APPENDIX A: THE CONSTITUTIVE EQUATIONS

A.1 INTRODUCTION

In order to provide a reasonably self-contained basis for the development of 1-D constitutive equations of thin-walled beam theory, in general, and of the theory presented in this work, in particular, and to avoid any interruption in the course of our exposition, several elements on the constitutive equations of the 3-D anisotropic elasticity theory will be summarized. For similar reasons, a number of basic elements which concern the 3-D constitutive equations of piezoelectric materials, strictly related to the developments in Chapter 3 will also be supplied.

A.2 LINEARLY ELASTIC 3-D ANISOTROPIC CONTINUUM

The kinematic equations and the equations of motion are valid for every continuous medium, irrespective of its physical properties.

For an elastic body, in addition to the mentioned equations, the ones defining its physical properties have to be considered. If the strains are small enough, the physical behavior of an elastic anisotropic material can be approximated by a linear relationship between stress and strain components; such a material is termed *Hookean*, (see Wempner, 1981).

Composite materials consist of a reinforcement, generally in the form of fibers of high thermal and mechanical performance, dispersed in a surrounding matrix in an appropriate pattern. Such materials exhibit a blend of properties superior to those of their individual constituents. The matrix bonds the reinforcement together and distributes loads among the fibers; the reinforcement supports the mechanical loads. Due to their high specific stiffness and strength, resistance to corrosion as well as due to their tailorability property, such structures are prime candidates in aeronautical, aerospace, naval, and civil applications; they can operate in severe and complex environmental conditions.

Changes in temperature are commonplace in composite materials, both during fabrication and use. There are two important effects emerging from the existence of the temperature field (see Daniel and Ishai, 1994; Vinson and Sierakowski, 2002). First, most materials expand when heated and contract when cooled, and secondly, the stiffness changes so that it becomes softer and weaker as it is heated. The combination of high temperature and high humidity has a detrimental effect on the structural performance of fiber-reinforced polymeric composites. To utilize in full the potential of composite materials, a method to predict their deformation and response characteristics due to changes in temperature and moisture absorption must be available (see Daniel and Ishai, 1994).

The ingestion of moisture leads to a linear variation of deformation with swelling, similar to the thermal loading. The constitutive equations incorporating changes in the temperature (T) and the specific moisture concentration (M) from some reference levels T_0 and M_0, respectively, at which the body is considered to be stress free, can be expressed in Cartesian tensor form as:

$$\sigma_{ij} = C_{ijmn}\epsilon_{mn} - \lambda_{ij}T - \mu_{ij}M \quad (i, j, m, n = 1, 2, 3), \tag{A.1}$$

where C_{ijmn}, λ_{ij} and μ_{ij} are the elasticity, the stress-temperature tensor and the stress-moisture tensor of the anisotropic body, respectively; these are fourth and second rank tensors, respectively.

This linear relationship characterizes the anisotropic Hookean solid and it represents an extension of the Duhamel-Neumann thermoelastic constitutive equations to the hygrothermoelastic case. We employ conventions of *free* (unrepeated) and *summation* (repeated) indices.

The components C_{ijmn}, λ_{ij} and μ_{ij}, satisfy the following symmetry conditions:

$$\begin{aligned} C_{ijmn} &= C_{jimn}, \quad C_{ijmn} = C_{ijnm}, \quad C_{ijmn} = C_{mnij}, \\ \lambda_{ij} &= \lambda_{ji}, \\ \mu_{ij} &= \mu_{ji}. \end{aligned} \tag{A.2a-e}$$

The first symmetry relation follows directly from the symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$, while the second arises from the symmetry of the strain tensor ϵ_{ij}. The third symmetry relation is a consequence of the interchangeability rule

$$\frac{\partial^2 \mathcal{W}}{\partial\epsilon_{ij}\partial\epsilon_{mn}} = \frac{\partial^2 \mathcal{W}}{\partial\epsilon_{mn}\partial\epsilon_{ij}}, \tag{A.3}$$

where, for the present case

$$\mathcal{W} = \frac{1}{2} C_{ijmn}\,\epsilon_{ij}\,\epsilon_{mn} - \lambda_{ij}\,\epsilon_{ij}T - \mu_{ij}\,\epsilon_{ij}\,M. \tag{A.4}$$

This form of (A.4) implies the existence of a free-reference-state characterized by $\sigma_{ij} = 0$ and $\epsilon_{ij} = 0$ at $T = T_0$ and $M = M_0$.

By virtue of these symmetries, the elastic and thermal coefficients can be represented in matrix form as:

$$\left[C_{ijmn}\right] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & Symm. & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix}, \tag{A.5a}$$

$$\left[\lambda_{ij}\right] \equiv \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ & \lambda_{22} & \lambda_{23} \\ Symm. & & \lambda_{33} \end{bmatrix}, \tag{A.5b}$$

$$\left[\mu_{ij}\right] \equiv \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ & \mu_{22} & \mu_{23} \\ Symm. & & \mu_{33} \end{bmatrix}. \tag{A.5c}$$

A body exhibiting such mechanical and thermal properties is termed *triclinic*.

For this general case, the matrices C_{ijmn}, λ_{ij}, and μ_{ij} are fully populated; there are 36 *nonzero* elastic coefficients, and 9 thermal and 9 hygrothermal coefficients. The symmetry properties in the general case of anisotropy, reduce the number of *independent* elastic coefficients to 21, and the thermal and hygrothermal ones to 6 each.

A.3 MATERIAL SYMMETRY

The various symmetries can reduce the number of independent coefficients C_{ijmn}, λ_{ij}, and μ_{ij}. We examine the effect on the strain energy density function of a rotation of axes.

A.3-1 One Surface of Symmetry (Monoclinic Hookean Material)

Suppose that through each point of the elastic body there is a surface $x_3 = 0$ with respect to which the energy functional remains invariant when x_3 is replaced by $-x_3$, then

$$\begin{aligned} &C_{1123} = C_{1131} = C_{2223} = C_{2231} = C_{2312} = C_{3112} = 0, \\ &C_{3323} = C_{3331}, \\ &\lambda_{13} = \lambda_{23} = 0, \\ &\mu_{13} = \mu_{23} = 0. \end{aligned} \tag{A.6a-d}$$

Within this kind of symmetry, the elastic coefficients containing the index 3 either once or thrice, and the hygrothermal coefficients containing the index 3 once should vanish. The matrices in Eqs. (A.5) reduce to

$$
\left[C_{ijmn}\right] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2331} & 0 \\ Symm. & & & & C_{3131} & 0 \\ & & & & & C_{1212} \end{bmatrix}, \qquad \text{(A.7a)}
$$

$$
\left[\lambda_{ij}\right] \equiv \begin{bmatrix} \lambda_{11} & \lambda_{12} & 0 \\ & \lambda_{22} & 0 \\ Symm. & & \lambda_{33} \end{bmatrix}, \qquad \text{(A.7b)}
$$

$$
\left[\mu_{ij}\right] \equiv \begin{bmatrix} \mu_{11} & \mu_{12} & 0 \\ & \mu_{22} & 0 \\ Symm. & & \mu_{33} \end{bmatrix}. \qquad \text{(A.7c)}
$$

Such a material, referred to as *monoclinic* or *monotropic* (see Wempner, 1981), is characterized by 13 *independent* elastic coefficients, 4 thermal and 4 hygro coefficients. The axis x_3 normal to the surface of symmetry is referred to as the *principal material direction*.

A.3-2 Three Planes of Symmetry (Orthotropic Material)

If the material exhibits elastic and thermal symmetries at each point with respect to the orthogonal surfaces x_3=0 and x_2=0, then *additional* coefficients, besides the ones given by Eqs. (A.6), will vanish:

$$
\begin{aligned} C_{1112} &= C_{3312} = C_{2331} = C_{2212} = 0, \\ \lambda_{12} &= 0, \quad \mu_{12} = 0. \end{aligned} \qquad \text{(A.8a-c)}
$$

For this material

(i) The elastic coefficients in Eqs. (A.6) and (A.8), containing the indices 1 and/or 3 once or thrice, and the thermal and the hygrothermal coefficients including the indices 1 and/or 3 once should vanish,

(ii) There are number of 9 *independent* elastic coefficients, as well as 3 thermal and 3 hygrothermal coefficients that characterize the orthotropic solid,

(iii) Being two orthogonal surfaces of material symmetry, the third mutually orthogonal surface will constitute another surface of symmetry, and,

(iv) The coordinate axes normal to the surfaces of elastic symmetry are referred to as *the principal material directions*.

For an orthotropic body, the thermoelastic matrices are

$$\left[C_{ijmn}\right] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{2323} & 0 & 0 \\ Symm. & & & & C_{3131} & 0 \\ & & & & & C_{1212} \end{bmatrix}, \tag{A.9a}$$

$$\left[\lambda_{ij}\right] \equiv \begin{bmatrix} \lambda_{11} & 0 & 0 \\ & \lambda_{22} & 0 \\ Symm. & & \lambda_{33} \end{bmatrix}, \tag{A.9b}$$

$$\left[\mu_{ij}\right] \equiv \begin{bmatrix} \mu_{11} & 0 & 0 \\ & \mu_{22} & 0 \\ Symm. & & \mu_{33} \end{bmatrix}. \tag{A.9c}$$

A.3-3 Transverse Isotropy

A transversely isotropic material is one in which, at each point there is one principal material direction about which there is rotational symmetry. If, the principal direction is parallel at each point to the axis x_3, and there is rotational symmetry about the x_3 axis, then the surface x_1 - x_2 is the surface of isotropy; the matrices of elastic, thermal and hygrothermal coefficients become

$$\left[C_{ijmn}\right] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ & C_{1111} & C_{1133} & 0 & 0 & 0 \\ & & C_{3333} & 0 & 0 & 0 \\ & & & C_{3131} & 0 & 0 \\ Symm. & & & & C_{3131} & 0 \\ & & & & & (C_{1111} - C_{1122})/2 \end{bmatrix}, \tag{A.10a}$$

$$\left[\lambda_{ij}\right] \equiv \begin{bmatrix} \lambda_{11} & 0 & 0 \\ & \lambda_{11} & 0 \\ Symm. & & \lambda_{33} \end{bmatrix}, \tag{A.10b}$$

$$\left[\mu_{ij}\right] \equiv \begin{bmatrix} \mu_{11} & 0 & 0 \\ & \mu_{11} & \\ Symm. & & \mu_{33} \end{bmatrix}. \tag{A.10c}$$

There are 5 *independent* elastic coefficients, 2 thermal hygrothermal coefficients. One of the materials exhibiting the transverse isotropy properties is the *pyrolytic graphite*, which due to its special thermo-mechanical properties is likely to be an excellent candidate for being used in the thermal protection of aerospace vehicles. Moreover, *unidirectional fiber reinforced composites* belong to the class of transverse isotropy; in such a case the isotropy plane is normal to the axis of the fibers. If the fibers are aligned in the x_1-direction, the x_2- x_3 plane is the plane of isotropy, and the indices 2 and 3 are interchangeable. For this case

$$\begin{aligned} C_{2222} &= C_{3333}, \quad C_{1133} = C_{1122}, \\ C_{3131} &= C_{1212}, \quad C_{2323} = (C_{2222} - C_{2233})/2, \\ \lambda_{22} &= \lambda_{33}, \quad \mu_{22} = \mu_{33}. \end{aligned} \tag{A.11a-f}$$

