^2} \geq 0$$

Equation (D.3) therefore is a circle centered at $z = z_0$ and radius $r$, where $z_0$ and $r$ are given by

$$\begin{cases}
z_0 = \frac{a-k^2b}{1-k^2} \\
r = \frac{k|a-b|}{|1-k^2|}
\end{cases}$$

The center of the circle lies on the line connecting the points $a$ and $b$, see Figure D.1, and the circle is called the harmonic circle to $a$ and $b$. If $k < 1$, the circle encircles $a$, and if $z > 1$, the circle encircles the point $b$ in the complex $z$-plane.
The harmonic circles in the complex $z$-plane. In the illustration $a = 1 + i$ and $b = -3 - i$. When $0 < k < 1$, the circles enclose the point $a$, and when $k > 1$, the circles enclose the point $b$.

**D.2 Equation of the ellipse**

The equation of the ellipse in the complex plane can take many forms. If the ellipse has foci at $z = z_1$ and $z = z_2$, and passes through $z = a$, one form of the equation is

$$|z - z_1| + |z - z_2| = |a - z_1| + |a - z_2|$$

This equation states that the sum of the distances from the point $z$ in the complex $z$-plane to the foci is constant and is equal to the sum of the distances from the point $a$ to the foci. In Theorem 8.2, we also use the following, less common, form of the ellipse:

**Lemma D.1.** Let

$$Z = (1 - z^{-1})^{1/2}, \quad A = (1 - a^{-1})^{1/2}$$

where the branches of the square roots are taken as the principal branch, $\text{Re}Z > 0$ and $\text{Re}A > 0$. Then, for a given $a$, the equation

$$\frac{|Z - 1|}{|Z + 1|} = \frac{|A - 1|}{|A + 1|}$$

defines an ellipse in the complex $z$-plane with foci at $z = 0$ and $z = 1$, and passing through the point $z = a$, see Figure D.2. Moreover, the inequality
defines the interior of the ellipse containing the origin in the z-plane, and the inequality
\[ \left| \frac{Z - 1}{Z + 1} \right| < \left| \frac{A - 1}{A + 1} \right| \]
defines the domain exterior to the ellipse in the z-plane.

Proof. The real constant
\[ C = \left| \frac{A - 1}{A + 1} \right| < 1 \]
since the complex number \( A = \left( 1 - a^{-1} \right)^{1/2} \) is located in the right-hand side of the complex A-plane, i.e., Re\( A > 0 \). For given A, the complex numbers Z satisfying
\[ |Z - 1| = C |Z + 1| \]
define a circle in the complex Z-plane with center at \( Z_0 \) and radius \( R \), where, see Section D.1.1,
\[
\begin{aligned}
Z_0 &= \frac{1 + C^2}{1 - C^2} \\
R &= \frac{2C}{|1 - C^2|}
\end{aligned}
\]
and the circle encircles the point \( Z = 1 \).

The parameter representation of the circle

Fig. D.2 The ellipse in Lemma D.1.
\[ Z = Z_0 + \text{Re}^{i\phi}, \quad \phi \in [0, 2\pi) \]

implies that the curve in the \( z \)-plane is

\[
\frac{1}{1-Z^2} = \frac{1}{(1+Z)(1-Z)} = \frac{1}{(1+Z_0 + \text{Re}^{i\phi})(1-Z_0 - \text{Re}^{i\phi})}
\]

However, this is an ellipse with foci at \( z = 0 \) and \( z = 1 \), since

\[
\begin{cases}
|z| = \frac{1}{|1+Z||1-Z|} = \frac{1}{C|1+Z|^2} \\
|z-1| = \frac{|Z|^2}{|1+Z||1-Z|} = \frac{|Z|^2}{C|1+Z|^2}
\end{cases}
\]

and

\[
\begin{cases}
|Z|^2 = (Z_0 + R \cos \phi)^2 + R^2 \sin^2 \phi = \frac{(1+C^2 + 2C \cos \phi)^2 + 4C^2 \sin^2 \phi}{(1-C^2)^2} \\
|1+Z|^2 = (Z_0 + 1 + R \cos \phi)^2 + R^2 \sin^2 \phi = \frac{4(1+C \cos \phi)^2 + C^2 \sin^2 \phi}{(1-C^2)^2}
\end{cases}
\]

