Appendix A

Calculation of M

We start by substituting Eq. (3.15') in the first two terms of Eq. (3.16a) to get

$$-f_a^{(0)}\frac{e_a}{m_a}(\vec{c}_a\times\vec{B})\cdot\left(\frac{\partial}{\partial\vec{c}_a}\vec{c}_a\mathbb{A}_a^{(1)}+(\vec{c}_a\times\vec{B})\mathbb{A}_a^{(2)}+\vec{B}(\vec{c}_a\cdot\vec{B})\mathbb{A}_a^{(3)}\right)$$

$$-f_a^{(0)}\frac{m_a}{\rho kT}\left\{\sum_j e_j\int d\vec{c}_j f_j^{(0)}\mathbb{A}_j(\vec{c}_j\times\vec{B})\right\}\vec{c}_a\equiv M$$

and examine term by term.

$$\text{a}):\quad(\vec{c}_a\times\vec{B})\cdot\frac{\partial}{\partial\vec{c}_a}(\vec{c}_a\mathbb{A}_a^{(1)})=(\vec{c}_a\times\vec{B})\mathbb{A}_a^{(1)}+(\vec{c}_a\times\vec{B})\cdot\vec{c}_a\frac{\partial\mathbb{A}_a^{(1)}}{\partial\vec{c}_a}$$

The second term vanishes so we keep only the first one

$$\text{b}):\quad(\vec{c}_a\times\vec{B})\cdot\left[\frac{\partial}{\partial\vec{c}_a}(\vec{c}_a\times\vec{B})\mathbb{A}_a^{(2)}\right]=(\vec{c}_a\times\vec{B})\cdot\left[\frac{\partial}{\partial\vec{c}_a}(\vec{c}_a\times\vec{B})\right]\mathbb{A}_a^{(2)}+$$

$$(\vec{c}_a\times\vec{B})\cdot\left[(\vec{c}_a\times\vec{B})\frac{\partial\mathbb{A}_a^{(2)}}{\partial\vec{c}_a}\right]$$

The second term vanishes since $\frac{\partial\mathbb{A}_a^{(2)}}{\partial\vec{c}_a}\sim\vec{c}_a$ and the first term we compute using the vector identity,

$$\text{grad}\left[(\vec{c}_a\times\vec{B})\cdot(\vec{c}_a\times\vec{B})\right]=2(\vec{c}_a\times\vec{B})\cdot\text{grad}(\vec{c}_a\times\vec{B})+2(\vec{c}_a\times\vec{B})\times\text{rot}(\vec{c}_a\times\vec{B})$$

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma,* Springer Series on Atomic, Optical, and Plasma Physics 53

where grad and rot are taken with respect to $\vec{c}_a$. Using well known vector identities, we get

$$2(\vec{c}_a \times \vec{B}) \cdot \text{grad}(\vec{c}_a \times \vec{B}) = \text{grad}\left(c_a^2 B^2 - (\vec{c}_a \cdot \vec{B})^2\right) -$$

$$2(\vec{c}_a \times \vec{B}) \times \left[\vec{c}_a \cdot \text{grad}B - \vec{B}\ \text{grad}\ \vec{c}_a + \vec{c}_a\ \text{div}\ \vec{B} - \vec{B}\ \text{div}\ \vec{c}_a\right]$$

and all second term vanishes, except $-\vec{B}$ grad $\vec{c}_a = -\vec{B}$, since $\vec{B}$ is constant in $\vec{c}_a$ space and $(\vec{c}_a \times \vec{B}) \times \vec{B} = (\vec{c}_a \cdot \vec{B})\vec{B} - B^2\vec{c}_a$. But $\text{grad}(c_a^2 B^2 - (\vec{c}_a \cdot \vec{B})^2) = 2B^2\vec{c}_a(\vec{c}_a \cdot \vec{B})\vec{B}$ so adding the two terms we get that

$$(\vec{c}_a \times \vec{B}) \cdot\ \text{grad}\ (\vec{c}_a \times \vec{B}) = 2(B^2\vec{c}_a - \vec{B}(\vec{c}_a \cdot \vec{B}))$$

c): The third term vanishes since $(\vec{c}_a \times \vec{B}) \cdot \vec{c}_a = 0$.
Thus the first term in M reads,

$$-f_a^{(0)}\frac{e_a}{m_a}(\vec{c}_a \times \vec{B})\mathbb{A}_a^{(1)} - f_a^{(0)}\frac{2e_a}{m_a}\left((B^2\vec{c}_a - \vec{B}(\vec{c}_a \cdot \vec{B}))\right)\mathbb{A}_a^{(2)} \tag{A.1}$$

We first simplify the second term in M to read

$$-f_a^{(0)}\frac{m_a}{\rho kT}\left\{\sum_j e_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j\vec{c}_j \cdot (\vec{c}_a \times \vec{B})\right\} =$$

$$f_a^{(0)}\frac{m_a}{\rho kT}(\vec{B}\times\vec{c}_a)\cdot\left\{\sum_j e_j \int d\vec{c}_j f_j^{(0)}\vec{c}_j\left(\mathbb{A}_j^{(1)}\vec{c}_j + (\vec{c}_j \times \vec{B})\mathbb{A}_j^{(2)} + \vec{B}(\vec{c}_j \cdot \vec{B})\mathbb{A}_j^{(3)}\right)\right\}$$

Again, proceed term by term,

$$\sum_j e_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j^{(1)}\vec{c}_j\vec{c}_j = \frac{1}{3}\sum_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j^{(1)}c_j^2\mathbb{I} \tag{A.2}$$

The second term is

$$\sum_j e_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j^{(2)}(\vec{c}_j \times \vec{B})\vec{c}_j = \sum_j e_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j^{(2)}\vec{c}_j\vec{c}_j \times \vec{B}$$

$$= \left(\frac{1}{3}\sum_j e_j \int d\vec{c}_j f_j^{(0)}\mathbb{A}_j^{(2)}c_j^2\mathbb{I}\right) \times \vec{B}$$

However

$$\left[(\vec{c}_a \times \vec{B}) \cdot \mathbb{I}\right] \times \vec{B} = (\vec{c}_a \times \vec{B}) \times \vec{B}$$

and this term yields

$$\left[B^2 \vec{c}_a - (\vec{c}_j \cdot \vec{B})\vec{B}\right] \frac{1}{3} \sum_j e_j \int d\vec{c}_j f_j^{(0)} \mathbb{A}_j^{(2)} c_j^2 \tag{A.3}$$

The third term vanishes since $(\vec{c}_a \times \vec{B}) \cdot \vec{B} = 0$

Using the identity

$$\int \left[c_j^2 - \frac{1}{B^2}(\vec{c}_a \cdot \vec{B})^2\right] f_j^{(0)} R(c) d\vec{c} = \frac{2}{3} \int d\vec{c}_j f_j^{(0)} R(c) c_j^2$$

which is valid for any arbitrary function of c, $R(c)$, we finally find combining Eqs. (A.2-A.3) that

$$M = -f_a^{(0)} \frac{e_a}{m_a} (\vec{c}_a \times \vec{B}) \mathbb{A}_a^{(1)} - f_a^{(0)} \frac{2e_a}{m_a} \left(B^2 \vec{c}_a - \vec{B}(\vec{c}_a \cdot \vec{B})\right) \mathbb{A}_a^{(2)} +$$

$$f_a^{(0)} \frac{m_a}{\rho k T} (\vec{c}_a \times \vec{B}) \mathbb{G}_B^{(1)} + f_a^{(0)} \frac{m_a}{\rho k T} \left(B^2 \vec{c}_a - \vec{B}(\vec{c}_a \cdot \vec{B})\right) \mathbb{G}_B^{(2)}$$

which are the results quoted in the text. $\mathbb{G}_B^{(1)}$ and $\mathbb{G}_B^{(2)}$ are defined in the text in page 31.

