Appendix A

The peak magnitude of a second-order transfer function is of importance in the scaling of the internal node voltage in a biquad realized using several active devices such as opamps or operational transconductance amplifiers. Closed-form solutions in terms of the coefficients of the numerator and denominator of the second-order transfer function will be beneficial in computer-aided design programs. Table A.1 gives these in the case of both analog and digital transfer functions.

The second-order digital transfer function considered is

$$H(z) = \frac{p\,z^2 - qz + r}{z^2 - uz + v} \tag{A.1}$$

Using bilinear transformation and considering without loss of generality T = 2 s, we have

$$H(s) = \frac{\left(\frac{p+q+r}{1+u+v}\right) s^2 + s\frac{2(p-r)}{1+u+v} + \frac{p-q+r}{1+u+v}}{s^2 + s\frac{2(1-v)}{1+u+v} + \frac{1-u+v}{1+u+v}} \tag{A.2}$$

or

$$H(s) = \frac{a\,s^2 + bs + c}{s^2 + ds + e} \tag{A.3}$$

The peak or valley of (A.3) can be found by equating the squared magnitude $|H(j\omega)|^2$ with K^2. The resulting quadratic equation in ω^2 will have equal roots in the case where the solution corresponds to a peak or valley. This yields the relationship:

$$K^4\,(d^4 - 4e\,d^2) + K^2\,(4\,b^2\,e - 8ace - 2\,b^2\,d^2 + 4ac\,d^2 + 4\,c^2 + 4\,a^2\,e^2) + (b^4 - 4\,b^2\,ac) = 0 \tag{A.4}$$

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Table A.1 (Adapted from [A.1] © IEEE 1983)

	Type of transfer function	Magnitude of the peak
1	Band-pass (bilinear) a = c = 0	$\frac{p-r}{1-v}$
2	Low-pass (bilinear) a = b = 0	$\frac{(p-q+r)(1+u+v)}{2(1-v)\sqrt{4v-u^2}}$
3	High-pass (bilinear) b = c = 0	$\frac{(p+q+r)(1-u+v)}{2(1-v)\sqrt{4v-u^2}}$
4	Notch b = 0	$\frac{2\sqrt{ac\,d^2+c^2+a^2\,e^2-2ace}}{d\sqrt{4e-d^2}}$
5	All-pass $u=\frac{q}{r}$ and $v=\frac{p}{r}$	r
6	General biquadratic function	$\frac{-y+\sqrt{y^2-4xz}}{2x}$

The quadratic equation in K^2 can be solved to obtain the peak and/or valley values of K. For special cases such as a band-pass transfer function, for example, $a = c = 0$, simple cases, the expression will be much simpler. Since the design of digital filters based on a bilinear s → z transformation starts with an analog prototype transfer function, (A.4) can be used directly to obtain K in terms of a, b, c, d, and e. In other cases, formulae in Table A.1 will be useful. Note that we have defined

$$x = d^4 - 4e\,d^2, y = 4\,b^2\,e - 8ace - 2\,b^2\,d^2 + 4ac\,d^2 + 4\,c^2 + 4\,a^2\,e^2, z = b^4 - 4\,b^2\,ac$$

in entry 6 in Table A.1.

Note also that from (A.1) directly, the peak or valley value can be found without using bilinear transformation. In this approach, $z = e^{j\omega T}$ is substituted to obtain first the relationship

$$K^2 = \frac{a_o\,\cos^2\omega T - b_o\,\cos\omega T + c_o}{\cos^2\omega T - d_o\,\cos\omega T + e_o} \tag{A.5}$$

where $a_o = \frac{pr}{v}, b_o = \frac{(p+r)q}{2v}, c_o = \frac{(p-r)^2+q^2}{4v}, d_o = \frac{(1+v)u}{2v}, e_o = \frac{(1-v)^2+u^2}{4v}$.

From (A.5), the peak or valley can be found by following a similar approach as described in the case of (A.3), from the equation

$$K^4\,(d_o^2 - 4\,e_o) + K^2\,(4\,c_o + 4\,a_o\,e_o - 2\,b_o\,d_o) + (b_o^2 - 4\,a_o\,c_o) = 0 \tag{A.6}$$

Laakso has pointed out that in both approaches described above, K is real. In the case where K is not real, the value of the transfer function at dc and infinite frequencies needs to be found and the higher among them is the maximum

magnitude. These are from (A.5) (at $\cos(\omega T) = 1$ and $\cos(\omega T) = -1$), respectively,

$$K^2 = \frac{a_o - b_o + c_o}{1 - d_o + e_o} \quad \text{and} \quad K^2 = \frac{a_o + b_o + c_o}{1 + d_o + e_o}$$

