Appendix

Appendix A

Table A.1 Physical properties of water

$T[^\circ C]$	$\rho[kg/m^3]$	$\eta_0[Pa \cdot s]$	$\upsilon[m^2/s]$	$p_{vapor}[kPa]$	$\sigma[N/m]$	$K[GPa]$
0	1000	1.75×10^{-3}	1.75×10^{-6}	0.611	0.0756	2.02
10	1000	1.30×10^{-3}	1.30×10^{-6}	1.23	0.0742	2.10
20	998	1.02×10^{-3}	1.02×10^{-6}	2.34	0.0728	2.18
30	996	8.00×10^{-4}	8.03×10^{-6}	4.24	0.0712	2.25
40	992	6.51×10^{-4}	6.56×10^{-7}	7.38	0.0696	2.28
50	988	5.41×10^{-4}	5.48×10^{-7}	12.3	0.0679	2.29
60	984	4.60×10^{-4}	4.67×10^{-7}	19.9	0.0662	2.28
70	978	4.02×10^{-4}	4.11×10^{-7}	31.2	0.0644	2.25
80	971	3.50×10^{-4}	3.60×10^{-7}	47.4	0.0626	2.20
90	965	3.11×10^{-4}	3.22×10^{-7}	70.1	0.0608	2.14
100	958	2.82×10^{-4}	2.94×10^{-7}	101.3	0.0589	2.07

Table A.2 Typical physical Properties of some common liquids at 1atm and 20°C

Liquid	$\rho[\mathrm{kg/m^3}]$	$\eta_0[\mathrm{Pa \cdot s}]$	$p_{\mathrm{vapor}}[\mathrm{kPa}]$	$\sigma[\mathrm{N/m}]$
Ammonia	829	2.20×10^{-4}	910	0.0213
Benzene	879	6.51×10^{-4}	10.1	0.0289
Ethanol	7887	1.20×10^{-3}	5.75	0.0228
Glycerine	1258	1.49	1.4×10^{-5}	0.0633
Kerosene	819	1.92×10^{-3}	3.11	0.0277
Methanol	788	5.98×10^{-4}	13.4	0.0226

Appendix B

Appendix B-1 Vector Tensor Operations

Write a vector $\boldsymbol{u}$ as a sum $\sum_i \hat{\boldsymbol{e}}_i u_i$ and a tensor $\mathbf{T}$ as a sum $\sum_i \hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j T_{ij}$, where $\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$ is the unit dyad. Note that the unit vectors $\hat{\boldsymbol{e}}_i$ are defined to give vectors and there are the scalar products $(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j)$ and vector products $(\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j)$. A third kind of product can be formed with the unit vector, namely the dyadic product $\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$, where the products $\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$ are the second order tensor. $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_j$ are of unit magnitude, so that the products $\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j$ are treated as unit dyads.

Based upon the dot and cross products of unit vector, which are performed by the geometrical definitions, the analogous operations for the unit dyads are defined by relating them to the operations for unit vectors

$$\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j : \hat{\boldsymbol{e}}_k \hat{\boldsymbol{e}}_l = (\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k)(\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_l) = \delta_{jk} \delta_{il} \tag{B.1-1}$$

$$\hat{\boldsymbol{e}}_i \hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k = \hat{\boldsymbol{e}}_i (\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k) = \hat{\boldsymbol{e}}_i \delta_{jk} \tag{B.1-2}$$

$$\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j \hat{\boldsymbol{e}}_k = (\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j) \hat{\boldsymbol{e}}_k = \delta_{ij} \hat{\boldsymbol{e}}_k \tag{B.1-3}$$

$$\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\hat{\boldsymbol{e}}_l = \hat{\boldsymbol{e}}_i\left(\hat{\boldsymbol{e}}_j \cdot \hat{\boldsymbol{e}}_k\right)\hat{\boldsymbol{e}}_l = \delta_{jk}\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_l \quad \text{(B.1-4)}$$

$$\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k = \hat{\boldsymbol{e}}_i\left(\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k\right) = \boldsymbol{\varepsilon}_{jkl}\,\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_l \quad \text{(B.1-5)}$$

$$\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j\hat{\boldsymbol{e}}_k = \left(\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j\right)\hat{\boldsymbol{e}}_k = \boldsymbol{\varepsilon}_{ijl}\,\hat{\boldsymbol{e}}_l\hat{\boldsymbol{e}}_k \quad \text{(B.1-6)}$$

