Appendix A
Poincaré, Agmon, and Other Basic Inequalities

In this appendix we review a few inequalities for basic Sobolev spaces of functions of one variable.

Let us first recall two elementary well-known inequalities:

Young’s inequality (most elementary version)

\[ ab \leq \frac{\gamma}{2} a^2 + \frac{1}{2\gamma} b^2 \quad \forall \gamma > 0. \quad \text{(A.1)} \]

Cauchy–Schwartz inequality

\[ \int_0^1 u(x)w(x) \, dx \leq \left( \int_0^1 u(x)^2 \, dx \right)^{1/2} \left( \int_0^1 w(x)^2 \, dx \right)^{1/2}. \quad \text{(A.2)} \]

The following lemma establishes the relationship between the $L_2$ norms of $w$ and $w_x$.

**Lemma A.1 (Poincaré inequality).** For any $w$, continuously differentiable on $[0, 1],$

\[ \int_0^1 w(x)^2 \, dx \leq 2w^2(1) + 4 \int_0^1 w_x(x)^2 \, dx, \]

\[ \int_0^1 w(x)^2 \, dx \leq 2w^2(0) + 4 \int_0^1 w_x(x)^2 \, dx. \quad \text{(A.3)} \]

**Proof.** We start with the $L_2$ norm,

\[ \int_0^1 w^2 \, dx = xw^2|_0^1 - 2 \int_0^1 xww_x \, dx \quad \text{(integration by parts)} \]

\[ = w^2(1) - 2 \int_0^1 xww_x \, dx \]

\[ \leq w^2(1) + \frac{1}{2} \int_0^1 w^2 \, dx + 2 \int_0^1 x^2w_x^2 \, dx. \]
Subtracting the second term from both sides of the inequality, we get the first inequality in (A.3):

\[
\frac{1}{2} \int_0^1 w^2 dx \leq w^2(1) + 2 \int_0^1 x^2 w_x^2 dx
\]

\[
\leq w^2(1) + 2 \int_0^1 x^2 dx.
\]  

(A.4)

The second inequality in (A.3) is obtained in a similar fashion. □

The inequalities (A.3) are conservative. The tight version of (A.3) is given next, which is sometimes called "a variation of Wirtinger’s inequality" [64].

**Lemma A.2.**

\[
\int_0^1 (w(x) - w(0))^2 dx \leq \frac{4}{\pi^2} \int_0^1 w^2(x) dx.
\]  

(A.5)

*Equality holds only for* \( w(x) = A + B \sin \frac{\pi x}{2} \).

The proof of (A.5) is far more complicated than the proof of (A.3).

Now we turn to reviewing the basic relationships between the \( L_2 \) and \( H_1 \) Sobolev norms and the maximum norm. The \( H_1 \) norm can be defined in more than one way. We define it as

\[
\|w\|_{H_1}^2 := \int_0^1 w^2 dx + \int_0^1 w_x^2 dx.
\]  

(A.6)

Note also that by using the Poincaré inequality, it is possible to drop the first integral in (A.6) whenever the function is zero at at least one of the boundaries.

**Lemma A.3 (Agmon’s inequality).** For a function \( w \in H_1 \), the following inequalities hold:

\[
\max_{x \in [0,1]} |w(x)|^2 \leq w(0)^2 + 2 \|w\||w_x|, \\
\max_{x \in [0,1]} |w(x)|^2 \leq w(1)^2 + 2 \|w\||w_x|.
\]  

(A.7)

*Proof.* We begin with

\[
\int_0^x w^2 dx = \int_0^x \frac{1}{2} d(2w_x^2) = \frac{1}{2} w^2 |x_0| = \frac{1}{2} w(x)^2 - \frac{1}{2} w(0)^2,
\]  

(A.8)

which gives

\[
\frac{1}{2} |w(x)|^2 \leq \int_0^x |w||w_x| dx + \frac{1}{2} w(0)^2.
\]  

(A.9)

Using the fact that an integral of a positive function is an increasing function of its upper limit, we can rewrite the last inequality as
\[ |w(x)|^2 \leq w(0)^2 + 2 \int_0^1 |w(x)||w_x(x)| \, dx. \tag{A.10} \]

The right-hand side of this inequality does not depend on \( x \); therefore,

\[
\max_{x \in [0,1]} |w(x)|^2 \leq w(0)^2 + 2 \int_0^1 |w(x)||w_x(x)| \, dx. \tag{A.11} 
\]

Using the Cauchy–Schwartz inequality, we get the first inequality of (A.7). The second inequality is obtained in a similar fashion.
Appendix B
Input–Output Lemmas for LTI and LTV Systems

In addition to a review of basic input–output stability results, we give several technical lemmas used in the book.

For a function $x : \mathbb{R}_+ \to \mathbb{R}^n$, we define the $L_p$ norm, $p \in [1, \infty]$, as

$$\|x\|_p = \begin{cases} \left( \int_0^\infty |x(t)|^p dt \right)^{1/p}, & p \in [1, \infty), \\ \sup_{t \geq 0} |x(t)|, & p = \infty, \end{cases}$$  \hspace{1cm} (B.1)

and the $L_{p,e}$ norm (truncated $L_p$ norm) as

$$\|x_t\|_p = \begin{cases} \left( \int_0^t |x(\tau)|^p d\tau \right)^{1/p}, & p \in [1, \infty), \\ \sup_{\tau \in [0,t]} |x(\tau)|, & p = \infty. \end{cases}$$  \hspace{1cm} (B.2)

**Lemma B.1 (Hölder’s inequality).** If $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|(fg)_t\|_1 \leq \|f_t\|_p \|g_t\|_q, \quad \forall t \geq 0.$$  \hspace{1cm} (B.3)

We consider an LTI causal system described by the convolution

$$y(t) = h \ast u = \int_0^t h(t-\tau)u(\tau) d\tau,$$  \hspace{1cm} (B.4)

where $u : \mathbb{R}_+ \to \mathbb{R}$ is the input, $y : \mathbb{R}_+ \to \mathbb{R}$ is the output, and $h : \mathbb{R} \to \mathbb{R}$ is the system’s impulse response, which is defined to be zero for negative values of its argument.

**Theorem B.1 (Young’s convolution theorem).** If $h \in L_{1,e}$, then

$$\|(h \ast u)_t\|_p \leq \|h_t\|_1 \|u_t\|_p, \quad p \in [1, \infty].$$  \hspace{1cm} (B.5)
Proof. Let \( y = h * u \). Then, for \( p \in [1, \infty) \), we have

\[
|y(t)| \leq \int_{0}^{t} |h(t - \tau)| |u(\tau)| d\tau
\]

\[
= \int_{0}^{t} |h(t - \tau)|^{\frac{p-1}{p}} |h(t - \tau)|^{\frac{1}{p}} |u(\tau)| d\tau
\]

\[
\leq \left( \int_{0}^{t} |h(t - \tau)| d\tau \right)^{\frac{p-1}{p}} \left( \int_{0}^{t} |u(\tau)|^p d\tau \right)^{\frac{1}{p}}
\]

\[
= \|h_t\|_1^{\frac{p-1}{p}} \left( \int_{0}^{t} |h(t - \tau)| |u(\tau)|^p d\tau \right)^{\frac{1}{p}}, \tag{B.6}
\]

where the second inequality is obtained by applying Hölder’s inequality. Raising (B.6) to power \( p \) and integrating from 0 to \( t \), we get

\[
\|y_t\|_p^p \leq \int_{0}^{t} \|h_t\|_1^{p-1} \left( \int_{0}^{\tau} |h(\tau - s)| |u(s)|^p ds \right) d\tau
\]

\[
= \|h_t\|_1^{p-1} \int_{0}^{t} \left( \int_{0}^{\tau} |h(\tau - s)| |u(s)|^p d\tau \right) ds
\]

\[
= \|h_t\|_1^{p-1} \int_{0}^{t} \left( \int_{0}^{\tau} |h(\tau - s)| |u(s)|^p d\tau \right) ds
\]

\[
= \|h_t\|_1^{p-1} \int_{0}^{t} |u(s)|^p \left( \int_{0}^{\tau} |h(\tau - s)| d\tau \right) ds
\]

\[
\leq \|h_t\|_1^{p-1} \int_{0}^{t} |u(s)|^p \left( \int_{0}^{\tau} |h(\tau)| d\tau \right) ds
\]

\[
\leq \|h_t\|_1^{p-1} \|h\|_1^{p-1} \|u_t\|_1^p
\]

\[
\leq \|h_t\|_1^{p-1} \|u_t\|_1^p, \tag{B.7}
\]

where the second line is obtained by changing the sequence of integration, and the third line by using the causality of \( h \). The proof for the case \( p = \infty \) is immediate by taking a supremum of \( u \) over \([0, t]\) in the convolution. \( \square \)

Lemma B.2. Let \( v \) and \( \rho \) be real-valued functions defined on \( \mathbb{R}_+ \), and let \( b \) and \( c \) be positive constants. If they satisfy the differential inequality

\[
\dot{v} \leq -cv + bp(t)^2, \quad v(0) \geq 0, \tag{B.8}
\]

(i) then the following integral inequality holds:

\[
v(t) \leq v(0)e^{-ct} + b \int_{0}^{t} e^{-c(t-\tau)} \rho(\tau)^2 d\tau. \tag{B.9}
\]

(ii) If, in addition, \( \rho \in L_2 \), then \( v \in L_1 \) and

\[
\|v\|_1 \leq \frac{1}{c} \left( v(0) + b\|\rho\|_2^2 \right). \tag{B.10}
\]
Proof. (i) Upon multiplication of (B.8) by $e^{ct}$, it becomes
\[
\frac{d}{dt} (v(t)e^{ct}) \leq b\rho(t)^2 e^{ct}.
\] (B.11)
Integrating (B.11) over $[0,t]$, we arrive at (B.9).

(ii) By integrating (B.9) over $[0,t]$, we get
\[
\int_0^t v(\tau)d\tau \leq \int_0^t v(0)e^{-ct}d\tau + b \int_0^t \left[ \int_0^\tau e^{-c(\tau-s)}\rho(s)^2ds \right] d\tau
\]
\[
\leq \frac{1}{c} v(0) + b \int_0^t \left[ \int_0^\tau e^{-c(\tau-s)}\rho(s)^2ds \right] d\tau.
\] (B.12)
Noting that the second term is $b\|h \ast \rho^2\|_1$, where
\[
h(t) = e^{-ct}, \quad t \geq 0,
\] (B.13)
we apply Theorem B.1. Since
\[
\|h\|_1 = \frac{1}{c},
\] (B.14)
we obtain (B.10).

