Miscellaneous Exercises

Abstract Landau Inequality

1. Let A be the generator of the C_o-semigroup $T(\cdot)$.

(a) Verify the identity

$$t^{-1}(T(t)x - x) = Ax + t^{-1}\int_0^t (t-s)T(s)A^2x\,ds \quad (x \in D(A^2). \tag{1}$$

(b) If $\|T(\cdot)\| \le M$, show that

$$\|Ax\| \le M(2\|x\|/t + (t/2)\|A^2x\|) \quad (t > 0;\ x \in D(A^2)). \tag{2}$$

(c) If $\|T(\cdot)\| \le M$, prove that

$$\|Ax\| \le 2M(\|A^2x\|\,\|x\|)^{1/2} \quad (x \in D(A^2)). \tag{3}$$

(Hint: minimize the right-hand side of (2) with respect to t.)

(d) Formulate (c) for the translation semigroup on $L^p(\mathbb{R})$, $1 < p < \infty$. (*This special case is known as Landau's inequality.*)

Variation on the Theme of Dissipativity

Notation. Given an operator A with domain $D(A)$, we shall denote by $D(A)_1$ the unit sphere of $D(A)$:

$$D(A)_1 := \{x \in D(A);\ \|x\| = 1\}.$$

2. Suppose the operator A is *weakly dissipative* (w.d.), that is,
For each $x \in D(A)_1$, there exists a unit vector $x^* \in X^*$ such that $x^*x = 1$ and $\Re x^*Ax \le 0$.

(a) If A is w.d., show that it is *bounded below* (b.b.), that is,

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \quad (x \in D(A);\ \lambda > 0). \tag{4}$$

(b) Denote $x_\lambda := (\lambda I - A)x$ for each $x \in D(A)$ and $\lambda > 0$. For each $x \in D(A)_1$ and $\lambda > 0$, there exists a unit vector $x_\lambda^* \in X^*$ such that

$$x_\lambda^* x_\lambda = \|x_\lambda\|. \tag{5}$$

If A is b.b., prove that such a vector x_λ^* satisfies necessarily the following inequalities:

$$\Re x_\lambda^* Ax \leq 0; \quad \Re x_\lambda^* x \geq 1 - \|Ax\|/\lambda. \tag{6}$$

Since the norm-closed unit ball of X^* is *weak**-compact, $\{x_\lambda^*\}$ has a *weak** limit point x^* as $\lambda \to \infty$. Verify that $\|x^*\| = 1$ and $\Re x^* Ax \leq 0$, that is, *A is w.d. iff it is b.b.*

(c) If A is b.b., then for each $\lambda > 0$, $\lambda I - A$ is injective and

$$\|\lambda(\lambda I - A)^{-1}x\| \leq \|x\| \quad (x \in D((\lambda I - A)^{-1})).$$

(d) If A is b.b. and $\lambda_0 I - A$ is surjective for some $\lambda_0 > 0$, prove that $\mathbb{R}^+ \subset \rho(A)$.

(e) Let A be b.b.; prove: (i) A is a closed operator iff $\lambda I - A$ has a closed range for some (hence for all) $\lambda > 0$. (ii) If A is densely defined, then it is *closable*, $\overline{A}$ is b.b., the range of $\lambda I - \overline{A}$ is the closure of the range of $\lambda I - A$ (for all $\lambda > 0$), and the latter range is dense in X for some $\lambda > 0$ iff $\overline{A}$ generates a contraction C_o-semigroup.

Resolvents of the Hille–Yosida Approximations

3. Let A be the generator of the C_o-semigroup $T(\cdot)$ with type ω, and let A_λ be its Hille–Yosida approximation ($\lambda > \omega$). Prove that

$$R(\mu; A) = \lim_{\lambda \to \infty} R(\mu; A_\lambda) \tag{7}$$

in the s.o.t., uniformly in μ on any vertical segment $\{\mu = c + it;\ -\tau \leq t \leq \tau\}$ with $c > \omega$ fixed. (Hint: $T_\lambda(\cdot) \to T(\cdot)$, cf. proof of Theorem 1.17.)

Adjoint Semigroup

4. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Since $D(A)$ is dense, the adjoint A^* is well-defined. Let W be *the closure of $D(A^*)$ in X^**. Prove:

(a) W is $T(t)^*$-invariant for all $t \geq 0$, and $R(\lambda; A^*)$-invariant for all $\lambda > \omega$ (where ω is the type of $T(\cdot)$).
Denote

$$S(t) := T(t)^*|_W; \quad R(\lambda) := R(\lambda; A^*)|_W \quad (t \geq 0; \lambda > \omega). \tag{8}$$

(b) Prove that $R(\cdot) : (\omega, \infty) \to W$ is a pseudo-resolvent with range dense in W, and

$$\|[(\lambda - \omega)R(\lambda)]^n\| \leq M \quad (\lambda > \omega;\ n \in \mathbb{N}). \tag{9}$$

Consequently (cf. Theorems 1.14 and 1.17), $R(\cdot)$ is the resolvent of the generator B of a C_o-semigroup $U(\cdot)$ on W. Prove that $U(\cdot) = S(\cdot)$. (Cf. Theorem 1.36.)
(c) Denote by A^*_W *the part of* A^* *in* W (cf. Definition 1.19). Prove that $B = A^*_W$.
(d) If X is reflexive, show that $W = X^*$, and $T(\cdot)^*$ is a C_o-semigroup whose generator is A^*.

Spectra of a Semigroup and its Generator

5. Let A be the generator of the C_o-semigroup $T(\cdot)$. Consider the *truncated Laplace transform* of $T(\cdot)$, defined by

$$L_r(\lambda)x = \int_0^r e^{-\lambda t} T(t)x\, dt \quad (r > 0; x \in X; \lambda \in \mathbb{C}). \tag{10}$$

Clearly $L_r(\lambda) \in B(X)$.

(a) Prove the identity

$$L_r(\lambda)(\lambda I - A) \subset (\lambda I - A)L_r(\lambda) = I - e^{-\lambda r}T(r). \tag{11}$$

(b) If $e^{\lambda r} \in \rho(T(r))$, then $\lambda \in \rho(A)$, and

$$R(\lambda; A) = e^{\lambda r} L_r(\lambda) R(e^{\lambda r}; T(r)). \tag{12}$$

In particular,

$$e^{r\sigma(A)} \subset \sigma(T(r)) \quad (r \geq 0). \tag{13}$$

(c) Prove that

$$e^{r\sigma_p(A)} \subset \sigma_p(T(r)) \quad (r \geq 0). \tag{14}$$

(Hint: apply Part (a).)
(d) Suppose $T(\cdot)$ is differentiable in the s.o.t. in (b, ∞) for some $b \geq 0$. Prove that $AT(t) \in B(X)$ and commutes with $T(s)$ for all $t > b$ and $s \geq 0$.

(e) Let $T(\cdot)$ be as in Part (d), and fix $r > b$. Prove that if $\lambda e^{r\lambda} \in \rho(AT(r))$, then $\lambda \in \rho(A)$ and

$$R(\lambda; A) = [T(r) + \lambda e^{r\lambda} L_r(\lambda)]\, R(\lambda e^{r\lambda}; AT(r)), \tag{15}$$

and the factors in the latter product may be interchanged.

(f) ($T(\cdot)$ as in Part (d).) Fix $r > b$, $\delta > 0$, and a, M such that $\|T(t)\| \le M\, e^{at}$ for all $t \ge 0$ (cf. Theorem 1.1). Denote $c = (1+\delta)\|AT(r)\|$ and

$$\Omega_{c,r} = \{\lambda \in \mathbb{C}; c\, e^{-r\Re\lambda} \le |\Im\lambda|\}. \tag{16}$$

Prove that $\Omega_{c,r} \subset \rho(A)$ and

$$\sup_{\lambda\in\Omega_{c,r};\Re\lambda\le a} \|(\Im\lambda)^{-1} R(\lambda; A)\| < \infty. \tag{17}$$

(*Conversely,* if there exist positive constants c, r with the latter property, then $T(\cdot)$ is differentiable in the s.o.t. in some ray (b, ∞). See [P].)

Compact Semigroups

6. Let $T(\cdot)$ be a C_o-semigroup such that $T(t)$ is a compact operator for each $t > t_0$. Fix $t > t_0$ and $\epsilon > 0$. Denote $M := \sup_{0\le s\le 1} \|T(s)\|$. By compactness of the set $\{T(t)x; \|x\| \le 1\}$, choose a finite open cover of it by open balls $B(T(t)x_j, \epsilon/(2(M+1)))$, $j = 1, \ldots, n$. By continuity of $T(\cdot)x_j$ at t, choose $0 < \delta < 1$ such that

$$\|T(t+h)x_j - T(t)x_j\| < \epsilon/2 \quad (0 < h < \delta;\ j = 0, \ldots, n). \tag{18}$$

(a) Prove that

$$\|T(t+h) - T(t)\| < \epsilon \quad (0 < h < \delta) \tag{19}$$

and conclude that $T(\cdot)$ is continuous in the u.o.t. for $t > t_0$ (for $h < 0$, use the argument in the proof of Theorem 1.1).

(b) If $t_0 = 0$, $R(\lambda; A)$ is compact for all $\lambda \in \rho(A)$. (Hint: for $\Re\lambda > \omega$, apply Theorem 1.15; for arbitrary $\lambda \in \rho(A)$, use then the resolvent identity.)

(c) If $T(\cdot)$ is continuous in the u.o.t. on $(0, \infty)$, show that

$$T(t) = \lim_{\lambda\to\infty} \lambda R(\lambda; A)\, T(t) \quad (t > 0) \tag{20}$$

in the $B(X)$-norm.

(d) Conclude: $T(\cdot)$ *is compact on* $(0, \infty)$ *if and only if it is continuous in the u.o.t. on* $(0, \infty)$ *and* $R(\lambda; A)$ *is compact for all* $\lambda \in \rho(A)$.

(e) If $t_0 = 0$ and $B \in B(X)$, then $A + B$ generates a C_o-semigroup $S(\cdot)$ which is compact on $(0, \infty)$. (Cf. Lemmas 2 and 5 in the proof of Theorem 1.38.)