Appendix B-2 Representative Operations

$$\mathbf{I}\cdot \boldsymbol{u} = \boldsymbol{u}\cdot \mathbf{I} = \boldsymbol{u} \quad \text{(B.2-1)}$$

$$\boldsymbol{uv}\cdot \boldsymbol{w} = \boldsymbol{u}(\boldsymbol{v}\cdot \boldsymbol{w}) \quad \text{(B.2-2)}$$

$$\boldsymbol{w}\cdot \boldsymbol{uv} = (\boldsymbol{w}\cdot \boldsymbol{u})\boldsymbol{v} \quad \text{(B.2-3)}$$

$$(\boldsymbol{uv} : \boldsymbol{wz}) = (\boldsymbol{uw} : \boldsymbol{vz}) = (\boldsymbol{w}\cdot \boldsymbol{v})(\boldsymbol{u}\cdot \boldsymbol{z}) \quad \text{(B.2-4)}$$

$$\mathbf{T} : \boldsymbol{uv} = (\mathbf{T}\cdot \boldsymbol{u})\cdot \boldsymbol{v} \quad \text{(B.2-5)}$$

$$(\boldsymbol{uv} : \mathbf{T}) = (\boldsymbol{u}\cdot [\boldsymbol{v}\cdot \mathbf{T}]) \quad \text{(B.2-6)}$$

Appendix B-3 Differential Operators

Cartesian Coordinates

$$\nabla = \hat{\boldsymbol{e}}_x \frac{\partial}{\partial x} + \hat{\boldsymbol{e}}_y \frac{\partial}{\partial y} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \quad \text{(B.3-1)}$$

Cylindrical Coordinates

$$\nabla = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{\partial}{\partial z} \quad \text{(B.3-2)}$$

Spherical Coordinates

$$\nabla = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(B.3-3)}$$

Appendix B-4 ∇ Operations

(i) Representative ∇ operations in Cartesian Coordinates (x, y, z)

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{B.4-1}$$

$$\nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} \tag{B.4-2}$$

$$\begin{aligned} \mathbf{T} : \nabla \boldsymbol{u} = {} & T_{xx}\left(\frac{\partial u_x}{\partial x}\right) + T_{xy}\left(\frac{\partial u_x}{\partial y}\right) + T_{xz}\left(\frac{\partial u_x}{\partial z}\right) \\ & + T_{yx}\left(\frac{\partial u_y}{\partial x}\right) + T_{yy}\left(\frac{\partial u_y}{\partial y}\right) + T_{yz}\left(\frac{\partial u_y}{\partial z}\right) \\ & + T_{zx}\left(\frac{\partial u_z}{\partial x}\right) + T_{zy}\left(\frac{\partial u_z}{\partial y}\right) + T_{zz}\left(\frac{\partial u_z}{\partial z}\right) \end{aligned} \tag{B.4-3}$$

(ii) Representative ∇ operations in Cylindrical Coordinates (r, θ, z)