Lemma B.3. Let $v, l_1,$ and $l_2$ be real-valued functions defined on $\mathbb{R}_+$, and let $c$ be a positive constant. If $l_1$ and $l_2$ are nonnegative and in $L_1$ and satisfy the differential inequality
\[
\dot{v} \leq -cv + l_1(t)v + l_2(t), \quad v(0) \geq 0,
\] (B.15)
then $v \in L_\infty \cap L_1$ and
\[
v(t) \leq \left( v(0)e^{-ct} + \|l_2\|_1 \right) e^{\|l_1\|_1 t},
\] (B.16)
\[
\|v\|_1 \leq \frac{1}{c} \left( v(0) + \|l_2\|_1 \right) e^{\|l_1\|_1 t}.
\] (B.17)
Proof. Using the facts that
\[
v(t) \leq w(t),
\] (B.18)
\[
\dot{w} = -cw + l_1(t)w + l_2(t),
\] (B.19)
\[
w(0) = v(0)
\] (B.20)
(the comparison principle), and applying the variation-of-constants formula, the differential inequality (B.15) is rewritten as
\[
v(t) \leq v(0)e^{\int_0^t [-c + l_1(s)]ds} + \int_0^t e^{\int_0^\tau [-c + l_1(s)]ds}l_2(\tau)d\tau
\]
\[
\leq v(0)e^{-ct}e^{\int_0^t l_1(s)ds} + \int_0^t e^{-c(t-\tau)}l_2(\tau)d\tau e^{\int_0^\tau l_1(s)ds}
\]
\[
\leq \left[ v(0)e^{-ct} + \int_0^t e^{-c(t-\tau)}l_2(\tau)d\tau \right] e^{\|l_1\|_1 t}.
\] (B.21)
By taking a supremum of $e^{-c(t-\tau)}$ over $[0, \infty]$, we obtain (B.16). Integrating (B.21) over $[0, \infty]$, we get
\[
\int_0^t v(\tau)d\tau \leq \left(\frac{1}{c}v(0) + \int_0^t \left[\int_0^\tau e^{-c(\tau-s)}l_2(s)ds\right]d\tau\right)e^{\|l_1\|_1}. \tag{B.22}
\]
Applying Theorem B.1 to the double integral, we arrive at (B.17). \(\Box\)

Remark B.1. An alternative proof that $v \in L_\infty \cap L_1$ in Lemma B.3 is using the Gronwall lemma (Lemma B.5). However, with the Gronwall lemma, the estimates of the bounds (B.16) and (B.17) are more conservative:
\[
v(t) \leq (v(0)e^{-ct} + \|l_2\|_1)\left(1 + \|l_1\|_1e^{\|l_1\|_1}\right), \tag{B.23}
\]
\[
\|v\|_1 \leq \frac{1}{c}(v(0) + \|l_2\|_1)\left(1 + \|l_1\|_1e^{\|l_1\|_1}\right), \tag{B.24}
\]
because
\[
e^x < (1+xe^x), \quad \forall x > 0. \tag{B.25}
\]
Note that the ratio between the bounds (B.23) and (B.16) and that between the bounds (B.24) and (B.17) are of the order $\|l_1\|_1$ when $\|l_1\|_1 \to \infty$. \(\Box\)

For cases where $l_1$ and $l_2$ are functions of time that converge to zero but are not in $L_p$ for any $p \in [1, \infty)$, we have the following lemma.

Lemma B.4. Consider the differential inequality
\[
\dot{v} \leq -(c - \beta_1(r_0,t))v + \beta_2(r_0,t) + \rho, \quad v(0) = v_0 \geq 0, \tag{B.26}
\]
where $c > 0$ and $r_0 \geq 0$ are constants, and $\beta_1$ and $\beta_2$ are class-$\mathcal{K}$ functions. Then there exist a class-$\mathcal{K}$ function $\beta_v$ and a class-$\mathcal{K}$ function $\gamma_v$ such that
\[
v(t) \leq \beta_v(v_0 + r_0,t) + \gamma_v(\rho), \quad \forall t \geq 0. \tag{B.27}
\]
Moreover, if
\[
\beta_i(r,t) = \alpha_i(r)e^{-\sigma_it}, \quad i = 1, 2, \tag{B.28}
\]
where $\alpha_i \in \mathcal{K}$ and $\sigma_i > 0$, then there exist $\alpha_v \in \mathcal{K}$ and $\sigma_v > 0$ such that
\[
\beta_v(r,t) = \alpha_v(r)e^{-\sigma_vt}. \tag{B.29}
\]
Proof. We start by introducing
\[
\tilde{v} = v - \frac{\rho}{c} \tag{B.30}
\]
and rewriting (B.26) as
\[
\dot{\tilde{v}} \leq -(c - \beta_1(r_0,t))\tilde{v} + \frac{\rho}{c}\beta_1(r_0,t) + \beta_2(r_0,t). \tag{B.31}
\]
It then follows that
\[
v(t) \leq v_0 e^{\int_{t}^{T_{c}(r)} [\beta_1(r, s) - c] ds} + \int_{0}^{t} \left[ \frac{P}{c} \beta_1(r_0, \tau) + \beta_2(r_0, \tau) \right] e^{\int_{t}^{T_{c}(r)} [\beta_1(r, s) - c] ds} d\tau + \frac{P}{c}. \tag{B.32}
\]

We note that
\[
e^{\int_{t}^{T_{c}(r)} [\beta_1(r, s) - c] ds} \leq k(r_0) e^{-\frac{c}{T_{c}}(t - \tau)}, \quad \forall \tau \in [0, t], \tag{B.33}
\]
where \(k\) is a positive, continuous, increasing function. To get an estimate of the overshoot coefficient \(k(r_0)\), we provide a proof of (B.33). For each \(c\), there exists a class-\(\mathcal{K}\) function \(T_{c} : \mathbb{R}_{+} \to \mathbb{R}_{+}\) such that
\[
\beta_1(r_0, s) \leq \frac{c}{2}, \quad \forall s \geq T_{c}(r_0). \tag{B.34}
\]

Therefore, for \(0 \leq \tau \leq T_{c}(r_0) \leq t\), we have
\[
\int_{\tau}^{t} [\beta_1(r_0, s) - c] ds \leq \int_{\tau}^{T_{c}(r_0)} [\beta_1(r_0, s) - c] ds + \int_{T_{c}(r_0)}^{t} \left( -\frac{c}{2} \right) ds
\leq (\beta_1(r_0, 0) - c)(T_{c}(r_0) - \tau) - \frac{c}{2} (t - T_{c}(r_0))
\leq T_{c}(r_0) \beta_1(r_0, 0) - \frac{c}{2} (t - \tau), \tag{B.35}
\]
so the overshoot coefficient in (B.33) is given by
\[
k(r_0) \triangleq e^{T_{c}(r_0) \beta_1(r_0, 0)}. \tag{B.36}
\]

For the other two cases, \(t \leq T_{c}(r_0)\) and \(T_{c}(r_0) \leq \tau\), getting (B.33) with \(k(r_0)\) as in (B.36) is immediate. Now substituting (B.33) into (B.32), we get
\[
v(t) \leq v_0 k(r_0) e^{-\frac{c}{T_{c}}t} + k(r_0) \int_{0}^{t} \left[ \frac{P}{c} \beta_1(r_0, \tau) + \beta_2(r_0, \tau) \right] e^{-\frac{c}{T_{c}}(t - \tau)} d\tau + \frac{P}{c}. \tag{B.37}
\]

To complete the proof, we show that a class-\(\mathcal{KL}\) function \(\beta\) convolved with an exponentially decaying kernel is bounded by another class-\(\mathcal{KL}\) function:
\[
\int_{0}^{t} e^{-\frac{c}{T_{c}}(t - \tau)} \beta(r_0, \tau) d\tau = \int_{0}^{t/2} e^{-\frac{c}{T_{c}}(t - \tau)} \beta(r_0, \tau) d\tau + \int_{t/2}^{t} e^{-\frac{c}{T_{c}}(t - \tau)} \beta(r_0, \tau) d\tau
\leq \beta(r_0, 0) \int_{0}^{t/2} e^{-\frac{c}{T_{c}}(t - \tau)} d\tau + \beta(r_0, t/2) \int_{t/2}^{t} e^{-\frac{c}{T_{c}}(t - \tau)} d\tau
\leq \frac{2}{c} \left[ \beta(r_0, 0) e^{-\frac{c}{T_{c}}t/2} + \beta(r_0, t/2) \right]. \tag{B.38}
\]
Thus, (B.37) becomes
\begin{equation}
 v(t) \leq k(r_0) \left\{ \left[ v_0 + \frac{2 \rho}{c^2} \beta_1(r_0, 0) + \frac{2}{c} \beta_2(r_0, 0) \right] e^{-\frac{c}{4} t} 
 + \frac{2 \rho}{c^2} \beta_1(r_0, t/2) + \frac{2}{c} \beta_2(r_0, t/2) \right\} + \frac{\rho}{c}. \tag{B.39}
\end{equation}

By applying Young’s inequality to the terms
\begin{equation}
 k(r_0) \frac{2 \rho}{c^2} \beta_1(r_0, 0) e^{-\frac{c}{4} t} \tag{B.40}
\end{equation}
and
\begin{equation}
 k(r_0) \frac{2 \rho}{c^2} \beta_1(r_0, t/2), \tag{B.41}
\end{equation}
we obtain (B.27) with
\begin{equation}
 \beta_v(r, t) = k(r) \left\{ \left[ r + \frac{k(r)}{c^2} \beta_1(r, 0)^2 + \frac{2}{c} \beta_2(r, 0) \right] e^{-\frac{c}{4} t} 
 + \frac{k(r)}{c^2} \beta_1(r, t/2)^2 + \frac{2}{c} \beta_2(r, t/2) \right\}, \tag{B.42}
\end{equation}
\begin{equation}
 \gamma_v(r) = \frac{r}{c} + \frac{r^2}{c^2}. \tag{B.43}
\end{equation}
The last statement of the lemma is immediate by substitution into (B.42). \qed

Now we give a version of Gronwall’s lemma.

**Lemma B.5 (Gronwall).** Consider the continuous functions \( \lambda : \mathbb{R}_+ \to \mathbb{R} \), \( \mu : \mathbb{R}_+ \to \mathbb{R}_+ \), and \( v : \mathbb{R}_+ \to \mathbb{R}_+ \), where \( \mu \) and \( v \) are also nonnegative. If a continuous function \( y : \mathbb{R}_+ \to \mathbb{R} \) satisfies the inequality
\begin{equation}
 y(t) \leq \lambda(t) + \mu(t) \int_{t_0}^{t} v(s) y(s) ds, \quad \forall t \geq t_0 \geq 0, \tag{B.44}
\end{equation}
then
\begin{equation}
 y(t) \leq \lambda(t) + \mu(t) \int_{t_0}^{t} \lambda(s) v(s) e^{\int_{t_0}^{s} \mu(\tau) v(\tau) d\tau} ds, \quad \forall t \geq t_0 \geq 0. \tag{B.45}
\end{equation}
In particular, if \( \lambda(t) \equiv \lambda \) is a constant and \( \mu(t) \equiv 1 \), then
\begin{equation}
 y(t) \leq \lambda e^{\int_{t_0}^{t} v(\tau) d\tau}, \quad \forall t \geq t_0 \geq 0. \tag{B.46}
\end{equation}
C.1 Lyapunov Stability and Class-$\mathcal{K}$ Functions

Consider the nonautonomous ODE system
\[
\dot{x} = f(x, t), \quad (C.1)
\]
where \( f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) is locally Lipschitz in \( x \) and piecewise continuous in \( t \).

**Definition C.1.** The origin \( x = 0 \) is the equilibrium point for (C.1) if
\[
f(0, t) = 0, \quad \forall t \geq 0. \quad (C.2)
\]

Scalar comparison functions are important stability tools.

**Definition C.2.** A continuous function \( \gamma : [0, a) \to \mathbb{R}_+ \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \gamma(0) = 0 \). It is said to belong to class \( \mathcal{K}_\infty \) if \( a = \infty \) and \( \gamma(r) \to \infty \) as \( r \to \infty \).