We simplify

\[
\begin{cases}
|Z|^2 = \frac{(1+C^2)^2 + 4C(1+C^2) \cos \phi + 4C^2}{(1-C^2)^2} \\
|1+Z|^2 = \frac{4 + 2C \cos \phi + C^2}{(1-C^2)^2}
\end{cases}
\]

The sum of the distances from \( z \) to the origin and 1 is

\[
|z| + |z-1| = \frac{(1-C^2)^2 + (1+C^2)^2 + 4C(1+C^2) \cos \phi + 4C^2}{4C(1+2C \cos \phi + C^2)}
\]

\[
= \frac{2(1+C^2)^2 + 4C(1+C^2) \cos \phi}{4C(1+2C \cos \phi + C^2)} = \frac{1+C^2}{2C}
\]

which is the equation of the ellipse with foci at \( z = 0 \) and \( z = 1 \). Notice that

\[
\frac{1+C^2}{2C} = \frac{|A+1|^2 + |A-1|^2}{2|A+1||A-1|} = \frac{|A|^2 + 1}{|A^2-1|} = \frac{|1-1/a| + 1}{1/|a|} = |a| + |a-1|
\]

by the parallelogram law \(|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2\), and we have proved that the

\[
|Z-1| = C|Z+1|
\]

defines an ellipse in the complex \( z \)-plane.
The complex number \( z = 1 \) lies inside the ellipse, and it corresponds to \( Z = (1 - z^{-1})^{1/2} = 0 \), and the complex number \( Z = 0 \) satisfies
\[
|Z - 1| > C|Z + 1|
\]
since
\[
C < 1
\]
The inequality
\[
|Z - 1| > C|Z + 1|
\]
therefore defines the domain inside the ellipse, and
\[
|Z - 1| < C|Z + 1|
\]
the outside. □
Appendix E
Elementary and special functions

A long list of elementary and special functions can be expressed in the hypergeometric function \( \binom{2}{1} F_1(\alpha, \beta; \gamma; z) \), and its two confluent versions, \( _1F_1(\alpha; \gamma; z) \) and \( _0F_1(\gamma; z) \), respectively. Some examples are given in this appendix. The Greek letters \( \alpha \) and \( \beta \) are arbitrary complex numbers, and \( n \) and \( m \) are non-negative integers.

E.1 Hypergeometric function \( \binom{2}{1} F_1(\alpha, \beta; \gamma; z) \)

Elementary functions

\[
\begin{align*}
(1+z)^\alpha &= F(-\alpha, \beta; \beta; -z) \\
\ln(1+z) &= z F(1, 1; 2; -z) \\
arctan z &= z F\left(\frac{1}{2}, 1; \frac{3}{2}; -z^2\right) \\
arcsin z &= z F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2\right)
\end{align*}
\]

Elliptic integrals

Complete elliptic integral of the first kind

\[
K(m) = \int_0^1 \left((1-t^2)(1-mt^2)\right)^{-1/2} \, dt = \int_0^{\pi/2} (1-m\sin^2 \theta)^{-1/2} \, d\theta = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; m\right)
\]
Complete elliptic integral of the second kind

\[
E(m) = \int_0^1 (1 - t^2)^{-1/2} (1 - mt^2)^{1/2} \, dt = \int_0^{\pi/2} (1 - m \sin^2 \theta)^{1/2} \, d\theta \\
= \frac{\pi}{2} F \left( -\frac{1}{2}, \frac{1}{2}; 1; m \right)
\]

**Jacobi polynomials**

\[
P_n^{(\alpha, \beta)}(x) = \binom{n + \alpha}{n} F \left( -n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2} \right)
\]

**Legendre functions**

Legendre functions of the first kind

\[
P_\nu(z) = F \left( -\nu, \nu + 1; 1 - \frac{z^2}{2} \right)
\]

Legendre functions of the second kind

\[
Q_\nu(z) = \frac{\sqrt{\pi}}{(2z)^{\nu + 1}} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 3/2)} F \left( 1 + \frac{\nu}{2}, 1 + \frac{\nu}{2}; \nu + \frac{3}{2}; \frac{1}{z^2} \right)
\]