Powers of the Generator

7. Let $-A$ be the generator of a C_o-semigroup $T(\cdot)$ which satisfies the growth condition

$$\|T(t)\| \le M\, e^{-at} \quad (t \ge 0) \tag{21}$$

for some $a > 0$ (note the minus sign!). Let $\alpha \ge 0$, and define the operator $A^{-\alpha}$ by $A^0 := I$ and

$$A^{-\alpha}x := \Gamma(\alpha)^{-1} \int_0^\infty t^{\alpha-1} T(t)x\, dt \quad (x \in X;\, \alpha > 0). \tag{22}$$

(a) Prove that the above integral converges uniformly on the closed unit ball of $B(X)$ (therefore $A^{-\alpha} \in B(X)$).

(b) Prove that

$$A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta} \quad (\alpha, \beta \ge 0). \tag{23}$$

(Hint: use the identity

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du.) \tag{24}$$

In particular, for all $n \in \mathbb{N}$, $A^{-n} = (A^{-1})^n$, the usual n-th power of $A^{-1} \in B(X)$ (cf. Theorem 1.15).

(c) For $0 < \alpha < 1$, prove the identity

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha} R(\lambda; -A)\, d\lambda, \tag{25}$$

where the integral converges in $B(X)$. Hint: use the identity

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}.$$

In particular,

$$\|A^{-\alpha}\| \le M(1 + a^{-1}) \quad (\alpha \in (0,1)). \tag{26}$$

(d) Prove that the function $t \in [0,\infty) \to A^{-t}$ is a C_o-semigroup.

(e) Prove that $A^{-\alpha}$ is injective for all $\alpha \ge 0$.

Using (e), we *define*

$$A^{\alpha} := (A^{-\alpha})^{-1}. \tag{27}$$

Prove that the (closed) operators A^α (with domain equal to the range of $A^{-\alpha}$) have the following properties:

(1) If $0 < \alpha \le \beta$, then $D(A^\alpha) \subset D(A^\beta)$.

(2) For each $\alpha \ge 0$, $D(A^\alpha)$ is dense in X.

(3) If $\alpha, \beta \in \mathbb{R}$, then

$$A^{\alpha+\beta}x = A^\alpha A^\beta x \tag{28}$$

for all $x \in D(A^\gamma)$, where $\gamma = \max(\alpha, \beta, \alpha+\beta)$.

C^∞-semigroups

8. Let A be the generator of a C_o-semigroup $T(\cdot)$ such that $T(t)X \subset D(A)$ for all $t > 0$.

(a) Prove the identity

$$T(t+h)x - T(t)x = \int_0^h T(s)AT(t)x\,ds \quad (x \in X;\, t,h > 0). \tag{29}$$

(b) Prove that $T(\cdot)$ is C^∞ on $(0,\infty)$ in the u.o.t. Hint: use (a) and the fact that for all $t > 0$, $AT(t) \in B(X)$ (why?). Conclude that $T'(t)$ exists (in the u.o.t.) and is equal to $AT(t) = T(t-a)AT(a)$ for all $t > a > 0$, etc.

Entire Vectors

9. Let $T(\cdot)$ be a C_o-group. Given $x \in X$, define

$$x_n := (n/2\pi)^{1/2} \int_{\mathbb{R}} e^{-nt^2/2} T(t)x\,dt \quad (n \in \mathbb{N}); \tag{30}$$

$$x_n(z) := (n/2\pi)^{1/2} \int_{\mathbb{R}} e^{-n(t-z)^2/2} T(t)x\,dt \quad (n \in \mathbb{N};\, x \in \mathbb{C}).$$

Prove that the above integrals converge (strongly) in X, $x_n(\cdot)$ is entire (for each $n \in \mathbb{N}$), $x_n \to x$, and $x_n(s) = T(s)x_n$ for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Conclude that x_n is an *entire vector* for $T(\cdot)$ (that is, $T(\cdot)x_n$ extends to an entire function), and that consequently *the entire vectors for* $T(\cdot)$ *are dense in* X.

Nonhomogeneous ACP

10. Let $T(\cdot)$ be a C_o-semigroup, A its generator, and let M, a be constants such that $\|T(t)\| \le M\,e^{at}$ for all $t \ge 0$ (cf. Theorem 1.1). Given a strongly C^1 function $f : [0,\infty) \to X$ and $x \in X$, define

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds \quad (t \ge 0). \tag{31}$$

(a) Prove that u is C^1 on $[0,\infty)$ with values in $D(A)$, and solves the (nonhomogeneous) abstract Cauchy problem

$$(ACP): \quad \frac{du}{dt} = Au + f \quad (t > 0); \quad u(0) = x. \tag{32}$$

(b) *Conversely*, if u is a solution of (ACP) with f strongly continuous, then

$$\frac{d}{ds}T(t-s)u(s) = T(t-s)f(s) \quad (0 \le s \le t), \tag{33}$$

so that u has necessarily the form (31).

In particular, if f is strongly C^1, (ACP) has the unique solution (31). (This is *Duhamel's formula.*)

The Graph Norm on $D(A)$

11. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Let $[D(A)]$ denote the normed space $D(A)$ normed by the *graph norm* of A

$$\|x\| := \|x\| + \|Ax\| \quad (x \in D(A)). \tag{34}$$

Prove:

(a) $[D(A)]$ is a Banach space.
(b) $S(\cdot) := T(\cdot)|_{D(A)}$ is a C_o-semigroup on $[D(A)]$.
(c) If B is the generator of $S(\cdot)$ (acting in the Banach space $[D(A)]$), then $D(B) = D(A^2)$ and $B = A|_{D(A^2)}$.

Commutativity

12. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X, and let $B \in B(X)$. Prove that the following statements are equivalent:

(a) $[B, T(\cdot)] = 0$ ($[P, Q]$ denotes here the Lie product $PQ - QP$ of not necessarily bounded operators).
(b) $[B, R(\lambda; A)] = 0$ for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$.
(c) $[B, R(\lambda; A)] = 0$ for all $\lambda > \omega$.
(d) $B\,D(A) \subset D(A)$ and $[B, A] = 0$ on $D(A)$.
(e) There exists a core $\mathcal{D}$ for A such that $B\,\mathcal{D} \subset D(A)$ and $[B, A] = 0$ on $\mathcal{D}$.

Square of the Generator

13. Let $T(\cdot)$ be a C_o-group of isometries on the Banach space X, and let A be its generator. Define $S(\cdot) : [0, \infty) \to B(X)$ by $S(0) = I$ and

$$S(t) = (2\pi t)^{-1/2} \int_{\mathbb{R}} e^{-s^2/(2t)} T(s)x\, ds \quad (t > 0). \tag{35}$$

Prove:

(a) $S(\cdot)$ is a C_o-semigroup of contractions. Denote its generator by B.
(b) For all $\lambda > 0$,

$$R(\lambda; B) = R(\lambda; (1/2)A^2). \tag{36}$$

Hint: Theorem 1.15 and the identities

$$(2\pi)^{-1/2} \int_0^\infty t^{-1/2} e^{-s^2/(2t)} e^{-\lambda t}\, dt = (2\lambda)^{-1/2} e^{-|s|(2\lambda)^{1/2}} \tag{37}$$

(for all $s \in \mathbb{R}$) and

$$R(\sqrt{2\lambda}; A) + R(\sqrt{2\lambda}; -A) = \sqrt{2\lambda} R(\lambda; (1/2)A^2) \tag{38}$$

(for all $\lambda > 0$). Cf. Theorem 1.11.
(c) Conclude that $B = (1/2)A^2$.
(d) $S(\cdot)$ has an analytic extension in $\mathbb{C}^+$.

Resolvents of Bounded Analytic Semigroups

14. Let A be the generator of the C_o-semigroup $T(\cdot)$, which is analytic in the sector S_θ (cf. Definition 1.53), and bounded in each subsector $S_{\theta-\epsilon}$ $(0 < \epsilon \leq \theta)$. Fix such an ϵ, and denote by M_ϵ the supremum of $\|T(\cdot)\|$ over $S_{\theta-\epsilon}$. As in the proof of Theorem 1.54, consider the C_o-semigroups $T_\alpha(\cdot)$ (with generator $A_\alpha := e^{i\alpha} A$) for each real α with $|\alpha| \leq \theta - \epsilon$. Prove:

(a) $\sigma(A_\alpha) \subset \{\lambda \in \mathbb{C}; \Re\lambda \leq 0\}$ and $\|R(\lambda; A_\alpha)\| \leq M_\epsilon / \Re\lambda$ for $\Re\lambda > 0$.
(b) $\sigma(A) \subset \{\mu \in \mathbb{C}; |\arg\mu| \geq \theta + \pi/2\}$ and $\|\mu\, R(\mu; A)\| \leq M_\epsilon$ for all $\mu \in S_{\theta-\epsilon}$.
(c) $\|\mu\, R(\mu; A)\|$ is bounded in each sector $S_{(\theta+\pi/2)-\epsilon}$.

A-boundedness

15. Let A be the generator of the contraction C_o-semigroup $T(\cdot)$ on the Banach space X, and let $\mathcal{D}$ be a core for A. Let $B : \mathcal{D} \to X$ be linear and satisfy the inequality

$$\|Bx\| \leq a\,\|Ax\| + b\,\|x\| \quad (x \in \mathcal{D}) \tag{39}$$

for some constants $a, b \geq 0$. Prove that B extends uniquely as an A-bounded operator with domain $D(A)$ (cf. Definition 1.28). Hint: show that $\mathcal{D}_1 := (I - A)\mathcal{D}$ is dense in X and extend $C := B\,R(1; A)$, originally defined and *bounded* on $\mathcal{D}_1$, (uniquely) to an element of $B(X)$. The wanted extension is $C(I - A)$.