**Definition C.3.** A continuous function \( \beta : [0, a) \times \mathbb{R}_+ \to \mathbb{R}_+ \) is said to belong to class \( \mathcal{K}\mathcal{L} \) if, for each fixed \( s \), the mapping \( \beta(r, s) \) belongs to class \( \mathcal{K} \) with respect to \( r \) and, for each fixed \( r \), the mapping \( \beta(r, s) \) is decreasing with respect to \( s \) and \( \beta(r, s) \to 0 \) as \( s \to \infty \). It is said to belong to class \( \mathcal{K}\mathcal{L}_\infty \) if, in addition, for each fixed \( s \), the mapping \( \beta(r, s) \) belongs to class \( \mathcal{K}_\infty \) with respect to \( r \).

The main list of stability definitions for ODE systems is given next.

**Definition C.4 (Stability).** The equilibrium point \( x = 0 \) of (C.1) is
- **uniformly stable** if there exist a class-$\mathcal{K}$ function \( \gamma(\cdot) \) and a positive constant \( c \), independent of \( t_0 \), such that
\[
|x(t)| \leq \gamma(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0) \text{ s.t. } |x(t_0)| < c; \quad (C.3)
\]
• **uniformly asymptotically stable** if there exist a class-\(\mathcal{KL}\) function \(\beta(\cdot, \cdot)\) and a positive constant \(c\), independent of \(t_0\), such that
\[
|x(t)| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall x(t_0) \text{ s.t. } |x(t_0)| < c;
\] (C.4)

• **exponentially stable** if (C.4) is satisfied with \(\beta(r, s) = kre^{-\alpha s}, k > 0, \alpha > 0\);

• **globally uniformly stable** if (C.3) is satisfied for any initial state \(x(t_0)\);

• **globally uniformly asymptotically stable** if (C.4) is satisfied for any initial state \(x(t_0)\) and \(x(t)\) are class-\(\mathcal{K}\) functions, and \(\beta(\cdot)\) is uniformly stable;

• **globally exponentially stable** if (C.4) is satisfied with \(\beta(r, s) = kre^{-\alpha s}, k > 0, \alpha > 0\).

The main Lyapunov stability theorem is then formulated as follows.

**Theorem C.1 (Lyapunov theorem).** Let \(x = 0\) be an equilibrium point of (C.1) and \(D = \{x \in \mathbb{R}^n \mid |x| < r\}\). Let \(V : D \times \mathbb{R}^n \rightarrow \mathbb{R}_+\) be a continuously differentiable function such that \(\forall t \geq 0, \forall x \in D\),

\[
\gamma_1(|x|) \leq V(x, t) \leq \gamma_2(|x|),
\] (C.5)

\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq -\gamma_3(|x|).
\] (C.6)

Then the equilibrium \(x = 0\) is

• uniformly stable if \(\gamma_1\) and \(\gamma_2\) are class-\(\mathcal{K}\) functions on \([0, r]\) and \(\gamma_3(\cdot) \geq 0\) on \([0, r]\);

• uniformly asymptotically stable if \(\gamma_1, \gamma_2,\) and \(\gamma_3\) are class-\(\mathcal{K}\) functions on \([0, r]\);

• exponentially stable if \(\gamma_i(\rho) = k_i\rho^\alpha\) on \([0, r]\), \(k_i > 0, \alpha > 0, i = 1, 2, 3\);

• globally uniformly stable if \(D = \mathbb{R}^n, \gamma_1\) and \(\gamma_2\) are class-\(\mathcal{K}_\infty\) functions, and \(\gamma_3(\cdot) \geq 0\) on \(\mathbb{R}_+\);

• globally uniformly asymptotically stable if \(D = \mathbb{R}^n\), \(\gamma_1\) and \(\gamma_2\) are class-\(\mathcal{K}_\infty\) functions, and \(\gamma_3\) is a class-\(\mathcal{K}\) function on \(\mathbb{R}_+\); and

• globally exponentially stable if \(D = \mathbb{R}^n\) and \(\gamma_i(\rho) = k_i\rho^\alpha\) on \(\mathbb{R}_+, k_i > 0, \alpha > 0, i = 1, 2, 3\).

In adaptive control our goal is to achieve convergence to a set. For time-invariant systems, the main convergence tool is LaSalle’s invariance theorem. For time-varying systems, a more refined tool is the LaSalle–Yoshizawa theorem. For pedagogical reasons, we introduce it via a technical lemma due to Barbalat. These key results and their proofs are of importance in guaranteeing that an adaptive system will fulfill its tracking task.

**Lemma C.1 (Barbalat).** Consider the function \(\phi : \mathbb{R}_+ \rightarrow \mathbb{R}\). If \(\phi\) is uniformly continuous and \(\lim_{t \rightarrow \infty} \int_0^t \phi(\tau)d\tau\) exists and is finite, then

\[
\lim_{t \rightarrow \infty} \phi(t) = 0.
\] (C.7)


Proof. Suppose that (C.7) does not hold; that is, either the limit does not exist or it is not equal to zero. Then there exists \( \varepsilon > 0 \) such that for every \( T > 0 \), one can find \( t_1 \geq T \) with \( |\phi(t_1)| > \varepsilon \). Since \( \phi \) is uniformly continuous, there is a positive constant \( \delta(\varepsilon) \) such that \( |\phi(t) - \phi(t_1)| < \varepsilon/2 \) for all \( t_1 \geq 0 \) and all \( t \) such that \( |t - t_1| \leq \delta(\varepsilon) \). Hence, for all \( t \in [t_1, t_1 + \delta(\varepsilon)] \), we have

\[
|\phi(t)| = |\phi(t) - \phi(t_1) + \phi(t_1)| \\
\geq |\phi(t_1)| - |\phi(t) - \phi(t_1)| \\
> \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},
\]

which implies that

\[
\left| \int_{t_1}^{t_1 + \delta(\varepsilon)} \phi(\tau) d\tau \right| = \int_{t_1}^{t_1 + \delta(\varepsilon)} |\phi(\tau)| d\tau > \frac{\varepsilon \delta(\varepsilon)}{2},
\]

where the first equality holds since \( \phi(t) \) does not change sign on \([t_1, t_1 + \delta(\varepsilon)]\). Noting that \( \int_{t_1}^{t_1 + \delta(\varepsilon)} \phi(\tau) d\tau = \int_0^{t_1} \phi(\tau) d\tau + \int_{t_1}^{t_1 + \delta(\varepsilon)} \phi(\tau) d\tau \), we conclude that \( \int_0^t \phi(\tau) d\tau \) cannot converge to a finite limit as \( t \to \infty \), which contradicts the assumption of the lemma. Thus, \( \lim_{t \to \infty} \phi(t) = 0 \).

\( \square \)

Corollary C.1. Consider the function \( \phi : \mathbb{R}_+ \to \mathbb{R} \). If \( \phi, \phi' \in \mathcal{L}_\infty \), and \( \phi \in \mathcal{L}_p \) for some \( p \in [1, \infty) \), then

\[
\lim_{t \to \infty} \phi(t) = 0.
\]

Theorem C.2 (LaSalle–Yoshizawa). Let \( x = 0 \) be an equilibrium point of (C.1) and suppose \( f \) is locally Lipschitz in \( x \) uniformly in \( t \). Let \( V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuously differentiable function such that

\[
\gamma_1(|x|) \leq V(x,t) \leq \gamma_2(|x|),
\]

\[
\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t) \leq -W(x) \leq 0,
\]

\( \forall t \geq 0, \forall x \in \mathbb{R}^n \), where \( \gamma_1 \) and \( \gamma_2 \) are class-\( \mathcal{X}_\infty \) functions and \( W \) is a continuous function. Then all solutions of (C.1) are globally uniformly bounded and satisfy

\[
\lim_{t \to \infty} W(x(t)) = 0.
\]

In addition, if \( W(x) \) is positive definite, then the equilibrium \( x = 0 \) is globally uniformly asymptotically stable.

Proof. Since \( \dot{V} \leq 0 \), \( V \) is nonincreasing. Thus, in view of the first inequality in (C.11), we conclude that \( x \) is globally uniformly bounded, that is, \( |x(t)| \leq B \), \( \forall t \geq 0 \). Since \( V(x(t),t) \) is nonincreasing and bounded from below by zero, we conclude that it has a limit \( V_\infty \) as \( t \to \infty \). Integrating (C.12), we have
\[
\lim_{t \to \infty} \int_{t_0}^{t} W(x(\tau)) \, d\tau \leq -\lim_{t \to \infty} \int_{t_0}^{t} \dot{V}(x(\tau), \tau) \, d\tau \\
= \lim_{t \to \infty} \{V(x(t_0), t_0) - V(x(t), t)\} \\
= V(x(t_0), t_0) - V_\infty, \tag{C.14}
\]
which means that \( \int_{t_0}^{\infty} W(x(\tau)) \, d\tau \) exists and is finite. Now we show that \( W(x(t)) \) is also uniformly continuous. Since \(|x(t)| \leq B\) and \(f\) is locally Lipschitz in \(x\) uniformly in \(t\), we see that for any \(t \geq t_0 \geq 0\),
\[
|x(t) - x(t_0)| = \left| \int_{t_0}^{t} f(x(\tau), \tau) \, d\tau \right| \leq L \int_{t_0}^{t} |x(\tau)| \, d\tau \\
\leq LB|t - t_0|, \tag{C.15}
\]
where \(L\) is the Lipschitz constant of \(f\) on \(\{|x| \leq B\}\). Choosing \(\delta(\epsilon) = \epsilon/LB\), we have
\[
|x(t) - x(t_0)| < \epsilon, \quad \forall |t - t_0| \leq \delta(\epsilon), \tag{C.16}
\]
which means that \(x(t)\) is uniformly continuous. Since \(W\) is continuous, it is uniformly continuous on the compact set \(\{|x| \leq B\}\). From the uniform continuity of \(W(x)\) and \(x(t)\), we conclude that \(W(x(t))\) is uniformly continuous. Hence, it satisfies the conditions of Lemma C.1, which then guarantees that \(W(x(t)) \to 0\) as \(t \to \infty\).

If, in addition, \(W(x)\) is positive definite, there exists a class-\(\mathcal{KL}\) function \(\beta\) and a class-\(\mathcal{K}\) function \(\gamma\) such that for any \(x(0)\) and for any input \(u(\cdot)\) continuous and bounded on \([0, \infty)\), the solution exists for all \(t \geq 0\) and satisfies

**C.2 Input-to-State Stability**

Input-to-state stability introduced by Sontag plays a crucial role in the analysis of nonlinear predictor feedback design.

**Definition C.5 (ISS).** The system
\[
\dot{x} = f(t, x, u), \tag{C.17}
\]
where \(f\) is piecewise continuous in \(t\) and locally Lipschitz in \(x\) and \(u\), is said to be input-to-state stable (ISS) if there exist a class-\(\mathcal{KL}\) function \(\beta\) and a class-\(\mathcal{K}\) function \(\gamma\) such that for any \(x(0)\) and for any input \(u(\cdot)\) continuous and bounded on \([0, \infty)\), the solution exists for all \(t \geq 0\) and satisfies
\[
|x(t)| \leq \beta(|x(t_0)|, t-t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right)
\]  
(C.18)

for all \( t_0 \) and \( t \) such that \( 0 \leq t_0 \leq t \).

The following theorem establishes the connection between the existence of a Lyapunov-like function and the input-to-state stability.