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Appendix B

The realized transfer function is affected by the nonideal passive and active components. The variation in the transfer function due to variations in pole-frequency and pole-Q can first be evaluated. These yield topology-independent expressions or curves. The variation in pole-frequency and pole-Q due to the active and passive components in the chosen topology is topology-dependent and can be estimated easily. Substituting the latter in the former expressions gives the total variation in the transfer function. Considering $H(s)$ as the desired biquadratic transfer function

$$H(s) = \frac{1}{D(s)} = \frac{1}{s^2 + s\frac{\omega_p}{Q_p} + \omega_p^2} \tag{B.1}$$

we can obtain

$$S_{\omega_p}^{|H(jw)|} = -\left(\frac{2(1-\gamma^2) + \frac{\gamma^2}{Q_p^2}}{(1-\gamma^2)^2 + \frac{\gamma^2}{Q_p^2}}\right) \tag{B.2}$$

and

$$S_{Q_p}^{|H(jw)|} = \left(\frac{\frac{\gamma^2}{Q_p^2}}{(1-\gamma^2)^2 + \frac{\gamma^2}{Q_p^2}}\right) \tag{B.3}$$

where $\gamma = \frac{\omega}{\omega_p}$. It can be seen from (B.2) and (B.3) that both sensitivities are functions of γ and hence universal curves as functions of γ are of interest. These are presented in Fig. B.1a, b [B.1]. It may be noted that $S_{\omega_p}^{|H(jw)|}$ reaches a maximum of about Q_p at the 3-dB frequencies $\gamma = 1 \pm \frac{1}{2Q_p}$ whereas the maximum value of $S_{Q_p}^{|H(jw)|}$ is unity. The ratio of $S_{\omega_p}^{|H(jw)|}$ to $S_{Q_p}^{|H(jw)|}$ is

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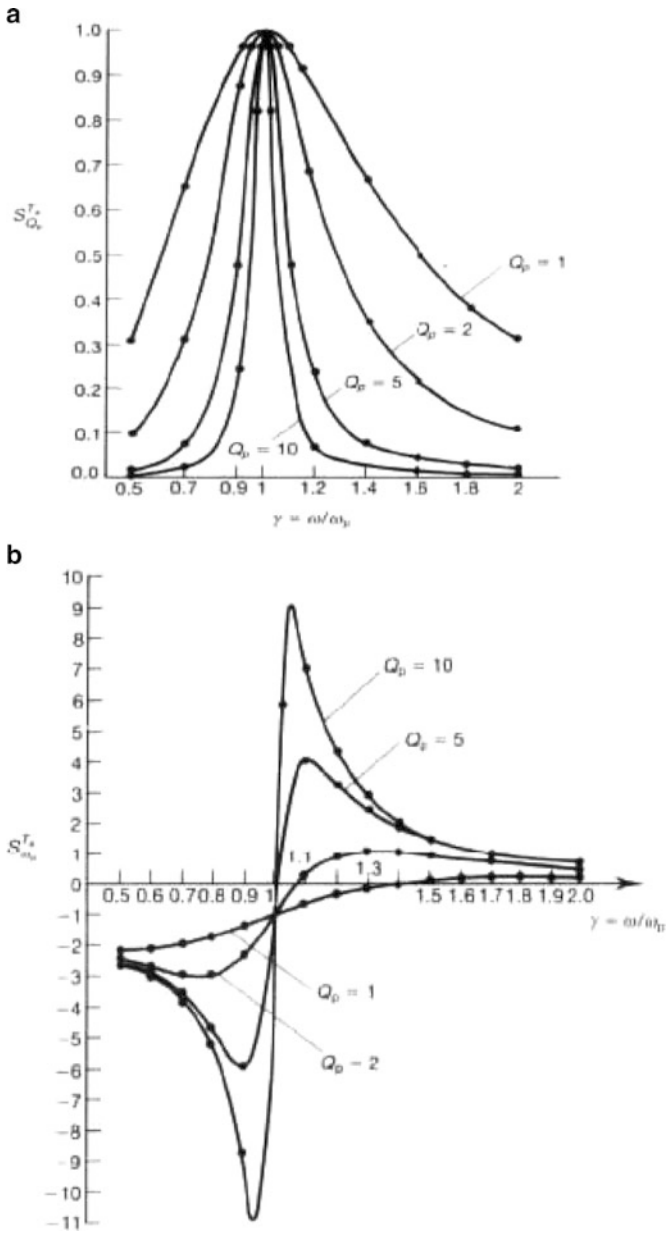

Fig. B.1 Universal *curves* showing sensitivity of transfer function to pole-frequency (**a**) and pole-Q (**b**) with pole-Q as a parameter (Adapted from [B.1] © IEEE 1973)

$$\frac{S_{\omega_p}^{|H(jw)|}}{S_{Q_p}^{|H(jw)|}} = -\left(1 + 2\,Q_p^2\left(\frac{1}{\gamma^2} - 1\right)\right) \tag{B.4}$$

At the 3-dB frequencies, this ratio is $2Q_p$. This is an important result showing that the magnitude of sensitivities to pole-frequency must be kept less than $2Q_p$ times the magnitude of sensitivities to pole-Q.

It can next be observed that the percentage variation in the transfer function can be expressed as

$$\frac{\Delta|H|}{|H|} = S_{\omega_p}^{|H|} \cdot \frac{\Delta\,\omega_p}{\omega_p} + S_{Q_p}^{|H|} \cdot \frac{\Delta\,Q_p}{Q_p} \tag{B.5}$$

We next note that

$$\frac{\Delta\,\omega_p}{\omega_p} = \sum_{i=1}^{k} S_{C_i}^{\omega_p} \cdot \frac{\Delta\,C_i}{C_i} + \sum_{i=1}^{l} S_{R_i}^{\omega_p} \cdot \frac{\Delta\,R_i}{R_i} \tag{B.6}$$

considering that the filter needs k capacitors and l resistors. Note that the active devices also need to be included in the above expressions since we have earlier described the evaluation of pole-Q and pole-frequency sensitivities to opamp finite gain and finite bandwidth.

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