16. Let A, B be (usually unbounded) operators on the Banach space X such that $D(A) \subset D(B)$. Suppose that the (bounded) operator $B\,R(\lambda; A)$ exists and is *compact* for some λ. Prove:

(a) $B\,R(\mu; A)$ is compact for all $\mu \in \rho(A)$.
(b) B is A-bounded.
(c) If X has the *approximation property* (that is, every compact operator on X is the $B(X)$-limit of finite rank operators, e.g., if X is a Hilbert space, cf. [RN, p. 204]), then the A-bound of B is zero (cf. Definition 1.28). Hint: given $\epsilon > 0$, there exist $x_k \in X$ and $x_k^* \in X^*$ ($k = 1, \ldots, n$) such that

$$\left\| B\,R(\lambda; A)x - \sum_k (x_k^* x)x_k \right\| < \epsilon \, \|x\| \quad (x \in X). \tag{40}$$

By density of $R(\lambda; A)^* X^*$ in X^*, we may take $x_k^* = R(\lambda; A)^* y_k^*$ for appropriate $y_k^* \in X^*$, and conclude that

$$\left\| By - \sum_k (y_k^* y)x_k \right\| < \epsilon \, \|(\lambda I - A)y\| \quad (y \in D(A). \tag{41}$$

Unitary Vectors

17. Let $T(\cdot)$ be a contraction C_o-semigroup on the *Hilbert* space X. A *unitary vector* for $T(\cdot)$ is a vector x such that $\|T(t)x\| = \|x\|$ for all $t \geq 0$. Denote by Y the set of all unitary vectors for $T(\cdot)$, and let

$$Z := \bigcap_{t \geq 0} T(t)Y. \tag{42}$$

Prove:

(a) Y is a closed invariant subspace for $T(\cdot)$.
(b) For each $s \geq 0$, $T(s)$ is an isometry of Z onto Z (i.e., $T(\cdot)|_Z$ is a unitary semigroup).
(c) If W is a closed $T(\cdot)$-invariant subspace of $Z^\perp$ such that $T(\cdot)|_W$ is unitary, then $W \subset Z$ and $T(t)W^\perp \subset W^\perp$ (in particular, Z is a reducing subspace for $T(t)$ for all $t \geq 0$, and the only $T(\cdot)$-invariant subspaces of $Z^\perp$ on which $T(\cdot)$ is a unitary semigroup are the trivial ones; one says in this case that $T(\cdot)$ is *completely nonunitary* on $Z^\perp$).

Markov Semigroups

18. Let K be a compact Hausdorff space, and let $C(K)$ be the Banach space of all complex continuous functions on K with the maximum norm. Let $C^+(K)$

be the positive cone in $C(K)$ ($:= \{f \in C(K); f \geq 0\}$). A Markov semigroup is a C_o-semigroup $T(\cdot)$ on $C(K)$ such that $T(\cdot)1 = 1$ and $T(\cdot)f \geq 0$ for all $f \in C^+(K)$. Let A be the generator of the Markov semigroup $T(\cdot)$. Prove:

(a) $\|T(\cdot)\| = 1$.
(b) If $f \in D(A)$, then $\overline{f} \in D(A)$ and $A\overline{f} = \overline{Af}$.
(c) $1 \in D(A)$ and $A1 = 0$.
(d) For all $\lambda > 0$, $\lambda R(\lambda; A)f \geq 0$ for all $f \in C^+(K)$, $\lambda R(\lambda; A)1 = 1$, and $\|\lambda R(\lambda; A)\| = 1$.

Translation Semigroup

19. Denote by $C_l([0,\infty))$ the Banach space of all complex continuous functions f on $[0,\infty)$ for which the limit $f(\infty) := \lim_{s\to\infty} f(s)$ exists in $\mathbb{C}$, with the supremum norm $\|f\| = \sup_{s\geq 0} |f(s)|$. If X is either $C_l([0,\infty))$ or $L^p(\mathbb{R}^+)$ ($1 \leq p < \infty$), consider the translation operators

$$T(t) : f(s) \to f(t+s) \quad (f \in X;\ t \geq 0).$$

Prove:

(a) $T(\cdot)$ is a C_o-semigroup on X with $\|T(\cdot)\| = 1$, and its generator A is the differentiation operator $f \to f'$ with "maximal domain"

$$D(A) = \{f \in X; \exists f',\ f' \in X\}. \tag{43}$$

(In the L^p case, $\exists f'$ means that the derivative f' exists almost everywhere, etc.)

(b) $\sigma(A) = \{\lambda \in \mathbb{C};\ \Re\lambda \leq 0\}$ and $\lambda R(\lambda; A)$ are contractions for all $\lambda > 0$.
(c) Consider the translation operators $T(t)$ ($t \in \mathbb{R}$) on X, where X is either $L^p(\mathbb{R})$ ($1 \leq p < \infty$) or the space $C_l(\mathbb{R})$ of all complex continuous functions f on $\mathbb{R}$ for which both $f(\infty)$ and $f(-\infty)$ exist in $\mathbb{C}$ (with the supremum norm $\|f\| = \sup_{\mathbb{R}} |f|$). Prove that $T(\cdot)$ is a C_o-group of isometries, whose generator A is the differentiation operator with maximal domain in X. Show also that $\sigma(A) = i\mathbb{R}$, and that $\mathcal{S}(\mathbb{R})$ is a core for A.
(d) Let B be any (usually unbounded) operator on a Banach space X with nonempty resolvent set $\rho(B)$. Prove that if λ and $-\lambda$ are both in $\rho(B)$, then $\lambda^2 \in \rho(B^2)$ and

$$R(\lambda^2; B^2) = -R(\lambda; B)\, R(-\lambda; B). \tag{44}$$

(e) If A is the generator of the translation group on $C_l(\mathbb{R})$, then A^2 generates a contraction C_o-semigroup $S(\cdot)$.
(f) Prove that the semigroup $S(\cdot)$ in Part (e) is the Gauss–Weierstrass semigroup (cf. Example 1.114).
(g) For A as in Part (e), prove that A^3 is *not* the generator of a C_o-semigroup.

The MacLaurin Formula for Semigroups

20. Let X be a Banach space, and consider the classical Volterra operator

$$V : u(t) \to \int_0^t u(s)\, ds \quad (t \geq 0), \tag{45}$$

defined on continuous X-valued functions u on $[0, \infty)$.

Let A generate the C_o-semigroup $T(\cdot)$ on X. Prove:

(a) For all $n \in \mathbb{N}$, $t > 0$, and $x \in D(A^n)$,

$$T(t)x = \sum_{k=0}^{n-1} (t^k/k!)A^k x + (V^n[T(\cdot)A^n x])(t). \tag{46}$$

(b) For all $n \in \mathbb{N}$ and $x \in D(A^n)$,

$$\lim_{t \to 0+} \left[T(t)x - \sum_{k=0}^{n-1} (t^k/k!)A^k x \right] = A^n x/n. \tag{47}$$

Restriction of Semigroup to Invariant Subspaces

21. Let A be the generator of the C_o-semigroup $T(\cdot)$ on the Banach space X. Suppose Y is a $T(\cdot)$-invariant linear manifold in X, which is a Banach space for a norm $\|\cdot\|_Y$, such that $(Y, \|\cdot\|_Y)$ is continuously embedded in X. Assume that the semigroup $T_Y(\cdot)$ defined by

$$T_Y(t) := T(t)\Big|_Y \quad (t \geq 0) \tag{48}$$

is of class C_o on the Banach space Y (with the norm $\|\cdot\|_Y$!). Prove that the generator of $T_Y(\cdot)$ is equal to A_Y, *the part of A in Y* (cf. Definition 1.19).

22. (Cf. Exercise 11.) Let A be the generator of the C_o-semigroup $T(\cdot)$, and suppose $0 \in \rho(A)$. For each $n \in \mathbb{N}$, consider the $T(\cdot)$-invariant linear manifold $D(A^n)$ with the norm

$$\|x\|_n := \|A^n x\| \quad (x \in D(A^n)). \tag{49}$$

Prove:

(a) $X_n := (D(A_n), \|\cdot\|_n)$ is a Banach space continuously embedded in X. Denote $T_n(\cdot) := T_{X_n}(\cdot)$ (cf. Exercise 21).

(b) $T_n(\cdot)$ is of class C_o (on X_n). (By Exercise 21, its generator A_n is equal to A_{X_n}, *the part of A in X_n*.

(c) (Write $X_0 := X$, $T_0(\cdot) := T(\cdot)$, and $A_0 := A$.) For all $n = 0, 1, 2, \ldots$, A_n is an isometry of X_{n+1} onto X_n, and

$$A_n T_{n+1}(\cdot) = T_n(\cdot) A_n. \tag{50}$$

(d) If X_{-n} are defined inductively for $n = 0, 1, 2, \ldots$ such that $X_{-(n+1)}$ is the completion of X_{-n} for the norm

$$\|x\|_{-(n+1)} := \|A_{-n}^{-1} x\|, \tag{51}$$

and $T_{-(n+1)}(t)$ is the continuous extension of $T_{-n}(t)$ to $X_{-(n+1)}$, then Parts (b) and (c) are valid also for all $n \in -\mathbb{N}$.

(e) $D(A_n) = X_{n+1}$ for all $n \in \mathbb{Z}$.

(f) If $m \geq n \in \mathbb{Z}$, then A_n is the (unique) continuous extension of the isometry A_m of X_{m+1} onto X_m to an isometry of X_{n+1} onto X_n.

Semigroups Arising from ACP

23. (Cf. Theorem 1.2.)

Let A be a closed operator on the Banach space X. Consider the associated Abstract Cauchy Problem (ACP) on $[0, \infty)$

$$\frac{du}{dt} = Au; \quad u(0) = x. \tag{ACP}$$

Assume that (ACP) has a unique C^1-solution (in the s.o.t.)

$$u : [0, \infty) \to D(A) \tag{52}$$

for each $x \in D(A)$.

Define $T(\cdot)$ on the Banach space $[D(A)]$ (the Banach space $D(A)$ with the graph norm $\|\cdot\|_A$ of A) by letting

$$T(\cdot)x := u \quad (x \in D(A)), \tag{53}$$

where u is the unique C^1-solution of (ACP). Prove:

(a) For each $t \geq 0$, $T(t)$ is a linear everywhere defined operator on $[D(A)]$, and satisfies the semigroup relations

$$T(s+t) = T(s)T(t) \quad (s.t \geq 0); \quad T(0) = I. \tag{54}$$

(I denotes here the identity operator in $[D(A)]$.)

(b) For each $x \in [D(A)]$, $T(\cdot)x$ is $[D(A)]$-continuous.