Appendix B

Linearized Boltzmann Collision Kernels

In what follows we shall discuss several important properties related to the nature and structure of the linearized Boltzmann collision kernels. Let $G_{ij} = G_i(\vec{\omega}_i, \vec{\omega}_j)$ and $H_{ij} = H_{ij}(\vec{\omega}_i, \vec{\omega}_j)$ be any two tensors functions of the velocities $\vec{\omega}_i$ and $\vec{\omega}_j$. Define

$$[G_{ij}, H_{ij}]_{ij} \equiv \frac{1}{n_i n_j} \int \cdots \int G_{ij} : (H'_{ij} - H_{ij}) f_i^{(0)} f_j^{(0)} g_{ij} \sigma(\Omega) d\Omega d\vec{v}_i d\vec{v}_j \quad \text{(B.1)}$$

where the subscript in the bracket denotes an integration over the variables $\vec{v}_i$ and $\vec{v}_j$ and the differential cross section $\sigma(\Omega)d\Omega = bdbd\epsilon$ where ϵ is the inclination of the orbit and b the impact parameter. Setting $\vec{v}_i \rightarrow \vec{v'}_i$, $\vec{v}_j \rightarrow \vec{v'}_j$ noticing that $g_{ij} = g'_{ij}$, $f_i^{(0)}(\vec{v}_i) f_j^{(0)}(\vec{v}_j) = f_i^{(0)}(\vec{v'}_i) f_j^{(0)}(\vec{v'}_j)$ and changing signs in (B.1) we get that

$$[G_{ij}, H_{ij}]_{ij} \equiv -\frac{1}{2n_i n_j} \int \cdots \int (G'_{ij} - G_{ij}) : (H'_{ij} - H_{ij}) f_i^{(0)} f_j^{(0)} g_{ij} \sigma(\Omega) d\Omega d\vec{v}_i d\vec{v}_j \quad \text{(B.2)}$$

The bracket is symmetrical with respect to an exchange of the indexes i and j in G_{ij} and H_{ij} and also between brackets. Hence the following set of equations holds:

$$[G_{ij}, H_{ij}]_{ij} = [H_{ij}, G_{ij}]_{ij} = [G_{ij}, H_{ij}]_{ji} = [H_{ij}, G_{ij}]_{ji} \quad \text{(B.3)}$$

It is also clear from its definition that $[\quad]_{ij}$ is a linear operator. Suppose that

$$G_{ij} = K_i + L_j \qquad H_{ij} = M_i + N_j$$

K_i and L_i depend only on $\vec{\omega}_i$ and L_j, N_j depend only on $\vec{\omega}_j$, then by inspection,

$$\begin{aligned}[K_i + L_j, M_i + N_j]_{ij} &= [K_i, M_i + N_j]_{ij} + [L_j, M_i + N_j]_{ij} \\ &= [K_i, M_i]_{ij} + [K_i, N_j]_{ij} + [L_j, M_i]_{ij} + [L_j, N_j]_{ij} \end{aligned} \tag{B.4}$$

In Eq. (B.4), $K_i \equiv K_i(\vec{\omega}_i)$, $L_j \equiv L_j(\vec{\omega}_j)$ and so on, and $[\quad]_{ij}$ indicates that the integral is evaluated for collisions between molecules of species i and j. Let us now consider two sets of tensor functions K_i and L_i and define

$$\{K; L\} = \sum_{ij} n_i n_j \left[K_i + K_j; L_i + L_j\right]_{ij} \tag{B.5}$$

From the properties of the square bracket it follows immediately that

$$\{K, L\} = \{L, K\} \tag{B.6a}$$

and

$$\{K; L + M\} = \{K, L\} + \{K, M\} \tag{B.6b}$$

Since $\{K; K\}$ represents integrals whose integrands are non-negative, it follows that

$$\{K, K\} \geq 0 \tag{B.7}$$

and further, for obvious reasons, the equality sign holds if K is a linear combination of the collisional invariants.

For a binary mixture, expanding (B.5) we get that

$$\{K, L\} = n_a^2 \left[K_a + K_a, L_a + L_a\right]_{aa} + 2n_a n_b \left[K_a + K_b, L_a + L_b\right]_{ab} + n_b^2 \left[K_b + K_b, L_b + L_b\right]_{bb}$$

which upon expansion and appropriate collection of terms yields

$$\{K, L\} = 2n_a^2 \left[K_a, L_a\right]_{aa} + 2n_b^2 \left[K_b, L_b\right]_{bb} + 2n_a n_b \left([K_a, L_a]_{ab} + [K_a, L_b]_{ab} + [K_b, L_a]_{ab} + [K_b, L_b]_{ab}\right) \tag{B.8}$$

Eqs. (B.7) and (B.8) are of importance in the solution to the integral equations.

Bibliography

[1] J. O. Hirschfelder, C. F. Curtiss and R. B. Byrd; *The Molecular Theory of Liquids and Gases*; John Wiley & Sons, New York (1964), $2^{\underline{nd}}$ printing.

Appendix C

The Case when $\vec{B} = \vec{0}$

There are at least two reasons to take the time and space to consider this case in some detail. Firstly and above all, the fact that the full thermodynamic theory can be derived including the explicit form of all transport for the mixture, and a rigorous proof showing that the transport matrix is symmetric in full agreement with Onsager's reciprocity theorem. Secondly, the explicit expressions for the thermal and electrical conductivities can be compared with those derived earlier by Spitzer [1]. Moreover the cross coefficients for the Soret and Dufour effects are also readily obtained which to the author's knowledge have never been appropriately assessed in the case of a ionized gas. Their values could be significant in some astrophysical systems such as cooling flows and planetary nebulae.

Many of the results to be given here arise simply from those in the main text just setting $\vec{B} = 0$. Others require some additional attention which will be offered in detail. The conservation equations are given by Eq. (2.17c) which remains unchanged. The momentum equation is

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \text{div } (\tau^k + \rho \vec{u}\vec{u}) = Q\vec{E} \tag{C.1}$$

which may be readily derived [c.f. Eq. (2.9)] and from Eq. (2.19)

$$\rho \frac{d}{dt}\left(\frac{e}{\rho}\right) + \text{div } \vec{J}_q + \tau_k : \text{grad } \vec{u} = \vec{J}_c \cdot \vec{E} \tag{C.2}$$

Nothing else changes with respect to the H-theorem, the validity of Eq. (2.26) giving the equilibrium distribution function nor the fact that the solution to the linearized homogeneous term of the Boltzmann equation $J(f^0 f^0) = 0$

is the local Maxwell-Boltzmann distribution function. The first substantial difference with the $\vec{B} \neq 0$ case is Eq. (2.29a) for the first order in gradients solution to the BE, namely,

$$\frac{\partial f_a^{(0)}}{\partial t} + \vec{v_a} \cdot \frac{\partial f_i^{(0)}}{\partial \vec{r}} + \frac{\vec{F_a}}{m_a} \cdot \frac{\partial f_a^{(0)}}{\partial \vec{v_a}} = f_a^{(0)} \left\{ C(\varphi_a^{(1)} + C(\varphi_a^{(1)} + \varphi_b^{(1)} \right\}$$

where $\vec{F_a} = \vec{F_a}^{(e)} + e_a \vec{E}$. Since both $\vec{F_a}^{(e)}$ and $e_a\vec{E}$ are conservative forces it may be readily identified with the linearized version of the Boltzmann equation in the traditional case, $\vec{F}$ conservative. Evaluation of the left hand side using the explicit form for $f^{(0)}$ and Eqs. (3.5)-(3.8) in the text where the last two terms in (3.7) are replaced by $Q\vec{E}$ leads to the result that

$$f_a^{(0)} \left\{ \frac{m_a}{kT} \overleftrightarrow{c_a {}^0 c_a} : \text{grad } \vec{u} + \left(\frac{m_a c_a^2}{2kT} - \frac{5}{2} \right) \text{grad } \ln T \cdot \vec{c_a} + \frac{n_a}{n} \vec{c_a} \cdot \vec{d_{ab}} = \right\} =$$

$$f_a^{(0)} \left\{ C(\varphi_a^{(1)}) + C(\varphi_a^{(1)} \varphi_b^{(0)}) \right\} \tag{C.3}$$

where

$$\vec{d_{ab}} = -\vec{d_{ba}} = \text{grad } \frac{n_a}{n} + \frac{n_a n_b (m_a - m_b)}{n\rho} \text{grad } \ln p$$
$$- \frac{\rho_a \rho_b}{p\rho} (\vec{F_a^e} - \vec{F_b^e}) - \frac{n_a n_b}{p\rho} (m_b e_a - m_a e_b) \vec{E} \tag{C.4}$$

Just as in the magnetic field free case all the theorems showing that Eq. (C.3) has no unique solution but an infinite number composed by a solution to the inhomogeneous part plus as arbitrary linear combination to the solution of the homogeneous part in these case being m_i, $m_i\vec{c_i}$, and $\frac{1}{2} m_i c_i^2$ where $(i = a, b)$, hold true. The latter solution is uniquely determined by the subsidiary conditions so one gets that

$$-\varphi_i^{(1)} = C_a{}^0 C_a \mathbb{B}_i : \text{grad } \vec{u} + \mathbb{A}_i \vec{c_i} \cdot \text{ grad } \ln T + \mathbb{D}_i \vec{d_{ij}} \cdot \vec{c_i} \tag{C.5}$$

where $\mathbb{A}_i$, $\mathbb{B}_i$ and $\mathbb{D}_i$ are scalar functions of c_i, n_i, T, etc. $\mathbb{A}_i$ and $\mathbb{D}_i$ are still subjected to the subsidiary condition that

$$\sum_i m_i \int f_i^{(0)} \left\{ \begin{array}{c} \mathbb{A}_i \\ \mathbb{D}_i \end{array} \right\} c_i^2 d\vec{c_i} = 0 \tag{C.6}$$

These functions depend on the particular interaction potential between the ions and the electrons in the gas and are solutions to the equations

$$-\left(\frac{m_i c_i^2}{2kT} - \frac{5}{2}\right) f_a^{(0)} \vec{c_i} = \{C(\mathbb{A}_i \vec{c_i}) + C(\mathbb{A}_i \vec{c_i} + \mathbb{A}_j \vec{c_j})\} f_i^{(0)} \quad \text{(C.6a)}$$

and

$$-\frac{n_i}{n} \vec{c_i} f_i^{(0)} = f_i^{(0)} \{C(\mathbb{D}_i \vec{c_i}) + C(\mathbb{D}_i \vec{c_i} + \mathbb{D}_i \vec{c_i} + \mathbb{D}_j \vec{c_j})\} \quad \text{(C.6b)}$$