**Theorem C.3.** Suppose that for the system (C.17), there exists a \( C^1 \) function \( V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^+ \) such that for all \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \),

\[
\gamma_1(|x|) \leq V(t,x) \leq \gamma_2(|x|),
\]

(C.19)

\[
|x| \geq \rho(|u|) \implies \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(t,x,u) \leq -\gamma_3(|x|),
\]

(C.20)

where \( \gamma_1, \gamma_2, \) and \( \rho \) are class-\( \mathcal{K}_\infty \) functions and \( \gamma_3 \) is a class-\( \mathcal{K} \) function. Then system (C.17) is ISS with \( \gamma = \gamma_1^{-1} \circ \gamma_2 \circ \rho \).

**Proof.** (Outline) If \( x(t_0) \) is in the set

\[
R_{t_0} = \left\{ x \in \mathbb{R}^n \left| |x| \leq \rho \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right) \right. \right\},
\]

(C.21)

then \( x(t) \) remains within the set

\[
S_{t_0} = \left\{ x \in \mathbb{R}^n \left| |x| \leq \gamma_1^{-1} \circ \gamma_2 \circ \rho \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right) \right. \right\}
\]

(C.22)

for all \( t \geq t_0 \). Define \( B = [t_0, T) \) as the time interval before \( x(t) \) enters \( R_{t_0} \) for the first time. In view of the definition of \( R_{t_0} \), we have

\[
\dot{V} \leq -\gamma_3 \circ \gamma_2^{-1}(V), \quad \forall t \in B.
\]

(C.23)

Then there exists a class-\( \mathcal{KL} \) function \( \beta_V \) such that

\[
V(t) \leq \beta_V(V(t_0), t-t_0), \quad \forall t \in B,
\]

(C.24)

which implies

\[
|x(t)| \leq \gamma_1^{-1}(\beta_V(\gamma_2(|x(t_0)|), t-t_0)) \overset{\triangle}{=} \beta(|x(t_0)|, t-t_0), \quad \forall t \in B.
\]

(C.25)

On the other hand, by (C.22), we conclude that

\[
|x(t)| \leq \gamma_1^{-1} \circ \gamma_2 \circ \rho \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right) \overset{\triangle}{=} \gamma \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right), \quad \forall t \in [t_0, \infty) \setminus B.
\]

(C.26)
Then, by (C.25) and (C.26),
\[
|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma \left( \sup_{\tau \geq t_0} |u(\tau)| \right), \quad \forall t \geq t_0 \geq 0.
\] (C.27)

By causality, it follows that
\[
|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} |u(\tau)| \right), \quad \forall t \geq t_0 \geq 0.
\] (C.28)

A function $V$ satisfying the conditions of Theorem C.3 is called an **ISS–Lyapunov function**. The inverse of Theorem C.3 is introduced next (stated here only for the time-invariant case for notational compactness and because we don’t need the time-varying case in this book), and an equivalent dissipativity-type characterization of ISS is also introduced.

**Theorem C.4 (Lyapunov characterization of ISS).** For the system
\[
\dot{x} = f(x, u),
\]
the following properties are equivalent:

1. the system is ISS;
2. there exists a smooth ISS–Lyapunov function;
3. there exist a smooth, positive-definite, radially unbounded function $V$ and class-$\mathcal{K}_\infty$ functions $\rho_1$ and $\rho_2$ such that the following dissipativity inequality is satisfied:
\[
\frac{\partial V}{\partial x} f(x, u) \leq -\rho_1(|x|) + \rho_2(|u|).
\]

The following lemma establishes a useful property that a cascade of two ISS systems is itself ISS.

**Lemma C.2.** Suppose that in the system
\[
\begin{align*}
\dot{x}_1 &= f_1(t, x_1, x_2, u), \\
\dot{x}_2 &= f_2(t, x_2, u),
\end{align*}
\] (C.29) (C.30)
the $x_1$-subsystem is ISS with respect to $x_2$ and $u$, and the $x_2$-subsystem is ISS with respect to $u$; that is,
\[
\begin{align*}
|x_1(t)| &\leq \beta_1(|x_1(s)|, t - s) + \gamma_1 \left( \sup_{s \leq \tau \leq t} \{|x_2(\tau)| + |u(\tau)|\} \right), \\
|x_2(t)| &\leq \beta_2(|x_2(s)|, t - s) + \gamma_2 \left( \sup_{s \leq \tau \leq t} |u(\tau)| \right),
\end{align*}
\] (C.31) (C.32)
where $\beta_1$ and $\beta_2$ are class-$\mathcal{K}_\infty$ functions and $\gamma_1$ and $\gamma_2$ are class-$\mathcal{K}$ functions. Then the complete $x = (x_1, x_2)$-system is ISS with

$$|x(t)| \leq \beta(|s(t)|, t - s) + \gamma \left( \sup_{\tau \leq t} |u(\tau)| \right), \quad \text{(C.33)}$$

where

$$\beta(r, t) = \beta_1(2\beta_1(r, t/2) + 2\gamma_1(2\beta_2(r, 0), t/2) + \gamma_2(2\beta_2(\gamma_2(r) + t/2), t/2),$$

$$\gamma(r) = \beta_1(2\gamma_1(2\gamma_2(r) + t/2), 0) + \gamma_1(2\gamma_2(r) + 2r) + \gamma_2(r). \quad \text{(C.35)}$$

**Proof.** With $(s, t) = (t/2, t), (C.31)$ is rewritten as

$$|x_1(t)| \leq \beta_1(|x_1(t/2)|, t/2) + \gamma_1 \left( \sup_{t/2 \leq \tau \leq t} \{ |x_2(\tau)| + |u(\tau)| \} \right). \quad \text{(C.36)}$$

From (C.32), we have

$$\sup_{t/2 \leq \tau \leq t} |x_2(\tau)| \leq \sup_{t/2 \leq \tau \leq t} \left\{ \beta_2(|x_2(0)|, \tau) + \gamma_2 \left( \sup_{0 \leq \sigma \leq \tau} |u(\sigma)| \right) \right\}$$

$$\leq \beta_2(|x_2(0)|, t/2) + \gamma_2 \left( \sup_{0 \leq \tau \leq t} |u(\tau)| \right), \quad \text{(C.37)}$$

and from (C.31), we obtain

$$|x_1(t/2)| \leq \beta_1(|x_1(0)|, t/2) + \gamma_1 \left( \sup_{0 \leq \tau \leq t/2} \{ |x_2(\tau)| + |u(\tau)| \} \right)$$

$$\leq \beta_1(|x_1(0)|, t/2) + \gamma_1 \left( \sup_{0 \leq \tau \leq t/2} \{ \beta_2(|x_2(0)|, \tau) + \gamma_2 \left( \sup_{0 \leq \sigma \leq \tau} |u(\sigma)| \right) + |u(\tau)| \} \right)$$

$$\leq \beta_1(|x_1(0)|, t/2) + \gamma_1 \left( \beta_2(|x_2(0)|, 0) + \sup_{0 \leq \tau \leq t/2} \{ \gamma_2 (|u(\tau)|) + |u(\tau)| \} \right)$$

$$\leq \beta_1(|x_1(0)|, t/2) + \gamma_1 \left( 2 \beta_2(|x_2(0)|, 0) \right)$$

$$\leq \beta_1(|x_1(0)|, t/2) + \gamma_1 \left( 2 \sup_{0 \leq \tau \leq t/2} \{ \gamma_2 (|u(\tau)|) + |u(\tau)| \} \right), \quad \text{(C.38)}$$

where in the last inequality we have used the fact that $\delta(a + b) \leq \delta(2a) + \delta(2b)$ for any class-$\mathcal{K}$ function $\delta$ and any nonnegative $a$ and $b$. Then, substituting (C.37) and (C.38) into (C.36), we get
Lemma C.3.\ Let $v$ and $\rho$ be real-valued functions defined on $\mathbb{R}_+$, and let $b$ and $c$ be positive constants. If they satisfy the differential inequality

$$\dot{v} \leq -cv + bp(t)^2, \quad v(0) \geq 0,$$

Combining (C.39) and (C.32), we arrive at (C.33) with (C.34)–(C.35). \qed

Since (C.29) and (C.30) are ISS, then there exist ISS–Lyapunov functions $V_1$ and $V_2$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \rho_1, \alpha_2, \rho_2$ such that

$$\frac{\partial V_1}{\partial x_1} f_1(t, x_1, x_2, u) \leq -\alpha_1(|x_1|) + \rho_1(|x_2|) + \rho_1(|u|), \quad (C.40)$$

$$\frac{\partial V_2}{\partial x_2} f_2(t, x_1, x_2, u) \leq -\alpha_2(|x_2|) + \rho_2(|u|). \quad (C.41)$$

The functions $V_1, V_2, \alpha_1, \rho_1, \alpha_2, \rho_2$ can always be found such that

$$\rho_1 = \alpha_2/2. \quad (C.42)$$

Then the ISS–Lyapunov function for the complete system (C.29)–(C.30) can be defined as

$$V(x) = V_1(x_1) + V_2(x_2), \quad (C.43)$$

and its derivative

$$\dot{V} \leq -\alpha_1(|x_1|) - \frac{1}{2} \alpha_2(|x_2|) + \rho_1(|u|) + \rho_2(|u|) \quad (C.44)$$

establishes the ISS property of (C.29)–(C.30) by part 3 of Theorem C.4.

In some applications of input-to-state stability, the following lemma is useful, as it is much simpler than Theorem C.3.

Lemma C.3. Let $v$ and $\rho$ be real-valued functions defined on $\mathbb{R}_+$, and let $b$ and $c$ be positive constants. If they satisfy the differential inequality

$$\dot{v} \leq -cv + bp(t)^2, \quad v(0) \geq 0, \quad (C.45)$$
then the following hold:

(i) If $\rho \in L_\infty$, then $v \in L_\infty$ and

$$v(t) \leq v(0)e^{-ct} + \frac{b}{c}\|\rho\|_\infty^2.$$  \hfill (C.46)

(ii) If $\rho \in L_2$, then $v \in L_\infty$ and

$$v(t) \leq v(0)e^{-ct} + b\|\rho\|_2^2.$$  \hfill (C.47)

**Proof.** (i) From Lemma B.2, we have

$$v(t) \leq v(0)e^{-ct} + b\int_0^t e^{-c(t-\tau)}\rho(\tau)^2d\tau$$

$$\leq v(0)e^{-ct} + b\sup_{\tau \in [0,t]}\{\rho(\tau)^2\}\int_0^t e^{-c(t-\tau)}d\tau$$

$$\leq v(0)e^{-ct} + b\|\rho\|_\infty^2\frac{1}{c}(1 - e^{-ct})$$

$$\leq v(0)e^{-ct} + \frac{b}{c}\|\rho\|_\infty^2.$$  \hfill (C.48)

(ii) From (B.9), we have

$$v(t) \leq v(0)e^{-ct} + b\sup_{\tau \in [0,t]}\left\{e^{-c(t-\tau)}\right\}\int_0^t \rho(\tau)^2d\tau$$

$$= v(0)e^{-ct} + b\|\rho\|_2^2.$$  \hfill (C.49)

**Remark C.1.** From Lemma C.3, it follows that if

$$\dot{v} \leq -cv + b_1\rho_1(t)^2 + b_2\rho_2(t)^2, \quad v(0) \geq 0,$$  \hfill (C.50)

and $\rho_1 \in L_\infty$ and $\rho_2 \in L_2$, then $v \in L_\infty$ and

$$v(t) \leq v(0)e^{-ct} + \frac{b_1}{c}\|\rho_1\|_\infty^2 + b_2\|\rho_2\|_2^2.$$  \hfill (C.51)

This, in particular, implies the input-to-state stability with respect to two inputs: $\rho_1$ and $\|\rho_2\|_2$.