**Associated Legendre functions**

\[
\begin{align*}
P_\nu^m(x) &= (1 - x^2)^{\nu/2} \frac{d^m}{dx^m} P_\nu(x) \\
Q_\nu^m(x) &= (1 - x^2)^{\nu/2} \frac{d^m}{dx^m} Q_\nu(x)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P}_\nu^m(z) &= (z^2 - 1)^{\nu/2} \frac{d^m}{dz^m} P_\nu(z) \\
\mathcal{Q}_\nu^m(z) &= (z^2 - 1)^{\nu/2} \frac{d^m}{dz^m} Q_\nu(z)
\end{align*}
\]
E.2 Confluent functions $\, _1 F_1 (\alpha; \gamma; z)$

**Tchebyseff polynomials**

Tchebyseff first kind

$$T_n (\cos \theta) = \cos n \theta = F \left( -n, n; \frac{1}{2}; \frac{1-x}{2} \right)$$

Tchebyseff second kind

$$U_n (\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta} = (n+1) F \left( -n, n+2; \frac{3}{2}; \frac{1-x}{2} \right)$$

E.2 Confluent functions $\, _1 F_1 (\alpha; \gamma; z)$

**Elementary functions**

\[
\begin{cases} 
    e^z = _1 F_1 (\alpha; \alpha; z) \\
    e^{-iz} \sin z = _1 F_1 (1; 2; -2iz) \\
    e^z \sinh z = _1 F_1 (1; 2; 2z) 
\end{cases}
\]

**Bessel functions**

$$J_{\nu} (z) = \frac{z^\nu e^{-iz} _1 F_1 (\nu + 1/2; 2\nu + 1; 2iz)}{2^\nu \Gamma (\nu + 1)}$$

**Laguerre polynomials**

$$L_n^{(\alpha)} (z) = \binom{n + \alpha}{n} _1 F_1 (-n; \alpha + 1; z)$$
Hermite polynomials

\[
\begin{align*}
H_{2m}(x) &= (-1)^m 2^m m! L_m^{(-1/2)}(x^2) \\
H_{2m+1}(x) &= (-1)^m 2^{m+1} (m+1)! L_m^{(1/2)}(x^2)
\end{align*}
\]
\[
x \in \mathbb{R}
\]

E.2.1 Error functions

\[
\text{erf}(z) = \frac{2z}{\sqrt{\pi}} _1F_1\left(1/2; 3/2; -z^2\right) = \frac{2z}{\sqrt{\pi}} e^{-z^2} _1F_1\left(1/3; 3/2; z^2\right)
\]

E.3 Confluent functions \( _0F_1(\gamma; z) \)

Bessel functions

\[
J_\nu(z) = z^\nu \frac{\Gamma(\nu + 1)}{2^\nu \Gamma(\nu + 1)} _0F_1(\nu + 1; -z^2/4)
\]
Appendix F
Notation

Most of the notation and symbols adopted in this textbook are traditional, and there is very little risk of confusion, but for the sake of completeness, we collect the symbols in this appendix.

- We use the notation \( \mathbb{Z} \) for all integers 0, ±1, ±2, ....
- The positive integers 1, 2, 3, 4, ... are denoted \( \mathbb{Z}_+ \).
- The negative integers \(-1, -2, -3, -4, \ldots \) are denoted \( \mathbb{Z}_- \).
- The natural, non-negative, integers 0, 1, 2, 3, ... are denoted \( \mathbb{N} \).
- The field of real numbers is denoted \( \mathbb{R} \).
- The field of complex numbers is denoted \( \mathbb{C} \). Sometimes the point at infinity is included, and it is then appropriate to view the field as the Riemann sphere.
- We use the symbols \( o \) and \( O \) defined by

\[
\begin{align*}
    f(x) &= o(g(x)), \quad x \to a \iff \lim_{x\to a} \frac{f(x)}{g(x)} = 0 \\
    f(x) &= O(g(x)), \quad x \to a \iff \frac{f(x)}{g(x)} \text{ bounded in a neighborhood of } a
\end{align*}
\]

- The symbol \( \Box \) is used to end a proof.
- The symbol \( [ \) is used to end an example or a comment.
- The dagger, \( ^\dagger \), in front of a problem denotes a more difficult problem.
- The star, \( ^\ast \), denotes complex conjugation of a complex number.
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