The solution for $\mathbb{B}_i$ associated with the tensor grad $\vec{u}$ will not couple with the vectorial fluxes $\vec{d_{ab}}$ and grad T so we shall simply ignore it in what follows. For isotropic systems there are no visco-electric effects. The minus sign in Eq. (C.5) has been included for convenience. Ignoring $\mathbb{B}_i$,

$$\varphi_i^{(1)} = -\mathbb{A}_i \vec{c_i} \cdot \frac{1}{T} \text{grad } T - \mathbb{D}_i \cdot \vec{c_i} \vec{d_{ij}} \quad \text{(C.5}'\text{)}$$

can now allow us to compute the different fluxes in the mixture. This we shall do in detail to clearly exhibit how does Onsager's reciprocity theorem holds true. There are essentially three currents, the heat flow, the mass flow and the electrical flow, the forces being grad T, $\vec{d}_{ij}^{(0)}$ and $\vec{d}_{ij}^{(e)}$ where

$$\vec{d}_{ab}^{(0)} = \text{grad } \frac{n_a}{n} + \frac{n_a n_b}{n\rho}(m_a - m_b) \text{ grad } \ln p \quad \text{(C.7a)}$$

$$\vec{d}_{ab}^{(e)} = \frac{n_a n_b}{p\rho}(m_b e_a - m_a e_b)\vec{E} \quad \text{(C.7b)}$$

for simplicity other external forces, $\vec{F}_i^{(e)} = 0$. Therefore, $\vec{d}_{ij}^{(0)}$ must be related to conventional diffusion processes and $\vec{d}_{ij}^{(e)}$ with electrical phenomena together with their respective cross effects. For this purpose we shall follow closely the treatment contained in references [2] and [3]. Onsager's symmetry arises then in a very simple way. We recall that

$$\vec{J_i} = m_i \int \vec{c_i} f_i(c_i) d\vec{c_i} = m_i n_i \langle \vec{c_i} \rangle = \rho_i \langle \vec{c_i} \rangle$$

$$\sum_i \vec{J_i} = 0 \quad (\vec{J_a} = -\vec{J_b})$$

Also

$$\vec{J_c} = n_a e_a \langle \vec{c_a} \rangle + n_b e_b \langle \vec{c_b} \rangle = \frac{e_a}{m_a} \vec{J_a} + \frac{e_b}{m_b} \vec{J_b} \quad \text{(C.8a)}$$

We now use the fact that the results of ref. [1] and [2] for a binary mixture do not depend on the explicit form of the diffusive force $\vec{d_{ab}}$. In particular, as shown in Chap. 2,

$$\vec{J} = \frac{1}{m_a}\vec{J_a} + \frac{1}{m_b}\vec{J_b} = \frac{m_b - m_a}{m_a m_b}\vec{J_a} \tag{C.8b}$$

$$\vec{J'}_q = \vec{J_q} - \frac{5}{2}kT\vec{J} \tag{C.8c}$$

as in the inert mixture and in addition we have the electric current

$$\vec{J_e} = \frac{e_a}{m_a}\vec{J_a} + \frac{e_b}{m_b}\vec{J_b} = \frac{e_b m_a + e_a m_b}{m_a m_b}\vec{J_a} \tag{C.8d}$$

On the other hand, as shown in Ref. [4], one may write that,

$$\vec{J_a} = -L_{aq} \text{ grad } \ln T - L_{ab}\vec{d_{ab}} \tag{C.9}$$

and L_{aq}, L_{ab} obey the OR theorem. It is also clear that by (C.8b) and (C.8d) the expressions for $\vec{J}$ and $\vec{J_e}$ will also fulfill such theorem so indeed we have the three linear relations,

$$\begin{aligned}
\vec{J} &= -L_{aa}\vec{d}^{(0)}_{ab} - L_{ae}\vec{d}^{(0)}_{ab} - L_{aq} \text{ grad } T \\
\vec{J'}_q &= -L_{qa}\vec{d}^{(0)}_{ab} - L_{qe}\vec{d}^{(0)}_{ab} - L_{qq} \text{ grad } T \\
\vec{J_c} &= -L_{ea}\vec{d}^{(0)}_{ab} - L_{eq} \text{ grad } T - L_{ee}\vec{d}^{(0)}_{ab}
\end{aligned} \tag{C.10}$$

and by its own structure, $\vec{d}^{(0)}_{ab} \propto -\text{grad } \phi$ since $\vec{E} = -\text{grad } \phi$.

The reader may now see that we may write the Onsager matrix in a more canonical form, namely

$$\begin{pmatrix} \vec{J} \\ \vec{J_c} \\ \vec{J_q} \end{pmatrix} = \begin{pmatrix} L_{aa} & L_{ae} & L_{aq} \\ L_{ea} & L_{ee} & L_{eq} \\ L_{qa} & L_{qe} & L_{qq} \end{pmatrix} \begin{pmatrix} -\vec{d}^{(0)}_{ab} \\ -\vec{d}^{(e)}_{ab} \\ -\text{ grad } T \end{pmatrix} \tag{C.11}$$

Although Eqs. (C.10) are those which follow from microscopic reversibility and are therefore the holders of symmetry, in Eq. (C.11) it is no longer obvious that indeed one can make $L_{ae} = L_{ea}$, etc. but only their ratio can be shown to be constant. Nevertheless in their comparison with experiment the relations following from Eq. (C.11) as well as (C.8b-d) will be used. This point has to be kept in mind. (See Ref. [5] Chap. 9). We shall come back to this point later on.

We now proceed with the evaluation of $\vec{J}_a (= -\vec{J}_b)$ and $\vec{J}_q$ in order to evaluate the corresponding transport coefficients. For this purpose we shall assume that

$$\begin{aligned} \mathbb{A}_i(c_i, \ldots) &= \textstyle\sum_{p=0}^{\infty} a_i^{(p)} S_{\frac{3}{2}}^{(p)}(c_i) \\ \mathbb{D}_i(c_i, \ldots) &= \textstyle\sum_{q=0}^{\infty} d_i^{(p)} S_{\frac{3}{2}}^{(q)}(c_i) \end{aligned} \tag{C.12}$$

$S_{\frac{3}{2}}^{(m)}(c_i)$ are Sonine polynomials whose properties are summarized in p. 35 of the main text. First of all the subsidiary conditions (C.6) now read,

$$\sum_{p=0}^{\infty} \left\{ \begin{array}{c} a_i^{(p)} \\ d_i^{(p)} \end{array} \right\} \int f_i^{(0)} c_i^2 S_{\frac{3}{2}}^{(p)}(c_i) d\vec{c}_i = 0$$

Using the dimensionless velocity,

$$\vec{\omega}_i = \vec{c}_i \sqrt{\frac{m_i}{2kT}}$$

and

$$f_i^{(0)} = n_i \left(\frac{m_i}{2\pi kT} \right)^{3/2} \exp^{-\omega_i^2}$$

dropping irrelevant constants we get that

$$\sum_i n_i \left\{ \begin{array}{c} a_i^{(p)} \\ d_i^{(p)} \end{array} \right\} \delta_{p,0} = 0$$

which yields

$$\begin{aligned} n_a a_a^{(0)} + n_b a_b^{(0)} &= 0 \\ n_a d_a^{(0)} + n_b d_b^{(0)} &= 0 \end{aligned} \tag{C.13}$$

the resulting conditions to be imposed on the expansions in (C.12).

Using Eq. (C.5′) and (C.12) in the expression for $\vec{J}_a$ we find that

$$\vec{J}_a = -n_a kT \frac{\partial \ln T a_a^{(0)}}{\partial \vec{r}} - n n_a kT d_a^{(0)} \vec{d}_{ab} \tag{C.14a}$$

Introducing this last equation into Eq. (C.8a) and Eq. (C.8b) we can calculate $\vec{J}_c$ and the mass flow, namely,

$$\vec{J}_c = -n_a kT \left(\frac{m_a + m_b}{m_a m_b} \right) e a_a^{(0)} \frac{\partial \ln T}{\partial \vec{r}} - n n_a kT \left(\frac{m_a + m_b}{m_a m_b} \right) e d_a^{(0)} \vec{d}_{ab} \quad \text{(C.14b)}$$

and

$$\vec{J} = \frac{m_b - m_a}{m_a m_b} \left(-n_a kT a_a^{(0)} \frac{\partial \ln T}{\partial \vec{r}} - n n_a kT d_a^{(0)} \vec{d}_{ab} \right) \quad \text{(C.14c)}$$

Finally, using these two last equations and Eq. (C.13) in Eq. (C.8c) we have that,

$$\vec{J'}_q = \frac{5}{2}(kT)^2 \left(\frac{n_a a_a^{(1)}}{m_a} + \frac{n_b a_b^{(1)}}{m_b} \right) \text{ grad } \ln T - \left(\frac{m_a + m_b}{m_a m_b} \right) n n_a (kT)^2 d_a^{(0)} \vec{d}_{ab} \quad \text{(C.14d)}$$

Since in these equations $\vec{d}_{ab} = \vec{d}_{ab}^{(0)} + \vec{d}_{ab}^{(e)}$ and $\vec{E} = -\text{grad } \phi$ we have the transport coefficients matrix in which the diffusive force $\vec{d}_{ab}$ reduces to grad n_a if p is constant.