In this book we study feedback design for forward-complete systems with input delay.

**Definition C.6 (Forward completeness).** A system

$$\dot{x} = f(x,u)$$  \hfill (C.52)
with a locally Lipschitz vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is said to be forward complete if, for every initial condition $x(0) = \xi$ and every measurable locally essentially bounded input signal $u : \mathbb{R}_+ \to \mathbb{R}$, the corresponding solution is defined for all $t \geq 0$; i.e., the maximal interval of existence of solutions is $T_{\text{max}} = +\infty$.

The following Lyapunov characterization of forward completeness was proved in [4].

**Theorem C.5.** System (C.52) is forward complete if and only if there exist a nonnegative-valued, radially unbounded, smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ and a class-$\mathcal{K}_\infty$ function $\sigma$ such that

$$\frac{\partial V(x)}{\partial x} f(x, u) \leq V(x) + \sigma(|u|) \quad (C.53)$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$. 


Appendix D
Bessel Functions

We review the definitions, basic properties, and graphical forms of Bessel functions.

D.1 Bessel Function $J_n$

The function (depicted in Fig. D.1)

$$y(x) = J_n(x)$$  \hspace{1cm} (D.1)

is a solution to the following ODE:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$  \hspace{1cm} (D.2)

Series representation

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!}$$  \hspace{1cm} (D.3)

Properties

$$2nJ_n(x) = x(J_{n-1}(x) + J_{n+1}(x))$$  \hspace{1cm} (D.4)

$$J_n(-x) = (-1)^n J_n(x)$$  \hspace{1cm} (D.5)

Differentiation

$$\frac{d}{dx} J_n(x) = \frac{1}{2} (J_{n-1}(x) - J_{n+1}(x)) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$  \hspace{1cm} (D.6)

$$\frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}, \quad \frac{d}{dx} (x^{-n} J_n(x)) = -x^{-n} J_{n+1}$$  \hspace{1cm} (D.7)
Asymptotic properties

\[ J_n(x) \approx \frac{1}{n!} \left( \frac{x}{2} \right)^n, \quad x \to 0 \quad \text{(D.8)} \]

\[ J_n(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\pi n}{2} - \frac{\pi}{4} \right), \quad x \to \infty \quad \text{(D.9)} \]

**D.2 Modified Bessel Function \( I_n \)**

The function (depicted in Fig. D.2)

\[ y(x) = I_n(x) \quad \text{(D.10)} \]

is a solution to the following ODE:

\[ x^2 y'' + xy' - (x^2 + n^2)y = 0 \quad \text{(D.11)} \]

Series representation

\[ I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(m+n)!} \quad \text{(D.12)} \]
D.2 Modified Bessel Function $I_n$

Relationship with $J_n(x)$

\[
I_n(x) = i^{-n}J_n(ix), \quad I_n(ix) = i^n J_n(x)
\]  

(D.13)

Properties

\[
2nI_n(x) = x(I_{n-1}(x) - I_{n+1}(x))
\]  

(D.14)

\[
I_n(-x) = (-1)^n I_n(x)
\]  

(D.15)

Differentiation

\[
\frac{d}{dx} I_n(x) = \frac{1}{2}(I_{n-1}(x) + I_{n+1}(x)) = \frac{n}{x} I_n(x) + I_{n+1}(x)
\]  

(D.16)

\[
\frac{d}{dx} (x^n I_n(x)) = x^n I_{n-1}, \quad \frac{d}{dx} (x^{-n} I_n(x)) = x^{-n} I_{n+1}
\]  

(D.17)

Asymptotic properties

\[
I_n(x) \approx \frac{1}{n!} \left( \frac{x}{2} \right)^n, \quad x \to 0
\]  

(D.18)

\[
I_n(x) \approx \frac{e^x}{\sqrt{2\pi x}}, \quad x \to \infty
\]  

(D.19)

Fig. D.2 Modified Bessel functions $I_n$. 
Appendix E

Parameter Projection

Our adaptive designs rely on the use of parameter projection in our identifiers. We provide a treatment of projection for a general convex parameter set. The treatment for some of our designs where projection is used for only a scalar estimate $\hat{D}$ is easily deduced from the general case.

Let us define the following convex set:

$$
\Pi = \{ \hat{\theta} \in \mathbb{R}^p \mid \mathcal{P}(\hat{\theta}) \leq 0 \},
$$

(E.1)

where by assuming that the convex function $\mathcal{P} : \mathbb{R}^p \rightarrow \mathbb{R}$ is smooth, we ensure that the boundary $\partial \Pi$ of $\Pi$ is smooth. Let us denote the interior of $\Pi$ by $\overset{o}{\Pi}$ and observe that $\nabla_{\hat{\theta}} \mathcal{P}$ represents an outward normal vector at $\hat{\theta} \in \partial \Pi$. The standard projection operator is

$$
\text{Proj}\{\tau\} = \begin{cases}
\tau, & \hat{\theta} \in \overset{o}{\Pi} \text{ or } \nabla_{\hat{\theta}} \mathcal{P}^T \tau \leq 0, \\
(I - \Gamma \nabla_{\hat{\theta}} \mathcal{P} \nabla_{\hat{\theta}} \mathcal{P}^T) \tau, & \hat{\theta} \in \partial \Pi \text{ and } \nabla_{\hat{\theta}} \mathcal{P}^T \tau > 0,
\end{cases}
$$

(E.2)

where $\Gamma$ belongs to the set $\mathcal{G}$ of all positive-definite symmetric $p \times p$ matrices. Although Proj is a function of three arguments, $\tau$, $\hat{\theta}$ and $\Gamma$, for compactness of notation, we write only $\text{Proj}\{\tau\}$.

The meaning of (E.2) is that when $\hat{\theta}$ is in the interior of $\Pi$ or at the boundary with $\tau$ pointing inward, then $\text{Proj}\{\tau\} = \tau$. When $\hat{\theta}$ is at the boundary with $\tau$ pointing outward, then $\text{Proj}$ projects $\tau$ on the hyperplane tangent to $\partial \Pi$ at $\hat{\theta}$.

In general, the mapping (E.2) is discontinuous. This is undesirable for two reasons. First, the discontinuity represents a difficulty for implementation in continuous time. Second, since the Lipschitz continuity is violated, we cannot use standard theorems for the existence of solutions. Therefore, we sometimes want to smooth the projection operator. Let us consider the following convex set:

$$
\Pi_\varepsilon = \{ \hat{\theta} \in \mathbb{R}^p \mid \mathcal{P}(\hat{\theta}) \leq \varepsilon \},
$$

(E.3)
which is a union of the set $\Pi$ and an $O(\varepsilon)$-boundary layer around it. We now modify (E.2) to achieve continuity of the transition from the vector field $\tau$ on the boundary of $\Pi$ to the vector field $\left( I - \Gamma \frac{\partial \hat{P}}{\partial \hat{\theta}} \frac{\partial \hat{P}^T}{\partial \hat{\theta}} \right) \tau$ on the boundary of $\Pi$:

$$\text{Proj}\{\tau\} = \begin{cases} 
\tau, & \hat{\theta} \in \overset{\circ}{\Pi} \text{ or } \nabla_{\hat{\theta}} P^T \tau \leq 0, \\
(1 - c(\hat{\theta}) \Gamma \frac{\partial \hat{P}}{\partial \hat{\theta}} \frac{\partial \hat{P}^T}{\partial \hat{\theta}}) \tau, & \hat{\theta} \in \Pi \setminus \overset{\circ}{\Pi} \text{ and } \nabla_{\hat{\theta}} P^T \tau > 0,
\end{cases}$$

(E.4)

$$c(\hat{\theta}) = \min \left\{ 1, \frac{P(\hat{\theta})}{\varepsilon} \right\}.$$  

(E.5)

It is helpful to note that $c(\partial \Pi) = 1$ and $c(\partial \Pi) = 1$.

In our proofs of stability of adaptive systems, we use the following technical properties of the projection operator (E.4).

**Lemma E.1 (Projection operator).** The following are the properties of the projection operator (E.4):

(i) The mapping $\text{Proj}: \mathbb{R}^p \times \Pi \times \mathcal{G} \rightarrow \mathbb{R}^p$ is locally Lipschitz in its arguments $\tau, \hat{\theta},$ and $\Gamma$.

(ii) $\text{Proj}\{\tau\}^T \Gamma^{-1} \text{Proj}\{\tau\} \leq \tau^T \Gamma^{-1} \tau, \forall \hat{\theta} \in \Pi.$

(iii) Let $\Gamma(t), \tau(t)$ be continuously differentiable and

$$\hat{\theta} = \text{Proj}\{\tau\}, \quad \hat{\theta}(0) \in \Pi.$$

Then, on its domain of definition, the solution $\hat{\theta}(t)$ remains in $\Pi$.

(iv) $-\hat{\theta}^T \Gamma^{-1} \text{Proj}\{\tau\} \leq -\hat{\theta}^T \Gamma^{-1} \tau, \forall \hat{\theta} \in \Pi, \theta \in \Pi.$

**Proof.** (i) The proof of this point is lengthy but straightforward and is omitted here.

(ii) For $\hat{\theta} \in \overset{\circ}{\Pi}$ or $\nabla_{\hat{\theta}} P^T \tau \leq 0$, we have $\text{Proj}\{\tau\} = \tau$ and (ii) trivially holds with equality. Otherwise, a direct computation gives

$$\text{Proj}\{\tau\}^T \Gamma^{-1} \text{Proj}\{\tau\} = \tau^T \Gamma^{-1} \tau - 2c(\hat{\theta}) \frac{\left(\nabla_{\hat{\theta}} P^T \tau\right)^2}{\nabla_{\hat{\theta}} P^T \Gamma \nabla_{\hat{\theta}} P} + c(\hat{\theta})^2 \left| \frac{\nabla_{\hat{\theta}} P \nabla_{\hat{\theta}} P^T \tau}{\nabla_{\hat{\theta}} P^T \Gamma \nabla_{\hat{\theta}} P} \right|^2$$

$$= \tau^T \Gamma^{-1} \tau - c(\hat{\theta}) \left( 2 - c(\hat{\theta}) \right) \frac{\left(\nabla_{\hat{\theta}} P^T \tau\right)^2}{\nabla_{\hat{\theta}} P^T \Gamma \nabla_{\hat{\theta}} P}$$

$$\leq \tau^T \Gamma^{-1} \tau,$$

(E.6)

where the last inequality follows by noting that $c(\hat{\theta}) \in [0, 1]$ for $\hat{\theta} \in \Pi \setminus \overset{\circ}{\Pi}$. 

\[418 \text{ Parameter Projection}\]
(iii) Using the definition of the Proj operator, we get

\[
\nabla_{\hat{\theta}} \mathcal{P}^T \text{Proj}\{\tau\} = \begin{cases} \\
\nabla_{\hat{\theta}} \mathcal{P}^T \tau, & \hat{\theta} \in \Pi \text{ or } \nabla_{\hat{\theta}} \mathcal{P}^T \tau \leq 0, \\
(1 - c(\hat{\theta})) \nabla_{\hat{\theta}} \mathcal{P}^T \tau, & \hat{\theta} \in \Pi_\varepsilon \setminus \Pi \text{ and } \nabla_{\hat{\theta}} \mathcal{P}^T \tau > 0,
\end{cases}
\]

(E.7)

which, in view of the fact that \(c(\hat{\theta}) \in [0, 1]\) for \(\hat{\theta} \in \Pi_\varepsilon \setminus \Pi\), implies that

\[
\nabla_{\hat{\theta}} \mathcal{P}^T \text{Proj}\{\tau\} \leq 0 \text{ whenever } \hat{\theta} \in \partial \Pi_\varepsilon;
\]

(E.8)

that is, the vector Proj\{\tau\} either points inside \(\Pi_\varepsilon\) or is tangential to the hyperplane of \(\partial \Pi_\varepsilon\) at \(\hat{\theta}\). Since \(\hat{\theta}(0) \in \Pi_\varepsilon\), it follows that \(\hat{\theta}(t) \in \Pi_\varepsilon\) as long as the solution exists.

