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List of symbols

\[ \langle n_1, \ldots, n_p \rangle \] - submonoid generated by \( \{n_1, \ldots, n_p\} \), p. 1.

\( \text{Arf}(S) \) - Arf closure of the numerical semigroup \( S \), p. 25.

\( \text{Ap}(S,m) \) - Apéry set of the element \( m \neq 0 \) in \( S \), p. 8.

\( \text{Ch}(S) \) - chain associated to the numerical semigroup \( S \), p. 92.

\( \text{Ch}(X) \) - union of the chains associated to the numerical semigroups in \( X \), p. 98.

\( \text{Cong}(\rho) \) - congruence generated by \( \rho \), p. 108.

\( d_A(a) \) - greatest common divisor of the elements in \( A \) less than or equal to \( a \in A \), p. 28.

\( D(X) \) - set of all positive divisors of the elements of \( X \), p. 51.

\( \Delta(X) = \{(x,x) \mid x \in X\} \) - the diagonal of \( X \times X \), p. 108.

\( e(S) \) - embedding dimension of the numerical semigroup \( S \), p. 9.

\( F(S) \) - Frobenius number of the numerical semigroup \( S \), p. 9.

\( \text{FG}(S) \) - set of fundamental gaps of \( S \), p. 52.

\( \text{Free}(X) \) - free monoid on \( X \), p. 107.

\( \mathcal{F}(X) \) - Frobenius variety generated by the set of numerical semigroups \( X \), p. 98.

\( g(S) \) - genus (or degree of singularity) of the numerical semigroup \( S \), p. 9.

\( \mathcal{G}(\mathcal{Y}) \) - graph associated to the Frobenius variety \( \mathcal{Y} \), pp. 92, 101.

\( \mathcal{I}(S) \) - set of irreducible numerical semigroups containing the numerical semigroup \( S \), p. 47.

\( \text{im}(f) \) - image of the homomorphism \( f \), p. 106.

\( \text{Irr}(\sigma) \) - set of irreducible elements of \( \sigma \), p. 109.

\( J(S) \) - for a numerical semigroup minimally generated by \( n_1 < \cdots < n_e \) is the set \( \{\lambda_2x_2 + \cdots + \lambda_ex_e \mid \lambda_2n_2 + \cdots + \lambda_en_e \not\in \text{Ap}(S,n_1)\} \), p. 138.

\( \text{ker}(f) \) the kernel congruence of the monoid homomorphism \( f \), p. 106.

\( m(S) \) - multiplicity of the numerical semigroup \( S \), p. 9.

\( M(f) \) - monoid associated to the subadditive function \( f \), p. 58.

Maximals_{\leq}(X) - maximal elements of \( X \) with respect to the ordering \( \leq \), p. 13.

Minimals_{\leq}(X) - minimal elements of \( X \) with respect to the ordering \( \leq \), p. 13.

\( a \mod b \) - is the quotient of the division of \( a \) by \( b \);

\( a \equiv b \mod m \) means \((a - b) \mod m = 0\), p. 20.

\( n(S) \) - cardinality of the set of elements in \( S \) less than its Frobenius number, p. 15.
N(S) - set of elements in S less than its Frobenius number, p. 15.
N - set of nonnegative integers, p. 1.
o(g) - order of an element g in a group G, p. 167.
O(S) - set of numerical semigroups containing the numerical semigroup S, p. 44.
PF(S) - set of pseudo-Frobenius numbers of the numerical semigroup S, p. 13.
Q_0^+ - set of nonnegative rational numbers, p. 59.
Q(S) - quotient group of S, p. 163.
\$\mathcal{S}\$ - set of all numerical semigroups, p. 47.
\$\mathcal{S}_m\$ - set of numerical semigroups with multiplicity m, p. 58.
\$\mathcal{S}(g_1,\ldots,g_t)\$ - set of numerical semigroups not cutting \{g_1,\ldots,g_t\}, p. 47.
\$\mathcal{S}(P)\$ - set of numerical semigroups admitting a pattern P, p. 96.
S(a,b,c) - set of integer solutions to \(ax \mod b \leq cx\), p. 58.
S(A) - with \(A \subset Q_0^+\), the set of integers of the submonoid \(\langle A \rangle\), p. 59.
Sat(S) - saturated closure of the numerical semigroup S, p. 30.
\$\mathcal{S}\mathcal{F}_m\$ - set of \(m\)-periodic subadditive functions, p. 58.
SG(S) - set of special gaps of S, p. 44.
R - relation defining the R-classes of the expressions of a given element in a numerical semigroup, p. 111.
t(S) - type of the numerical semigroup S, p. 13.
\$\mathcal{V}\$ - a Frobenius variety, p. 99.
\$\mathcal{V}(A)\$ - \$\mathcal{V}\$-monoid generated by A, p. 99. Used also as a prefix to denote systems of generators and monoids relative to this variety; see Chapter 6.
Z(n) - set of factorizations or expression of n in a numerical semigroup S, p. 111.
Z_B(n) - set of factorizations or expression of n in a numerical semigroup S relative to the set of generators B, p. 111.
Index