Insertion of the explicit form for $\vec{d}_{ab}$ in Eqs. (C.14b-d) leads precisely to the general set of equations (C.10). Notice should be made of the fact that Onsager's symmetry cannot be expected any longer. The proof of the ORT requires the full form of $\phi_i^{(1)}$ of which only four coefficients survive for the explicit calculation of the transport coefficients namely, $a_a^{(0)}$, $d_a^{(0)}$, $a_a^{(1)}$ and $a_b^{(1)}$. These will be obtained by solving the integral equations which arise when (C.5′) is inserted back in the linearized Boltzmann equation, Eq. (C.3).

Indeed, if this substitution is performed, we get that

$$\begin{aligned} -\left(\frac{m_a c_a^2}{2kT} - \frac{5}{2} \right) f_a^{(0)} \vec{c}_a &= \left\{ C(\mathbb{A}_a \vec{c}_a) + C(\mathbb{A}_a \vec{c}_a + \mathbb{A}_b \vec{c}_b) \right\} f_a^{(0)} \\ -\frac{n_a}{n} f_a^{(0)} \vec{c}_a &= \left\{ C(\mathbb{D}_a \vec{c}_a) + C(\mathbb{D}_a \vec{c}_a + \mathbb{D}_b \vec{c}_b) \right\} f_a^{(0)} \end{aligned} \quad \text{(C.15)}$$

and two identical equations for species b. These equations, far simpler than those arising in the $B \neq 0$ case will be solved by a rather straightforward variational method using the results of Appendix B. The argument goes as follows: Eqs. (C.15) are of the form,

$$R_i(\vec{v}) = \sum_{j=a}^{b} \int d\vec{v'}_i g \sigma(\Omega) d\Omega f_i^{(0)}(\vec{v}) f_j^{(0)}(\vec{v}_1) \left[T_i(\vec{v'}_i) + T_j(\vec{v'}_j) - T_i(\vec{v}_i) + T_j(\vec{v}_j) \right] \quad \text{(C.16)}$$

where $R_i(\vec{v})$ is known, T_i is unknown and $\vec{v'}_i$ and $\vec{v'}_j$ are the velocities of the particles after collision, and for collisions among of the same species $(\vec{v'}, \vec{v'}_1 \rightarrow \vec{v}, \vec{v}_1)$. Also,

$$C(\mathbb{A}_a\vec{c}_a) = \int d\vec{v'}_a g\sigma(\Omega)d\Omega \left\{\vec{c'}_a\mathbb{A}_a(\vec{c'}_a) + \vec{c'}_{a_1}\mathbb{A}_a(\vec{c'}_{a_1}) - \vec{c}_a\mathbb{A}_a(\vec{c}_a) \right. \\ \left. -\vec{c}_{a1}\mathbb{A}_a(\vec{c}_{a1})\right\} f_a^{(0)}(\vec{c}_a)f_a^{(0)}(\vec{c'}_a)$$

$$C(\mathbb{A}_a\vec{c}_a + \mathbb{A}_b\vec{c}_b) = \int d\vec{v'}_a g\sigma(\Omega)d\Omega \left\{\vec{c'}_a\mathbb{A}_a(\vec{c'}_a) + \vec{c'}_b\mathbb{A}_b(\vec{c'}_b) - \vec{c}_a\mathbb{A}_a(\vec{c}_a) \right. \\ \left. -\vec{c}_b\mathbb{A}_b(\vec{c}_b)\right\} f_a^{(0)}(\vec{c}_a)f_a^{(0)}(\vec{c}_b)$$

and in each case $g =\mid \vec{v}_i - \vec{v}_j \mid$, the relative velocity of the colliding particles.

Let now $t_i(\vec{v})$ be a trial function proposed as solution to Eq. (C.15). Integrating over $\vec{v}_i$ after multiplication by $t_i(\vec{v})$, one gets

$$\sum_i \int d\vec{v}_i R_i(\vec{v}) : t_i(\vec{v}) = \sum_{ij} \int d\vec{v}_i\vec{v}_j g\sigma(\Omega)d\Omega f_i^{(0)}(\vec{v}_i)f_j^{(0)}(\vec{v}_j) \\ t_i(\vec{v}) : \left[t_i(\vec{v'}_i) + t_j(\vec{v'}_j) - t_i(\vec{v}_i) + t_j(\vec{v}_j)\right]$$

This equation according to the definition of collisional brackets and Eq. (B.8) yield that

$$\sum_i \int d\vec{v}_i R_i(\vec{v}) : t_i(\vec{v}) = -\frac{1}{2}\left\{t_i, t_i\right\} =$$

$$-n_a^2\left[t_a, t_a\right]_{aa} - n_a n_b\left[t_a + t_b, t_a + t_b\right]_{ab} - n_b^2\left[t_b, t_b\right]_{bb} \tag{C.17}$$

Clearly, from Eq. (C.16)

$$\sum_i \int d\vec{v}_i R_i(\vec{v}) : t_i(\vec{v}) = -\frac{1}{2}\left\{t_i, T_i\right\}$$

so if $t_i(\vec{v})$ in fact satisfies the integral equation (C.15) we would have that

$$\left\{t_i, t_i\right\} = \left\{t_i, T_i\right\}$$

By Eq. (B.7),

$$\left\{t_i - T_i; t_i - T_i\right\} = \left\{t_i, t_i\right\} - 2\left\{t_i, T_i\right\} + \left\{T_i, T_i\right\} \geq 0$$

then

$$-2\sum_i \int d\vec{v}_i R_i(\vec{v}) : t_i(\vec{v}) = \left\{t_i, t_i\right\} \leq \left\{T_i, T_i\right\} \tag{C.18}$$

Eq. (C.18) is the basis of the variational principle. The proposed solution $t_i(\vec{v})$ must be such that it maximizes the collision integral $\{T_i, T_i\}$.

We start with the second of Eqs. (C.15) which after multiplication by $f_a^{(0)}$ on both sides and summation over the two species, reads

$$-2\sum_i \int d\vec{c}_i t_i : R_i = -\frac{2}{n}\sum_i n_i \int d\vec{c}_i \vec{c}_i \vec{c}_i f_i^{(0)}(\vec{c}_i) \sum_{p=0}^{\infty} d_p^{(i)} S_{\frac{3}{2}}^{p}(\omega_i)$$

after substitution of the trial function

$$\vec{t}_i = \vec{c}_i \sum_{p=0}^{\infty} d_i^{(p)} S_{\frac{3}{2}}^{p}(\vec{c}_i)$$

Introducing the dimensionless velocity $\vec{\omega}_i$ and carrying the integrals recalling that

$$\int_0^{\infty} d\omega \omega^4 \exp^{-\omega^4} S_{\frac{3}{2}}^{p} S_{\frac{3}{2}}^{0} = \frac{3}{8}\sqrt{\pi}\delta_{p,0}$$

and introducing the subsidiary condition given by Eq. (C.13) we finally arrive at the result that

$$-2\sum_i \int d\vec{c}_i t_i : R_i = -6kT\frac{n_a}{n}\frac{(n_a m_b - n_b m_a)}{m_a m_b} d_a^{(0)} \tag{C.19}$$

The inequality (C.18) thus yields that

$$-6kT\frac{n_a}{n}\frac{(n_a m_b - n_b m_a)}{m_a m_b} d_a^{(0)} = \left\{ \vec{c}_a \sum_p d_a^{(p)} S_{\frac{3}{2}}(\omega_a); \vec{c}_a \sum_p d_a^{(p)} S_{\frac{3}{2}}(\omega_a) \right\}$$

If we now use Eq. (B.8) for the binary mixture, to first approximation, noticing that $[\vec{c}_a, \vec{c}_a]_{aa} = [\vec{c}_b, \vec{c}_b]_{bb} = 0$ we have that $(d_a^{(p)} = 0,\ p > 0)$ so that

$$-6kT\frac{n_a}{n}\left(\frac{n_a m_b - n_b m_a}{m_a m_b}\right) d_a^{(0)} = 2n_b \left[d_a^{(0)2} [\vec{\omega}_a, \vec{\omega}_a]_{aa} + 2 d_a^{(0)} d_b^{(0)} [\vec{\omega}_a, \vec{\omega}_b]_{ab} + d_b^{(0)2} [\vec{\omega}_b, \vec{\omega}_b]_{bb} \right]$$