(iv) For \(\hat{\theta} \in \Pi\), (iv) trivially holds with equality. For \(\hat{\theta} \in \Pi_\varepsilon \setminus \Pi\), since \(\theta \in \Pi\) and \(\mathcal{P}\) is a convex function, we have

\[
(\theta - \hat{\theta})^T \nabla_{\hat{\theta}} \mathcal{P} \leq 0 \text{ whenever } \hat{\theta} \in \Pi_\varepsilon \setminus \Pi.
\]

(E.9)

With (E.9), we now calculate

\[
-\tilde{\theta}^T \Gamma^{-1} \text{Proj}\{\tau\} = -\tilde{\theta}^T \Gamma^{-1} \tau
\]

\[
+ \begin{cases} \\
0, & \hat{\theta} \in \Pi \text{ or } \nabla_{\hat{\theta}} \mathcal{P}^T \tau \leq 0 \\
c(\hat{\theta}) \frac{(\tilde{\theta}^T \nabla_{\hat{\theta}} \mathcal{P})(\nabla_{\hat{\theta}} \mathcal{P}^T \tau)}{\nabla_{\hat{\theta}} \mathcal{P}^T \Gamma \nabla_{\hat{\theta}} \mathcal{P}}, & \hat{\theta} \in \Pi_\varepsilon \setminus \Pi \text{ and } \nabla_{\hat{\theta}} \mathcal{P}^T \tau > 0
\end{cases}
\]

\[
\leq -\tilde{\theta}^T \Gamma^{-1} \tau,
\]

(E.10)

which completes the proof. \(\square\)
Appendix F
Strict-Feedforward Systems: A General Design

In this appendix and the next two appendices we give an extensive review of tools for the design of explicitly computable feedback laws for the stabilization of strict-feedforward systems. The emphasis is on a subclass of strict-feedforward systems that are feedback linearizable. This property yields a closed-form expression for the inverse backstepping transformation for the infinite-dimensional actuator state in predictor-based feedback laws. In addition, for all strict-feedforward systems, the direct backstepping transformation is obtainable in closed form, making them the most interesting class of systems from the point of view of predictor-based feedback design.

\section*{F.1 The Class of Systems}

Consider the class of \textit{strict-feedforward systems}

\begin{equation}
\begin{aligned}
\dot{x}_1 &= x_2 + \psi_1(x_2, x_3, \ldots, x_n) + \phi_1(x_2, x_3, \ldots, x_n)u, \\
\dot{x}_2 &= x_3 + \psi_2(x_3, \ldots, x_n) + \phi_2(x_3, \ldots, x_n)u, \\
&\quad \vdots \\
\dot{x}_{n-2} &= x_{n-1} + \psi_{n-2}(x_{n-1}, x_n) + \phi_{n-2}(x_{n-1}, x_n)u, \\
\dot{x}_{n-1} &= x_n + \phi_{n-1}(x_n)u, \\
\dot{x}_n &= u,
\end{aligned}
\tag{F.1}
\end{equation}

or, for short,

\begin{equation}
\begin{aligned}
\dot{x}_i &= x_{i+1} + \psi_i(x_{i+1}) + \phi_i(x_{i+1})u, \quad i = 1, 2, \ldots, n,
\end{aligned}
\tag{F.2}
\end{equation}

where

\begin{equation}
\bar{x}_j = [x_j, x_{j+1}, \ldots, x_n]^{T},
\tag{F.3}
\end{equation}

421
\[ x_{n+1} = u, \quad (F.4) \]
\[ \phi_n = 1, \quad (F.5) \]
\[ \phi_i(0) = 0, \quad (F.6) \]
\[ \psi_i(x_{i+1}, 0, \ldots, 0) \equiv 0, \quad (F.7) \]
\[ \frac{\partial \psi_i(0)}{\partial x_j} = 0 \quad (F.8) \]

for \( i = 1, 2, \ldots, n - 1, j = i + 1, \ldots, n \).

Relative to the class of systems in [195], we make a trade of generality for conceptual clarity by requiring that the drift term be of the form \( x_{i+1} + \psi_i(x_{i+1}) \), where the \( \psi_i \)'s, in addition to being higher-order, vanish whenever \( x_{i+2}, \ldots, x_n \) vanish. At the end of Section H.1 we show that this restriction can be relaxed in some cases; however, we keep it throughout most of the present appendix and the next two appendices for notational and conceptual convenience. We note that condition (F.7) means, in particular, that \( \psi_n = 0 \) and \( \psi_{n-1}(x_n) \equiv 0 \).

### F.2 The Sepulchre–Jankovic–Kokotovic Algorithm

The control law for this class of systems is designed as follows. Let

\[ \beta_{n+1} = 0, \quad (F.9) \]
\[ \alpha_{n+1} = 0. \quad (F.10) \]

For \( i = n, n - 1, \ldots, 2, 1 \), the designer needs to symbolically (preferably) or numerically calculate

\[ z_i = x_i - \beta_{i+1}, \quad (F.11) \]
\[ w_i(x_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j} \phi_j - \frac{\partial \beta_{i+1}}{\partial x_n}, \quad (F.12) \]
\[ \alpha_i(x_i) = \alpha_{i+1} - w_i z_i, \quad (F.13) \]
\[ \beta_i(x_i) = -\int_0^\infty \left[ \varepsilon_i^{[i]}(\tau, x_i) + \psi_{i-1} \left( \frac{\varepsilon_i^{[i]}(\tau, x_i)}{\alpha_i} \right) \right. \]
\[ \left. + \phi_{i-1} \left( \frac{\varepsilon_i^{[i]}(\tau, x_i)}{\alpha_i} \right) \right] d\tau, \quad (F.14) \]

where the notation in the integrand of (F.14) refers to the solutions of the (sub)system(s)

\[ \frac{d}{d\tau} \varepsilon_j^{[i]} = \varepsilon_{j+1}^{[i]} + \psi_j \left( \frac{\varepsilon_j^{[i]}}{\alpha_j} \right) + \phi_j \left( \frac{\varepsilon_j^{[i]}}{\alpha_j} \right), \quad (F.15) \]
for $j = i, i + 1, \ldots, n$, at time $\tau$, starting from the initial condition $x_i$. The control law is

$$u = \alpha_1.$$  \hfill (F.16)

It is important to first understand the meaning of the integral in (F.14). Clearly, the solution $\xi_j(\tau, x_i)$ is impossible to obtain analytically in general but, when possible, will lead to an implementable control law. Note that the last of the $\beta_i$'s that needs to be computed is $\beta_2$ ($\beta_1$ is not defined).

The stability analysis of the closed-loop system is straightforward. Starting with the observation that

$$x_{i+1} + \psi_i + \phi_i \alpha_{i+1} = \sum_{j=i+1}^{n} \frac{\partial \beta_{i+1}}{\partial x_j} \left(x_{j+1} + \psi_j + \phi_j \alpha_{i+1}\right),$$  \hfill (F.17)

it is easy to verify that

$$\dot{z}_i = w_i \left(u + \sum_{j=i+1}^{n} w_j z_j\right).$$  \hfill (F.18)

Noting from (F.16) and (F.13) that

$$u = -\sum_{i=1}^{n} w_i z_i,$$  \hfill (F.19)

we get

$$\dot{z}_i = -w_i^2 z_i - \sum_{j=1}^{i-1} w_i w_j z_j$$  \hfill (F.20)

(note that this notation implies that $\dot{z}_1 = -w_1^2 z_1$). Taking the Lyapunov function

$$V = \frac{1}{2} \sum_{i=1}^{n} z_i^2,$$  \hfill (F.21)

one obtains

$$\dot{V} = -\frac{1}{2} \sum_{i=1}^{n} w_i^2 z_i^2 - \frac{1}{2} \left(\sum_{i=1}^{n} z_i w_i\right)^2.$$  \hfill (F.22)

**Theorem F.1 ([195]).** The feedback system (F.2), (F.16) is globally asymptotically stable at the origin.

**Proof.** Although the proof of this theorem is available in [195], we provide some of its elements here for two reasons—one is to ease a nonexpert reader into the topic of forwarding, and the other is that some of our further arguments mimic those used in the proof of this theorem (and we shall not repeat them). First, a careful inspection of the design algorithm reveals that

$$\beta_i(0) = 0,$$  \hfill (F.23)
which means that the triangular coordinate transformation $z(x)$ is a global diffeomorphism with

$$z(0) = 0.$$  \hfill (F.24)

From (F.22), it then follows that the equilibrium $x = 0$ is globally stable. LaSalle’s theorem guarantees that $z_i w_i \to 0$ as $t \to \infty$. Since $w_n \equiv 1$ and $z_n = x_n$, it follows that $x_n(t) \to \infty$. One can verify recursively that $w_i(0) = 1$ for all $i$ [this is a consequence of the fact that $x_{n+1} = u$ and of the presence of the linear term $\xi_i^{[i]}$ in (F.14)]. Thus, it follows that $w_{n-1}(x_n(t)) \to 1$, which, along with $\beta_n(0) = 0$, implies that $x_{n-1}(t) \to \infty$. Continuing in this fashion, one recursively shows that $w_i(t) \to 1, \beta_{i+1}(t) \to 0$ for each $i$ and, thus, that $x(t) \to 0$ as $t \to \infty$.  \hfill \Box
Appendix G
Strict-Feedforward Systems: A Linearizable Class

In this appendix we focus on a linearizable subclass of strict-feedforward systems and specialize the SJK algorithm to this class. For linearizable systems, explicit formulas for the control laws are obtained.

G.1 Linearizability of Feedforward Systems

The main interest from the application’s point of view is making the forwarding control law explicit, namely, making the closed-form computation of the integral in (F.14) tractable. Toward that end, let us start by noting that system (F.15), which needs to be solved analytically, can be written in the \( z \)-coordinates\(^1\) as

\[
\frac{d}{d\tau} \xi_i^j = -w_j^2 \xi_i^j - \sum_{l=1}^{j-1} w_j w_l \xi_l^j, \quad j = i, i+1, \ldots, n, \tag{G.1}
\]

which is obtained with \( \dot{\xi}_i^j = w_j \alpha_i \).

Suppose now that (somehow) all of the \( w_l \)'s were equal to 1 [for all values of their arguments, rather than just \( w_l(0) = 1 \)]. We would have a lower-triangular linear system

\[
\frac{d}{d\tau} \xi_i^j = -\xi_i^j - \sum_{l=1}^{j-1} \xi_l^j, \quad j = i, i+1, \ldots, n, \tag{G.2}
\]

which is easily solvable in closed form. Then the only difficulty remaining would be the integration with respect to \( \tau \) of the integral (F.14) (using an appropriate coordinate change from \( \xi_i^j \) to \( \xi_i^j \)). Calculating the integral is by no means trivial, but

\(^1\) We point out that, analogous to (F.15), we use \( \xi \), a Greek version of \( z \), to denote the solution of the \( z \)-subsystem, under the control \( \alpha_i \), starting from initial condition \( z \). It should also be understood that \( w_j \) stands for \( w_j \left( \xi_i^j (\Sigma_{j+1}) \right) \), where \( \xi_k^j = \xi_k^j + \beta_{k+1} \left( \xi_{k+1}^j \right) \), and so on (i.e., expressing \( w_j \) as a function of \( \xi_i^j \)).
it is a much easier task than solving the nonlinear ODE (F.15) and calculating the integral.