(c) The linear operator

$$W : x \in [D(A)] \to T(\cdot)x \in C([0,r],[D(A)]) \tag{55}$$

is a *closed* operator. (The space $C(\cdots)$ denotes the Banach space of all $[D(A)]$-valued $[D(A)]$-continuous functions on $[0,r]$, $r > 0$ arbitrary, with the obvious norm.) Hint: use the integral equation equivalent to (ACP). Conclude that $T(t) \in B([D(A)])$ for all $t \geq 0$ (and consequently $T(\cdot)$ is a C_o-semigroup on $[D(A)]$).

(d) Prove that the generator of the C_o-semigroup $T(\cdot)$ (on $[D(A)]$) is *the part of* A *in* $D(A)$. (Cf. Exercise 11.)

Bounded Below Semigroups

24. A C_o-semigroup $T(\cdot)$ is *bounded below* if $\inf_{t>0} \|T(t)\| > 0$. Prove that $T(\cdot)$ is bounded below iff $\|T(t)\| \geq 1$ for all $t > 0$.

Natural Operational Calculus for Groups

25. Let $T(\cdot)$ be a C_o-group on the Banach space X. Denote by $\mathcal{M}_T$ the space of all complex Borel measures μ on $\mathbb{R}$ such that

$$\|\mu\|_T := \int_{\mathbb{R}} \|T(\cdot)\| \, d|\mu| < \infty. \tag{56}$$

(a) Prove that $\mathcal{M}_T$ is a Banach algebra (with convolution of measures as multiplication) with respect to the norm $\|\cdot\|_T$, and is a Banach subspace of $C_0(\mathbb{R})^*$ if $T(\cdot)$ is bounded below on $(0,\infty)$ (cf. Exercise 24).

(b) Denote the Fourier–Stieltjes transform of the Borel measure μ by $\hat{\mu}$:

$$\hat{\mu}(s) := \int_{\mathbb{R}} e^{ist} d\mu(t) \quad (s \in \mathbb{R}). \tag{57}$$

Let

$$\mathcal{A}_T := \{\hat{\mu};\ \mu \in \mathcal{M}_T\}. \tag{58}$$

Prove that $\mathcal{A}_T$ is a Banach algebra for the pointwise operations between functions, with respect to the norm

$$\|\hat{\mu}\|_T := \|\mu\|_T \quad (\mu \in \mathcal{M}_T). \tag{59}$$

(c) For $f \in \mathcal{A}_T$ (say $f = \hat{\mu}$ for the unique $\mu \in \mathcal{M}_T$), define the operator $\tau(f)$ on X by

$$\tau(f)x := \int_{\mathbb{R}} T(t)x \, d\mu(t) \quad (x \in X). \tag{60}$$

Prove that τ is a continuous homomorphism of the Banach algebra $\mathcal{A}_T$ into $B(X)$ with norm 1, such that

$$\tau(f_t) = T(t) \quad (t \in \mathbb{R}), \tag{61}$$

where $f_t(s) = e^{ist}$, $s, t \in \mathbb{R}$.

(d) If $S(\cdot)$ and $T(\cdot)$ are C_o-groups on X, then $\mathcal{A}_S \subset \mathcal{A}_T$ if and only if $\frac{\|T(\cdot)\|}{\|S(\cdot)\|}$ is bounded on $\mathbb{R}$.

(e) The C_o-group $T(\cdot)$ is *temperate* if $\|T(t)\| = O(|t|^k)$ for some non-negative integer k. In that case, prove that the Schwartz space $\mathcal{S}(\mathbb{R})$ is topologically contained in the Banach algebra $\mathcal{A}_T$, and for all $x \in X$,

$$\tau(f)x = \int_{\mathbb{R}} T(t)x\, \hat{f}(t)\, dt \quad (f \in \mathcal{S}(\mathbb{R})), \tag{62}$$

where $\hat{f}$ is the Fourier transform of f,

$$\hat{f}(t) := (2\pi)^{-1} \int_{\mathbb{R}} e^{-ist} f(s)\, ds \quad (t \in \mathbb{R}).$$

Construction of Analytic Semigroups

26. Let A be a closed densely defined operator on the Banach space X. Suppose there exists $0 < \delta \le \pi/2$ such that the open sector $S_{\delta+\pi/2}$ is contained in $\rho(A)$ and $\lambda R(\lambda; A)$ is bounded in each subsector $S_{\delta-\epsilon+\pi/2}$ with $0 < \epsilon < \delta$.

Fix $\delta' \in (0, \delta)$ and $z \in S_{\delta'}$. Denote $b = 1/|z|$, choose $\epsilon = (\delta - \delta')/2$, set $\alpha = \delta - \epsilon + \pi/2$, and let Γ be the "positively oriented" path consisting of the circular arc

$$\{w = b\, e^{i\theta};\ |\theta| \le \alpha\},$$

complemented by the two rays

$$\{w = r\, e^{\pm i\alpha};\ r \ge b\}.$$

Define

$$T(z) = (1/2\pi i) \int_{\Gamma} e^{zw} R(w; A)\, dw. \tag{63}$$

Prove:

(a) The integral converges in operator norm, and there exists a constant $C_{\delta'}$ such that $\|T(z)\| \le C_{\delta'}$ for all $z \in S_{\delta'}$.

(b) $T(\cdot)$ is analytic in S_δ.

(c) $T(z + u) = T(z)T(u)$ for all $z, u \in S_\delta$. Hint: by Cauchy's integral theorem, the integration path in the definition of $T(u)$ can be translated to the right. Use the resolvent identity and replace the paths by closed paths, using circular arcs as above with radii tending to infinity.

(d) $\lim_{z\in S_{\delta'};\, z\to 0} T(z)x = x$ for all $x \in X$ and $\delta' \in (0,\delta)$. (In particular, the restriction of $T(\cdot)$ to $\mathbb{R}^+$, complemented with the definition $T(0) = I$, is a C_o-semigroup, which admits an analytic extension in S_δ that is bounded in each proper subsector. Denote its generator by B.)

(e) $B = A$. Hint: prove that $R(\lambda; B) = R(\lambda; A)$ for some $\lambda > 0$; cf. Theorem 1.15. (*This construction shows that an operator A satisfying the hypothesis of the present exercise is the generator of an analytic semigroup, whose analytic extension in a suitable sector is bounded in every proper subsector.* The converse is also true.)

Approximation of C_o-semigroups by Uniformly Continuous Semigroups

27. Let $T(\cdot)$ be a C_o-semigroup, and let A be its generator. Suppose B is a *bounded* (everywhere defined) operator *commuting* with $T(\cdot)$. Prove:

(a) For all $x \in D(A)$ and $t \geq 0$, the following identity holds:

$$T(t)x - e^{tB}x = \int_0^t e^{(t-s)B}T(s)(A-B)x\,ds. \tag{64}$$

Hint: integrate with respect to s over $[0,t]$ the expression for

$$\frac{d}{ds}[e^{(t-s)B}T(s)x]$$

when $x \in D(A)$.

(b) If the semigroups $T(\cdot)$ and e^{tB} are bounded (say, by the constants H, K, respectively), then

$$\|T(t)x - e^{tB}x\| \leq HKt\,\|(A-B)x\| \quad (x \in D(A);\ t \geq 0). \tag{65}$$

(c) Suppose $T(\cdot)$ is bounded, B_n, $n \in \mathbb{N}$ are *bounded* operators *commuting* with $T(\cdot)$, and $\|e^{tB_n}\| \leq K$ for all $t \geq 0$ and $n \in \mathbb{N}$ (with a constant K independent of n). Prove that if $B_n x \to Ax$ for all x in a core for A, then

$$T(t) = \lim_n e^{tB_n} \quad (t \geq 0) \tag{66}$$

in the s.o.t., uniformly for t in bounded intervals.

(d) If $T(\cdot)$ is a contraction C_o-semigroup, and

$$B_n := n[T(1/n) - I] \quad (n \in \mathbb{N}), \tag{67}$$

then (66) is valid.

(e) If $T(\cdot)$ is a bounded C_o-semigroup, and B_n are chosen as the Hille–Yosida approximations of A

$$B_n := n[n\,R(n;A) - I] \quad (n \in \mathbb{N}), \tag{68}$$

then (66) is valid. (Cf. Lemma 1.16 and the beginning of the proof of Theorem 1.17.)

Stability in the u.o.t

28. Let ω be the type of the C_o-semigroup $T(\cdot)$. Prove that *the following statements are equivalent*:

(a) $\omega < 0$ (and consequently there exist constants $a > 0$ and $M \geq 1$ such that $\|T(t)\| \leq M\, e^{-at}$ for all $t \geq 0$, that is, $\|T(t)\|$ *decays exponentially to zero* as $t \to \infty$).
(b) $\lim_{t\to\infty} \|T(t)\| = 0$.
(c) $\|T(c)\| < 1$ for some $c > 0$.
(d) $r(T(c)) < 1$ for some $c > 0$.

(Hint: Theorem 1.4.)

29. *Fix* $p \in [1, \infty)$. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X such that

$$\|T(\cdot)x\| \in L^p(\mathbb{R}^+) \tag{69}$$

for all $x \in X$. (The norm on $L^p(\mathbb{R}^+)$ will be denoted by $\|\cdot\|_p$.) Fix $a > 0$ and $M \geq 1$ such that $\|T(t)\| \leq M\, e^{at}$ for all $t \geq 0$ (cf. Theorem 1.1). Define the map

$$V : X \to L^p(\mathbb{R}^+)$$

by

$$V(x) = \|T(\cdot)x\| \quad (x \in X). \tag{70}$$

Prove:

(a) There exists a constant $K > 0$ such that $\|V(x)\|_p \leq K\, \|x\|$ for all $x \in X$.
(b) For all $t > 0$,

$$\|T(t)\| \leq (ap)^{1/p} MK. \tag{71}$$

(Hint: integrate with respect to s over $[0, t]$ the trivial identity

$$e^{-aps}\|T(t)x\|^p = e^{-aps}\|T(s)T(t-s)x\|^p$$

and estimate the integral on the right-hand side.)
(c) For all $t > 0$,

$$\|T(t)\| \leq (ap)^{1/p} M\, K^2 t^{-1/p}. \tag{72}$$

Hint: integrate (with respect to s over $[0, t]$) the trivial identity

$$\|T(t)x\|^p = \|T(t-s)T(s)x\|^p,$$

and estimate the integral on the right-hand side.)
(d) Conclude that the equivalent relations in Exercise 28 are also equivalent to the property (69) for some (*or all*) $p \in [1, \infty)$.