Once more, since $d_b^{(0)} = -(n_a/n_b) d_a^{(0)}$ we have

$$-3kT\frac{(n_a m_b - n_b m_a)}{n(m_a m_b)} d_a^{(0)} = n_b d_a^{(0)2} \left[[\vec{\omega}_a, \vec{\omega}_a]_{aa} - \frac{n_a}{n_b} [\vec{\omega}_a, \vec{\omega}_b]_{ab} + \frac{n_a^2}{n_b^2} [\vec{\omega}_b, \vec{\omega}_b]_{bb} \right]$$

so that the trivial solution is $d_a^{(0)} = 0$ and the sought one is given by

$$d_a^{(0)} = -\frac{3kT}{n}\left(\frac{n_a m_b - n_b m_a}{m_a m_b}\right)\left(1 - 2\frac{n_a}{n_b}\sqrt{\frac{m_a}{m_b}} + \frac{n_a^2}{n_b^2}\right)^{-1}\frac{1}{[\vec{\omega}_a, \vec{\omega}_b]_{ab}} \quad \text{(C.20a)}$$

As we see everything may be expressed in terms of a single collision integral using the results of Appendix D, namely

$$[\vec{\omega}_a, \vec{\omega}_b]_{ab} = -\sqrt{\frac{m_a}{m_b}}\,[\vec{\omega}_a, \vec{\omega}_a]_{ab} \quad \text{(C.20)}$$

and

$$[\vec{\omega}_b, \vec{\omega}_b]_{ab} = \frac{m_a}{m_b}\,[\vec{\omega}_a, \vec{\omega}_a]_{ab}$$

where

$$[\vec{\omega}_a, \vec{\omega}_a]_{ab} = \frac{1}{(4\pi\epsilon_0)^2}\frac{\sqrt{2\pi}e^4}{(kT)^{\frac{3}{2}}}\frac{1}{\sqrt{m_a}}\ln\left(1 + \left(\frac{4kTd}{e^2}\right)^2\right) \quad \text{(C.20b)}$$

and

$$d = \sqrt{\frac{kT\epsilon_0}{e^2 n}} \quad \text{(C.20c)}$$

is Debye's length. Eqs (C.20a-b-c) are the final result of this calculation. Improvement on the values for d_0^a can be obtained by considering more terms in the evaluation of $\{\quad\}$ but we leave that to the reader if and when he considers it necessary.

For the first of the equations (C.15) we have that

$$-2\sum_i \int d\vec{c}_i t_i : R_i = -2\sum_i \int \sum_{p=0}^{\infty} a_i^{(p)} n_i \frac{m_i}{2\pi kT}^{\frac{3}{2}} 4\pi\left(\frac{m_i c_i^2}{2kT} - \frac{5}{2}\right)$$

$$\exp^{-\frac{m_i c_i^2}{2kT}} c_i^4 dc_i S_{\frac{3}{2}}^{(p)}(\omega_i)$$

where the trial function used is the one in the previous case with different coefficients. Introducing the velocity ω_i, recalling that

$$\int_0^{\infty} S_{\frac{3}{2}}^{(1)}(\omega) S_{\frac{3}{2}}^{(p)}(\omega)\omega^4 \exp^{-\omega^2} d\omega = \frac{15}{16}\sqrt{\pi}\delta_{p,1}$$

we wind up with the result that

$$15kT\left(\frac{n_a}{m_a}a_a^{(1)} + \frac{n_b}{m_b}a_b^{(1)}\right) = \{t, t\}$$

The calculation of $\{t, t\}$ is a rather cumbersome procedure. One resorts to Eq. (B.3) setting

$$K_a = L_a = a_0^{(a)}\vec{c}_a + a_a^{(1)}S_{\frac{3}{2}}(\vec{c}_a)$$

$$K_b = L_b = a_0^{(b)}\vec{c}_b + a_b^{(1)}S_{\frac{3}{2}}(\vec{c}_b)$$

and makes use of all the values for the corresponding collision integrals which are given in Appendix D. After a lengthy but straight forward algebraic procedure one finds that $\{t_i, t_i\}\,\tau$ is given precisely by the terms arising from the collisional integrals which are written in the right hand side of Eq. (5.8) after the ω-dependent terms namely the second and third rows are ignored. We find unnecessary to repeat that equation here. The remaining steps of the variational procedure lead precisely to Eqs. (5.9) and (5.10) when $\omega_a = \omega_b = 0$. As expected the results in this case are those quoted in Eqs. (5.11). The procedure to obtain the d_i's is completely analogous.

Bibliography

[1] L. Spitzer; *The Physics of Fully Ionized Gases*, Wiley-Interscience, New York (1962).

[2] S. Chapman and T. G. Cowling; loc. cit. Chap. 1.

[3] G. W. Ford and G. E. Uhlenbeck; loc. cit. Chap. 1.

[4] P. Goldstein and L. García-Colín; *J. Non-Equilib. Thermodyn.* **30**, 173 (2005).

[5] S. R. de Groot and P. Mazur; *Non-Equilib. Thermodynamics*, Dover Publications Inc., Mineola, N.Y. (1984), Chap. XI.

[6] J. O. Hirschfelder, C. F. Curtiss and R. B. Byrd; *The Molecular Theory of Liquids and Gases*, John Wiley & Sons, New York (1964). 2^{nd} ed.

Appendix D

The Collision Integrals

In order to determine the coefficients of viscosity, thermal conduction, and diffusion of a gas, it is necessary first to evaluate the collision integrals. In this Appendix we shall consider collisions and the evaluation of the various collision integrals in detail and how the cross section can be simply expressed for Coulomb interactions.

Suppose we have two particles colliding, the first of mass m_1, charge e_1, velocity v_1 and the second mass m_2, charge e_2 and velocity v_2. For Coulomb forces between the particles the equation of motion is given by

$$\mu \frac{d^2\vec{r}}{dt^2} = \frac{k_e e_1 e_2}{r^3}\vec{r} \tag{D.1}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is the reduced mass, and $k_e = 1/4\pi\varepsilon_0$ where ε_0 the permitivity of vacuum ($\varepsilon_0 = 8.85 \times 10^{-12}\ \ \mathrm{C^2/Jm}$).

The geometry of the collision is shown in Figure D.1. Initially particle 1 has a relative velocity

$$\vec{g} = \vec{v}_1 - \vec{v}_2 \tag{D.2}$$

and asymptotic distance of approach b. We suppose that its relative position makes an angle β with the direction $-\vec{g}$, so initially $\beta = 0$. Finally $\beta = \pi - \chi$ where χ is the scattering angle we wish to find as a function of g and b.

The differential cross section for scattering into unit solid angle shall be

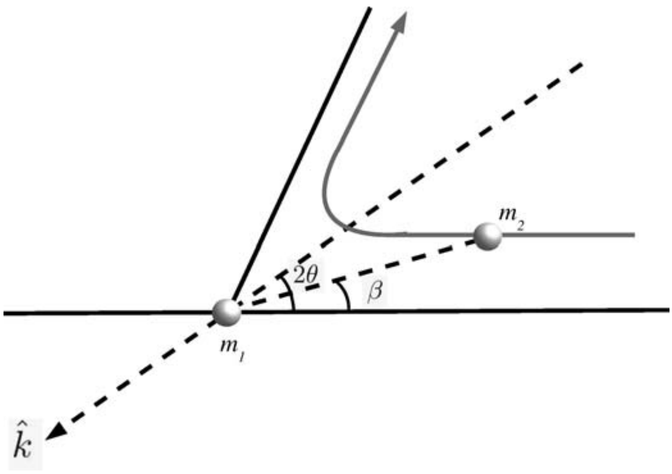

Figure D.1: Geometry of a collision.

replaced by the well known Rutherford formula, namely,

$$\sigma(\chi, \varepsilon) = \left(\frac{k_e e_1 e_2}{2\mu g^2} \operatorname{cosec}^4 \frac{1}{2}\chi \right)^2 \tag{D.3}$$

The angles of this expression are depicted in Figure D.1.

From Figure D.1 we see that,

$$\begin{aligned} \chi &= \pi - 2\theta \\ \cot \tfrac{1}{2}\chi &= \frac{\mu b g^2}{k_e e_1 e_2} \end{aligned} \tag{D.4}$$

is convenient to make use of the unit vector $\hat{k}$ drawn as show in Figure D.1. Clearly

$$\hat{k} \cdot \vec{g} = \vec{g} \cos\theta = -\hat{k} \cdot \vec{g}\,' \tag{D.5}$$

where

$$\vec{g}\,' = \vec{v}\,'_2 - \vec{v}\,'_1 = \vec{c}\,'_2 - \vec{c}\,'_1 \tag{D.6}$$

and it is the relative velocity after the collision. Also

$$\begin{aligned} \vec{g}\,' &= \vec{g} - 2(\vec{g} \cdot \hat{k}) \\ &= \vec{g} - 2g\cos\theta\hat{k} \end{aligned} \tag{D.7}$$

It is also convenient to introduce the dimensionless numbers

$$M_1 = \frac{m_1}{m_1 + m_2}$$
$$M_2 = \frac{m_2}{m_1 + m_2} \tag{D.8}$$

The center of gravity velocity is, relative to the drift velocity $\vec{u}$

$$\vec{G} = M_1\vec{c}_1 + M_2\vec{c}_2 \tag{D.9}$$

and from (D.8) and (D.9) the following equations can be derived

$$\vec{c}_1 = \vec{G} + M_2\vec{g}$$
$$\vec{c}_2 = \vec{G} + M_1\vec{g}$$
$$\vec{c}'_1 = \vec{G} + M_2\vec{g}\,'$$
$$\vec{c}'_2 = \vec{G} + M_1\vec{g}\,' \tag{D.10}$$