Before we start exploring the conditions under which one would get

\[ w_i(x_{i+1}) = \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j} \phi_j - \frac{\partial \beta_{i+1}}{\partial x_n} = 1, \]  

(G.3)

let us note another consequence of this. In this case the coordinate change, before applying the feedback, would yield

\[
\dot{z} = \begin{bmatrix}
0 & 1 & 1 & \cdots & 1 \\
0 & 0 & 1 & \vdots & \\
\vdots & 0 & 0 & \ddots & \\
\vdots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 0 & 0
\end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} u.
\]  

(G.4)

We refer to this as the Teel [218] canonical form. This is a completely controllable linear system. Hence, the systems that satisfy condition (G.3) are linearizable (into this linear form and, ultimately, into the Brunovsky canonical form).

Thus, the exploration of analytical computability of control laws for strict-feedforward systems amounts, to a large extent, to a study of linearizability. Clearly, merely checking the coordinate-free conditions for linearizability [67] won’t get us any closer to actually finding the control laws. Such a test would lead to conditions on the \( \phi_i \)'s in the form of partial differential equations that they have to satisfy (these conditions would arise from the involutivity test).

Until now we have used the word “linearizable” somewhat loosely. In the next definition we make this notion precise.

**Definition G.1.** If there exists a diffeomorphism

\[
y_i = x_i - \theta_{i+1}(x_{i+1}), \quad i = 1, \ldots, n - 1, \]  

(G.5)
\[
y_n = x_n,
\]  

(G.6)

where

\[
\theta_i(0) = \frac{\partial \theta_i(0)}{\partial x_j} = 0, \quad i = 2, \ldots, n, \quad j = i, \ldots, n,
\]  

(G.7)

transforming the strict-feedforward system (F.2)–(F.7) into a system of the form

\[
\dot{y}_i = y_{i+1}, \quad i = 1, 2, \ldots, n - 1, \]  

(G.8)
\[
\dot{y}_n = u,
\]  

(G.9)

then system (F.2)–(F.7) is said to be *diffeomorphically equivalent to a chain of integrators (DECI).*
We point out that the term “DECI” does not reflect that (G.5), (G.6) restrict the class of admissible diffeomorphisms to a “triangular” form. In the next theorem we give sufficient conditions for characterizing DECI strict-feedforward systems.

**Theorem G.1.** All strict-feedforward systems (F.2)–(F.7) with

\[
\phi_{n-1}(x_n) = \theta'_n(x_n), \quad \psi_{n-1}(x_n) = 0,
\]

and

\[
\phi_i(x_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(x_{i+1})}{\partial x_j} \phi_j(x_{j+1}) + \frac{\partial \theta_{i+1}(x_{i+1})}{\partial x_n},
\]

\[
\psi_i(x_{i+1}) = \sum_{j=i+1}^{n-1} \frac{\partial \theta_{i+1}(x_{i+1})}{\partial x_j} \left( x_{j+1} + \psi_j(x_{j+1}) \right) - \theta_{i+2}(x_{i+2})
\]

for \( i = n - 2, \ldots, 1 \), using some \( C^1 \) scalar-valued functions \( \theta_i(x_i) \) satisfying (G.7), are DECI.

**Proof.** Straightforward to verify using (G.5), (G.6). \qed

Theorem G.1 is not a substitute for a geometric test of linearizability, nor is it a control design tool. It is just a parametrization of a subclass of strict-feedforward systems that are DECI.

For instance, all third-order strict-feedforward systems of the form

\[
\dot{x}_1 = x_2 + \frac{\partial \theta_2(x_2, x_3)}{\partial x_2} x_3 - \theta_3(x_3)
\]

\[
+ \left( \frac{\partial \theta_2(x_2, x_3)}{\partial x_2} \theta'_3(x_3) + \frac{\partial \theta_2(x_2, x_3)}{\partial x_3} \right) u,
\]

\[
\dot{x}_2 = x_3 + \theta'_3(x_3) u,
\]

\[
\dot{x}_3 = u
\]

are linearizable, where any two locally quadratic \( C^1 \) functions \( \theta_2(x_2, x_3) \) and \( \theta_3(x_3) \) are the “parameters.”

In the next section we show that the SJK procedure greatly simplifies for DECI strict-feedforward systems and, in particular, directly leads to (13.96) for (13.91) without having to solve nonlinear ODEs of the form (F.15).
G.2 Algorithms for Linearizable Feedforward Systems

**General Algorithm**

For linearizable strict-feedforward systems, we present the following design algorithm, which eliminates the requirement to solve the ODEs (F.15) and reduces the problem to calculating a set of integrals with respect to time. Let

\[ \beta_{n+1} = 0, \tag{G.17} \]
\[ \alpha_{n+1} = 0. \tag{G.18} \]

For \( i = n, n-1, \ldots, 2, 1, \)

\[ \alpha_i(x_i) = -\sum_{j=i}^{n} (x_j - \beta_{j+1}(x_{j+1})) , \tag{G.19} \]

\[ \xi_n^{[i]}(\tau, x_i) = e^{-\tau} \sum_{k=0}^{n-i} \frac{(-\tau)^k}{k!} (x_{n-k} - \beta_{n-k+1}(x_{n-k+1})) , \tag{G.20} \]

\[ \xi_j^{[i]}(\tau, x_i) = e^{-\tau} \sum_{k=0}^{j-i} \frac{(-\tau)^k}{k!} (x_{j-k} - \beta_{j-k+1}(x_{j-k+1})) + \beta_{j+1} \left( \xi_{j+1}^{[i]}(\tau, x_i) \right) , \tag{G.21} \]

\[ \beta_i(x_i) = -\int_{0}^{\infty} \left[ \xi_i^{[i]}(\tau, x_i) + \psi_{i-1} \left( \xi_i^{[i]}(\tau, x_i) \right) \right] d\tau , \tag{G.22} \]

\[ + \phi_{i-1} \left( \xi_i^{[i]}(\tau, x_i) \right) \alpha_i \left( \xi_i^{[i]}(\tau, x_i) \right) d\tau. \tag{G.23} \]

The control law is

\[ u = \alpha_1. \tag{G.24} \]

We stress that, due to linearizability, the ODEs (F.15) are solved in closed form, and the only calculation remaining is the integrals (G.23), which can be obtained with symbolic software (coded in Mathematica or Maple/MATLAB). This calculation is particularly straightforward (and can be done, in principle, by hand) when the nonlinearities \( \psi_i(\cdot), \phi_i(\cdot) \) are polynomial. In that case, the following identity is useful in calculating (G.23):

\[ \int_{0}^{\infty} \tau^p e^{-q\tau} d\tau = \frac{p!}{q^{p+1}}, \quad \forall p, q \in \mathbb{N}. \tag{G.25} \]

**Theorem G.2.** If the strict-feedforward plant (F.2)–(F.7) is DECI, then the feedback system (F.2), (G.24) is globally asymptotically stable at the origin.

**Proof.** One can verify that in the coordinates

\[ z_i = x_i - \beta_{i+1}(x_{i+1}) \tag{G.26} \]
the control system becomes (G.4), and under the feedback control (G.24), the resulting system is
\[
\dot{z} = \begin{bmatrix}
-1 & 0 & 0 & \cdots & 0 \\
-1 & -1 & 0 & \ddots & \\
\vdots & -1 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & 0 \\
-1 & \cdots & \cdots & -1 & -1
\end{bmatrix} \begin{bmatrix}
z \\
\end{bmatrix}.
\] (G.27)

The rest of the proof is as in Theorem F.1.

As we indicated in Section G.1, checking the geometric conditions for linearizability is easy, whereas actually constructing the linearizing coordinates is not. The algorithm (G.19)–(G.23) constructs the coordinate change into the (non-Brunovsky) Teel canonical form (G.4). The next theorem gives the coordinate change into the Brunovsky/chain-of-integrators form.

**Theorem G.3.** If the strict-feedforward plant (F.2)–(F.7) is DECI, it has a relative degree\(^2\) \(n\) with respect to the output
\[
y_1 = \sum_{j=1}^{n} \left( n - \frac{1}{j - 1} \right) (-1)^{j-1} \left( x_j - \beta_{j+1}(x_{j+1}) \right).
\] (G.28)

Furthermore, the coordinate change (G.19)–(G.23), (G.26), and
\[
y_i = \sum_{j=i}^{n} \left( n - \frac{i}{j - i} \right) (-1)^{j-i} z_j, \quad i = 1, 2, \ldots, n,
\] (G.29)
converts system (F.2) into the chain of integrators (G.8)–(G.9).

**Proof.** By verification.

Inverse optimality, proved for the general case in [195], becomes particularly meaningful in the linearizable case.

**Theorem G.4 (Inverse optimality).** The control law
\[
u^* = 2\alpha_1(x) = -2 \sum_{j=1}^{n} \left( x_j - \beta_{j+1}(x_{j+1}) \right),
\] (G.30)
where \(\alpha_1(x)\) is defined via (G.19)–(G.23), minimizes the cost functional
\[
J = \int_0^\infty \left( l(x(t)) + u(t)^2 \right) dt
\] (G.31)

\(^2\) As defined in [67].
along the solutions of (F.2), where
\[
I(x) = \sum_{j=1}^{n} (x_j - \beta_{j+1}(\chi_{j+1}))^2 + \left( \sum_{j=1}^{n} (x_j - \beta_{j+1}(\chi_{j+1})) \right)^2 \tag{G.32}
\]
is a positive-definite, radially unbounded function. Furthermore, the control law (G.30) remains globally asymptotically stabilizing at the origin in the presence of input-unmodeled dynamics of the form
\[
a(I + \mathcal{P}), \tag{G.33}
\]
where \(a \geq 1/2\) is a constant, \(\mathcal{P}u\) is the output of any strictly passive nonlinear system\(^3\) with \(u\) as its input, and \(I\) denotes the identity operator.

**Proof.** It follows from Theorem 2.8, Theorem 2.17, and Corollary 2.18 in [109]. \(\square\)

The main result of this section was a control algorithm that eliminates the requirement to solve the ODEs (F.15) and reduces the problem to calculating only the integrals (G.23). In the next two sections we present algorithms that eliminate even the need to calculate the integrals (G.23) for two subclasses of DECI strict-feedforward systems.

**Linearizable Feedforward Systems of Type I**

Consider the class of strict-feedforward systems given by

\[
\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \tag{G.34}
\]
\[
\dot{x}_i = x_{i+1}, \quad i = 2, \ldots, n-1, \tag{G.35}
\]
\[
\dot{x}_n = u, \tag{G.36}
\]
where \(\pi_j(0) = 0\). Any system in this class is DECI.

**Theorem G.5.** The diffeomorphic transformation

\[
y_1 = x_1 - \sum_{j=2}^{n} \int_{0}^{x_j} \pi_j(s)ds, \tag{G.37}
\]
\[
y_i = x_i, \quad i = 2, \ldots, n, \tag{G.38}
\]
converts the strict-feedforward system (G.34)–(G.36) into the chain of integrators (G.8)–(G.9). The feedback law

\(^3\) With possibly nonzero initial conditions.
$$u = \alpha_1(x) = -\sum_{i=1}^{n} \left( \frac{n}{i-1} \right) y_i$$  \hspace{1cm} (G.39)$$
globally asymptotically stabilizes the origin of (G.34)–(G.36).