The matrices of elastic and thermal coefficients become

$$[C_{ijmn}] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{2222} & C_{2233} & 0 & 0 & 0 \\ & & C_{2222} & 0 & 0 & 0 \\ & & & (C_{2222} - C_{2233})/2 & 0 & 0 \\ Symm. & & & & C_{1212} & 0 \\ & & & & & C_{1212} \end{bmatrix}, \tag{A.12a}$$

$$[\lambda_{ij}] \equiv \begin{bmatrix} \lambda_{11} & 0 & 0 \\ & \lambda_{22} & 0 \\ Symm. & & \lambda_{22} \end{bmatrix}, \tag{A.12b}$$

$$[\mu_{ij}] \equiv \begin{bmatrix} \mu_{11} & 0 & 0 \\ & \mu_{22} & 0 \\ Symm. & & \mu_{22} \end{bmatrix}. \tag{A.12c}$$

A.3-4 Isotropic Hookean Material

If there are no preferred directions, the role of the indices 1,2 and 3 are fully interchangeable. In this case

$$\begin{aligned} C_{1111} &= C_{2222} = C_{3333}, \\ C_{1122} &= C_{1133} = C_{2233}, \\ C_{1212} &= C_{3131} = C_{2323} \equiv (C_{1111} - C_{1122})/2, \\ \lambda_{11} &= \lambda_{22} = \lambda_{33}, \\ \mu_{11} &= \mu_{22} = \mu_{33}. \end{aligned} \tag{A.13a-e}$$

As a result, the matrices of elastic and hygrothermal coefficients become

$$[C_{ijmn}] \equiv \begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ & C_{1111} & C_{1122} & 0 & 0 & 0 \\ & & C_{1111} & 0 & 0 & 0 \\ & & & (C_{1111}-C_{1122})/2 & 0 & 0 \\ & & & & (C_{1111}-C_{1122})/2 & 0 \\ Symm. & & & & & (C_{1111}-C_{1122})/2 \end{bmatrix}, \tag{A.14a}$$

$$[\lambda_{ij}] \equiv \begin{bmatrix} \lambda_{11} & 0 & 0 \\ & \lambda_{11} & 0 \\ Symm. & & \lambda_{11} \end{bmatrix}, \tag{A.14b}$$

$$[\mu_{ij}] \equiv \begin{bmatrix} \mu_{11} & 0 & 0 \\ & \mu_{11} & 0 \\ Symm. & & \mu_{11} \end{bmatrix}. \tag{A.14c}$$

The body has two independent elastic coefficients, and one thermal and one hygrothermal coefficient. An account of the elastic and hygrothermal coefficients for various types of elastic symmetries is presented in Table A.1.

Table A.1: Summary of 3-D elastic and hygrothermal coefficients

Type of material symmetry	Number of elastic independent coefficients	Number of thermal (hygro) independent coefficients	Number of nonzero elastic coefficients	Number of nonzero thermal (hygro) coefficients
Triclinic	21	6	36	9
Monoclinic	13	4	20	5
Orthotropic	9	3	12	3
Transversely Isotropic	5	2	12	3
Isotropic	2	1	12	3

For orthotropic, transversely isotropic, and isotropic materials the shear and normal components of stress and strain tensors are not coupled.

Triclinic and isotropic materials represent two extreme types of anisotropy: the former has one surface of symmetry, the latter an infinity. For a thorough discussion of the various types of anisotropy, see Bogdanovich and Pastore (1996).

A.4 ALTERNATIVE FORM OF THE 3-D CONSTITUTIVE EQUATIONS

Recall the general constitutive equations of elastic anisotropic Hookean solids:

$$\sigma_{ij} = C_{ijmn}\epsilon_{mn} - \lambda_{ij}T - \mu_{ij}M, \tag{A.15a}$$

$$\epsilon_{ij} = S_{ijmn}\sigma_{mn} + \alpha_{ij}T + \beta_{ij}M. \tag{A.15b}$$

Equation (A.15b) represents the inverted counterpart of Eq. (A.15a). Here S_{ijmn} denotes the compliance tensor, while α_{ij} and β_{ij} denote the thermal expansion and the moisture-swelling tensors, respectively. Their components follow the same pattern of *nonzero* and number of independent coefficients in the various symmetry cases as their inverted counterparts. In order to establish the relationship between (S_{ijmn}, C_{ijmn}), (λ_{ij}, μ_{ij}), and $(\alpha_{ij}, \beta_{ij})$ we will replace σ_{mn} in Eq (A.15b) by its expression in Eq. (A.15a). This yields (Librescu, 1990)

$$\epsilon_{ij} = S_{ijmn}\left[C_{mnpq}\epsilon_{pq} - \lambda_{mn}T - \mu_{mn}M\right] + \alpha_{ij}T + \beta_{ij}M. \tag{A.16}$$

Collection and identification of the coefficients associated with ϵ_{ij}, T and M results in :

$$\begin{aligned} &S_{ijmn}C_{mnpq} = (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})/2, \\ &\alpha_{ij} - S_{ijmn}\lambda_{mn} = 0, \\ &\beta_{ij} - S_{ijmn}\mu_{mn} = 0, \end{aligned} \tag{A.17a-c}$$

where δ_{ij} denotes the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$). Equation (A.17a) expresses the fact that C_{ijmn} and S_{mnpq} are inverse tensors, whereas the remaining two equations establish a relationship between α_{ij} and λ_{ij}, and between β_{ij} and μ_{ij}, respectively. Multiplication of equations $(A.17b)$ and $(A.17c)$ by C_{mnpq} and employment of Eq. $(A.17a)$ yields

$$\lambda_{pq} = \alpha_{ij}C_{ijpq}, \quad \mu_{pq} = \beta_{ij}C_{ijpq}. \tag{A.18a,b}$$

In light of Eqs. (A.18) and $(A.17b, c)$, the constitutive equations (A.15) become

$$\begin{aligned} \sigma_{ij} &= C_{ijmn}\left[\epsilon_{mn} - \alpha_{mn}T - \beta_{mn}M\right], \\ \epsilon_{ij} &= S_{ijmn}\left[\sigma_{mn} + \lambda_{mn}T + \mu_{mn}M\right]. \end{aligned} \tag{A.19a,b}$$

This form of the constitutive equations will be used later.

A.5 TRANSFORMATION OF MATERIAL COEFFICIENTS

In Section A.3.2 the characteristics of an orthotropic material were described assuming that its axes of material symmetry coincide with the geometrical axes (i.e. with the global axes of convenience).

However, the principal directions of orthotropy of the constituent layer materials often do not coincide with the geometrical axes whose orientation is dictated by the character of the problem to be investigated or by the boundary conditions.

In addition, for a laminated composite structure the principal directions of orthotropy are different in each constituent layer, this being part of the implementation of the tailoring technique.

In such cases, the transformation of the constitutive relations with respect to the global coordinate system is required.

If the constitutive behavior in the principal material coordinate system (x_1', x_2',$x_3'(\equiv x_3)$) is known, the constitutive law in the axes of convenience, (x_1, x_2, x_3), as represented in Fig. A.1, is found by applying the tensor transformation rules.

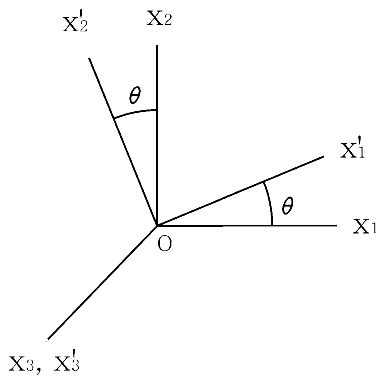


Figure A.1: Original and rotated material coordinate systems

The constitutive equations referred to the primed coordinate system, after an in-plane rotation of the x_1 - x_2 axes about the x_3-axis, assume the form:

$$\sigma_{i'j'} = C_{i'j'k'l'}(\epsilon_{k'l'} - \alpha_{k'l'}T - \beta_{k'l'}M), \tag{A.20}$$

where

$$\begin{aligned} \sigma_{i'j'} &= a_{i'l}a_{j'm}\sigma_{lm}, \\ \epsilon_{k'l'} &= a_{k'p}a_{l'q}\epsilon_{pq}, \\ \alpha_{k'l'} &= a_{k'p}a_{l'q}\alpha_{pq}, \\ \beta_{k'l'} &= a_{k'p}a_{l'q}\beta_{pq}. \end{aligned} \tag{A.21a-d}$$

In Eqs. (A.21), $a_{i'j}$ denotes the direction cosine of the angle between the $x_{i'}$ and x_j axes, so that

$$[a] \equiv a_{i'j} = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A.22}$$

where $m \equiv \cos\theta, n \equiv \sin\theta, \theta \in [0,\ 2\pi]$ being the counterclockwise rotation angle about the positive x_3-axis. Note that $[a]$ is an orthogonal matrix which satisfies the relationship $[a]\,[a]^T = \mathbf{I} = [a]^T\,[a]$, $([a]^T \equiv [a]^{-1})$ where superscript T denotes the matrix transpose, while $\mathbf{I}$ is the identity matrix whose diagonal elements are the unites.

In matrix form Eq. $(A.21a)$ is given by

$$\left\{\sigma'\right\} = [T_3]\,\{\sigma\}\,. \tag{A.23}$$

Here

$$T_3(\theta) = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix} \tag{A.24}$$

is the transformation matrix, while $\left\{\sigma'\right\}$ and $\{\sigma\}$ denote the stress tensors in the primed and unprimed coordinates, respectively. Similarly, the transformation of strains is given by

$$\left\{\gamma'\right\} = [\tilde{T}_3]\,\{\gamma\}\,, \tag{A.25}$$

where

$$\tilde{T}_3(\theta) = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & m^2 - n^2 \end{bmatrix}, \tag{A.26}$$

and $\gamma_{ij} \equiv \epsilon_{ij}$ for $i = j$ and $\gamma_{ij} \equiv 2\epsilon_{ij}$ for $i \neq j$. Here ϵ_{ij} denote the tensorial strain components. It is apparent that the following relationship between $\tilde{T}_3$ and T_3 holds valid

$$\tilde{T}_3(\theta) = [T_3^{-1}]^T = [T_3(-\theta)]^T\,. \tag{A.27}$$

Replacement in Eq. (A.20) of Eqs. (A.21) and (A.25) results, in view of Eqs. (A.24), (A.26) and (A.27), in the constitutive equations expressed in the global coordinates (x_1, x_2, x_3), where the elastic and hygrothermal properties in the principal material coordinates (x'_1, x'_2, x'_3) are assumed to be known. In matrix form these are given as:

$$\{\sigma\} = [C]\,(\{\gamma\} - \{\alpha\}\,T - \{\beta\}\,M), \tag{A.28}$$

where

$$\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}\}^T ,$$
$$\{\gamma\} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\}^T ,$$
$$\{\alpha\} \equiv [T_3]^{-1} \left\{\alpha'\right\} = \{\alpha_{11}, \alpha_{22}, \alpha_{33}, 0, 0, \alpha_{12}\}^T , \tag{A.29a-d}$$
$$\{\beta\} \equiv [T_3]^{-1} \left\{\beta'\right\} = \{\beta_{11}, \beta_{22}, \beta_{33}, 0, 0, \beta_{12}\}^T ,$$

while

$$[C] = [T_3]^{-1} \left[C'\right] \left[\tilde{T}_3\right], \tag{A.30a}$$

where

$$\left[C'\right] \equiv \left[C_{i'j'k'l'}\right] \text{ and } [T_3(\theta)]^{-1} = [T_3(-\theta)] . \tag{A.30b,c}$$

Expressed in full, Eq. (A.28) becomes

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & C_{3312} \\ 0 & 0 & 0 & C_{2323} & C_{2331} & 0 \\ 0 & 0 & 0 & C_{3123} & C_{3131} & 0 \\ C_{1211} & C_{1222} & C_{1233} & 0 & 0 & C_{1212} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} - \alpha_{11}T - \beta_{11}M \\ \epsilon_{22} - \alpha_{22}T - \beta_{22}M \\ \epsilon_{33} - \alpha_{33}T - \beta_{33}M \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} - \alpha_{12}T - \beta_{12}M \end{Bmatrix}. \tag{A.31}$$

where the symmetry of the stiffness matrix $[C]$ is implied. In Eqs. (A.30a) and (A.29c,d), $[C]$, $\{\alpha\}$ and $\{\beta\}$ denote, respectively, the matrices of transformed elastic and hygrothermal coefficients in the global coordinates.