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \tag{B.4-4}$$

$$\nabla^2 s = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2} \tag{B.4-5}$$

$$\begin{aligned} (\mathbf{T} : \nabla \boldsymbol{u}) = {} & T_{rr}\left(\frac{\partial u_r}{\partial r}\right) + T_{r\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) + T_{rz}\left(\frac{\partial u_r}{\partial z}\right) \\ & + T_{\theta r}\left(\frac{\partial u_\theta}{\partial r}\right) + T_{\theta\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) + T_{\theta z}\left(\frac{\partial u_\theta}{\partial z}\right) \\ & + T_{zr}\left(\frac{\partial u_z}{\partial r}\right) + T_{z\theta}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) + T_{zz}\left(\frac{\partial u_z}{\partial z}\right) \end{aligned} \tag{B.4-6}$$

(iii) Representative ∇ operations in Spherical Coordinates (r, θ, ϕ)

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} \tag{B.4-7}$$

$$\nabla^2 s = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial s}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 s}{\partial\phi^2} \qquad \text{(B.4-8)}$$

$$\begin{aligned}\mathbf{T}:\nabla\boldsymbol{u} = {} & T_{rr}\left(\frac{\partial u_r}{\partial r}\right) + T_{r\theta}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right) + T_{r\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi}{r}\right) \\ & + T_{\theta r}\left(\frac{\partial u_\theta}{\partial r}\right) + T_{\theta\theta}\left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}\right) + T_{\theta\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi} - \frac{u_\phi}{r}\cot\theta\right) \\ & + T_{\phi r}\left(\frac{\partial u_\phi}{\partial r}\right) + T_{\phi\theta}\left(\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\right) \\ & + T_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{u_\theta}{r}\cot\theta\right) \qquad \text{(B.4-9)}\end{aligned}$$

Appendix B-5 Representative Differential Relations

$$\nabla\times\nabla s = 0 \qquad \text{(B.5-1)}$$

$$\nabla\cdot(\nabla\times\boldsymbol{u}) = 0 \qquad \text{(B.5-2)}$$

$$\nabla rs = r\nabla s + s\nabla r \qquad \text{(B.5-3)}$$

$$\nabla\cdot s\boldsymbol{u} = (\nabla s\cdot\boldsymbol{u}) + s(\nabla\cdot\boldsymbol{u}) \qquad \text{(B.5-4)}$$

$$\nabla\times s\boldsymbol{u} = \nabla s\times\boldsymbol{u} + s\nabla\times\boldsymbol{u} \qquad \text{(B.5-5)}$$

$$\begin{aligned}\nabla\cdot(\boldsymbol{u}\cdot\boldsymbol{w}) = {} & \boldsymbol{u}\cdot\nabla\boldsymbol{w} + \boldsymbol{w}\cdot\nabla\boldsymbol{u} \\ & + \boldsymbol{u}\times(\nabla\times\boldsymbol{w}) + \boldsymbol{w}\times(\nabla\times\boldsymbol{u})\end{aligned} \qquad \text{(B.5-6)}$$

$$\nabla\cdot(\boldsymbol{u}\times\boldsymbol{w}) = \boldsymbol{w}\cdot(\nabla\times\boldsymbol{u}) - \boldsymbol{u}\cdot(\nabla\times\boldsymbol{w}) \qquad \text{(B.5-7)}$$

$$\begin{aligned}\nabla\times(\boldsymbol{u}\times\boldsymbol{w}) = {} & \boldsymbol{u}(\nabla\cdot\boldsymbol{w}) - \boldsymbol{w}(\nabla\cdot\boldsymbol{u}) \\ & + \boldsymbol{w}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{w}\end{aligned} \qquad \text{(B.5-8)}$$

$$\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u} \qquad \text{(B.5-9)}$$

$$\nabla\cdot\nabla\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u}) \qquad \text{(B.5-10)}$$

$$\nabla \cdot (\nabla s \times \nabla r) = 0 \tag{B.5-11}$$

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{2}\nabla(\boldsymbol{u} \cdot \boldsymbol{u}) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) \tag{B.5-12}$$

$$\nabla \cdot \boldsymbol{uw} = \boldsymbol{u} \cdot \nabla \boldsymbol{w} + \boldsymbol{w}(\nabla \cdot \boldsymbol{u}) \tag{B.5-13}$$