Semigroups on Hilbert Space

30. (This exercise is a preliminary for the next one.)

Let X be a (complex) Hilbert space. Denote by $C_c^\infty(\mathbb{R}, X)$, $\mathcal{S}(\mathbb{R}, X)$, $L^2(\mathbb{R}, X)$, etc. the "usual" spaces C_c^∞, $\mathcal{S}$, L^2, etc. (respectively) of X-valued functions u over $\mathbb{R}$, with definitions adequately modified so that the norm $\|u(\cdot)\|$ replaces the usual absolute values. Fix an orthonormal basis $\{x_j;\, j \in J\}$ for X. Prove:

(a) $u \in L^2(\mathbb{R}; X)$ if and only if $(u(\cdot), x_j) \in L^2(\mathbb{R})$ for all $j \in J$. When this is the case, one has the identity

$$\|u\|^2_{L^2(\mathbb{R},X)} = \sum_{j\in J} \|(u(\cdot), x_j)\|^2_{L^2(\mathbb{R})}. \tag{73}$$

(b) If $u \in L^2(\mathbb{R}, X)$ and $\phi \in L^1(\mathbb{R})$, then the convolution $u * \phi$ is a "well-defined" element of $L^2(\mathbb{R}, X)$, and

$$\|u * \phi\|_{L^2(\mathbb{R},X)} \le \|u\|_{L^2(\mathbb{R},X)} \|\phi\|_{L^1(\mathbb{R})}. \tag{74}$$

(c) $C_c^\infty(\mathbb{R}, X)$ (and therefore $\mathcal{S}(\mathbb{R}, X)$) is dense in $L^2(\mathbb{R}, X)$. (Cf. proof of Theorem II.1.2 in [K17].)

(d) Define the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbb{R}, X)$ by

$$(\mathcal{F}u)(s) = (1/\sqrt{2\pi}) \int_{\mathbb{R}} e^{-ist} u(t)\, dt. \tag{75}$$

(In the special case $X = \mathbb{C}$, it is well-known that $\mathcal{F}$ is a linear isometry of $\mathcal{S}(\mathbb{R})$ onto itself with respect to the $L^2(\mathbb{R})$ norm on $\mathcal{S}(\mathbb{R})$; cf. for example [K17, p. 379].) Prove that, for any Hilbert space X, $\mathcal{F}$ is a (linear) *isometry* of $\mathcal{S}(\mathbb{R}, X)$ onto itself with respect to the $L^2(\mathbb{R}, X)$ norm, that is,

$$\int_{\mathbb{R}} \|\mathcal{F}u\|^2\, ds = \int_{\mathbb{R}} \|u\|^2\, dt \tag{76}$$

for all $u \in \mathcal{S}(\mathbb{R}, X)$, and conclude that $\mathcal{F}$ *extends uniquely as a linear isometry of* $L^2(\mathbb{R}, X)$ *onto itself.*

Stability in the u.o.t. on Hilbert Space

31. (This exercise is related to Exercises 28 and 29.) Let $T(\cdot)$ be a C_o-semigroup on the *Hilbert* space X, let A be its generator and ω its type. Fix $M \ge 1$ such that $\|T(t)\| \le M\, e^{(|\omega|+1)t}$ for all $t \ge 0$ (cf. Theorem 1.1) and $a > |\omega| + 1$. Define $S(\cdot) : \mathbb{R} \to B(X)$ by setting $S(t) = 0$ for $t < 0$ and

$$S(t) = e^{-at} T(t) \quad (t \ge 0). \tag{77}$$

Prove:

(a) For each $x \in X$, one has $S(\cdot)x \in L^2(\mathbb{R}, X)$ and

$$R(a + i\cdot; A)x = \sqrt{2\pi}\mathcal{F}[S(\cdot)x]. \tag{78}$$

(b) For each $x \in X$,

$$\int_{\mathbb{R}} \|R(a + is; A)x\|^2 \, ds = 2\pi \int_0^\infty \|S(t)x\|^2 \, dt. \tag{79}$$

(Cf. Exercise 30.)

(c) There exists a constant $K > 0$ such that

$$\|R(a + i\cdot)x\|_{L^2(\mathbb{R},X)} \le K \|x\| \quad (x \in X). \tag{80}$$

(d) *Assume in the sequel that $R(\cdot; A)$ exists and is (uniformly) bounded in $\mathbb{C}^+$.* Observe that the imaginary axis is then contained in $\rho(A)$ (cf. Theorem 1.11). (Therefore the resolvent is uniformly bounded in $\overline{\mathbb{C}^+}$; denote $L := \sup_{\Re z \ge 0} \|R(z; A)\|$). Prove that for all $x \in X$ and $s \in \mathbb{R}$

$$\|R(is; A)x\| \le (1 + aL) \|R(a + is)x\|. \tag{81}$$

(e) Prove that for all $x \in X$

$$\|R(i\cdot; A)x\|_{L^2(\mathbb{R},X)} \le (1 + aL)K \|x\|, \tag{82}$$

and similarly (by considering the adjoint semigroup, cf. Exercise 4),

$$\|R(i\cdot; A^*)x\|_{L^2(\mathbb{R},X)} \le (1 + aL)K \|x\|. \tag{83}$$

(f) Apply Theorem 1.15(3) and integration by parts to prove that for all $x \in X$ and $t > 0$,

$$2\pi i \, tT(t)x = \int_{\mathbb{R}} e^{(a+is)t} R(a + is; A)^2 x \, ds. \tag{84}$$

(g) Prove the inequality

$$2\pi |(tT(t)x, y)| \le \|R(i\cdot; A)x\|_{L^2(\mathbb{R},X)} \|R(-i\cdot; A^*)y\|_{L^2(\mathbb{R},X)} \tag{85}$$

for all $x, y \in X$, and conclude that

$$\|T(t)\| \le H/t \quad (t > 0), \tag{86}$$

with $H = (1 + aL)^2 K^2/(2\pi)$. In particular, $\lim_{t\to\infty} \|T(t)\| = 0$, that is, *the assumption in Part (d) implies that $T(\cdot)$ satisfies the equivalent properties in Exercise 28.* Conversely, if $\omega < 0$, $R(\lambda; A)$ exists and is uniformly bounded in $\mathbb{C}^+$ (cf. Theorem 1.11), that is, the latter property is *equivalent* to the properties in Exercise 28 (when X is a Hilbert space).

Hille–Yosida Space, Semi-Simplicity Space, etc.

32. (The following exercises give a unified construction of the Hille–Yosida space, the semi-simplicity space, the Laplace–Stieltjes space, etc.)

Let X be a Banach space, and let $\mathcal{A}$ be a subset of $B(X)$ containing the identity I in its strong closure. Denote

$$\|x\|_{\mathcal{A}} := \sup_{T\in\mathcal{A}} \|Tx\| \quad (x \in X) \tag{87}$$

and

$$Z = Z(\mathcal{A}); = \{x \in X;\ \|x\|_{\mathcal{A}} < \infty\}. \tag{88}$$

Prove:

(a) Z with the norm $\|\cdot\|_{\mathcal{A}}$ is a Banach subspace of X.
(b) For any S in the commutant $\mathcal{A}'$ of $\mathcal{A}$ in $B(X)$, $SZ \subset Z$ and $S|_Z \in B(Z)$ with $\|S|_Z\|_{B(Z)} \leq \|S\|$.
(c) If $\mathcal{A}$ is a semigroup (for operator multiplication), then Z is $\mathcal{A}$-invariant, and $\mathcal{A}|_Z$ $(:= \{T|_Z;\ T \in \mathcal{A}\})$ is contained in the closed unit ball $B_1(Z)$ of $B(Z)$.
(d) (Hypothesis as in Part (c).) If W is an $\mathcal{A}$-invariant Banach subspace of X such that $\mathcal{A}|_W \subset B_1(W)$, then W is a Banach subspace of Z. (This is the *maximality* of Z with respect to the property in Part (c).)

33. Let A be an operator with domain $D(A)$ in the Banach space X, whose resolvent set contains a ray (a,∞) $(a \in \mathbb{R})$. Let $\mathcal{A}$ be the unital semigroup generated by the operator family

$$(\lambda - a)R(\lambda; A) \quad (\lambda > a). \tag{89}$$

Denote by A_Z the *part of A in* $Z = Z(\mathcal{A})$, and let W be *the closure of $D(A_Z)$ in the Banach subspace Z* (cf. Exercise 32). Prove:

(a) A_W generates a C_o-semigroup $T(\cdot)$ on W such that $\|T(t)\|_{B(W)} \leq e^{at}$.
(b) If V is a "resolvent-invariant" (i.e., $R(\lambda; A)V \subset V$ for all $\lambda > a$) Banach subspace of X such that A_V generates a C_o-semigroup $S(\cdot)$ on V such that $\|S(t)\|_{B(V)} \leq e^{at}$, then V is a Banach subspace of W and $S(\cdot) = T(\cdot)|_V$.
(Cf. Definition 1.22 and Theorem 1.23; the Banach subspace W is precisely the Hille–Yosida space for A, denoted there by Z. Part (b) is the "maximality" property of the Hille–Yosida space. Cf. Theorem 1.17.)

34. Let A be a (generally unbounded) operator on the Banach space X, whose resolvent set contains the (open) sector S_θ, for some $\theta \in (0, \pi/2]$. Let $\mathcal{A}$ be the unital semigroup of $B(X)$ generated by the operator family

$$\{\lambda R(\lambda; A);\ \lambda \in S_\theta\}. \tag{90}$$

As in Exercise 33, denote by A_Z the part of A in $Z = Z(\mathcal{A})$, and let W be the closure of $D(A_Z)$ in the Banach subspace Z. Prove:

(a) A_W generates an analytic C_o-semigroup of contractions on W in the sector S_θ.

(b) If V is a resolvent-invariant Banach subspace of X such that A_V generates an analytic C_o-semigroup of contractions on V in the sector S_θ, then V is a Banach subspace of W.

(The Banach subspace W is the *analytic Hille–Yosida space for* A; Part (b) states its "maximality." Cf. Corollary 3 in Section C of Part I.)