We will use Voigt's contracted notation in place of the tensor indicial notation. This consists of replacing in the elastic and hygrothermal coefficients of a pair of indices by a single one, through the replacement rules: $11 \rightarrow 1; 22 \rightarrow 2; 33 \rightarrow 3$; 23 (or 32) $\rightarrow 4$; 31 (or 13) $\rightarrow 5$; 12 (or 21) $\rightarrow 6$.

On the basis of Eqs. (A.30b) and (A.25), Eq. $(A.30a)$ can be written in full as:

$$\begin{aligned} C_{11} &= C_{1'1'}m^4 + 2(C_{1'2'} + 2C_{6'6'})m^2n^2 + C_{2'2'}n^4, \\ C_{12} &= (C_{1'1'} + C_{2'2'} - 4C_{6'6'})m^2n^2 + C_{1'2'}(m^4 + n^4), \\ C_{13} &= C_{1'3'}m^2 + C_{2'3'}n^2, \\ C_{16} &= -C_{2'2'}mn^3 + C_{1'1'}m^3n - (C_{1'2'} + 2C_{6'6'})mn(m^2 - n^2), \\ C_{22} &= C_{2'2'}m^4 + 2(C_{1'2'} + 2C_{6'6'})m^2n^2 + C_{1'1'}n^4, \\ C_{23} &= C_{1'3'}n^2 + C_{2'3'}m^2, \\ C_{26} &= C_{1'1'}\mathrm{mn}^3 - C_{2'2'}\mathrm{m}^3\mathrm{n} + (C_{1'2'} + 2C_{6'6'})mn(m^2 - n^2), \\ C_{33} &= C_{3'3'}, \end{aligned} \tag{A.32a-m}$$

$$C_{36} = (C_{1'3'} - C_{2'3'})mn,$$
$$C_{44} = C_{4'4'}m^2 + C_{5'5'}n^2,$$
$$C_{45} = (C_{5'5'} - C_{4'4'})mn,$$
$$C_{55} = C_{4'4'}n^2 + C_{5'5'}m^2,$$
$$C_{66} = (C_{1'1'} + C_{2'2'} - 2C_{1'2'})m^2n^2 + C_{6'6'}(m^2 - n^2)^2.$$

In addition, the elements of matrices α and β are, respectively:

$$\begin{aligned} \alpha_1 &= \alpha_{1'}m^2 + \alpha_{2'}n^2, \\ \alpha_2 &= \alpha_{2'}m^2 + \alpha_{1'}n^2, \\ \alpha_3 &= \alpha_{3'}, \\ \alpha_6 &= 2(\alpha_{1'} - \alpha_{2'})mn, \end{aligned} \tag{A.33a-d}$$

and

$$\begin{aligned} \beta_1 &= \beta_{1'}m^2 + \beta_{2'}n^2, \\ \beta_2 &= \beta_{2'}m^2 + \beta_{1'}n^2, \\ \beta_3 &= \beta_{3'}, \\ \beta_6 &= 2(\beta_{1'} - \beta_{2'})mn. \end{aligned} \tag{A.34a-d}$$

In Eqs. (A.32) through (A.34), $C_{I'J'}$, $\alpha_{I'}$ and $\beta_{I'}$ denote the elastic and hygrothermal coefficients of an orthotropic material expressed in principal material directions. The relationships in Eqs. (A.31) through (A.34) reveal that in non-principal coordinates (or off-axis coordinates), an orthotropic body is characterized, likewise a monoclinic material, by 13 independent elastic constants and 4 independent thermal/hygrothermal coefficients. In this case, the coupling between normal and shear effects is present. However, C_{16}, C_{26}, C_{36}, α_6, and β_6 are not independent, but merely linear combinations of the remaining constants. Since the material, in its principal material directions is orthotropic, it is called a *generally orthotropic* material.

Equations (A.32) through (A.34) show that when the principal and the global (geometric) coordinates coincide (implying $\theta = 0$), C_{ij}, α_i, and β_i reduce to $C_{i'j'}$, $\alpha_{i'}$, and $\beta_{i'}$, respectively. In such a case the material behaves as a truly orthotropic one, and the equations (A.31) become:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 - \alpha_1 T - \beta_1 M \\ \epsilon_2 - \alpha_2 T - \beta_2 M \\ \epsilon_3 - \alpha_3 T - \beta_3 M \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{Bmatrix}. \tag{A.35}$$

It should be noticed that when the coordinate transformations have to be implemented, the tensor indices must be employed.

A.6 ALTERNATIVE REPRESENTATIONS

Equations (A.32) through (A.34) show that the stiffness quantities of a generally orthotropic material depend on the angle of orientation of its principal material directions. The technology taking advantage of this property, in the sense of modifying the response of the structure according to predetermined goals by changing fiber orientation angle, is referred to as *structural tailoring*. It has been used successfully in the aeronautical industry and is likely to play a significant role in the design of future generations of aeronautical/aerospace vehicles, of helicopter and turbine blades, of space stations, satellites, etc. For practical purposes an alternative representation of equations (A.32) through (A.34) will be presented. In this representation a separation of the quantities that are invariant to the rotation of axis (and which represent the truly composite material properties) and of those depending on the rotation of the principal material directions is applied.

To this end, and following Tsai and Hahn (1980) we use the elementary trigonometric identities:

$$\begin{aligned} m^4 &= (3 + 4\cos 2\theta + \cos 4\theta)/8, \\ m^3 n &= (2\sin 2\theta + \sin 4\theta)/8, \\ mn^3 &= (2\sin 2\theta - \sin 4\theta)/8, \\ m^2 n^2 &= (1 - \cos 4\theta)/8, \\ n^4 &= (3 - 4\cos 2\theta + \cos 4\theta)/8. \end{aligned} \tag{A.36a-e}$$

Direct substitution of Eqs. (A.36) into Eqs. (A.32) through (A.34) yields:

$$\begin{aligned} C_{11} &= U_1' + U_2' \cos 2\theta + U_3' \cos 4\theta, \\ C_{22} &= U_1' - U_2' \cos 2\theta + U_3' \cos 4\theta, \\ C_{12} &= U_4' - U_3' \cos 4\theta, \\ C_{66} &= U_5' - U_3' \cos 4\theta, \\ C_{16} &= (U_2' \sin 2\theta)/2 + U_3' \sin 4\theta, \\ C_{26} &= (U_2' \sin 2\theta)/2 - U_3' \sin 4\theta, \end{aligned} \tag{A.37a-l}$$

$$\begin{aligned} C_{13} &= U_8' + U_9' \cos 2\theta, \\ C_{23} &= U_8' - U_9' \cos 2\theta, \\ C_{55} &= U_{10}' + U_{11}' \cos 2\theta, \end{aligned}$$

$$
\begin{aligned}
C_{45} &= U'_{11} \sin 2\theta, \\
C_{44} &= U'_{10} - U'_{11} \cos 2\theta, \\
C_{36} &= U'_{9} \sin 2\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_1 &= V'_1 + V'_2 \cos 2\theta, \\
\alpha_2 &= V'_1 - V'_2 \cos 2\theta, \\
\alpha_6 &= V'_2 \sin 2\theta, \\
\alpha_3 &= \alpha_{3'},
\end{aligned}
\tag{A.38a-d}
$$

and

$$
\begin{aligned}
\beta_1 &= P'_1 + P'_2 \cos 2\theta, \\
\beta_2 &= P'_1 - P'_2 \cos 2\theta, \\
\beta_6 &= P'_2 \sin 2\theta, \\
\beta_3 &= \beta_{3'}.
\end{aligned}
\tag{A.39a-d}
$$

The quantities U'_i, V'_i, and P'_i depend only on the properties of the orthotropic material and are independent of ply-angle θ).

Their expressions are:

$$
\begin{aligned}
U'_1 &= (3C_{1'1'} + 3C_{2'2'} + 2C_{1'2'} + 4C_{6'6'})/8, \\
U'_2 &= (C_{1'1'} - C_{2'2'})/2, \\
U'_3 &= (C_{1'1'} + C_{2'2'} - 2C_{1'2'} - 4C_{6'6'})/8, \\
U'_4 &= (C_{1'1'} + C_{2'2'} + 6C_{1'2'} - 4C_{6'6'})/8, \\
U'_5 &= (C_{1'1'} + C_{2'2'} - 2C_{1'2'} + 4C_{6'6'})/8, \\
U'_8 &= (C_{1'3'} + C_{2'3'})/2, \\
U'_9 &= (C_{1'3'} - C_{2'3'})/2, \\
U'_{10} &= (C_{5'5'} + C_{4'4'})/2, \\
U'_{11} &= (C_{5'5'} - C_{4'4'})/2, \\
V'_1 &= (\alpha_{1'} + \alpha_{2'})/2, \\
V'_2 &= (\alpha_{1'} - \alpha_{2'})/2, \\
P'_1 &= (\beta_{1'} + \beta_{2'})/2, \\
P'_2 &= (\beta_{1'} - \beta_{2'})/2.
\end{aligned}
\tag{A.40a-m}
$$

In Eqs. (A.37) through (A.39), the positive sign of θ is opposite to that considered in the tensorial transformation, i.e. it is positive when measured from the on-axis (i.e. the symmetry axes) of a ply to the laminate coordinate system. This sign reversal of the angle θ, enables one to emphasize the contribution of an off-axis ply-angle to the laminate. This representation is useful in the tailoring analysis of laminates. Each constituent lamina could have a different ply orientation.

The coordinate transformation (A.32) through (A.34) yield the following invariants

$$\begin{aligned}
&C_{11} + C_{22} + C_{33} + 2(C_{12} + C_{23} + C_{31}) \\
&\qquad = C_{1'1'} + C_{2'2'} + C_{3'3'} + 2(C_{1'2'} + C_{2'3'} + C_{3'1'}), \\
&C_{11} + C_{22} + C_{33} + 2(C_{44} + C_{55} + C_{66}) \\
&\qquad = C_{1'1'} + C_{2'2'} + C_{3'3'} + 2(C_{4'4'} + C_{5'5'} + C_{6'6'}), \\
&\alpha_1 + \alpha_2 + \alpha_3 = \alpha_{1'} + \alpha_{2'} + \alpha_{3'}, \\
&\beta_1 + \beta_2 + \beta_3 = \beta_{1'} + \beta_{2'} + \beta_{3'}.
\end{aligned} \tag{A.41a-d}$$

A.7 ELASTIC COEFFICIENTS OF ORTHOTROPIC MATERIALS IN TERMS OF ENGINEERING CONSTANTS

In practice, the coefficients C_{ij} for the orthotropic material are defined in terms of the engineering constants. To this end, as a first step, from Eq. (A.17a) the relationships between C_{ij} and S_{ij} for an orthotropic body are derived. These are:

$$\begin{aligned}
C_{11} &= (S_{22}S_{33} - S_{23}^2)/S, \\
C_{12} &= (S_{13}S_{23} - S_{12}S_{33})/S, \\
C_{22} &= (S_{33}S_{11} - S_{13}^2)/S, \\
C_{13} &= (S_{12}S_{23} - S_{13}S_{22})/S, \\
C_{33} &= (S_{11}S_{22} - S_{12}^2)/S, \\
C_{23} &= (S_{12}S_{13} - S_{23}S_{11})/S, \\
C_{44} &= 1/S_{44}, \\
C_{55} &= 1/S_{55}, \\
C_{66} &= 1/S_{66},
\end{aligned} \tag{A.42a-i}$$

where

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}. \tag{A.43}$$

The components of the compliance tensor are expressed in terms of the engineering constants as

$$\left[S_{ij}\right] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}. \tag{A.44}$$