$$s\hat{\boldsymbol{e}} : \nabla \boldsymbol{u} = s(\nabla \cdot \boldsymbol{u}) \tag{B.5-14}$$

$$\nabla \cdot s\hat{\boldsymbol{e}} = \nabla s \tag{B.5-15}$$

$$\nabla \cdot s\,\mathbf{T} = \nabla s \cdot \mathbf{T} + s\nabla \cdot \mathbf{T} \tag{B.5-16}$$

$$\nabla \cdot \mathbf{T}_a = -\frac{1}{2}\nabla \times \boldsymbol{A},\ \boldsymbol{A} = \text{vec}\,\mathbf{T} \tag{B.5-17}$$

$$\text{where } \mathbf{T}_a = \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right)$$

Appendix B-6 Equation of Continuity

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \boldsymbol{u}) = 0$$

Cartesian Coordinates (x, y, z)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{B.6-1}$$

Cylindrical Coordinates (r, θ, z)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{B.6-2}$$

Spherical Coordinates (r, θ, ϕ)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho u_\phi) = 0 \tag{B.6-3}$$

When the fluid is assumed to have constant mass density ρ, the equation simplifies to $\nabla \cdot \boldsymbol{u} = 0$

Appendix B-7 Equation of Motion in Terms of Stress Tensor $\boldsymbol{\tau}$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{Dt}} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{g}$$

(i) Cartesian Coordinates (x, y, z)

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right) + \rho g_x \qquad \text{(B.7-1)}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right) + \rho g_y \qquad \text{(B.7-2)}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right) + \rho g_z \qquad \text{(B.7-3)}$$

(ii) Cylindrical Coordinates (r, θ, z)

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^{\,2}}{r}\right) = -\frac{\partial p}{\partial r} + \left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right\} + \rho g_r \qquad \text{(B.7-4)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta}\right\} + \rho g_\theta \qquad \text{(B.7-5)}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rz}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}+\frac{\partial}{\partial z}\tau_{zz}\right\}+\rho g_z \qquad \text{(B.7-6)}$$

(iii) Spherical Coordinates (r,θ,ϕ)

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{u_\theta{}^2+u_\phi{}^2}{r}\right)$$
$$=-\frac{\partial p}{\partial r}+\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{r\theta}\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{r\phi}-\frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r}\right\}$$
$$+\rho g_r \qquad \text{(B.7-7)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta-u_\phi{}^2\cot\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\tau_{\theta\theta}\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\theta}+\frac{\tau_{r\theta}-\tau_{\phi\phi}\cot\theta}{r}\right\}$$
$$+\rho g_\theta \qquad \text{(B.7-8)}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r+u_\theta u_\phi\cot\theta}{r}\right)$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\phi}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\phi}+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\phi}+\frac{\left(\tau_{r\phi}-\tau_{\phi r}\right)+2\tau_{\theta\phi}\cot\theta}{r}\right\}$$
$$+\rho g_\phi \qquad \text{(B.7-9)}$$

Appendix B-8 Equation of Motion for a Newtonian Fluid with Constant ρ and η_0

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{Dt}} = -\nabla p + \eta_0 \nabla^2 \boldsymbol{u} + \rho \boldsymbol{g}$$

(i) Cartesian Coordinates (x, y, z)

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta_0\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x \tag{B.8-1}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \eta_0\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \rho g_y \tag{B.8-2}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x \frac{\partial u_z}{\partial x} + u_y \frac{\partial u_z}{\partial y} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \eta_0\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \tag{B.8-3}$$

(ii) Cylindrical Coordinates (r, θ, z)

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^{\,2}}{r}\right) = -\frac{\partial p}{\partial r} + \eta_0\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right\} + \rho g_r \tag{B.8-4}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \eta_0\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right\} + \rho g_\theta \tag{B.8-5}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)$$
$$=-\frac{\partial p}{\partial z}+\eta_0\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right\}+\rho g_z \qquad \text{(B.8-6)}$$