Also,

$$\vec{c}'_1 = \vec{c}_1 + 2\vec{g}M_2\cos\theta\hat{k}$$
$$\vec{c}'_2 = \vec{c}_2 + 2\vec{g}M_1\cos\theta\hat{k} \tag{D.11}$$

Now define new variables $\vec{x}$ and $\vec{y}$ by

$$\vec{x} = \vec{g}\sqrt{\frac{\mu}{2kT}}$$
$$\vec{y} = \vec{G}\sqrt{\frac{m_1 + m_2}{2kT}} \tag{D.12}$$

Then

$$\begin{aligned}
\vec{w}_1 &= M_1^{\frac{1}{2}}\vec{y} + M_2^{\frac{1}{2}}\vec{x} \\
\vec{w}_2 &= M_2^{\frac{1}{2}}\vec{y} - M_1^{\frac{1}{2}}\vec{x} \\
\vec{w}\,'_1 &= w_1 - 2M_2^{\frac{1}{2}}\vec{x}\cos\theta\hat{k} \\
\vec{w}\,'_2 &= w_2 - 2M_1^{\frac{1}{2}}\vec{x}\cos\theta\hat{k}
\end{aligned} \tag{D.13}$$

where

$$\vec{w}_i = \sqrt{\frac{m_i}{2kT}} \tag{D.14}$$

Since the Jacobian of the transformation is,

$$J = \frac{\partial(\vec{x},\vec{y})}{\partial(\vec{c}_1,\vec{c}_2)} = \frac{\sqrt{m_1 m_1}}{2kT} \tag{D.15}$$

and from Eq. (D.4) in terms of this new variables, the collision integral can be written as,

$$[G_1, H_2; K_1 L_2]_{12} = -\left(\frac{2kT}{\mu}\right)^{\frac{1}{2}} \frac{1}{\pi^3} \int d\vec{x}d\vec{y}d\varepsilon bdb\vec{x}\exp(-x^2-y^2)$$
$$[G_1(\vec{w}_1) + H_2(\vec{w}_2)] : [K_1(\vec{w}\,'_1) + L_2(\vec{w}\,'_2) - K_1(\vec{w}_1) + L_2(\vec{w}_2)] \tag{D.16}$$

Using this formula all the integrals for collisions between an electron and an ion can be worked out. Then the integrals for collisions between like particles can be obtained by setting the masses equal. We shall give the details of just one calculation, namely of $[\vec{w}_1, \vec{w}_1]_{12}$. From Eq. (D.16) this is

$$[\vec{w}_1, \vec{w}_1]_{12} = -\left(\frac{2kT}{\mu}\right)^{\frac{1}{2}} \frac{1}{\pi^3} \int\cdots\int d\vec{x}d\vec{y}d\varepsilon bdb\vec{x}\exp(-x^2-y^2)\vec{w}_1\cdot(\vec{w}\,'_1 - \vec{w}_1) \tag{D.17}$$

Now from Eq. (D.11)

$$\vec{w}_1\cdot(\vec{w}\,'_1 - \vec{w}_1) = -\left[M_1^{\frac{1}{2}}\vec{y} + M_2^{\frac{1}{2}}\vec{x}\right] 2M_2^{\frac{1}{2}}\vec{x}\cos\theta\cdot\hat{k} \tag{D.18}$$

Since the y integral is odd, it averages to zero on integrating over $\vec{y}$, and finally

$$[\vec{w}_1, \vec{w}_1]_{12} = 2M_2\left(\frac{2kT}{\mu}\right)^{\frac{1}{2}} \frac{2\pi}{\pi^{\frac{3}{2}}} \int d\vec{x}x^3\exp(-x^2)\int bdb\cos^2\theta \tag{D.19}$$

Using Eq. (D.4) this last integral is

$$\int_0^d b db \cos^2\theta = \int_0^d \frac{b db}{1+\left(\frac{2kTx^2}{k_e e_1 e_2}\right)^2 b^2} = \frac{1}{2}\left(\frac{k_e e_1 e_2}{2kT}\right)^2 \frac{1}{x^4} \ln\left\{1+d^2\left(\frac{2kTx^2}{k_e e_1 e_2}\right)^2\right\} \tag{D.20}$$

This expression diverges if d goes to infinity but the divergence is only logarithmic and is therefore very slow. Whence, it does not matter very much what choice we make for d within reasonable limits. Because the value we should use for d is not fixed precisely and because the answer is insensitive anyway we might just as well replace x^2 where it appears inside the logarithm by its average value which is 2. Hence Eq. (D.20) becomes

$$\frac{1}{2}\left(\frac{k_e e_1 e_2}{2kT}\right)^2 \frac{1}{x^4}\psi$$

where ψ is the logarithm factor

$$\psi = \ln\left\{1+\left(\frac{4kTd}{k_e e_1 e_2}\right)^2\right\} \tag{D.21}$$

So Eq. (D.19) becomes

$$[\vec{w}_1,\vec{w}_1]_{12} = M_2\left(\frac{2kT}{\mu}\right)^{\frac{1}{2}}\left(\frac{k_e e_1 e_2}{2kT}\right)^2 \frac{2\psi}{\sqrt{\pi}}\int_0^\infty d\vec{x}\frac{1}{x}\exp(-x^2) = \sqrt{2\pi}\frac{(k_e e_1 e_2)^2\psi}{(kT)^{\frac{3}{2}}}\left\{\frac{m_2}{m_1(m_1+m_2)}\right\}^{\frac{1}{2}} \tag{D.22}$$

If m_2 it is much grater than m_1

$$[\vec{w}_1,\vec{w}_1]_{12} \simeq \sqrt{2\pi}\frac{(k_e e_1 e_2)^2\psi}{\sqrt{m_1}(kT)^{\frac{3}{2}}} = \varphi \tag{D.23}$$

Interchanging the masses in Eq. (D.23) gives

$$[\vec{w}_2,\vec{w}_2]_{12} \simeq \sqrt{2\pi}\frac{(k_e e_1 e_2)^2\psi}{(kT)^{\frac{3}{2}}}\left\{\frac{m_1}{m_2(m_1+m_2)}\right\}^{\frac{1}{2}} = \frac{m_1}{m_1}[\vec{w}_1,\vec{w}_1]_{12} \tag{D.24}$$

Now we shall calculate $[\vec{w}_1, \vec{w}_2]_{12}$ as follows. By definition

$$[\vec{w}_1, \vec{w}_2]_{12} = -\left(\frac{2kT}{\mu}\right)^{\frac{1}{2}} \frac{1}{\pi^3} \int d\vec{x}d\vec{y}d\varepsilon bdb\vec{x} \exp(-x^2 - y^2)\vec{w}_1 \cdot (\vec{w}\,'_2 - \vec{w}_2) \tag{D.25}$$

and from Eq. (D.13)

$$\vec{w}\,'_2 - \vec{w}_2 = -\left(\frac{m_1}{m_2}\right)^{\frac{1}{2}} (\vec{w}\,'_1 - \vec{w}_1) \tag{D.26}$$

Comparing Eqs. (D.25) and (D.26) with Eq. (D.22) we therefore see that

$$[\vec{w}_1, \vec{w}_2]_{12} = -\left(\frac{m_1}{m_2}\right)^{\frac{1}{2}} [\vec{w}_1, \vec{w}_1]_{12} \tag{D.27}$$

From Eq. (D.22) and Eq. (D.27)

$$[\vec{w}_1, 0; \vec{w}_1, \vec{w}_2]_{12} = \left\{1 - \left(\frac{m_1}{m_2}\right)^{\frac{1}{2}}\right\} \sqrt{2\pi} \frac{(k_e e_1 e_2)^2 \psi}{(kT)^{\frac{3}{2}}} \left\{\frac{m_2}{m_1(m_1 + m_2)}\right\}^{\frac{1}{2}} \tag{D.28}$$

We can now get an expression for $[\vec{w}_1, \vec{w}_1]_1$ by setting $m_1 = m_2$ in Eq. (D.28). This gives

$$[\vec{w}_1, \vec{w}_1]_{11} = 0 \tag{D.29}$$

Similarly

$$[\vec{w}_2, \vec{w}_2]_{22} = 0 \tag{D.30}$$

In a similar way all other integrals we need can be evaluated and the results are given in the following list.