Substitution of the expressions of S_{ij} as given by (A.44) into the right-hand side members of Eqs. (A.42) and (A.43) gives the relationships between C_{ij} and the engineering coefficients as:

$$\begin{aligned} C_{11} &= E_1(1 - \nu_{23}\nu_{32})/\Delta, \\ C_{22} &= E_2(1 - \nu_{31}\nu_{13})/\Delta, \\ C_{33} &= E_3(1 - \nu_{12}\nu_{21})/\Delta, \\ C_{12} &= E_1(\nu_{21} + \nu_{31}\nu_{23})/\Delta, \\ C_{13} &= E_1(\nu_{31} + \nu_{21}\nu_{32})/\Delta, \\ C_{23} &= E_2(\nu_{32} + \nu_{12}\nu_{31})/\Delta, \\ C_{44} &= G_{23}, \\ C_{55} &= G_{31}, \\ C_{66} &= G_{12}. \end{aligned} \tag{A.45a-i}$$

Here E_1, E_2, E_3 denote the Young's moduli along the principal material directions; G_{12}, G_{23}, G_{31} denote the shear moduli, ν_{12}, ν_{23}, ν_{31} denote the Poisson's ratios while

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{32}\nu_{23} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{13}\nu_{32}. \tag{A.46}$$

From symmetry considerations of the compliance matrix $[S_{ij}]$, the following reciprocal relationship hold

$$\frac{E_i}{\nu_{ij}} = \frac{E_j}{\nu_{ji}} \quad (i, j = 1, 2, 3) \tag{A.47}$$

For a *unidirectional composite reinforced in the x_1-direction* we have

$$\begin{aligned} &E_2 = E_3 \equiv E, \quad E_1 \equiv E', \\ &\nu_{13} = \nu_{12} \equiv \nu', \quad \nu_{23} = \nu_{32} \equiv \nu, \\ &G_{23} \equiv G = E/2(1+\nu), \quad G_{12} = G_{13} \equiv G', \\ &\nu_{21} = \nu_{31} = \nu' E/E'. \end{aligned} \tag{A.48a-g}$$

For a transversely isotropic solid whose plane of isotropy is parallel at each point to the plane x_1-x_2, the elastic constants are

$$\begin{aligned} &E_1 = E_2 \equiv E; E_3 = E'; \quad G_{12} \equiv G(\equiv E/2(1+\nu)); \quad G_{23} = G_{31} \equiv G', \\ &\nu_{31} = \nu_{32} \equiv \nu'; \quad \nu_{12} = \nu_{21} \equiv \nu; \quad \nu_{13} = \nu_{23} \equiv \nu' E/E'. \end{aligned} \tag{A.49a-g}$$

For these two cases, the associated stiffness coefficients C_{ij} are obtainable by replacing Eqs. (A.48) and (A.49) in (A.45).

For completeness, the elastic stiffness coefficients C_{ij} for this type of transverse isotropy are

$$\begin{aligned} C_{11} &= \frac{E(E\nu'^2 - E')}{\Delta} = C_{22}, \\ C_{12} &= -\frac{E(E\nu'^2 + E'\nu)}{\Delta}, \\ C_{13} &= -\frac{EE'\nu'(1+\nu)}{\Delta} = C_{23}, \\ C_{33} &= -\frac{E'^2\,(1-\nu^2)}{\Delta}, \\ C_{44} &= C_{55} = G', \end{aligned} \tag{A.50a-e}$$

where $\Delta = (1+\nu)(2E\nu'^2 + E'\nu - E')$.

The pyrolytic graphite and its alloys feature this type of transverse isotropy. Due to its very low thermal conductivity coefficient in the thickness direction as compared to that in the tangential direction, it is used in the design of thermal protection of aerospace vehicle systems.

A.8 ANISOTROPIC THIN-WALLED BEAMS

A.8-1 Introduction

A thin-walled composite beam consists of a number of laminae bonded and/or cured together. The theory of thin-walled beams is essentially a 1-D approximation of the 3-D theory of elasticity. This should result in a system of ordinary differential governing equations expressed in terms of the 1-D displacements measures $u_0(x_2)$, $v_0(x_2)$, $w_0(x_2)$, the 1-D rotational measures $\theta_1(x_2)$, $\theta_2(x_2)$, $\phi(x_2)$ and the warping measure W_M. To obtain the system of 1-D governing equations, the constitutive equations have to feature the same 1-D character. An intermediate stage, consisting of the derivation of 2-D constitutive equations will be considered; the 2-D stress resultants and stress couples should relate the 2-D strain measures, where $x_1(\equiv s)$ and $x_2(\equiv z)$ are the independent coordinates involved in these equations.

A.8-2 3-D Equations for a Lamina

Consider a laminated composite thin-walled beam consisting of N orthotropic laminae referenced to a nonprincipal material coordinate system. For the k-th constituent lamina of the laminate, the constitutive equations are

$$\left\{\begin{matrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{nn} \\ \sigma_{zn} \\ \sigma_{sn} \\ \sigma_{sz} \end{matrix}\right\}_{(k)} = [C]_{(k)} \left\{\begin{matrix} \epsilon_{ss} - \alpha_s T - \beta_s M \\ \epsilon_{zz} - \alpha_z T - \beta_z M \\ \epsilon_{nn} - \alpha_n T - \beta_n M \\ \gamma_{zn} \\ \gamma_{sn} \\ \gamma_{sz} - \alpha_{sz} T - \beta_{sz} M \end{matrix}\right\}_{(k)}, \tag{A.51a}$$

where

$$[C]_{(k)} \equiv \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}_{(k)}. \tag{A.51b}$$

Since the material properties in Eqs. (A.51) are affiliated to the k-th layer, for the sake of identification, the matrices are associated with the subscript (k). Unless otherwise specified, the indices s, z, and n will replace the former indices 1, 2, and 3, respectively.

As in the theory of plates and shells (see e.g. Librescu, 1975, 1990), we will invoke the fact that σ_{nn} is negligible when compared with the other stress components. The condition $\sigma_{nn} = 0$ leads to the following transverse normal strain

$$\begin{aligned} \epsilon_{nn} = & -\frac{C_{13}}{C_{33}}(\epsilon_{ss} - \alpha_s T - \beta_s M) - \frac{C_{23}}{C_{33}}(\epsilon_{zz} - \alpha_z T - \beta_z M) \\ & -\frac{C_{36}}{C_{33}}(\gamma_{sz} - \alpha_{sz} T - \beta_{sz} M) + \alpha_n T + \beta_n M. \end{aligned} \tag{A.52}$$

As a result, Eq. (A.51a) is recast in the form:

$$\begin{Bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{zn} \\ \sigma_{sn} \\ \sigma_{sz} \end{Bmatrix}_{(k)} = [\bar{Q}]_{(k)} \begin{Bmatrix} \varepsilon_{ss} - \bar{\alpha}_s T - \bar{\beta}_s M \\ \varepsilon_{zz} - \bar{\alpha}_z T - \bar{\beta}_z M \\ \gamma_{zn} \\ \gamma_{sn} \\ \gamma_{sz} - \bar{\alpha}_{sz} T - \bar{\beta}_{sz} M \end{Bmatrix}_{(k)}, \tag{A.53}$$

where $[\bar{Q}]$ is the matrix of the *reduced elastic coefficients* given by

$$[\bar{Q}] = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} & 0 \\ 0 & 0 & \bar{Q}_{54} & \bar{Q}_{55} & 0 \\ \bar{Q}_{61} & \bar{Q}_{62} & 0 & 0 & \bar{Q}_{66} \end{bmatrix}, \tag{A.54a}$$

where

$$\bar{Q}_{IJ} = C_{IJ} - \frac{C_{I3}C_{J3}}{C_{33}} = \bar{Q}_{JI}, \quad (I, J = 1, 2, 6), \tag{A.54b}$$

$$\bar{Q}_{LM} \equiv C_{LM}, \quad (L, M = 4, 5) \tag{A.54c}$$

while

$$\overline{\alpha}_I = \alpha_I - \frac{C_{I3}}{C_{33}}\alpha_3, \quad \overline{\beta}_I = \beta_I - \frac{C_{I3}}{C_{33}}\beta_3, \tag{A.54d}$$

are the *reduced* stress-temperature and stress-moisture coefficients, respectively.

For an orthotropic body:

$$\bar{Q}_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}; \quad \bar{Q}_{12} = \frac{E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} = \frac{E_2\nu_{12}}{1 - \nu_{12}\nu_{21}}$$
$$\bar{Q}_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}; \quad \bar{Q}_{66} = \bar{C}_{66} = G_{12}; \quad \bar{Q}_{44} = G_{23}; \quad \bar{Q}_{55} = G_{31}. \tag{A.55a-f}$$

A.8-3 2-D Stress-Resultants and Stress-Couples

In order to obtain a 2-D variant of the constitutive equations, the 2-D stress resultants and stress couples have to be defined. As in the theory of plates and shells, these quantities are defined so that these 2-D quantities are statically equivalent to their 3-D counterparts. Suppose that the kth lamina occupies the domain $n_{(k-1)} < n < n_{(k)}$, $k = 1 \dots N$, we define:

(a) *The membrane stress resultants*

$$\left\{ \begin{array}{c} N_{ss} \\ N_{zz} \\ N_{sz} \end{array} \right\} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \left\{ \begin{array}{c} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{array} \right\}_{(k)} dn, \tag{A.56}$$

(b) *The transverse shear stress resultants*

$$\left\{ \begin{array}{c} N_{zn} \\ N_{sn} \end{array} \right\} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \left\{ \begin{array}{c} \sigma_{zn} \\ \sigma_{sn} \end{array} \right\}_{(k)} dn, \tag{A.57}$$

and

(c) *The stress couples*

$$\left\{ \begin{array}{c} L_{zz} \\ L_{sz} \end{array} \right\} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \left\{ \begin{array}{c} \sigma_{zz} \\ \sigma_{sz} \end{array} \right\}_{(k)} ndn. \tag{A.58}$$

Here $n_{(k)}$ and $n_{(k-1)}$ denote the distances from the middle surface of the cross-section to the upper and lower surfaces of the k-th layer, respectively. The $n_{(k)}$ values are signed numbers varying from $n_{(0)} = -h/2$ to $n_{(N)} = h/2$ in a laminate of N layers and of total thickness h (see Fig. A.2). Equations (A.56) through (A.58) show that the stress resultants and stress couples do not depend on n, but are functions of (s, z) coordinates.