(iii) Spherical Coordinates (r,θ,ϕ)

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial \phi}-\frac{{u_\theta}^2+{u_\phi}^2}{r}\right)=-\frac{\partial p}{\partial r}$$
$$+\eta_0\left\{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 u_r\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial \phi^2}\right\} \qquad \text{(B.8-7)}$$
$$+\rho g_r$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}+\frac{u_r u_\theta-{u_\phi}^2\cot\theta}{r}\right)$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\eta_0\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta\sin\theta\right)\right)\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}-\frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right\}+\rho g_\theta \qquad \text{(B.8-8)}$$

$$\rho\left(\frac{\partial u_\phi}{\partial t}+u_r\frac{\partial u_\phi}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}+\frac{u_\phi u_r+u_\theta u_\phi\cot\theta}{r}\right)$$
$$=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+\eta_0\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(u_\phi\sin\theta\right)\right)\right.$$
$$\left.+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial \phi}+\frac{2}{r^2}\frac{\cot\theta}{\sin\theta}\frac{\partial u_\theta}{\partial \phi}\right\}+\rho g_\phi \qquad \text{(B.8-9)}$$

Appendix B-9 Equation of Energy in Terms of q (b=0)

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{q} + \left(\frac{\partial \ln \frac{1}{\rho}}{\partial \ln T} \right)_p \frac{\mathrm{D}p}{\mathrm{D}t} + \boldsymbol{\tau} : \nabla \boldsymbol{u}$$

(i) Cartesian Coordinates (x, y, z)

$$\rho c_p \left(\frac{\partial T}{\partial t} + u_x \frac{\partial T}{\partial x} + u_y \frac{\partial T}{\partial y} + u_z \frac{\partial T}{\partial z} \right)$$

$$= -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) + \left(\frac{\partial \ln \frac{1}{\rho}}{\partial \ln T} \right)_p \frac{\mathrm{D}p}{\mathrm{D}t} + \boldsymbol{\tau} : \nabla \boldsymbol{u} \qquad \text{(B.9-1)}$$

(ii) Cylindrical Coordinates (r, θ, z)

$$\rho c_p \left(\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r} \frac{\partial T}{\partial \theta} + u_z \frac{\partial T}{\partial z} \right)$$

$$= -\left(\frac{1}{r} \frac{\partial}{\partial r} (r q_r) + \frac{1}{r} \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} \right) + \left(\frac{\partial \ln \frac{1}{\rho}}{\partial \ln T} \right)_p \frac{\mathrm{D}p}{\mathrm{D}t} + \boldsymbol{\tau} : \nabla \boldsymbol{u} \qquad \text{(B.9-2)}$$

(iii) Spherical Coordinates (r, θ, ϕ)

$$\rho c_p \left(\frac{\partial T}{\partial t} + u_r \frac{\partial T}{\partial r} + \frac{u_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{u_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right)$$

$$= -\left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (q_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial q_\phi}{\partial \phi} \right)$$

$$+ \left(\frac{\partial \ln \frac{1}{\rho}}{\partial \ln T} \right)_p \frac{\mathrm{D}p}{\mathrm{D}t} + \boldsymbol{\tau} : \nabla \boldsymbol{u} \qquad \text{(B.9-3)}$$