35. Let $T(\cdot)$ be a holomorphic C_o-semigroup on $\mathbb{C}^+$, acting in the Banach space X. Let

$$\mathcal{A} = \{T(\lambda);\ \lambda \in \mathbb{C}^+\}. \tag{91}$$

($\mathcal{A}$ is a semigroup whose strong closure contains the identity, by the C_o condition.) Let $Z = Z(\mathcal{A})$ ($T(\cdot)$-invariant!), and define

$$Z_b := \{x \in Z;\ \lim_{t\to 0+} \|[T(t) - I]x\|_{\mathcal{A}} = 0\}. \tag{92}$$

Prove:

(a) Z_b is a $T(\cdot)$-invariant Banach subspace of X, and $T(\cdot)|_{Z_b}$ is a holomorphic contraction C_o-semigroup on $\mathbb{C}^+$ in the Banach space Z_b. (Therefore $T(\cdot)|_{Z_b}$ possesses a *boundary group*, which is a C_o-group of isometries on Z_b. Cf. Theorem 1.105.)

(b) Z_b is a "maximal" Banach subspace of X with the property in Part (a).

36. Let K be a compact nowhere dense set in $\mathbb{C}$ with connected complement. Let T be a bounded operator on the Banach space X, with spectrum contained in K. As usual, $C(K)$ denotes the Banach algebra of all complex continuous functions f on K with the norm $\|f\|_K := \sup_K |f|$. By Lavrentiev's theorem, the subalgebra $\mathcal{P}(K)$ of polynomials (restricted to K) is dense in $C(K)$ (cf. [Ga, Theorem 8.7]). Let

$$\mathcal{A} = \{p(T);\ p \in \mathcal{P}(K), \|p\|_K \le 1\}, \tag{93}$$

and let $Z = Z(\mathcal{A})$ be the associated Banach subspace (cf. Exercise 32). Prove:

(a) There exists a unique continuous representation $\tau : C(K) \to B(Z)$ such that $\|\tau(f)\|_{B(Z)} \le \|f\|_K$ for all $f \in C(K)$ and $\tau(p) = p(T)$ for all $p \in \mathcal{P}(K)$.

(b) If W is a T-invariant Banach subspace of X with the property in Part (a) (with W replacing Z), then W is a Banach subspace of Z.

(c) If X is reflexive, there exists a unique spectral measure E on Z supported by K, which commutes with every $S \in B(X)$ commuting with T, such that $\|E(\cdot)\|_{B(Z)} \le 1$ and

$$\tau(f)x = \int_K f(\lambda)\, E(d\lambda)x \quad (x \in Z;\ f \in C(K)). \tag{94}$$

(The Banach subspace Z in this case is the semi-simplicity space for T.)

37. (The preceding exercise does not apply for example to the case $K = \Gamma := \{\lambda \in \mathbb{C};\ |\lambda| = 1\}$. The present exercise indicates how to modify the construction in this case.)

Let T be a bounded operator on the Banach space X with spectrum on the unit circle Γ.

Let $\mathcal{R}(\Gamma)$ denote the algebra of restrictions to Γ of all complex polynomials in λ and λ^{-1} $(= \overline{\lambda}$ on $\Gamma)$.

If $p \in \mathcal{R}(\Gamma)$, we may write

$$p(\lambda) = \sum_{k\in\mathbb{Z}} \alpha_k \lambda^k, \tag{95}$$

where $\alpha_k \in \mathbb{C}$ and only finitely many α_k are nonzero. We then define

$$p(T) = \sum_{k\in\mathbb{Z}} \alpha_k T^k. \tag{96}$$

(Note that T is invertible, since $\sigma(T) \subset \Gamma$!) Set

$$\mathcal{A} := \{p(T);\ p \in \mathcal{R}(\Gamma),\ \|p\|_\Gamma \le 1\}, \tag{97}$$

and let $Z = Z(\mathcal{A})$ be the associated Banach subspace of X. Prove:

(a) There exists a continuous representation $\tau : C(\Gamma) \to B(Z)$ such that $\tau(p) = p(T)$ for all $p \in \mathcal{R}(\Gamma)$ and $\|\tau(f)\|_{B(Z)} \le \|f\|_\Gamma$ for all $f \in C(\Gamma)$.
(b) Z is "maximal" in the "usual sense" with respect to the property in Part (a) (cf. preceding exercise).
(c) If X is reflexive, Part (c) of Exercise 36 is valid (with $K = \Gamma$).

38. (This "exercise" is merely a remark on Exercise 32 and Section B of Part II.) Let X be a Banach space, and let

$$F : [0,\infty) \to B(X)$$

be a strongly continuous function such that $F(0) = I$. Choose

$$\mathcal{A} = \left\{ \int_0^\infty \phi(s) F(s)\, ds;\ \phi \in C_c^\infty(\mathbb{R}^+),\ \|\phi\|_\infty = 1 \right\}. \tag{98}$$

The Banach subspace $Z := Z(\mathcal{A})$ in the present case is the Laplace–Stieltjes space for F (cf. Section B in Part II). When F is a contraction C_o-semigroup $T(\cdot)$, $\mathcal{A}$ is a semigroup, and if X is reflexive, Z coincides topologically with the semi-simplicity space for the generator $-A$ of $T(\cdot)$ as defined in Section A of Part II (when $\mathbb{R}^+ \subset \rho(-A)$).

39. Let $T(\cdot)$ be a C_o-group of operators on the Banach space X, and denote its generator by iA. Let

$$\mathcal{A} = \left\{ \sum_k \alpha_k T(t_k);\ \alpha_k \in \mathbb{C},\ t_k \in \mathbb{R},\ \left| \sum_k \alpha_k e^{it_k s} \right| \le 1 \right\}, \tag{99}$$

where the sums in (99) are finite and $s \in \mathbb{R}$.

(In this case, $\mathcal{A}$ is a unital semigroup, so that Exercise 32 applies in all its parts.) If X is reflexive, prove that the Banach subspace $Z = Z(\mathcal{A})$ coincides with the semi-simplicity space for $T(\cdot)$. (Cf. Section B, Part I.)

40. (Cf. Section B in Part II and Chapter 10 in [17].) Let A be an unbounded operator with *real* spectrum on the Banach space X, and denote by T its Cayley transform

$$T := (iI - A)(iI + A)^{-1} = -2iR(-i; A) - I. \tag{100}$$

Since $\sigma(T)$ lies on the unit circle Γ (cf. [DS I–III, Lemma VII.9.2]), we may consider the Banach subspace $Z = Z(\mathcal{A})$ associated with the semigroup $\mathcal{A}$ defined in (97). Prove:

(a) There exists a continuous representation

$$\tau : C(\overline{\mathbb{R}}) \to B(Z)$$

such that

$$\|\tau(f)\|_{B(Z)} \le \|f\|_\infty \quad (f \in C(\overline{\mathbb{R}}))$$

and $\tau(\phi) = T|_Z$ for $\phi(s) = \frac{i-s}{i+s}$.

(b) If X is reflexive, then there exists a spectral measure on Z

$$F : \mathcal{B}(\mathbb{R}) \to B(Z),$$

such that $\|F(\cdot)\|_{B(Z)} \le 1$, F commutes with every $U \in B(X)$ which commutes with A, and

(i) $D(A_Z)$ is the set of all $x \in Z$ such that the integral

$$\int_{\mathbb{R}} s\, F(ds)x := \lim_{a \to -\infty,\, b \to \infty} \int_a^b s\, F(ds)x$$

exists in X and belongs to Z;

(ii) for all $x \in D(A_Z)$,

$$Ax = \int_{\mathbb{R}} s\, F(ds)x,$$

and

(iii) for all nonreal $\lambda \in \mathbb{C}$ and $x \in Z$,

$$R(\lambda; A)x = \int_{\mathbb{R}} \frac{1}{\lambda - s} F(ds)x.$$

(c) Formulate (and prove) a "maximality and uniqueness" statement for the Banach subspace Z with respect to the properties in Parts (a) and (b).

(d) Let $P(\cdot,\cdot)$ be the *Poissonian* of A

$$P(t,s) := \frac{1}{2\pi i}[R(t-is;A) - R(t+is;A)] \quad (t \in \mathbb{R};\, s > 0).$$

Consider the operators

$$U_s : C_c(\mathbb{R}) \to B(X) \quad (s > 0)$$

defined by

$$U_s h = \int_{\mathbb{R}} h(t)P(t,s)\,dt \quad (s > 0;\, h \in C_c(\mathbb{R})).$$

Let

$$\mathcal{A}_1 := \{I\} \cup \{U_s h;\, s > 0,\, h \in C_c(\mathbb{R}),\, \|h\|_\infty \le 1\},$$

and let $Z_1 := Z(\mathcal{A}_1)$ be the associated Banach subspace as in Exercise 32. Set

$$Z' := \{x \in Z_1;\, \lim_{|u|\to\infty} R(\cdot + iu; A)x = 0\}.$$

For X reflexive, prove that Z' is a closed subspace of Z_1 which coincides with Z, with equality of norms.

Approximation Formula for the Integrated Semigroup

41. Let $T(\cdot)$ be a *bounded* C_o-semigroup on the Banach space X, and let A be its generator. Since $T(\cdot)x$ is continuous on $[0,\infty)$ for each $x \in X$ (by the C_o-condition, cf. Theorem 1.1), the X-valued function $f\,T(\cdot)x$ is Bochner integrable for each $f \in L^1(\mathbb{R}^+)$ and $x \in X$, and we may then define a map

$$\tau : L^1(\mathbb{R}^+) \to B(X)$$

by

$$\tau(f)x = \int_0^\infty f(s)T(s)x\,dt \quad (f \in L^1(\mathbb{R}^+);\, x \in X). \tag{101}$$

(Clearly $\|\tau\| \le M := \sup \|T(\cdot)\|$.)

(a) Let $f_{n,t}(s) := 1 - \exp\Big(-e^{n(t-s)}\Big)$ $(t > 0)$. Prove that $f_{n,t} \to \chi_{[0,t]}$ in $L^1(\mathbb{R}^+)$ as $n \to \infty$, uniformly in t on bounded intervals in $\mathbb{R}^+$ ($\chi_{[0,t]}$ denotes the characteristic function of $[0,t]$). Consequently,

$$\tau(f_{n,t})x \to \int_0^t T(s)x\,ds \quad (x \in X;\, t > 0) \tag{102}$$

as $n \to \infty$, strongly in X and uniformly in t on bounded intervals.