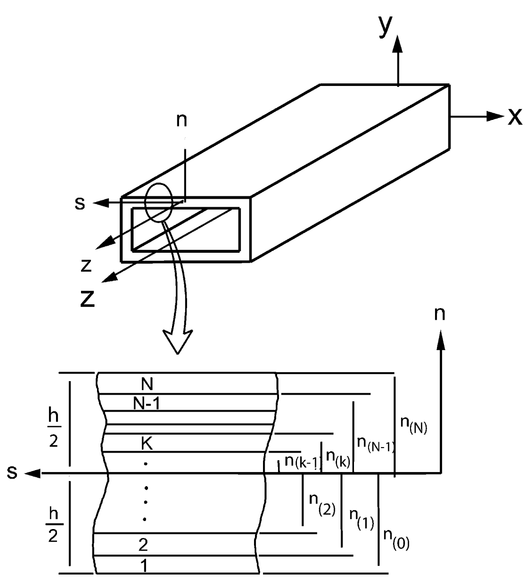
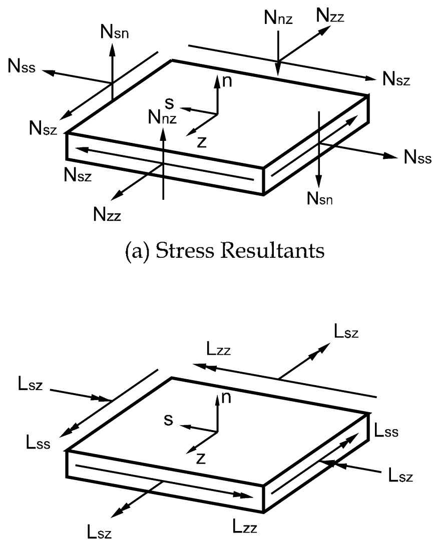


Figure A.2: Geometry of an N-ply laminate



(a) Stress Resultants

(b) Stress Couples

Figure A.3: Stress resultants and stress couples on a beam element

Equations (A.56) through (A.58) involve the assumption that the reference surface of the thin-walled beam is *shallow* in the sense of $h/R_1 << 1$, where R_1 denotes the radius of curvature of the mid-surface in the s-direction. The stress resultants have the units of force per unit length, while the stress couples have the units of moment per unit length. The positive sign convention for the stress resultants and stress couples is displayed in Fig. A.3.

The symmetry of the stress tensor and of the shallowness of beam middle surface, implies $N_{sz} = N_{zs}$ and $L_{sz} = L_{zs}$.

A.8-4 A First Step Toward Obtaining the Constitutive Equations of Thin-Walled Open Beams

Upon substituting Eqs. (A.53) into (A.56) through (A.58), we obtain the 2-D constitutive equations. As an example, consider the circumferential (hoop) membrane stress resultant N_{ss}. In conjunction with Eq. (A.53), (see Song, 1990)

$$
\begin{aligned}
N_{ss}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \sigma_{ss}^{(k)} dn \\
&= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} [\bar{Q}_{11}(\epsilon_{ss} - \bar{\alpha}_s T - \bar{\beta}_s M) + \bar{Q}_{12}(\epsilon_{zz} - \bar{\alpha}_z T - \bar{\beta}_z M) \\
&\quad + \bar{Q}_{16}(\gamma_{sz} - \bar{\alpha}_{sz} T - \bar{\beta}_{sz} M)]_{(k)} dn. \qquad \text{(A.59)}
\end{aligned}
$$

With the definitions of *stretching* and *bending-stretching coupling* stiffness quantities A_{ij} and B_{ij}, respectively

$$A_{ij} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \bar{Q}_{ij}^{(k)} dn, \quad B_{ij} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \bar{Q}_{ij}^{(k)} n dn, \tag{A.60a,b}$$

as well as those of hygrothermal hoop forces (per unit length)

$$\begin{aligned} N_{ss}^{T}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} T(s,z,n)[\bar{Q}_{11}\bar{\alpha}_s + \bar{Q}_{12}\bar{\alpha}_z + \bar{Q}_{16}\bar{\alpha}_{sz}]_{(k)} dn, \\ N_{ss}^{M}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} M(s,z,n)[\bar{Q}_{11}\bar{\beta}_s + \bar{Q}_{12}\bar{\beta}_z + \bar{Q}_{16}\bar{\beta}_{sz}]_{(k)} dn, \end{aligned} \tag{A.61a,b}$$

Eq. (A.59) becomes:

$$\begin{aligned} N_{ss}(s,z) = &A_{11}\epsilon_{ss} + A_{12}\epsilon_{zz}^{(0)} + B_{12}\epsilon_{zz}^{(1)} \\ &+ A_{16}\gamma_{sz}^{(0)} + 2B_{16}W_M - N_{ss}^{T} - N_{ss}^{M}. \end{aligned} \tag{A.62}$$

Proceeding thus for the other stress resultant quantities, we obtain

$$\begin{Bmatrix} N_{ss} \\ N_{zz} \\ N_{sz} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{ss} \\ \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{12} & B_{16} \\ B_{22} & B_{26} \\ B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(1)} \\ 2W_M \end{Bmatrix} - \begin{Bmatrix} N_{ss}^{T} \\ N_{zz}^{T} \\ N_{sz}^{T} \end{Bmatrix} - \begin{Bmatrix} N_{ss}^{M} \\ N_{zz}^{M} \\ N_{sz}^{M} \end{Bmatrix}, \tag{A.63}$$

where, in addition to Eqs. (A.61), we have also

$$\begin{aligned} N_{zz}^{T}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} T(s,z,n)[\bar{Q}_{12}\bar{\alpha}_s + \bar{Q}_{22}\bar{\alpha}_z + \bar{Q}_{26}\bar{\alpha}_{sz}]_{(k)} dn, \\ N_{sz}^{T}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} T(s,z,n)[\bar{Q}_{16}\bar{\alpha}_s + \bar{Q}_{26}\bar{\alpha}_z + \bar{Q}_{66}\bar{\alpha}_{sz}]_{(k)} dn. \end{aligned} \tag{A.64a,b}$$

N_{zz}^{M} and N_{sz}^{M}, are obtainable from Eq (A.64) by applying the substitutions

$$T \rightarrow M; \;\; \bar{\alpha}_s \rightarrow \bar{\beta}_s, \;\; \bar{\alpha}_z \rightarrow \bar{\beta}_z, \;\; \bar{\alpha}_{sz} \rightarrow \bar{\beta}_{sz} \tag{A.65a-d}$$

The transverse shear stress resultants, N_{zn} and N_{sn}, are

$$\begin{aligned} N_{zn}(s,z) &= A_{44}\gamma_{zn} + A_{45}\gamma_{sn}, \\ N_{sn}(s,z) &= A_{45}\gamma_{zn} + A_{55}\gamma_{sn}. \end{aligned} \tag{A.66a,b}$$

By virtue of in-plane cross-section non-deformability implying $\gamma_{sn} = 0$, Eqs. (A.66) reduce to

$$\begin{aligned} N_{zn}(s,z) &= A_{44}\gamma_{zn}, \\ N_{sn}(s,z) &= A_{45}\gamma_{zn}. \end{aligned} \tag{A.67a,b}$$

The transverse shear stiffness quantities A_{LM} are

$$A_{LM} = \sum_{k=1}^{N} k_{LM}^2\ \bar{Q}_{LM}^{(k)}(n_{(k)} - n_{(k-1)}), \quad (L, M = 4, 5) \tag{A.68}$$

where k_{LM}^2 are transverse shear correction factors.

If we assume that transverse shear stresses have a parabolic distribution across the laminate thickness, and define a continuous weighting function (see Vinson and Sierakowski, 2002), we obtain

$$f(n) = \frac{5}{4}\left[1 - \left(\frac{n}{h/2}\right)^2\right], \tag{A.69}$$

an alternative expression for A_{LM}:

$$A_{LM} = \frac{5}{4}\sum_{k=1}^{N} \bar{Q}_{LM}^{(k)}\left[n_{(k)} - n_{(k-1)} - \frac{4}{3h^2}(n_{(k)}^3 - n_{(k-1)}^3)\right]. \tag{A.70}$$

For a single layered beam, $N = 1, n_{(k)} = h/2,\ n_{(k-1)} = -h/2$, and

$$A_{LM} = \frac{5}{6}\bar{C}_{LM}h \quad (L, M = 4, 5). \tag{A.71}$$

This form can also be obtained from Eq. (A.68), by prescribing Reissner's transverse shear correction factor $k_{LM}^2 \equiv k^2 = 5/6$.

Similarly, the stress couples are expressible in matrix form as:

$$\begin{Bmatrix} L_{zz} \\ L_{sz} \end{Bmatrix} = \begin{bmatrix} B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{ss} \\ \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{22} & D_{26} \\ D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(1)} \\ 2W_M \end{Bmatrix} - \begin{Bmatrix} L_{zz}^T \\ L_{sz}^T \end{Bmatrix} - \begin{Bmatrix} L_{zz}^M \\ L_{sz}^M \end{Bmatrix}. \tag{A.72}$$

In these relationships, both the stiffness quantities B_{ij} and D_{ij} occur. The latter as well as the *thermal and hygric moments* (per unit length) appearing in Eqs. (A.72) are defined as:

$$D_{ij} = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} \bar{Q}_{ij}^{(k)} n^2 dn. \tag{A.73}$$

$$\begin{aligned}
L_{zz}^{T}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} T(s,z,n)(\bar{Q}_{12}\bar{\alpha}_s + \bar{Q}_{22}\bar{\alpha}_z + \bar{Q}_{26}\bar{\alpha}_{sz})_{(k)} n dn,\\
L_{sz}^{T}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} T(s,z,n)(\bar{Q}_{16}\bar{\alpha}_s + \bar{Q}_{26}\bar{\alpha}_z + \bar{Q}_{66}\bar{\alpha}_{sz})_{(k)} n dn,\\
L_{zz}^{M}(s,z) &= \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} M(s,z,n)(\bar{Q}_{12}\bar{\beta}_s + \bar{Q}_{22}\bar{\beta}_z + \bar{Q}_{26}\bar{\beta}_{sz})_{(k)} n dn,
\end{aligned} \tag{A.74a-d}$$

$$L_{sz}^{M}(s,z) = \sum_{k=1}^{N} \int_{n_{(k-1)}}^{n_{(k)}} M(s,z,n)(\bar{Q}_{16}\bar{\beta}_s + \bar{Q}_{26}\bar{\beta}_z + \bar{Q}_{66}\bar{\beta}_{sz})_{(k)} n dn.$$

A.8-5 Remarks on Stiffness Quantities

Equations (A.63), (A.67), and (A.72) are the 2-D thermoelastic constitutive equations for the laminated composite anisotropic thin-walled beams. In their expression, in contrast to their isotropic single-layer counterpart, a variety of coupling effects are present. These couplings constitute new degrees of freedom in the hands of the designer. Among them, the stiffness quantities B_{ij} induce a coupling between membrane stress resultants (see Eqs. (A.63)) and the extension strain $\epsilon_{zz}^{(1)}$ varying through the beam thickness, and the warping W_M, and between the stress couples (see Eqs. (A.72)) and the in-plane strain components.

The stiffness terms A_{16}, A_{26} introduce a coupling between the circumferential (hoop) and the axial membrane stress resultants and the membrane shear strain components. Since the material properties are piecewise constant through the thickness of the beam, the stiffness quantities can be expressed as:

$$A_{ij} = \sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[n_{(k)} - n_{(k-1)}], \quad B_{ij} = \frac{1}{2}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[n_{(k)}^2 - n_{(k-1)}^2],$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{ij}^{(k)}[n_{(k)}^3 - n_{(k-1)}^3]. \tag{A.75a-c}$$

For uniform thermal and moisture fields throughout the thickness [i.e, when $T \equiv T(s, z)$ and $M \equiv M(s, z)$], the expressions for the associated thermal and hygric forces and moments, become

$$
\begin{aligned}
N_{ss}^{T}(s, z) =& T(s, z) \sum_{k=1}^{N} \{\bar{Q}_{11}\bar{\alpha}_s + \bar{Q}_{12}\bar{\alpha}_z + \bar{Q}_{16}\bar{\alpha}_{sz}\}_{(k)}[n_{(k)} - n_{(k-1)}],\\
N_{zz}^{T}(s, z) =& T(s, z) \sum_{k=1}^{N} \{\bar{Q}_{12}\bar{\alpha}_s + \bar{Q}_{22}\bar{\alpha}_z + \bar{Q}_{26}\bar{\alpha}_{sz}\}_{(k)}[n_{(k)} - n_{(k-1)}],\\
N_{sz}^{T}(s, z) =& T(s, z) \sum_{k=1}^{N} \{\bar{Q}_{16}\bar{\alpha}_s + \bar{Q}_{26}\bar{\alpha}_z + \bar{Q}_{66}\bar{\alpha}_{sz}\}_{(k)}[n_{(k)} - n_{(k-1)}],\\
L_{zz}^{T}(s, z) =& \frac{1}{2} T(s, z) \sum_{k=1}^{N} \{\bar{Q}_{11}\bar{\alpha}_s + \bar{Q}_{12}\bar{\alpha}_z + \bar{Q}_{16}\bar{\alpha}_{sz}\}_{(k)}[n_{(k)}^2 - n_{(k-1)}^2],\\
L_{sz}^{T}(s, z) =& \frac{1}{2} T(s, z) \sum_{k=1}^{N} \{\bar{Q}_{16}\bar{\alpha}_s + \bar{Q}_{26}\bar{\alpha}_z + \bar{Q}_{66}\bar{\alpha}_{sz}\}_{(k)}[n_{(k)}^2 - n_{(k-1)}^2].
\end{aligned}
\tag{A.76a-e}
$$

N_{ss}^{M}, N_{zz}^{M}, N_{sz}^{N}, L_{zz}^{M} and L_{sz}^{M}, are obtainable from their thermal counterparts, Eqs. (A.76), by using the substitutions, Eq. (A.65). The laminate stacking sequence code specifies the ply composition, the exact location and sequence of various plies.