Appendix C

Appendix C-1 Buckingham Π-Theorem

The Buckingham π -theorem is used in the study of dimensional analysis and similitude, which is based on the notion of dimensional homogeneity. The theorem is examined in a given fluid system where variables of q_1, q_2, …, q_n are chosen so that they are pertinent to a physical phenomena. Then, we will express the phenomena by a functional form as

$$f(q_1, q_2, q_3, ..., q_n) = 0 \tag{C.1-1}$$

where n represents the total number of variables. If there are m basic dimensions involved in the variables of $q_1 \sim q_n$, the Buckingham π -theorem states that the same physical phenomena can be correlated by $(n-m)$ nondimensional numbers (independent from nondimensional groups), called π -parameters, which are given as a functional form

$$g(\pi_1, \pi_2, \pi_3, ..., \pi_{n\text{-}m}) = 0 \tag{C.1-2}$$

When a given fluid system contains a dependent variable, say q_1, the physical phenomenon can be expressed similarly in the form

$$q_1 = h(q_2, q_3, ..., q_n) \tag{C.1-3}$$

and

$$\pi_1 = s(\pi_2, \pi_3, ..., \pi_{n\text{-}m}) \tag{C.1-4}$$

where π_1 includes the dependent variable and the remaining π -parameters include the rest of independent variables. The procedure adopted for determining the nondimensional π -parameters are as follows;

Step (1) In having written the functional form of either Eq. (C.1-1) or Eq. (C.1-3), select m repeating the variables from n -independent variables in Eq. (C.1-1) or $(n-1)$-independent variables in Eq. (C.1-3). The repeating variables must include all of the basic dimensions, but they must not form π -parameters by themselves. In order to obtain the most significant π -parameters, it is desirable to choose one variable with geometric characteristics, second variable with flow characteristics and another variable with fluid properties, such as l, U and ρ respectively, with reference to Table C.1. For example, writing Eq. (C.1-1)

$$f(q_1, q_2, q_3, ..., q_n) = 0 \tag{C.1-5}$$

we may choose m repeating variables, say if $m = 3$, select three variables to meet the criteria, say q_2, q_4 and q_5 among n variables.

Step (2) Write π -parameters of $\pi_1, \pi_2, ..., \pi_{n\text{-}m}$ in the power form for the repeating variables, for example with q_2, q_4 and q_5 in Step (1) with each of the remaining variables as

$$\pi_1 = q_2^a q_4^b q_5^c q_1$$

$$\pi_2 = q_2^a q_4^b q_5^c q_3$$

$$\pi_3 = q_2^a q_4^b q_5^c q_6$$

$$\vdots$$

$$\pi_{n-m} = q_2^a q_4^b q_5^c q_n$$

Step (3) Apply the dimensional analysis to obtain the power constants for each π -parameter subjecting that the π -parameters are all dimensionless.

Step (4) Write the functional form using π -parameters to describe the physical phenomenon of the fluid system.

Step (5) In correlating experimental results, one dependent π -parameter (say π_1) can be expressed by a function likewise

$$\pi_1 = s\left(c_2 \pi_2^{\alpha_2}, c_3 \pi_3^{\alpha_3}, ... c_{n-m} \pi_{n-m}^{\alpha_{n-m}}\right) \tag{C.1-6}$$

where $c_2, c_3, ... c_{n-m}$ and $\alpha_2, \alpha_3, ... \alpha_{n-m}$ are constants determined from the results of experiments. Note that if some dimensionless variable, such as the length ratio l_1/l_2, the roughness $\varepsilon \cdots$ etc, are contained in the primary variables $(q_1, q_2, \cdots q_n)$, they are themselves treated as π -parameters and to be excluded from the procedure by simply adding as π -parameters in the resultant functional form, for example

$$f(q_1, q_2, \cdots q_n, \theta, l_1/l_2) = 0 \tag{C.1-7}$$

$$s(\pi_1 \cdots \pi_{n-m}, \varepsilon, l_1/l_2) = 0 \tag{C.1-8}$$

Table C.1 Dimensions of quantities frequently used in fluid flow problems

Quantity	Symbol	Dimensions
Length	l	L
Time	t	T
Mass	m	M
Area	A	L^2
Volume	V	L^3
Force	F	ML/T^2
Velocity	U	L/T
Acceleration	a	L/T^2
Angular frequency	ω	$1/T$
Gravity	g	L/T^2
Flow rate	Q	L^3/T
Mass flux	$\dot{m}$	M/T
Pressure	p	M/LT^2
Stress	$\mathbf{T}$	M/LT^2
Work	W	ML^2/T^2
Power, heat flux	P, q	ML^2/T^3
Density	ρ	M/L^3
Specific weight	γ	M/L^2T^2
Viscosity	η_0	M/LT
Kinematic viscosity	υ	L^2/T
Surface tension	σ	M/T^2
Bulk modulus	K	M/LT^2