Symbols

0-matrix 150

A

adjacent fractions 66
Apéry set 8
with respect to a set 124
Archimedean element 155
Arf closure
of a semigroup 25
of a set 25

B

Bézout sequence 61
end of a 61
length of a 61
proper 65
binary relation 106
equivalence 106
inverse relation 108

C

chain associated to a numerical semigroup 92
complete intersection 129
conductor 9
congruence
cancellative 109
diagonal congruence 108
finitely generated 108
generated by a set 108
minimal relation 110
monoid 106

D

degree of singularity 9
Dickson’s Lemma 109
directed graph 91
of all numerical semigroups 92
edges 91
path 91
vertex 91
dominant 17

E

embedding dimension 9

F

factorization 122
catenary degree 122
distance 122
greatest common divisor 122
length 122
Frobenius number 9
Frobenius variety 93
generated by a family of numerical semigroups 98

G

gap 9
fundamental 52
special 44
genus 9
gluing 124
of numerical semigroups 130
Index

graph 111
  associated to an element in a numerical semigroup 113
  connected 111
  generating tree 111
  connected component 113
directed 91
directed 111
path 111
vertices 111

I

ideal 17
  canonical 18
  maximal 17
  principal 23
  relative 17
idempotent element 155
irreducible element of a congruence 108

M

minimal presentation 111
minimal system of generators 8
modular Diophantine inequality 69
monoid 6
cancellative 109
epimorphism 6
finitely generated 6
finitely presented 108
free 107
free of units 159
half-factorial 122
hereditarily finitely generated 161
homomorphism 6
  image 106
  kernel congruence 106
  isomorphism 6
  monomorphism 6
  N-monoid 163
quasi-Archimedean 159
quotient 106
weakly cancellative 161
multiplicity 9

N

numerical semigroup 6
  acute 17
  admitting a pattern 96
  almost symmetric 55
  Arf 23
  arithmetic 88
  associated to a set of rational numbers 59
  free 133
  half-line 17
  irreducible 33
  maximal embedding dimension 20
  modular 69
  ordinary 17
  proportionally modular 59
  pseudo-symmetric 34
  saturated 28
  simple 154
  symmetric 34
  telescopic 136
  with Apéry set of unique expression 138

O

order 167
oversemigroup 44

P

pattern 96
  admissible 97
  strongly admissible 97
  presentation 108
  proportionally modular Diophantine inequality 59
  factor of a 59
  modulus of a 59
  proportion 59
  pseudo-Frobenius number 13

Q

quotient group 163
quotient of a numerical semigroup by an integer 78
quotient set 106

R

ratio 17
R-class 111

S

Saturated closure 30
semigroup 5
  Archimedean 155
  epimorphism 6
  finitely generated 6
  homomorphism 6
  isomorphism 6
  monomorphism 6
  multiple joined 155
  quotient 157
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>system of generators of a torsion free</td>
<td>6</td>
</tr>
<tr>
<td>set of expressions</td>
<td>111</td>
</tr>
<tr>
<td>strongly positive sequence</td>
<td>150</td>
</tr>
<tr>
<td>subadditive function period of a</td>
<td>58</td>
</tr>
<tr>
<td>submonoid</td>
<td>6</td>
</tr>
<tr>
<td>trivial</td>
<td>6</td>
</tr>
<tr>
<td>subsemigroup</td>
<td>5</td>
</tr>
<tr>
<td>subsemigroup generated by a set</td>
<td>6</td>
</tr>
<tr>
<td>system of generators system of a congruence</td>
<td>108</td>
</tr>
<tr>
<td>of a monoid</td>
<td>6</td>
</tr>
<tr>
<td>root</td>
<td>91</td>
</tr>
<tr>
<td>son</td>
<td>92</td>
</tr>
<tr>
<td>undirected</td>
<td>111</td>
</tr>
<tr>
<td>type</td>
<td>13</td>
</tr>
<tr>
<td>U</td>
<td></td>
</tr>
<tr>
<td>unique expression element</td>
<td>138</td>
</tr>
<tr>
<td>unit</td>
<td>159</td>
</tr>
<tr>
<td>V</td>
<td></td>
</tr>
<tr>
<td>( \gamma )-monoid</td>
<td>99</td>
</tr>
<tr>
<td>W</td>
<td></td>
</tr>
<tr>
<td>weight</td>
<td></td>
</tr>
<tr>
<td>modular Diophantine inequality</td>
<td>75</td>
</tr>
</tbody>
</table>