For an unsymmetric laminate this prescription is done from the bottom of the laminate, to the top, while for a symmetric laminate, from the middle surface to the top surface. Several standard notations are given next: $[0_3^o/90_2^o/45^o/45^o/90_2^o/0_3^o]_T \equiv [0_3^o/90_2^o/45^o]_S$, where subscripts T and S stand for total number of plies and symmetric sequence respectively, $[\pm\theta°]_n = [+\theta°/-\theta°/+\theta°/-\theta° \cdots]$; $[\pm 45°]_n = [+45°/-45°/+45°/-45°]$, where, the subscript n indicates that the sub-laminate $[\pm\theta°]$ is repeated n times.

There are a number of special laminate configurations that, in addition to their technological importance, lead to simplifications of the 2-D constitutive law. These simplifications depend basically on the *odd* and *even* character of stiffness $\bar{Q}_{ij}$ and of thermal $\bar{\alpha}_i$ and hygrothermal $\bar{\beta}_i$ expansion coefficients, as well as on the stacking sequence characteristics of the laminate. The odd and even features of $\bar{Q}_{ij}$, $\bar{\alpha}_i$, and $\bar{\beta}_i$ (i.e. the change and the invariance in sign of the respective quantities respectively, when ply-angle $\theta \to -\theta$) can easily be revealed when using Eqs. (A.32) through (A.34) and are summarized by

$$
\bar{Q}_{ij}(\theta) = \begin{cases} -\bar{Q}_{ij}(-\theta) \text{ if } (ij) = (16), (26), (45), (36) \\ \bar{Q}_{ij}(-\theta) \text{ if } (ij) = (11), (12), (13), (23), (22), (44), (55), (66) \end{cases}
$$

$$\bar{\alpha}(\theta) = \begin{cases} -\bar{\alpha}(-\theta) \text{ if } i = 6 \\ \bar{\alpha}(-\theta) \text{ if } i = 1, 2, 3 \end{cases} \quad \text{(A.77a,b)}$$

whereas for $\bar{\beta}_i(\theta)$ a similar behavior as for $\bar{\alpha}_i(\theta)$ is obtained. Based on the odd and even properties of a generally orthotropic material, special classes of laminate configurations can be distinguished.

A.8-6 Selected Classes of Laminate Configurations

(a) *Symmetric Laminates*

This case requires both geometric and material symmetry properties as well as symmetry of the hygrothermal distributions about the middle surface of the beam. For the conditions of symmetry stated above,

$$B_{ij} = 0. \quad \text{(A.78)}$$

For the same case, when the thermal and hygrothermal fields are uniform across the laminate thickness, Eqs. (A.74) imply

$$L_{zz}^T = L_{sz}^T = L_{zz}^M = L_{sz}^M = 0, \quad \text{(A.79)}$$

which means that there are no hygrothermal induced moments (per unit length).

(b) *Balanced Laminates*

To every lamina with ply-orientation $+\theta$, there exists a companion laminae of identical material properties and thickness of orientation $-\theta$ *somewhere* within the laminate. For this case

$$A_{16} = A_{26} = 0, \quad \text{(A.80a)}$$

and

$$N_{sz}^T = N_{sz}^M = 0. \quad \text{(A.80b)}$$

For a balanced and symmetric laminate, B_{ij}=0, $A_{16} = A_{26} = 0$, $N_{sz}^T = N_{sz}^M = 0$ and $L_{sz}^T = L_{sz}^M = 0$. It clearly appears that the use of *unbalanced* angle-ply laminates (i.e, of such laminates for which A_{16}, A_{26} and D_{26} are different from zero), is essential when the tailoring technique should be implemented. A balanced laminate can be symmetric, antisymmetric or asymmetric.

(c) *Antisymmetric Laminates*

For this case, to every lamina with orientation $+\theta$ and location n with respect to the middle surface of the beam, there is another lamina with orientation $-\theta$ at the location $-n$. This laminate consists of an even number of plies. This configuration yields the simplification

$$A_{16} = A_{26} = D_{16} = D_{26} = 0. \quad \text{(A.81)}$$

Moreover, if the antisymmetric laminate consist of pairs of $+\theta_i$ and $-\theta_i$ orientations, symmetrically located about the middle surface and having the same thickness and elastic properties, besides Eq. (A.81), we have

$$B_{11} = B_{22} = B_{12} = B_{66} = 0. \quad \text{(A.82)}$$

Additional simplifications can occur for cross-ply laminated composite thin-walled beam structures. For this case the reader is referred to the monographs by Daniel and Ishai (1994), Bogdanovich and Pastore (1996), Vinson and Sierakowski (2002) and Reddy (2004), where other ply sequence configurations of practical importance are also considered.

A.8-7 Equations for Open Cross-Section Beams

One of the assumptions in general use within the theory of thin-walled beams is that the stress component σ_{ss} is negligible compared with the remaining stress components. This means that N_{ss} can be assumed negligible in the constitutive equations. In light of this assumption and in conjunction with Eq. (A.62)

$$\epsilon_{ss} = \frac{1}{A_{11}}(-A_{12}\epsilon_{zz}^{(0)} - A_{16}\gamma_{sz}^{(0)} - 2B_{16}W_M - B_{12}\epsilon_{zz}^{(1)} + N_{ss}^T + N_{ss}^M). \quad \text{(A.83)}$$

Replacement of Eq.(A.83) into the constitutive equations (A.63b,c) yields the final form of 2-D constitutive equations for thin-walled open beams:

$$\begin{Bmatrix} N_{zz} \\ N_{sz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ W_M \\ \epsilon_{zz}^{(1)} \end{Bmatrix} - \begin{Bmatrix} \hat{N}_{zz}^T \\ \hat{N}_{sz}^T \end{Bmatrix} - \begin{Bmatrix} \hat{N}_{zz}^T \\ \hat{N}_{sz}^M \end{Bmatrix},$$

$$\begin{Bmatrix} L_{zz} \\ L_{sz} \end{Bmatrix} = \begin{bmatrix} K_{41} & K_{42} & K_{43} & K_{44} \\ K_{51} & K_{52} & K_{53} & K_{54} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ W_M \\ \epsilon_{zz}^{(1)} \end{Bmatrix} - \begin{Bmatrix} \hat{L}_{zz}^T \\ \hat{L}_{sz}^T \end{Bmatrix} - \begin{Bmatrix} \hat{L}_{zz}^M \\ \hat{L}_{sz}^M \end{Bmatrix}.$$

(A.84a,b)

The constitutive equations (A.67a,b) associated with the transverse shear stress resultants, remain unchanged. Within this variant of constitutive equations, the modified stiffness quantities K_{ij} are defined by:

$$\begin{aligned}
&K_{11} \equiv A_{22} - A_{12}^2/A_{11},\\
&K_{12} \equiv A_{26} - A_{12}A_{16}/A_{11} = K_{21},\\
&K_{13} \equiv 2(B_{26} - A_{12}B_{16}/A_{11}),\\
&K_{14} \equiv B_{22} - A_{12}B_{12}/A_{11} = K_{41},\\
&K_{22} \equiv A_{66} - A_{16}^2/A_{11},\\
&K_{23} \equiv 2(B_{66} - A_{16}B_{16}/A_{11}),\\
&K_{24} \equiv B_{26} - A_{16}B_{12}/A_{11} = K_{42},\\
&K_{43} \equiv 2(D_{26} - B_{12}B_{16}/A_{11}),\\
&K_{44} \equiv D_{22} - B_{12}^2/A_{11},\\
&K_{51} \equiv B_{26} - B_{16}A_{12}/A_{11},\\
&K_{52} \equiv B_{66} - B_{16}A_{16}/A_{11},\\
&K_{53} \equiv 2(D_{66} - B_{16}^2/A_{11}),\\
&K_{54} \equiv D_{26} - B_{12}B_{16}/A_{11},
\end{aligned} \tag{A.85a-m}$$

while the modified hygrothermal forces and moments are:

$$\begin{aligned}
&\hat{N}_{zz}^T \equiv N_{zz}^T - A_{12}N_{ss}^T/A_{11},\\
&\hat{N}_{zz}^M \equiv N_{zz}^M - A_{12}N_{ss}^M/A_{11},\\
&\hat{N}_{sz}^T \equiv N_{sz}^T - A_{16}N_{ss}^T/A_{11},\\
&\hat{N}_{sz}^M \equiv N_{sz}^M - A_{16}N_{ss}^M/A_{11},\\
&\hat{L}_{zz}^T \equiv L_{zz}^T - B_{12}N_{ss}^T/A_{11},\\
&\hat{L}_{zz}^M \equiv L_{zz}^M - B_{12}N_{ss}^M/A_{11},\\
&\hat{L}_{sz}^T \equiv L_{sz}^T - B_{16}N_{ss}^T/A_{11},\\
&\hat{L}_{sz}^M \equiv L_{sz}^M - B_{16}N_{ss}^M/A_{11}.
\end{aligned} \tag{A.86a-h}$$

A.8-8 2-D Constitutive Equations for Closed Cross-Section Beams

The procedure of deriving the constitutive relationships for closed cross-section beams is similar to that used for open beams. For thin-walled beams of closed cross-section, the difference lies in the expression of the torsional function. In conjunction with Eq. (2.1-62) the expression of the shear strain component is

$$\gamma_{sz} = \gamma_{sz}^{(0)} + \gamma_{sz}^{(t)}, \tag{A.87a}$$

where

$$\gamma_{sz}^{(0)}(s,z) = \gamma_{xz}(z)\frac{dx}{ds} + \gamma_{yz}(z)\frac{dy}{ds}, \tag{A.87b}$$

and

$$\gamma_{sz}^{(t)} = \Psi W_M. \tag{A.87c}$$

Herein

$$\Psi \equiv \psi + 2n; \quad W_M \equiv \phi', \tag{A.88a,b}$$

where ψ is the torsional function that is expressed by Eq. (2.1-44c) considered in conjunction with (2.1-36c) and (2.1-39a), or in an inclusive form by Eq. (9.1-2).