(*)Basic Dimensions; L (length), M (Mass), and T (Time), i.e. $m = 3$

Appendix C-2 Example of π-Analysis

In order to illustrate π -theorem, suppose that force F acting in a fluid system is supposed to be dependent on the velocity U, density ρ, gravity g, viscosity η_0, surface tension σ, angular frequency (velocity) ω, bulk modulus K, surface roughness ε, characteristic length l and another representative linear dimension l_1. For this fluid system a physical phenomena would be described with nondimensional numbers by applying π -theorem as demonstrated below.

(i) Functional form

$$f\left(F, l, U, \rho, \eta_0, g, \sigma, K, \omega, \frac{l_1}{l}, \varepsilon\right) = 0 \tag{C.2-1}$$

We have eleven variables, i.e. $n = 11$, which contain three basic dimensions L, M and T, i.e. $m = 3$. According to π - theorem, we can find eight $(n - m = 11 - 3 = 8)$ π -parameters, so that we have

$$g(\pi_1\ \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8) = 0 \tag{C.2-2}$$

(ii) Choice of repeating variables

Choose three repeating variables such as l, U and ρ, and set $\pi_7 = l_1/l$ and $\pi_8 = \varepsilon$, since they are two already dimensionless parameters. Thus we will find π_1 to π_6, i.e. six π - parameters.

(iii) Conduct dimensional analysis

$$\pi_1 = l^a U^b \rho^c F \tag{C.2-3}$$

$$\left[M^0 L^0 T^0\right]\pi_1 = L^a\left(LT^{-1}\right)^b\left(ML^{-3}\right)^c MLT^{-2}$$

$$0 = c + 1, 0 = a + b - 3c + 1, 0 = -b - 2$$

$$\therefore a = -2, b = -2, c = -1 \tag{C.2-4}$$

$$\pi_1 = F/\left(\rho\ u^2\ l^2\right)$$

$$\tag{C.2-5}$$

$$\pi_2 = l^a U^b \rho^c \eta_0$$

$$\left[M^0 L^0 T^0\right]\pi_2 = L^a\left(LT^{-1}\right)^b\left(ML^{-3}\right)^c ML^{-1}T^{-1}$$

$$0 = c + 1, 0 = a + b - 3c - 1, 0 = -b - 1$$

$$\therefore a = -1,\ b = -1, c = -1 \tag{C.2-6}$$

$$\pi_2 = \frac{\eta_0/\rho}{lU} \tag{C.2-7}$$

Similarly for $\pi_3 \sim \pi_6$, we can obtain

$$\pi_3 = \frac{lg}{U^2}, \pi_4 = \frac{\sigma}{U^2 lg}, \pi_5 = \frac{K}{U^2\ \rho} \text{ and } \pi_6 = \frac{l\,\omega}{U} \tag{C.2-8}$$

(iv) Set functional form

Each of the π - parameters for $\pi_2 \sim \pi_6$ in this expression is *a* common nondimensional number, which is derived from similitude (Section 6.2), so that we have

$$f\left(\frac{F}{\rho U^2 l^2}, \frac{1}{Re}, \frac{1}{Fr^2}, \frac{1}{M^2}, St, \frac{l_1}{l}, \varepsilon\right) = 0 \tag{C.2-9}$$

and for the dependent variable F, the functional form is given where

$$c_f = \frac{F}{1/2\,\rho U^2 l^2} = f\left(Re, Fr, M, St, \frac{l_1}{l}, \varepsilon\right) \tag{C.2-10}$$

Appendix D

Appendix D-1 Invariant of Second Order Tensor

The invariants of a tensor are scalar quantities, which remain unchanged for the coordinate transformation of rotation. There are three principal invariants for second order tensors.