$$[\vec{w}_1, \vec{w}_1]_{11} = 0$$

$$[\vec{w}_2, \vec{w}_2]_{22} = 0$$

$$[\vec{w}_1, \vec{w}_1]_{12} = \varphi$$

$$[\vec{w}_1, \vec{w}_2]_{12} = -M_1^{\frac{1}{2}} \varphi$$

$$[\vec{w}_2, \vec{w}_2]_{12} = M_1 \varphi$$

$$[\vec{w}_1, \vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}_1^2)]_{12} = \frac{3}{2}\varphi$$

$$[\vec{w}_2, \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_{12} = \frac{3}{2} M_1^2 \varphi$$

$$[\vec{w}_1, \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_{12} = -\frac{3}{2} M_1^{\frac{3}{2}} \varphi$$

$$[\vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1), \vec{w}_2]_{12} = -\frac{3}{2} M_1^{\frac{1}{2}} \varphi$$

$$[\vec{w}_2, \vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1)]_1 = 0$$

$$[\vec{w}_2, \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_2 = 0$$

$$[\vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1), \vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1)]_{12} = \frac{13}{4} \varphi$$

$$[\vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2), \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_{12} = \frac{15}{2} M_1^{\frac{1}{2}} \varphi$$

$$[\vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1), \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_{12} = -\frac{27}{4} M_1^{\frac{3}{2}} \varphi \left(1 + \frac{16}{27}\delta\right)$$

$$[\vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1), \vec{w}_1 S^1_{\frac{3}{2}}(\vec{w}^2_1)]_1 = \sqrt{2}\varphi(1-\delta)$$

$$[\vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2), \vec{w}_2 S^1_{\frac{3}{2}}(\vec{w}^2_2)]_2 = \sqrt{2M_1}\varphi(1-\delta)$$

where

$$\delta = \frac{1}{\psi} \frac{\left(\frac{4kTd}{k_e e_1 e_2}\right)^2}{1 + \left(\frac{4kTd}{k_e e_1 e_2}\right)^2}$$

which, for all except very extreme conditions is $1/\psi$ and very small compared to unity. Hence under almost all conditions δ can be set equal to zero in this list.

It remains to examine the validity of Eq. (D.1) which assumed that while the particles were interacting all forces other than their Coulomb interaction could be ignored. This will be valid provided the Debye distance, d, is smaller than the gyromagnetic radius, namely, provided

$$d = \left(\frac{kT}{4\pi(k_e e_1 e_2)}\right)^{\frac{1}{2}} << \frac{mv}{eB}$$

This inequality holds very well except for low densities $\leq 10^{12}$ m^{-3} and high fields ($H \geq 5000$ gauss).

Appendix E

Calculation of the Coefficients $a_i^{(0)}$, $a_i^{(1)}$, $d_i^{(0)}$ and $d_i^{(1)}$

We here outline the details of two calculations leading to the coefficients $a_i^{(1)} = a_i^{(1)(1)} + iBa_i^{(1)(2)}$, $i = a, b$; the same for $a_i^{(0)}$ and their analogs for $d_i^{(0)}$, $d_i^{(1)}$. The former ones are simply the solution to the inhomogeneous system of three equations defined in Eq. (5.9) in the text. This system can be solved by hand with some approximations which are thereafter compared with the ones obtained with a computer program. After separating their real and complex parts one obtains that

$$\begin{aligned}
Re\left(a_a^{(0)}\right) = a_a^{(1)(0)} &= \frac{5\tau}{\Delta_1}\left(1.956 - 19.8x^2 - 4.5x^4\right)\\
Im\left(a_a^{(0)}\right) = a_a^{(2)(0)} &= \frac{5\tau}{\Delta_1}\left(21.3x + 4.78x^3\right)\\
Re\left(a_a^{(1)}\right) = a_a^{(1)(1)} &= \frac{5\tau}{\Delta_1}\left(1.305 + 68.28x^2 + 16.14x^4\right)\\
Im\left(a_a^{(1)}\right) = a_a^{(1)(2)} &= \frac{5\tau}{\Delta_1}\left(5.31x + 82.452x^3 + 18x^5\right)\\
Re\left(a_b^{(1)}\right) = a_b^{(1)(1)} &= \frac{1}{M_1}\frac{5\tau}{\Delta_1}\left(0.35154 + 7.21x^2 + 37.99x^4\right)\\
Im\left(a_b^{(1)}\right) = a_b^{(1)(2)} &= \frac{1}{M_1}\frac{5\tau}{\Delta_1}\left(0.036x - 16.45x^3 - 18x^5\right)
\end{aligned} \tag{E.1}$$

where

$$\Delta_1 = 3.744 + \times 10^{10}x^2 + 4.59 \times 10^{10}x^4 + 90x^6 \tag{E.2}$$

As stated in page 49 the only difference between the previous case and the one corresponding to the $\mathbb{D}_i$ functions in Eq. (3.15b) is that in constructing

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma,* Springer Series on Atomic, Optical, and Plasma Physics 53

the variational procedure the inhomogeneous terms in Eqs. (5.10) change to the values given in Eqs. (5.9). Thus the algebraic system of inhomogeneous equations may be solved to yield the following results:

$$\begin{aligned}
Re\left(d_a^{(0)}\right) &= d_a^{(1)(0)} = & -\tfrac{3\tau}{2\Delta_2}\left(2.86 + 6.616x^2 + 1.415x^4\right)\\
Im\left(d_a^{(0)}\right) &= d_a^{(2)(0)} = & -\tfrac{3\tau}{2\Delta_2}\left(27.745x + 40.05x^3 + 7.5x^5\right)\\
Re\left(d_a^{(1)}\right) &= d_a^{(1)(1)} = & \tfrac{3\tau}{2\Delta_2}\left(0.6523 - 6.604x^2 - 1.5x^4\right)\\
Im\left(d_a^{(1)}\right) &= d_a^{(1)(2)} = & -\tfrac{3\tau}{2\Delta_2}\left(7.095x + 1.598x^3\right)\\
Re\left(d_b^{(1)}\right) &= d_b^{(1)(1)} = & \tfrac{3\tau}{2\Delta_2}\left(0.5426 + 0.339x^2 + 1.5x^4\right)\\
Im\left(d_b^{(1)}\right) &= d_b^{(1)(2)} = & \tfrac{3\tau}{2\Delta_2}\left(5.343x + 4.265x^3\right)
\end{aligned} \tag{E.3}$$

where $\Delta_2 = \Delta_1$.

Appendix F

The proof of the equalities given in Eq. (8.7) is just a matter of a careful lengthy manipulation. All follow the same procedure. For instance,

$$2\overleftrightarrow{\tau}_i^{(5)} = 2\overleftrightarrow{Q}_i^{(7)}$$

Using the form for the right hand and shown in Eq. (8.6) we take, say component xy. Next we sum over repeated indices γ, φ and λ. Take γ first and then φ keeping only the non zero terms according to the structure of ϵ_{ijk}. In the final summation over λ, the resulting expression is identical to the one obtained for the left hand side according to Eq. (8.2). To show an easier example, this same procedure gives that

$$2(Q_i^{(2)})_{xy} = B_x C_x C_z - B_y C_y C_z - B_z(C_x^2 - C_y^2)$$

which is identical to $2(\tau_i^{(2)})_{xy}$. Thus

$$(Q_i^{(2)})_{\alpha\beta} = (\tau_i^{(2)})_{\alpha\beta} \qquad \text{for} \;\; \alpha \neq \beta$$

and for $\alpha = \beta$ the result follows at once.

The two middle expressions in Eq. (8.7) are also straight forward. Since $(Q_i^{(4)})_{\alpha\beta} = 0$ if $\alpha = \beta$ one readily sees that after expansion

$$(\overleftrightarrow{\tau}_i^{(3)})_{xy} = (\overleftrightarrow{Q}_i^{(3)})_{xy} - \frac{1}{3}B^2(\overleftrightarrow{Q}_i^{(1)})_{xy} - \frac{1}{3}(\overleftrightarrow{Q}_i^{(5)})_{xy}$$

and for diagonal components,

$$(Q_i^{(4)})_{\alpha\alpha} = (\vec{C}\cdot\vec{B})^2 - B^2C^2$$

accounts for the extra term. The same holds for $(\overleftrightarrow{\tau}_i^{(4)})$.

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma,* Springer Series on Atomic, Optical, and Plasma Physics 53

Appendix G

List of useful integrals

$$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi$$

$$\int_0^{\pi} \sin\theta \cos^2\theta d\theta = \frac{2}{3}$$

$$\int_0^{\pi} \sin^3\theta \cos^2\theta d\theta = \frac{4}{15}$$

$$\int_0^{2\pi} \sin^4\theta \cos^4\theta d\theta = \frac{3}{64}\pi$$

$$\int_0^{\pi} \sin^5\theta d\theta = \frac{16}{15}$$

$$\int_0^{2\pi} \sin^2\theta \cos^2 d\theta = \frac{\pi}{4}$$

$$\int_0^{\pi} \sin^4\theta d\theta = \int_0^{\pi} \cos^4\theta d\theta = \frac{3}{8}\pi$$

$$\int_0^{\pi} \sin^3\theta d\theta = \frac{4}{3}$$

$$\int_0^{2\pi} \sin^2\theta \cos^4\theta d\theta = \frac{\pi}{8}$$

$$\int_0^{\pi} \sin\theta \cos^6\theta d\theta = \frac{2}{7}$$

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma,*
Springer Series on Atomic, Optical, and Plasma Physics 53