Proceeding as for open cross-section beams, and assuming that h and G_{sz} are uniform along the beam circumference we obtain the following 2-D constitutive equations

$$\begin{Bmatrix} N_{ss} \\ N_{zz} \\ N_{sz} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{ss} \\ \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ +2\frac{\Omega}{\beta} W_M \end{Bmatrix} + \begin{Bmatrix} B_{12} \\ B_{22} \\ B_{26} \end{Bmatrix} \epsilon_{zz}^{(1)} + 2\begin{Bmatrix} B_{16} \\ B_{26} \\ B_{66} \end{Bmatrix} W_M - \begin{Bmatrix} N_{ss}^T \\ N_{zz}^T \\ N_{sz}^T \end{Bmatrix} - \begin{Bmatrix} N_{ss}^M \\ N_{zz}^M \\ N_{sz}^M \end{Bmatrix},$$

$$N_{zn} = A_{44}\gamma_{zn},$$

$$N_{sn} = A_{45}\gamma_{zn}, \tag{A.89a-d}$$

$$\begin{Bmatrix} L_{zz} \\ L_{sz} \end{Bmatrix} = \begin{bmatrix} B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{ss} \\ \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} + 2\frac{\Omega}{\beta} W_M \end{Bmatrix} + \begin{Bmatrix} D_{22} \\ D_{66} \end{Bmatrix} \epsilon_{zz}^{(1)} + 2\begin{Bmatrix} D_{26} \\ D_{66} \end{Bmatrix} W_M - \begin{Bmatrix} L_{zz}^T \\ L_{sz}^T \end{Bmatrix} - \begin{Bmatrix} L_{zz}^M \\ L_{sz}^M \end{Bmatrix}.$$

Here

$$\Omega \equiv \frac{1}{2}\oint r_n ds, \quad \beta \equiv \oint ds. \tag{A.89e,f}$$

Note that the warping measure W_M strongly intervenes in the constitutive equations of both open and closed cross-section beams.

Remark:

One should observe that due to cross-section nondeformability implying that $\gamma_{sn} = 0$, the term $N_{sn}\ \gamma_{sn}$ in the strain energy becomes immaterial. For this reason, the stiffness A_{45} does not appear in the 1-D stiffnesses a_{ij}.

A.8-9 Final Form of 2-D Constitutive Equations

The smallness of the hoop stress resultant N_{ss} combined with Eq (A.89a) yields

$$\begin{aligned}\epsilon_{ss} = & -\frac{1}{A_{11}}\bigg(A_{12}\epsilon_{zz}^{(0)} + A_{16}\gamma_{sz}^{(0)} + 2A_{16}\frac{\Omega}{\beta}W_M \\ & + 2B_{16}W_M + B_{12}\epsilon_{zz}^{(1)} - N_{ss}^T - N_{ss}^M\bigg).\end{aligned} \tag{A.90}$$

Replacing Eq. (A.90) in Eqs. (A.89a,d) leads to

$$\left\{\begin{array}{c} N_{zz} \\ N_{sz} \end{array}\right\} = \left[\begin{array}{cccc} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \end{array}\right]\left\{\begin{array}{c} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ W_M \\ \epsilon_{zz}^{(1)} \end{array}\right\} - \left\{\begin{array}{c} \hat{N}_{zz}^T \\ \hat{N}_{sz}^T \end{array}\right\} - \left\{\begin{array}{c} \hat{N}_{zz}^M \\ \hat{N}_{sz}^M \end{array}\right\},$$

$$\left\{\begin{array}{c} L_{zz} \\ L_{sz} \end{array}\right\} = \left[\begin{array}{cccc} K_{41} & K_{42} & K_{43} & K_{44} \\ K_{51} & K_{52} & K_{53} & K_{54} \end{array}\right]\left\{\begin{array}{c} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ W_M \\ \epsilon_{zz}^{(1)} \end{array}\right\} - \left\{\begin{array}{c} \hat{L}_{zz}^T \\ \hat{L}_{sz}^T \end{array}\right\} - \left\{\begin{array}{c} \hat{L}_{zz}^M \\ \hat{L}_{sz}^M \end{array}\right\}. \tag{A.91a,b}$$

As expected, the constitutive equations (A.89b,c) remain unchanged. The stiffness quantities that are affected by this change are $K_{13},\ K_{23},\ K_{43}$, and K_{53}. Their expressions are

$$\begin{aligned}K_{13} &= 2\left(A_{26} - \frac{A_{12}A_{16}}{A_{11}}\right)\frac{\Omega}{\beta} + 2\left(B_{26} - \frac{A_{12}B_{16}}{A_{11}}\right), \\ K_{23} &= 2\left(A_{66} - \frac{A_{16}^2}{A_{11}}\right)\frac{\Omega}{\beta} + 2\left(B_{66} - \frac{A_{16}B_{16}}{A_{11}}\right), \\ K_{43} &= 2\left(B_{26} - \frac{B_{12}A_{16}}{A_{11}}\right)\frac{\Omega}{\beta} + 2\left(D_{26} - \frac{B_{12}B_{16}}{A_{11}}\right), \\ K_{53} &= 2\left(B_{66} - \frac{B_{16}A_{16}}{A_{11}}\right)\frac{\Omega}{\beta} + 2\left(D_{66} - \frac{B_{16}^2}{A_{11}}\right).\end{aligned} \tag{A.92a-d}$$

The remaining stiffness quantities K_{ij} defined by Eq. (A.85) remain unaltered. Note that the constitutive equations for multicell closed section beams are similar to those for single cell counterparts. Thus, Eqs.(A.91) and (A.89b,c) can be applied to multicell beam structures as well.

A.8-10 Unified Form of 2-D Constitutive Equations

A unified expression of 2-D constitutive equations, applicable to both open and closed cross-section beams will be displayed next. This is given by

$$\begin{Bmatrix} N_{zz} \\ N_{sz} \\ L_{zz} \\ L_{sz} \\ N_{zn} \\ N_{sn} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & 0 \\ K_{51} & K_{52} & K_{53} & K_{54} & 0 \\ 0 & 0 & 0 & 0 & A_{44} \\ 0 & 0 & 0 & 0 & A_{45} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \\ W_M \\ \epsilon_{zz}^{(1)} \\ \gamma_{zn} \end{Bmatrix} - \begin{Bmatrix} \hat{N}_{zz}^T \\ \hat{N}_{sz}^T \\ \hat{L}_{zz}^T \\ \hat{L}_{sz}^T \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \hat{N}_{zz}^M \\ \hat{N}_{sz}^M \\ \hat{L}_{zz}^M \\ \hat{L}_{sz}^M \\ 0 \\ 0 \end{Bmatrix}. \tag{A.93}$$

With the exception of K_{13}, K_{23}, K_{43}, and K_{53} whose expressions are different for open and closed cross-section beams, the remaining stiffness quantities are formally similar for both cases; the constitutive equations, Eqs. (A.93), can be used for both open and closed section beams.

The constitutive equations of thin-walled beam theory used in the literature are also displayed here. In present notations, these are

$$\begin{Bmatrix} N_{zz} \\ N_{sz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} \epsilon_{zz}^{(0)} \\ \gamma_{sz}^{(0)} \end{Bmatrix}. \tag{A.94}$$

Clearly, Eqs. (A.93) are much more comprehensive than (A.94) that do not include bending and the thickness-wise contributions, and as such, are likely to describe more accurately the behavior of thin-walled beams.

A.8-11 Two Structural Coupling Configurations – CUS and CAS

We consider two structural configurations which exhibit special structural couplings. Considered first by Rehfield and Atilgan (1989), these structural configurations are referred to as *circumferentially uniform stiffness* (CUS) and *circumferentially asymmetric stiffness* (CAS) configurations. For a thin-walled beam of rectangular cross-section,(CUS) implies the ply-angle distribution $\theta(y) = \theta(-y)$ of the top and bottom walls of the box beam (*flanges*) and of $\theta(x) = \theta(-x)$ of the lateral walls (*webs*). The latter one, (CAS), implies $\theta(y) = -\theta(-y)$, and $\theta(x) = -\theta(-x)$ (see Fig. A.4).

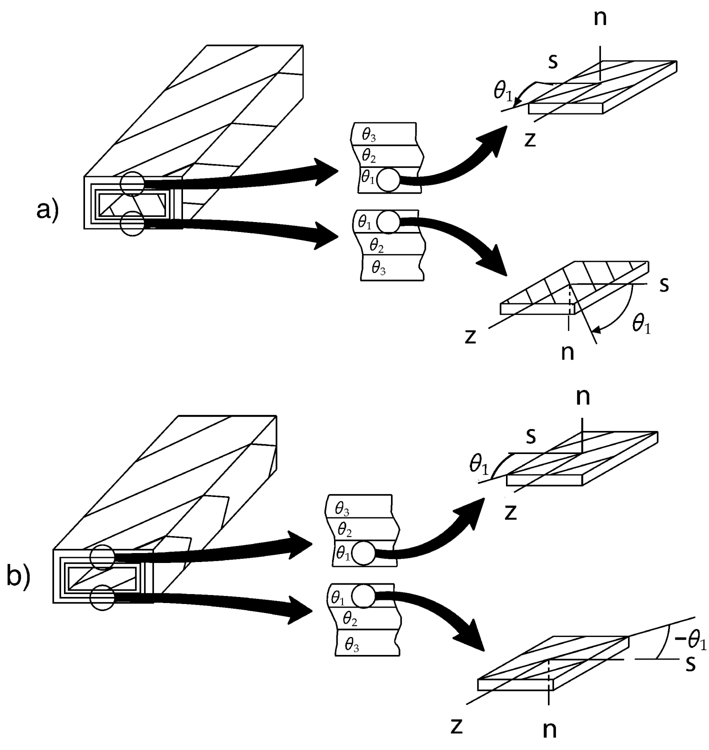


Figure A.4: a) Circumferential uniform stiffness (CUS) configuration. b) Circumferentially asymmetric stiffness (CAS) configuration

For the CAS configuration (see Eq. A.77), the stiffness quantities $\bar{Q}_{16}$, $\bar{Q}_{26}$, $\bar{Q}_{36}$, $\bar{Q}_{45}$ of each of the layers in the bottom flanges are the negative value of the layers counterpart in the top flanges, that is:

$$\bar{Q}_{16}^{(U)} = -\bar{Q}_{16}^{(L)};\ \bar{Q}_{26}^{(U)} = -\bar{Q}_{26}^{(L)};\ \bar{Q}_{36}^{(U)} = -\bar{Q}_{36}^{(L)};\ \bar{Q}_{45}^{(U)} = -\bar{Q}_{45}^{(L)}. \tag{A.95a-d}$$

This implies that in this case the following relationships hold

$$A_{16}^{(U)} = -A_{16}^{(L)};\ A_{26}^{(U)} = -A_{26}^{(L)};\ A_{36}^{(U)} = -A_{36}^{(L)};\ A_{45}^{(U)} = -A_{45}^{(L)};$$
$$D_{16}^{(U)} = -D_{16}^{(L)};\ D_{26}^{(U)} = -D_{26}^{(L)}\ D_{36}^{(U)} = -D_{36}^{(L)};\ D_{45}^{(U)} = -D_{45}^{(L)}. \tag{A.96a-h}$$

Similar types of relationships appear also in the opposite webs of the beam.

For CUS, the stiffness quantities $\bar{C}_{16}$, $\bar{C}_{26}$, $\bar{C}_{36}$, $\bar{C}_{45}$ in the opposite members exhibit the same sign, a trend which is valid also for the same components of the matrices $[A]$ and $[D]$. The CUS and CAS configurations are also referred to as the *antisymmetric* and *symmetric* configurations, respectively, see Smith and Chopra (1991).

Similar ply-angle configurations can be implemented in open beams, as well. As an example, in Figs. (13.1-1a) and (13.1-1b), the CAS and CUS are shown in a I beam.

A.8-12 Additional Remarks

The derivation of 2-D form of the constitutive equations is but an intermediate step. The goal is to convert these constitutive equations to a 1-D form, where only the single independent variable z (representing the longitudinal axis of the beam) will be involved. This step is carried out in Chapter 3.