Consider a second order tensor $\mathbf{T}$, whose components are a_{ij} for unit dyads $\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$, and b_{ij} for unit dyads $\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j$ after the coordinate transformation. When λ is a scalar and is an eigenvalue of $\mathbf{T}$, we have the following relationship

$$\psi_B(\lambda) = \left|\lambda\delta_{ij} - a_{ij}\right| = \left|\lambda\delta_{ij} - b_{ij}\right| = \psi_A(\lambda) \tag{D.1-1}$$

where ψ_A and ψ_B are the characteristic equations of a_{ij} and b_{ij}. Eq. (D.1-1) shows that the characteristic equations are equal to both frames ($\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ and $\hat{\boldsymbol{e}}'_i\hat{\boldsymbol{e}}'_j$), so that they are unaffected by the coordinate transformation. They are also given when

$$\psi(\lambda) = \psi_A(\lambda) = \psi_B(\lambda) = \lambda_3 - I_1\lambda^2 + I_2\lambda - I_3 \tag{D.1-2}$$

where I_1, I_2 and I_3 are principal invariants, which are also unaffected by the coordinate transformation. They are respectively defined as

$$I_1 = a_{11} + a_{22} + a_{33} \tag{D.1-3}$$

$$I_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{(D.1-4)}$$

$$I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{(D.1-5)}$$

and alternating, I_1, I_2 and I_3 are written where

$$I_1 = t_r A \ , I_2 = \frac{1}{2}\left(t_r{}^2 A - t_r A^2\right) \text{and } I_3 = \det A \quad \text{(D.1-6)}$$

where $t_r{}^2 A = (t_r A)^2$ and $t_r A^2 = t_r(AA)$. Another set of invariants is defined by the so-called moment, as follows

$$\overline{I}_k = t_r A^k \quad \text{(D.1-7)}$$

With the moment, given by Eq. (D.1-7), the invariants are representatively given for k =1,2 and 3 as

$$\overline{I_1} = t_r A = I_1 \quad \text{(D.1-8)}$$

$$\overline{I_2} = t_r A^2 = I_1^2 - 2I_2 \quad \text{(D.1-9)}$$

$$\overline{I_3} = t_r A^3 = \frac{1}{2}\left(6I_3 + 2I_1^3 - 6I_1 I_2\right) \quad \text{(D.1-10)}$$

It is useful to note that there are relationships between two principal invariants I_k and the moments $\overline{I_k}$ as follows

$$I_1 = \overline{I_1} \quad \text{(D.1-11)}$$

$$I_2 = \frac{1}{2}\left(\overline{I}_1{}^2 - \overline{I}_2{}^2\right) \quad \text{(D.1-12)}$$

$$I_3 = \frac{1}{6}\overline{I}_1^3 - \frac{1}{2}\overline{I}_1\overline{I}_2 + \frac{1}{3}\overline{I}_3 \quad \text{(D.1-13)}$$

For a symmetric tensor $\mathbf{T}$, eigenvalues of $\mathbf{T}$ are real and $\mathbf{T}$ is diagonalizable. For example, if eigenvalues λ_1, λ_2 and λ_3 are of a symmetric tensor $\mathbf{T}$, $\mathbf{T}$ can be transformed to a diagonal matrix

$$\mathbf{T}' = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \tag{D.1-14}$$

Then we have invariants

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 \tag{D.1-15}$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \tag{D.1-16}$$

and

$$I_3 = \lambda_1\lambda_2\lambda_3 \tag{D.1-17}$$

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