Appendix H

The collision integrals used to arrive at Eq. (8.33) in the text are,

$$\left[\overleftrightarrow{w_a\,{}^0w_a}, \overleftrightarrow{w_a\,{}^0w_a}\right]_{aa} = \sqrt{2}\varphi$$

$$\left[\overleftrightarrow{w_a\,{}^0w_a}, \overleftrightarrow{w_a\,{}^0w_a}\right]_{ab} = 2\varphi$$

$$\left[\overleftrightarrow{w_b\,{}^0w_b}, \overleftrightarrow{w_b\,{}^0w_b}\right]_{bb} = \sqrt{2}M_1\varphi$$

$$\left[\overleftrightarrow{w_b\,{}^0w_b}, \overleftrightarrow{w_b\,{}^0w_b}\right]_{ab} = \frac{10}{3}M_1\varphi$$

$$\left[\overleftrightarrow{w_a\,{}^0w_a}, \overleftrightarrow{w_b\,{}^0w_b}\right]_{ab} = -\frac{4}{3}M_1\varphi$$

where

$$M_1 = \frac{m_a}{m_a + m_b}$$

Now we start from Eq. (8.33) in the text. Calculating $\delta\mathfrak{D}(\mathfrak{I}_i) = 0$, collecting terms in $\delta g_a^{(0)}$ and $\delta g_b^{(0)}$, performing obvious arithmetical simplifications, introducing the Larmor frequencies ω_i, we arrive at two equations,

$$-\frac{10\tau m_a}{4kT} = (40i\omega_a\tau + 9.66)\, g_a^{(0)} + \frac{8}{3}M_1^2 g_b^{(0)}$$

$$-\frac{10\tau m_b}{kTM_1} = (-40i\omega_a\tau + 243.98)\, g_b^{(0)} - \frac{8}{3}M_1^2 g_a^{(0)}$$

Noticing that $\omega_a = M_1\omega_b$ if $m_b >> m_a$ and neglecting obvious small terms we find that the determinant for this system of equations is, setting $x = \omega_a\tau$

$$\Delta = \left(47 + 1.6x^2 + 9.842x\right) \times 10^3$$

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma,* Springer Series on Atomic, Optical, and Plasma Physics 53

Trivially then

$$g_a^{(0)} = \frac{10\tau m_a}{kT}\frac{258.4 - 40ix}{\Delta} \tag{H.1}$$

$$g_b^{(0)} = \frac{10\tau m_b}{M_1 kT}\frac{9.66 + 40ix}{\Delta} \tag{H.2}$$

When G_i is substituted by $P_i/2$ and B by $2B$ we obtain the solution to Eq. (8.20),

$$p_a^{(0)} = \frac{20m_a\tau}{kT}\frac{258.4 - 40ix'}{\Delta'} \tag{H.3}$$

$$p_b^{(0)} = \frac{20\tau m_a}{M_1 kT}\frac{9.66 + 40ix'}{\Delta'}m_b \tag{H.4}$$

where $\omega_a' = 2\omega_a$ and $\Delta' = \Delta$ when $\omega_a = 2\omega_a$. Finally, the solution to Eqs. (8.15a) is obtained when $B = 0$, $\omega_a = \omega_b = 0$

$$\begin{aligned} l_a^{(0)} &= 1.046\tfrac{m_a\tau}{kT} \\ l_b^{(0)} &= 0.039\tfrac{m_b\tau}{M_1 kT} \end{aligned} \tag{H.5}$$

and τ, the mean free collision time is $\tau = 1/\varphi n$.

Omitting the unnecessary superscript naught, the relation between these results and the Γ_i's is now summarized as follows,

$$\begin{aligned} l_i &= \Gamma_i^1 + B^2\Gamma_i^3 \\ Rep_i &= 2\Gamma_i^1 + B^2\Gamma_i^4 \\ Imp_i &= B(\Gamma_i^2 + B^2\Gamma_i^5) \\ Reg_i &= \Gamma_i^1 + B^2\Gamma_i^3 \\ Img_i &= \Gamma_i^2 \end{aligned} \tag{H.6}$$

so that $B^2\Gamma_i^5 = Imp_i - Img_i$, whereas the shear viscosity η is determined by l_i when $B = 0$, in this case all other coefficients vanish. This completes the solution to the problem.

Appendix I

List of Marshall's Equations and Notation

I.1 Equations

$$-\vec{d}_2 = \vec{d}_1 = \nabla(\frac{n_1}{n}) + \frac{n_1 n_2 (m_2 - m_1)}{p n \rho} \nabla p - \frac{\rho_1 \rho_2}{p \rho}(X_1 - X_2) - \frac{n_1 n_2}{p \rho}(e_1 m_2 - e_2 m_1)\vec{E}' \tag{M3.14}$$

$$\vec{\jmath} = -\{\nabla \log T\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} a_i^{I,0} - \{\vec{H} \times \nabla \log T\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} a_i^{II,0}$$
$$-\vec{H}\{\vec{H} \cdot \nabla \log T\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} a_i^{III,0} - \{n\vec{d}_i\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} e_i^{I,0}$$
$$\{\vec{H} \times n\vec{d}_i\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} e_i^{II,0} - \{\vec{H}(\vec{H} \cdot n\vec{d}_1)\}\frac{1}{2}(2kT)^{\frac{1}{2}} \sum_i \frac{n_i e_i}{\sqrt{m_i}} e_i^{III,0} \tag{M3.61}$$

$$\vec{\jmath} = \sigma^I \vec{D}'' + \sigma^{II} \vec{D}^{\perp} + \sigma^{III} \vec{h} \times \vec{D}^{\perp} + \varphi^I \{\nabla T\}'' + \varphi^{II} \{\nabla T\}^{\perp} + \varphi^{III} \vec{h} \times \{\nabla T\}^{\perp} \tag{M3.62}$$

$$D = \vec{E} + \frac{1}{c}\vec{u} \times \vec{H} - \frac{m_2 - m_1}{n(e_1 m_2 - e_2 m_1)} \nabla p - \frac{m_1 m_2}{e(m_1 + m_2)}(X_1 - X_2) - \frac{p\rho}{n_1 n_2 (e_1 m_2 - e_2 m_1)} \nabla(\frac{n_1}{n}) \tag{M3.63}$$

L.S. García-Colín, L. Dagdug, *The Kinetic Theory of a Dilute Ionized Plasma*,
Springer Series on Atomic, Optical, and Plasma Physics 53

$$\sigma^I = \frac{ne^2\tau}{2m_1}1.931$$

$$\sigma^{II} = \frac{ne^2\tau}{2m_1}\frac{\omega^2\tau^2 + 1.802}{\omega^4\tau^4 + 6.282\omega^2\tau^2 + 0.933} \tag{M7.8}$$

$$\sigma^{III} = \frac{ne^2\tau}{2m_1}\frac{-\omega\tau(\omega^2\tau^2 + 4.382)}{\omega^4\tau^4 + 6.282\omega^2\tau^2 + 0.933}$$

where

$$\omega = -\frac{eH}{cm_1}$$

and

$$\tau = \frac{3}{\sqrt{2\pi}}\frac{\sqrt{m_1}(kT)^{\frac{3}{2}}}{ne^4\psi} \tag{M7.10}$$

$$\vec{q} = -\theta^I\{\nabla T\}'' + \theta^{II}\{\nabla T\}^\perp + \theta^{III}\vec{h}\times\{\nabla T\}^\perp + \xi^I\vec{D}'' + \xi^{II}\vec{D}^\perp + \xi^{III}\vec{h}\times\vec{D}^\perp \tag{M7.16}$$

$$\theta^I = \tfrac{n\tau k^2T}{m_1}3.59$$

$$\theta^{II} = \tfrac{n\tau k^2T}{m_1}\left\{\tfrac{0.458\omega^2\tau^2+3.01}{\omega^4\tau^4+6.282\omega^2\tau^2+0.933} + \tfrac{4.458}{\omega^2\tau^2+12.716}\right\} \tag{M7.19}$$

$$\theta^{III} = \tfrac{n\tau k^2T}{m_1}1.25\left\{\tfrac{\omega\tau}{\omega^2\tau^2+12.716} - \tfrac{\omega\tau(\omega^2\tau^2+6.2)}{\omega^4\tau^4+6.282\omega^2\tau^2+0.933}\right\}$$

I.2 Notation

$\vec{E}' = \vec{E} + \frac{1}{c}\vec{u}\times\vec{H}$

c : Velocity of light.

e_i^m : coefficients of the expansions.

$\vec{H}$: The magnetic field.

$\vec{h}$: A unit vector in the direction of $\vec{H}$.

i, j : Subscripts labeling electrons and ions, 1 for electrons and 2 for ions.

$\vec{j}$: The conduction current.

k : Boltzmann′s constant.

m_i : The mass of particle i.

n_i : The number density of particles i.

$n = n_1 + n_2$: The total number density.

$\vec{q}$: The heat flux vector.

T : Temperature.

$\vec{u}$: The drift velocity.

X_i : any non-electromagnetic force per unit mass which acts on particles i.

ξ^n : Coefficients giving the contribution to the heat flux from the generalized electric field $\vec{D}$.

φ^n : Thermal diffusion coefficients.

ψ : Logarithmic cut-off factor.

θ^n : Thermal conduction coefficients.

σ^n : Coefficients of electric conductivity.

ρ_i : Density of particles i.

ρ : Total density.

τ : A collision time for electrons.

ω: The gyromagnetic frequency for electrons.

Index