A.9 PIEZOELECTRIC CONSTITUTIVE EQUATIONS

A.9-1 Preliminaries

As is well known, piezoelectric materials generate an electrical charge in response to a mechanical deformation or, conversely, provide mechanical strain when an electric field is applied across them. As a result, piezoelectric materials are excellent candidates for the role of sensors and actuators in the technology of smart structures (see e.g. Bailey and Hubbard, 1985; Tzou and Zhong, 1992; Tzou, 1993; Tzou and Anderson, 1992; Hanagud et al., 1992; Sunar and Rao, 1999; Chopra, 2002). In applications, piezoceramic materials are sandwiched between conductive surfaces (electrodes) and polarized in a suitable direction (see Fig. A.5). To model such structures, the constitutive equations of piezoelectric media have to be considered. In the sequel, several basic elements concerning these equations will be presented; for details, see e.g. Maugin (1988) and Eringen and Maugin (1990).

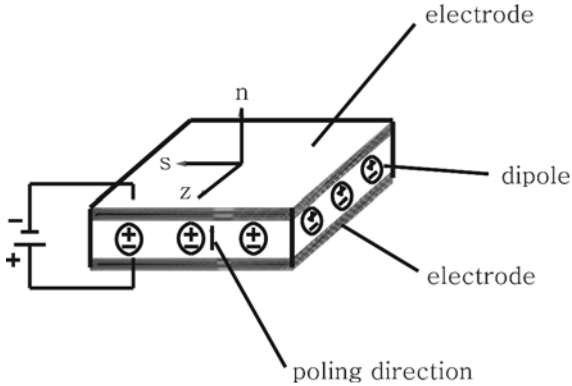


Figure A.5: Actuator layer

A.9-2 Piezoelectric Medium

In contracted indicial notations, the constitutive equations of a 3-D piezoelectric continuum are

$$\begin{aligned} \sigma_i &= C_{ij}^{\mathcal{E}} \, S_j - e_{ri} \, \mathcal{E}_r, \\ D_r &= e_{rj} \, S_j + \epsilon_{r\ell}^{S} \, \mathcal{E}_\ell, \qquad (r, \ell = 1, 2, 3) \end{aligned} \tag{A.97a,b}$$

In Eqs. (A.97) σ_i and $S_j(i, j = \overline{1, 6})$ denote the stress and strain components, respectively, where

$$S_j = \begin{cases} S_{pr} \text{ for } p = r, \, j = 1, 2, 3 \\ 2S_{pr} \text{ for } p \neq r, \, j = 4, 5, 6. \end{cases} \tag{A.98}$$

$C_{ij}^{\mathcal{E}}$, e_{ri} and $\epsilon_{r\ell}^{S}$ denote the elastic coefficients (measured under constant electric field), the piezoelectric tensor and the dielectric tensor (measured under constant strain), while $\mathcal{E}_r$ and D_r denote the electric field intensity and electric displacement vector, respectively. In Eqs. (A.97) the summation convention over repeated indices is implied.

While Eq. (A.97a) describes the **converse** piezoelectric effect consisting of the generation of mechanical stress or strain when an electric field is applied, Eq. (A.97b) describes the **direct** piezoelectric effect consisting of generation of an electrical charge under a mechanical force. In a piezoelectric adaptive structure the direct effect is used for distributed sensing, while the converse effect is used for the active distributed control.

Equations (A.97) are valid for the most general case of anisotropy, i.e., for triclinic crystals. In this case, the piezoelectric continuum is characterized by 21 independent elastic constants, 18 independent piezoelectric constants, and 6 independent dielectric constants.

For a thickness-polarized continuum, the constitutive equations are the same as for a piezocrystal of the hexagonal 6*mm* symmetry class. This represents a transversely isotropic piezoelectric, the n-axis being a sixfold axis of symmetry.

In this case (see Eringen and Maugin, 1990), the piezoelectric continuum is characterized by five independent elastic coefficients $C_{11} = C_{22}$, $C_{13} = C_{23}$, C_{33}, $C_{44} = C_{55}$, $C_{66}(\equiv (C_{11} - C_{12})/2)$, three independent piezoelectric coefficients $e_{15} = e_{24}$, $e_{31} = e_{32}$ and e_{33} and two independent dielectric constants, $\epsilon_{11} = \epsilon_{22}$ and ϵ_{33}. We consider only this case.

The electric field vector $\boldsymbol{\mathcal{E}}$ is represented in terms of the transverse normal component $\mathcal{E}_3$ (implying that $\mathcal{E}_1 = \mathcal{E}_2 = 0$).

Note due to the uniform voltage distribution, $\mathcal{E}_3$ is independent of space (but possibly dependent on time).

A.9-3 2-D Piezoelectric Constitutive Equations

We will assume that the master (or host) structure is composed of m layers, while the actuator is composed of ℓ piezoelectric layers. The actuators can be distributed over the entire span of the beam (see Figs. 6.6-1a and 9.2-1), or can be in the form of patches. Along the circumferential s, spanwise z and thickness n directions, they are distributed according to the law (see Fig. A.6):

$$\begin{aligned} R_{(k)}(n) &= Y\left(n - n_{(k-1)}\right) - Y\left(n - n_{(k)}\right), \\ R_{(k)}(s) &= Y\left(s - s_{k1}\right) - Y\left(s - s_{k2}\right), \\ R_{(k)}(z) &= Y\left(z - z_{k1}\right) - Y\left(z - z_{k2}\right), \end{aligned} \tag{A.99a-c}$$

where $Y(\cdot)$ denotes Heaviside's distribution, $R(\cdot)$ is a spatial function describing the distribution of actuator patches, while $(n_{(k-1)}, n_{(k)})$, $(s_{k1},\ s_{k2})$ and $(z_{k1},\ z_{k2})$ denote the top and bottom heights of the piezopatch measured across the beam thickness, and its location along the beam circumference and span, respectively.

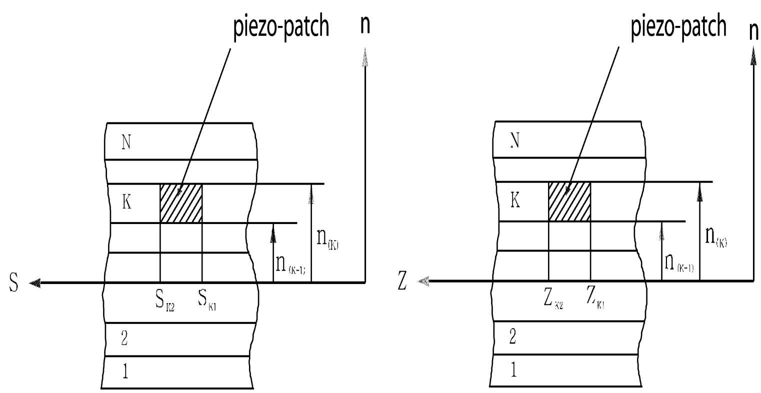


Figure A.6: Piezopatch location.

Proceeding as for Eq. (A.53), we find the 3-D constitutive equations for the transversely isotropic piezoelectric layers expressed in the coordinates (s, z, n):

$$\left\{\begin{array}{c}\sigma_{ss}\\ \sigma_{zz}\\ \sigma_{sz}\end{array}\right\}_{(k)} = \left[\begin{array}{ccc}\bar{Q}_{11} & \bar{Q}_{12} & 0\\ \bar{Q}_{12} & \bar{Q}_{11} & 0\\ 0 & 0 & \bar{Q}_{66}\end{array}\right]_{(k)} \left\{\begin{array}{c}S_{ss}\\ S_{zz}\\ S_{sz}\end{array}\right\}_{(k)} - \left\{\begin{array}{c}\bar{e}_{31}^{(k)}\,\mathcal{E}_3^{(k)}\;R_{(k)}(n)\;R_{(k)}(s)R_{(k)}(z)\\ \bar{e}_{31}^{(k)}\,\mathcal{E}_3^{(k)}\;R_{(k)}(n)\;R_{(k)}(s)R_{(k)}(z)\\ 0\end{array}\right\}, \tag{A.100a}$$

$$\sigma_{zn}^{(k)} = C_{44}^{(k)}\;S_{zn}^{(k)}\,. \tag{A.100b}$$

where

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{E}{1-\nu^2};\quad \bar{Q}_{12} = \frac{E\nu}{1-\nu^2},$$
$$\bar{Q}_{66} = G_{12} = \frac{E}{2(1+\nu)};\quad C_{44} = G_{23} \equiv G', \tag{A.101a-d}$$
$$\bar{e}_{31} = e_{31} - \frac{E\nu'}{E'(1-\nu)}e_{33}. \tag{A.101e}$$

In Eq. (A.100a), the last terms identify the actuation stresses induced by the applied electric field.

The 2-D stress resultants and stress couples of the global structure can be obtained via the integration of the 3-D stress components through the total laminate thickness, i.e. through the thickness of the host structure and of piezoelectric layers. Assuming that the hoop stress is small, and replacing the notation for the strain components used in the piezoelectricity theory by the standard one in frequent usage in beam theory (see Eqs. (A.91)), we express the 2-D stress-resultants and stress couples as:

$$\left\{\begin{array}{c}N_{zz}\\ N_{sz}\end{array}\right\} = \left[\begin{array}{cccc}K_{11} & K_{12} & K_{13} & K_{14}\\ K_{21} & K_{22} & K_{23} & K_{24}\end{array}\right]\left\{\begin{array}{c}\epsilon_{zz}^{(0)}\\ \gamma_{sz}^{(0)}\\ W_M\\ \epsilon_{zz}^{(1)}\end{array}\right\} - \left\{\begin{array}{c}\tilde{N}_{zz}\\ 0\end{array}\right\},$$

(A.102a,b)

$$\left\{\begin{array}{c}L_{zz}\\ L_{sz}\end{array}\right\} = \left[\begin{array}{cccc}K_{41} & K_{42} & K_{43} & K_{44}\\ K_{51} & K_{52} & K_{53} & K_{54}\end{array}\right]\left\{\begin{array}{c}\epsilon_{zz}^{(0)}\\ \gamma_{sz}^{(0)}\\ W_M\\ \epsilon_{zz}^{(1)}\end{array}\right\} - \left\{\begin{array}{c}\tilde{L}_{zz}\\ 0\end{array}\right\},$$

and

$$N_{zn} = A_{44}\gamma_{zn}, \quad N_{sn} = A_{45}\gamma_{zn}. \tag{A.102c,d}$$

In these equations K_{ij} denote the modified local stiffness coefficients of the adaptive structure, while $\tilde{N}_{zz}$ and $\tilde{L}_{zz}$ denote the piezoelectrically induced stress resultant and stress couple, respectively. Their expressions are:

$$\tilde{N}_{zz}(s,z) = \mathcal{E}_3\left(1 - \frac{A_{12}}{A_{11}}\right)(n_2 - n_1)\,\overline{e}_{31}R(s,z),$$
$$\tilde{L}_{zz}(s,z) = \mathcal{E}_3\overline{e}_{31}(s,z)\,(n_2 - n_1)\left[\frac{1}{2}(n_1 + n_2) - \frac{B_{12}}{A_{11}}\right]R(s,z). \tag{A.103a,b}$$

Note that in Eqs. (A.102) and (A.103), the stiffness quantities A_{ij}, B_{ij} and D_{ij} as given by Eqs.(A.75) and K_{ij} by Eqs. (A.85) and (A.92),have to be based on the **total** number of constituent layers $N = m + \ell$, for the host (m) and the piezoelectric layers (ℓ). One additional remark is in order here: while the material of the host structure can exhibit any type of anisotropy, the piezoactuators/sensors as considered here feature transversely isotropic properties, the surface of isotropy being parallel at each point to the beam mid-surface.

In Eqs. (A.103) the following notation was used:

$$R_{(k)}(s,z) \equiv R_{(k)}(s)R_{(k)}(z). \tag{A.104}$$

For $z_1 = 0$ and $z_2 = L$, the piezoactuator is spread over the entire beam span.

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