(b) Prove that

$$\int_0^t T(s)x\,ds = \lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}e^{tkn}R(kn;A)x \quad (x\in X;\, t>0), \tag{103}$$

strongly in X and uniformly in t on bounded intervals in $\mathbb{R}^+$.

Semigroup Induced on Quotient Space

42. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Suppose Y is a closed $T(\cdot)$-invariant subspace, and let $\pi : X \to X/Y$ be the canonical map. Define $V(t) \in B(X/Y)$ by

$$V(t)(\pi x) = \pi(T(t)x) \quad (t \geq 0;\, x \in X). \tag{104}$$

Prove:

(a) $V(\cdot)$ is a C_o-semigroup on X/Y.
(b) If B denotes the generator of $V(\cdot)$, then

$$D(B) = \pi(D(A))$$

and

$$B(\pi x) = \pi(Ax) \quad (x \in D(A)). \tag{105}$$

Semigroup Induced on $l^\infty(X)$

43. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Let

$$\begin{aligned} Y &= l^\infty(X) \\ &:= \{y := \{y_n\};\ y_n \in X,\ \|y\|_Y := \sup_n \|y_n\| < \infty\}. \end{aligned} \tag{106}$$

For each $t \geq 0$, define

$$S(t)y := \{T(t)y_n\} \quad (y \in Y). \tag{107}$$

Set

$$W := \{y \in Y;\ \lim_{t\to 0+} \|S(t)y - y\|_Y = 0\}. \tag{108}$$

Prove:

(a) $S(\cdot)$ is a C_o-semigroup on the Banach space W.
(b) If B denotes the generator of $S(\cdot)$, then

$$D(B) = \{y \in W;\ y_n \in D(A),\ \textit{and}\ \{Ay_n\} \in W\ \textit{for all}\ n\}$$

and

$$By = \{Ay_n\} \quad (y \in D(B)). \tag{109}$$

Semigroup Induced on a Tensor Space

44. Let X, Y be Banach spaces, and let $X \otimes Y$ be their algebraic tensor product (its typical element is a finite sum $\sum x_k \otimes y_k$, with $x_k \in X$, $y_k \in Y$). If $\| \cdot \|$ is a *cross norm* on $X \otimes Y$ (that is, $\|x \otimes y\| = \|x\| \, \|y\|$ for all $x \in X$, $y \in Y$), denote by $X \hat{\otimes} Y$ the completion of $X \otimes Y$ under the given cross norm. Let $T(\cdot)$, $S(\cdot)$ be C_o-semigroups on X, Y (respectively), with respective generators A, B. For each $t \geq 0$, define

$$V(t) = T(t) \otimes S(t),$$

where the right-hand side is defined as the unique continuous extension of the operator defined on $X \otimes Y$ by

$$V(t) \left(\sum x_k \otimes y_k \right) = \sum (T(t)x_k) \otimes (S(t)y_k).$$

Prove:

(a) $V(\cdot)$ is a C_o-semigroup on $X \times Y$, and $\|V(t)\| = \|T(t)\| \, \|S(t)\|$ for all $t \geq 0$.

(b) The (algebraic) tensor product $D(A) \otimes D(B)$ is a core for the generator C of $V(\cdot)$, and C is equal to the closure of the operator

$$A \otimes I + I \otimes B$$

defined on $D(A) \otimes D(B)$.

Infinite Product of Semigroups

45. Let $\{T_k(\cdot)\}$ be a sequence of pairwise commuting contraction C_o-semigroups on the Banach space X. Let A_k be the generator of $T_k(\cdot)$, $k \in \mathbb{N}$. For each $n \in \mathbb{N}$, set

$$P_n(\cdot) := \prod_{k=1}^{n} T_k(\cdot).$$

Prove:

(a) For each $n \in \mathbb{N}$, $P_n(\cdot)$ is a C_o-semigroup, and its generator is the closure of the operator $\sum_{k=1}^{n} A_k$ with domain $\bigcap_{k=1}^{n} D(A_k)$.

(b) A *convergence vector* for the given sequence is a vector

$$x \in \bigcap_{k=1}^{\infty} D(A_k)$$

such that

$$\sum_{k=1}^{\infty} \|A_k x\| < \infty. \tag{110}$$

Assume that the subspace D of all convergence vectors for the given sequence is dense in X. Prove that for each $t \geq 0$, $P_n(t)$ converges strongly to an operator $P(t) \in B(X)$, uniformly with respect to t in compact intervals, and $P(\cdot)$ is a contraction C_o-semigroup, called the *infinite product* of the given semigroups (denoted $\prod_{k=1}^{\infty} T_k(\cdot)$).

(c) The subspace D of convergence vectors is a core for the generator A of $P(\cdot)$, and $Ax = \sum_{k=1}^{\infty} A_k x$ for all $x \in D$.
Hints: for Part (b), observe that

$$\|P_{n+m}(t)x - P_n(t)x\| \leq \left\| \prod_{k=n+1}^{n+m} T_k(t)x - x \right\|$$

and

$$\left\| \prod_{k=n+1}^{p} T_k(t)x - \prod_{k=n+1}^{p-1} T_k(t)x \right\| \leq \|T_p(t)x - x\|$$

for all $p \geq n+1$, with an "empty product" equal to I by definition.
For Part (c), note that D is $T(\cdot)$-invariant and dense in X (by hypothesis).

Perturbation of Generator by $B \in B([D(A)])$

46. Let $T(\cdot)$ be a C_o-semigroup on the Banach space X, and let A be its generator. Denote by $[D(A)]$ the Banach space defined by the domain of A with the graph norm. Let $B \in B([D(A)])$ and fix $\lambda_0 \in \rho(A)$. Since

$$P := (\lambda_0 I - A)B\,R(\lambda_0; A) \in B(X) \tag{111}$$

(why?), we may choose $\lambda > \omega$ (the type of $T(\cdot)$!) so that

$$\|PR(\lambda; A)\| < 1. \tag{112}$$

Define

$$Q := (\lambda_0 I - A)B\,R(\lambda; A)\ (\in B(X)), \tag{113}$$

$$R := (\lambda I - A)B\,R(\lambda; A)\ (\in B(X)), \tag{114}$$

and

$$U := I - B\,R(\lambda; A). \tag{115}$$

(a) Prove that U is invertible in $B(X)$, and $U\,D(A) = D(A)$.
Hint: in any Banach algebra, the sets of nonzero elements of $\sigma(ab)$ and $\sigma(ba)$ coincide (cf. for example [K17, p. 173]). In the Banach algebra $B(X)$, take $a = Q$ and $b = R(\lambda_0; A)$, so that $ba = B\,R(\lambda; A)$ and $ab = P\,R(\lambda; A)$.

(b) Prove that

$$U(A+R)U^{-1} = A+B.$$

(c) Conclude that $A+B$ (with domain $D(A)$) generates a C_o-semigroup on X.

Intertwining and Spectrum

47. Let $T(\cdot)$ and $S(\cdot)$ be C_o-semigroups on the Banach spaces X and Y, respectively, and let $A,\ B$ be their respective generators. Suppose $C \in B(Y,X)$ intertwines the given semigroups, that is,

$$CS(\cdot) = T(\cdot)C.$$

Prove:

(a) $CB \subset AC$.
(b) $CR(\lambda;B) = R(\lambda;A)C$ for all $\lambda \in \rho(A) \cap \rho(B)$.
(c) If CY is dense in X, then $CD(B)$ is a core for A.
(d) Suppose $0 \in \rho(B)$, and let then Ω be an open disk with radius $r > 0$ centered at 0 such that $\overline{\Omega} \subset \rho(B)$. Suppose also that $R(\lambda;A)$ exists and $\|R(\lambda;A)\| \le |\Re\lambda|^{-1}$ for $\Re\lambda \neq 0$. Finally, suppose that CY is dense in X. Fix $x \in X$ and a sequence $\{y_n\} \subset Y$ such that $Cy_n \to x$. Define

$$y_n(\lambda) := [1+(\lambda/r)^2]CR(\lambda;B)y_n \quad (\lambda \in \overline{\Omega}).$$

Prove

$$\|y_n(re^{i\theta})\| \le (2/r)\|Cy_n\| \quad (\theta \in [0,2\pi),\ \cos\theta \neq 0),$$

and conclude that

$$\|y_n(\lambda)\| \le (2/r)\|Cy_n\| \quad (\lambda \in \Omega),$$

and therefore

$$\|R(\lambda;A)x\| \le \frac{2r}{|r^2+\lambda^2|}\|x\| \quad (\lambda \in \Omega,\ \Re\lambda \neq 0),$$

hence

$$\|R(\lambda;A)\| \le (8/3r) \quad (|\lambda| < r/2,\ \Re\lambda \neq 0).$$

Consequently, $0 \in \rho(A)$. Conclude that

$$\sigma(A) \subset \sigma(B) \cap i\mathbb{R}.$$

Mining Lemma 2.16

48. Let (S, Σ, σ) be a σ-finite positive measure space, and let Ω be a locally compact Hausdorff space. Denote by $M(\Omega)$ the space of all regular complex Borel measures on Ω. Let

$$T : L^1(S, \Sigma, \sigma) \to C_0(\Omega)$$

be a bounded linear operator. Prove:

(a) Given $\phi \in L^\infty(S, \Sigma, \sigma)$ and a constant $K > 0$, there exists $\mu \in M(\Omega)$ such that

$$\phi = T^*\mu; \quad \|\mu\| \le K \tag{116}$$

if and only if

$$\left| \int_S f\phi \, d\sigma \right| \le K \, \|Tf\|_\infty \tag{117}$$

for all f in a dense subset of $L^1(S, \Sigma, \sigma)$. (Cf. Lemma 2.16.)

(b) Suppose $\phi_n = T^*\mu_n$ with $\mu_n \in M(\Omega)$, $\|\mu_n\| \le K$, $n = 1, 2, \ldots$, and $\phi_n \to \phi$ pointwise almost everywhere on S. Then $\phi = T^*\mu$ for some $\mu \in M(\Omega)$ with $\|\mu\| \le K$. If T^* is injective, then $\mu = w^* - \lim \mu_n$. (This may be interpreted as a general version of the Paul Levy "continuity theorem.")

(c) A sequence $\phi = \{\phi_n\} \in l^\infty$ is the moments sequence of a measure $\mu \in M([0,1])$ with $\|\mu\| \le K$ if and only if

$$\left| \sum c_n \phi_n \right| \le K \max_{[0,1]} \left| \sum c_n t^n \right| \tag{118}$$

for all finite complex vectors $(c_1, \ldots, c_n)$. (Hint: choose $T : l^1 \to C([0,1])$ in an appropriate way.) State an analogous criterion for the trigonometric moments problem.

(d) A bounded continuous function ϕ on $\mathbb{R}$ is the cosine-Stieltjes transform

$$\phi(t) = \int_{\mathbb{R}} \cos(st)\mu(ds) \quad (t \in \mathbb{R})$$

of a measure $\mu \in M(\mathbb{R})$ with $\|\mu\| \le K$ if and only if

$$\left| \int_{\mathbb{R}} f\phi \, dt \right| \le K \|\tilde{f}\|_\infty, \tag{119}$$

where $\tilde{f}$ denotes the *cosine transform* of f

$$\tilde{f}(s) := \int_{\mathbb{R}} \cos(st) f(t) \, dt. \tag{120}$$

(e) Let $k \in C_b(\mathbb{R})$ have a convergent improper Riemann integral on $\mathbb{R}$. Given $\phi \in C_b(\mathbb{R})$ and $K > 0$, one has

$$\phi(t) = (k * \mu)(t) := \int_{\mathbb{R}} k(t-s)\mu(ds) \quad (t \in \mathbb{R}) \tag{121}$$

for some $\mu \in M(\mathbb{R})$ with $\|\mu\| \le K$ if and only if

$$\left| \int_{\mathbb{R}} f\phi \, dt \right| \le K \, \|\check{k} * f\|_\infty \tag{122}$$

for all f in a dense subset of $L^1(\mathbb{R})$.
(We used the following notation: $\check{k}(t) := k(-t)$; if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $f * g$ is their convolution.)
Classical kernels k for which Part (e) applies are the *Dirichlet kernel* $k(t) = (\sin t)/t$, the *Fejer kernel* $k(t) = [(\sin t)/t]^2$, and the *Poisson kernel*

$$k_\epsilon(t) = \frac{\epsilon}{\pi(\epsilon^2 + t^2)}, \tag{123}$$

where $\epsilon > 0$. In the latter case, prove the *uniqueness* of the representation of ϕ, when it exists.

49. Let A be a commutative (complex) Banach algebra, and let $\mathcal{M}$ be its *structure space* (that is, the space of all regular maximal ideals with the Gelfand topology). Denote by $G : A \to C_0(\mathcal{M})$ the Gelfand transform ($Gx = \hat{x}$). Given $x^* \in A^*$ and a constant $K > 0$, prove that there exists a measure $\mu \in M(\mathcal{M})$ such that

$$x^* = G^*\mu; \quad \|\mu\| \le K \tag{124}$$

if and only if

$$|x^* x| \le K \, r(x) \tag{125}$$

for all x in a dense subset of A. ($r(x)$ denotes the spectral radius of x. Cf. Lemma 2.16.)

The Eberlein and Schoenberg Criteria for Fourier–Stieltjes Transforms

50. Let G be a locally compact Abelian group. Denote by $L^1(G)$ its *group algebra* (that is, its L^1 space with respect to the Haar measure dt on G, with convolution as multiplication). Let Γ be the dual group $\hat{G}$. Given a bounded continuous function ϕ on G and a constant $K > 0$, prove that ϕ is the Fourier–Stieltjes transform

$$\phi(t) = \int_\Gamma (t, \gamma)\mu(d\gamma) \quad (t \in G) \tag{126}$$

of some $\mu \in M(\Gamma)$ with $\|\mu\| \le K$ if and only if

$$\left| \int_G f\phi \, dt \right| \le K \, \|\hat{f}\|_\infty \tag{127}$$

for all f in a dense subset of $L^1(G)$.

Cf. Exercise 49; (t, γ) denotes the value of the character $\gamma \in \Gamma$ at the point $t \in G$, and $\hat{f}$ denotes the Fourier transform of $f \in L^1(G)$, that is,

$$\hat{f}(\gamma) := \int_G (t, \gamma) f(t) \, dt \quad (\gamma \in \Gamma). \tag{128}$$

(The above statement is known as *the Eberlein criterion* for Fourier–Stieltjes transforms on locally compact Abelian groups; the special case for $G = \mathbb{R}$ is the *Schoenberg criterion.*)

51. Let X, Y be Banach spaces, and $T \in B(X, Y)$. Prove that the restriction of T^* to the (strongly) closed unit ball of Y^* has *weak**-closed range. (Cf. Lemma 2.16.)

52. Suppose iS generates a C_o-group $S(\cdot) : \mathbb{R} \to B(X)$ on the Banach space X, and $V \in B(X)$ leaves $D(S)$ invariant and $[S, V] \subset V^2$. Let $T_a(\cdot)$ be the C_o-group generated by iT_a, where $T_a = (S - V) + V_a$ and $V_a = S(a)V\,S(-a)$, $a \in \mathbb{R}$ (cf. Corollary of Theorem 1.38). Prove:

(a) For all $a, t \in \mathbb{R}$,

$$T_a(t) = S(t) - atV\,S(t)V_a.$$

(b) For all $a, t \in \mathbb{R}$ such that $t \neq -a$,

$$T_a(t) = S(t) + i\frac{at}{a+t}[S(a+t), V]\,S(-a).$$

(Cf. Theorem 3.18 and its proof.)

Notes and References

The theory of operator semigroups was essentially started by the 1932 paper of M. H. Stone on groups of unitary operators in Hilbert space [St]. The general Banach space theory was established in the following two decades, and is detailed in the classical 1957 monograph [HP]. Later results are included and/or referred to in many more recent books, some of which are cited in our bibliography.

Part I. General Theory

Standard books on the general theory of operator semigroups are [D3, EN1, EN2, G, HP, P]. Chapters on the subject are also included in general texts on Functional Analysis such as [DS I–III, Kat1, RS, Y].

A. Basic Theory

Most of the material concerning the interplay between a semigroup and its generator, culminating with the Hille–Yosida theorem, was developed in the 1930s and 1940s, and is necessarily found in any text on operator semigroups.

We comment below on some more recent results included in this section.

The Hille–Yosida space. The terminology and Theorem 1.23 are from [K5].

The Trotter–Kato convergence theorem. Theorem 1.32 goes back to [Tr].

Exponential formulas. The treatment here follows [D3, P], and is based on work by [Kat3, Ch1, Ch2, Tr].

Perturbations of generators. Theorem 1.38 is due to Hille–Phillips. The proof given here is essentially the one in [DS I–III].

Groups of operators. Theorem 1.40 is from [N]. Theorem 1.41 is the classical Stone theorem [St].

B. The Semi-simplicity Space for Groups

Theorem 1.49 and the following analysis are from [K3].

C. Analyticity

Theorem 1.54 is a variation on a result of [Liu]. The proof given here is from [K9].

D. The Semigroup as a Function of its Generator

Noncommutative Taylor formula. The results of this section are from [K7].
Analytic families of semigroups. The results are from [K10].

E. Large Parameter

The results of this section are (in the exposition order) from [K12, K13, K14, K15], [KP], and [AB]. See also [LV].

F. Boundary Values

The main facts about "regular semigroups" are contained in [HP, Theorems 17.9.1 and 17.9.2]; the "converse" part (and the corollaries) are from [K16].

G. Pre-Semigroups

The concept appears in germinal form in [DaP] (under the name of "regularizable semigroups"). In [DP], the name "C-semigroup" is coined, and the detailed analysis of these families is started (see [DL1, DL2, DL3, M1, M2, MT1, MT2, MT3, T1, T2], as a partial list for this subject). Since a C-semigroup is *not* a semigroup (unless $C = I$), we call it here a *pre-semigroup.*
Theorems 1.119–1.121 are from [DL1].
Theorem 1.124 is from [DL2] (but we coined the term *exponentially tamed* as a reference to Property 3).

Part II. Integral Representations

A. The Semi-Simplicity Space

The concept goes back to [K1] for a single *bounded* operator, with extensions to unbounded operators appearing in [K1, K2, KH2, KH3]. Theorem 2.3 is from [KH2]. A variant of this theorem is found in [KH3]. Theorem 2.12 is from [K2] (see also [K4]). Lemma 2.16 is from [KH3] (see also [DLK]).

B. The Laplace–Stieltjes Space

The concepts of the *Laplace–Stieltjes space* and of the *Integrated Laplace space* for a family of closed operators were introduced and studied in [DLK]. Theorems 2.15, 2.20, 2.21, and 2.23 are from [DLK] (with some modification of the proofs). Theorem 2.28 is a special case of the main result of [DL3]. *Integrated semigroups* were introduced in [Neu].

C. Families of Unbounded Symmetric Operators

Semigroups of unbounded symmetric operators. First results on this subject were obtained in [De] and [Nus]. A general theory of semigroups of unbounded operators in Banach space was developed in [H1, H2]. Theorem 2.29 is from [KL], as well as the proof of Theorem 2.31 (the latter theorem appeared originally in [Nel], with a different proof). Another proof of Theorem 2.29 is found in [Fr], and serves as a model for the proofs of Theorems 2.35 and 2.37 (first published in [KH3]) for local cosine families of symmetric operators. Frohlich's proof was modified in [V1] to generalize the Frohlich–Klein–Landau theorem to local symmetric semigroups defined on *semigroups* of $\mathbb{R}^+$. For local symmetric semigroups defined on general topological semigroups, see [V2]. The results on local semigroups are generalized to a Banach space setting in [KH1] (see also [K4]).
Local cosine families of unbounded symmetric operators. The results are from [KH3]. The concept of *semi-analytic vectors* is due to Nussbaum, as well as Theorem 2.39 (with a proof independent of the result on local cosine families of symmetric operators; see [RS]).

Part III. A Taste of Applications

A. Dependence on Parameters

This section is based on [KM].

B. Similarity (etc.)

The results are from [K18, K19, K20], [K23], and [KPe], with some modifications. See also [K4], [KH5], and [VK] for related results and generalizations.

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(The bibliography lists mostly works which are related somehow to the material of the book or which are needed as references for results used in proofs, and which appeared after 1975. For papers published before 1975, we refer to the vast bibliography of [G].)

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