**Proof.** The first part is by verification. In the second part we note that the y-system has \( n \) closed-loop poles at \(-1\) and use that fact that the coordinate change is diffeomorphic. \( \square \)

We note that in the design (G.37), (G.38), and (G.39) we have completely circumvented the SJK procedure. It is therefore worth noting that, following the SJK procedure, one would have obtained

$$\alpha_i(x_i) = -\sum_{j=i}^{n} \left( \frac{n-i+1}{j-i} \right) x_j + \delta_{i,1} \sum_{j=2}^{n} \int_{0}^{x_j} \pi_j(s)ds,$$

\hspace{1cm} (G.40)

$$w_i = 1,$$

\hspace{1cm} (G.41)

where \( \delta_{i,1} \) denotes the Kronecker delta.\(^4\) However, the most important product of the SJK procedure is the coordinate shift \( \beta_i \) (from \( x \) to \( z \)), which is given in the context of the following result.

**Corollary G.1.** The control law (G.30), with \( \alpha_1(x) \) defined in (G.39), applied to the plant (G.34)–(G.36) achieves the result of Theorem G.4 with

$$\beta_{i+1}(x_{i+1}) = -\sum_{j=i+1}^{n} \left( \frac{n-i}{j-i} \right) x_j + \delta_{i,1} \sum_{j=2}^{n} \int_{0}^{x_j} \pi_j(s)ds \hspace{1cm} (G.42)$$

for \( i = 1, \ldots, n-1 \).

While in Section G.2 we showed that one can avoid having to solve the nonlinear ODEs (F.15), in Theorem G.5 we showed that for the feedforward subclass (G.34)–(G.36), one can also avoid having to calculate the integrals (G.23). In the next result we go even further and show that not only does one have a closed-form formula for the control law (G.39), but one can even get a closed-form formula for the solutions of the system under that control law. This is not just an aesthetically pleasing result—it will allow us, in Section H.1, to extend the constructive methodology to a class of strict-feedforward systems that are not linearizable.

To prevent confusion about the notation in the theorem, before its statement we emphasize that \( x \), which denotes the initial condition, is constant. This notation is important for a seamless use of the theorem in subsequent results. We also point out that, relative to the notation in Sections F.2 and G.1, \( \xi(\tau,x) \) and \( \zeta(\tau,z) \) should be understood, respectively, as \( \xi^{[1]}(\tau,x) \) and \( \zeta^{[1]}(\tau,z) \).

**Lemma G.1.** Starting from the initial condition denoted by \( x \), the solution \( \xi_i(\tau,x) \) of the feedback system (G.34)–(G.36), (G.37)–(G.39) at time \( \tau \) is

\( \xi \)

\( ^4 \) Note that (G.40) for \( i = 1 \) is the same as (G.39).
\[ \xi_i(\tau, x) = e^{-\tau} \left[ \sum_{j=1}^{n} \binom{n-i}{j-i} (-1)^{j-i} \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l \right. \\
\left. + (-1)^i \sum_{j=i}^{n-i} \binom{n-i}{j-i} \frac{\tau^{j-1}}{(j-1)!} \left( \sum_{m=2}^{n} \int_0^{x_m} \pi_m(s) \, ds \right) \right], \]
\[ i = 2, \ldots, n, \]  
\tag{G.43}

for \( i = 2, \ldots, n \), and
\[ \xi_1(\tau, x) = e^{-\tau} \left[ \sum_{j=1}^{n-1} \binom{n-1}{j-1} (-1)^{j-1} \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l \right. \\
\left. - \sum_{j=1}^{n-1} \binom{n-1}{j-1} \frac{\tau^{j-1}}{(j-1)!} \left( \sum_{m=2}^{n} \int_0^{x_m} \pi_m(s) \, ds \right) \right] \\
\left. + \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) \, ds, \right) \tag{G.44}

whereas the control signal is
\[ u = \tilde{\alpha}_1(\tau, x) = -e^{-\tau} \sum_{i=1}^{n} \binom{n}{i-1} \left[ \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \right. \\
\left. \times \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} x_l \right. \\
\left. + (-1)^i \sum_{j=i}^{n-i} \binom{n-i}{j-i} \frac{\tau^{j-1}}{(j-1)!} \left( \sum_{m=2}^{n} \int_0^{x_m} \pi_m(s) \, ds \right) \right]. \tag{G.45}

\textbf{Proof.} By using (G.37), (G.38), their inverse,
\[ x_1 = y_1 - \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) \, ds, \tag{G.46} \]
\[ x_i = y_i, \quad i = 2, \ldots, n, \tag{G.47} \]
the transformation (G.29), and its inverse,
\[ z_i = \sum_{j=i}^{n} \binom{n-i}{j-i} y_j, \quad i = 1, 2, \ldots, n, \tag{G.48} \]
the explicit form of the solution of (G.27),
\[ \zeta_j(\tau, z) = \sum_{k=0}^{j-1} \frac{(-\tau)^k}{k!} e^{-\tau} z_{j-k}, \tag{G.49} \]
and (G.39). \( \square \)


**Linearizable Feedforward Systems of Type II**

Consider the subclass of the strict-feedforward systems (F.2) given by

\[
\dot{x}_i = x_{i+1} + \phi_i(x_{i+1})u, \quad i = 1, \ldots, n - 1, \tag{G.50}
\]

\[
\dot{x}_n = u, \tag{G.51}
\]

where \(\phi_i(0) = 0\). In this section we construct control laws for a linearizable subclass of (G.50), (G.51).

To characterize the linearizable subclass, let us consider the functions \(\phi_{n-1}(x_n)\) and \(\phi_i(0, \ldots, 0, x_n), i = 1, \ldots, n - 2\), as given and introduce the following sequence of functions:

\[
\mu_n(x_n) = \int_0^{x_n} \phi_{n-1}(s) ds, \tag{G.52}
\]

\[
\mu_i(x_n) = \frac{1}{x_n} \int_0^{x_n} \left[ \phi_{i-1}(0, \ldots, 0, s) - \sum_{j=i+1}^{n} \mu_j(s) \gamma_{i-n-j}(0, \ldots, 0, s) \right] ds \tag{G.53}
\]

for \(i = n - 1, n - 2, \ldots, 2\), and

\[
\gamma_1(x_n) = \mu'_n(x_n), \tag{G.54}
\]

\[
\gamma_k(x_n) = \sum_{l=1}^{k-1} \gamma_l(x_n) \mu_{n+1-k}(x_n) + \frac{d \mu_{n+1-k}(x_n)}{dx_n} \tag{G.55}
\]

for \(k = 2, \ldots, n - 2\).

**Theorem G.6.** If

\[
\phi_i(x_{i+1}) = \sum_{j=i+1}^{n-1} \gamma_{j-i}(x_n)x_j + \phi_i(0, \ldots, 0, x_n), \tag{G.56}
\]

\(\forall x, i = 1, \ldots, n - 2\), then the diffeomorphic transformation

\[
y_i = x_i - \sum_{j=i+1}^{n} \mu_{i+1-n-j}(x_n)x_j, \quad i = 1, \ldots, n - 1, \tag{G.57}
\]

\[
y_n = x_n \tag{G.58}
\]

converts the strict-feedforward system (G.50)–(G.51) into the chain of integrators (G.8)–(G.9). The feedback law

\[
u = \alpha_1(x) = - \sum_{i=1}^{n} \binom{n}{i-1} y_i \tag{G.59}
\]

globally asymptotically stabilizes the origin of (G.50)–(G.51).
Proof. The first part is by (lengthy) verification. The rest is as in the proof of Theorem G.5. □

As in Section G.2, we point out that, following the SJK procedure, one would have obtained

$$\alpha_i(x) = -x_i - \sum_{m=i+1}^{n} x_m \left[ \binom{n-i+1}{m-i} - \sum_{j=i}^{m} \binom{n-i+1}{j-i} \right] \mu_{j+1+n-m}(x_n),$$  

$$w_i = 1,$$  

and the coordinate shift $\beta_i$ is given in the context of the following result.

Corollary G.2. The control law (G.30), with $\alpha_1(x)$ defined in (G.59), applied to the plant (G.50)–(G.51), (G.52), (G.53), (G.54), (G.55), (G.56), achieves the result of Theorem G.4 with

$$\beta_{i+1}(x_{i+1}) = - \sum_{m=i+1}^{n} x_m \left[ \binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \right] \mu_{j+1+n-m}(x_n),$$  

$$i = 1, \ldots, n-1.$$  

Example G.1. To illustrate the above concepts (and notation), let us consider a fourth-order example of a Type II feedforward system:

$$\dot{x}_1 = x_2 + \left( \frac{x_2^2}{2} - \frac{x_3 x_4}{12} \right) u,$$  

$$\dot{x}_2 = x_3 + \frac{x_3^2}{2} u,$$  

$$\dot{x}_3 = x_4 + x_4 u,$$  

$$\dot{x}_4 = u.$$  

The control law

$$u = -y_1 - 4y_2 - 6y_3 - 4y_4$$  

$$= -z_1 - z_2 - z_3 - z_4,$$  

where

$$y_1 = x_1 - \frac{x_4 x_2}{2} + \frac{x_4^2 x_3}{6} - \frac{x_4^4}{24},$$  

$$y_2 = x_2 - \frac{x_4 x_3}{2} + \frac{x_4^3}{6},$$  

$$y_3 = x_3 - \frac{x_4^2}{2},$$  

$$y_4 = x_4,$$
which is obtained with

\[ \mu_2 = \frac{x_4^3}{24}, \]  
\[ \mu_3 = -\frac{x_4^2}{6}, \]  
\[ \mu_4 = \frac{x_4}{2}, \]

and

\[ z_i = x_i - \beta_{i+1}, \]

with

\[ \beta_4 = \left( \frac{x_4}{2} - 1 \right) x_4, \]  
\[ \beta_3 = \left( \frac{x_4}{2} - 2 \right) x_3 - x_4 + x_4^2 - \frac{x_4^3}{6}, \]  
\[ \beta_2 = \left( \frac{x_4}{2} - 3 \right) x_2 + \left( -3 + \frac{3}{2} x_4 - \frac{x_4^2}{6} \right) x_3 \]
\[ - x_4 - \frac{3}{2} x_4^2 + \frac{1}{2} x_4^3 - \frac{1}{24} x_4^4, \]

achieves (G.4) for \( n = 4, \)

\[ \dot{z} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix} z \]

and

\[ (s + 1)^4 y_1(s) = 0. \]

To use the results of this section for control designs beyond the Type II class of systems, we need the inverse of the coordinate transformation (G.57). The explicit form of the inverse transformation is given in the following theorem.

**Lemma G.2.** Consider the series of functions

\[ \lambda_n(x_n) = \mu_n(x_n), \]  
\[ \lambda_i(x_n) = \frac{1}{x_n} \int_0^{x_n} \left( s \sum_{i=1}^n \gamma_i(s) \lambda_i(s) + \phi_{i-1}(0, \ldots, 0, s) \right) ds \]
for \( i = n - 1, \ldots, 2 \). The inverse of the diffeomorphic transformation (G.57) is

\[
x_i = y_i + \sum_{j=i+1}^{n} \lambda_{i+1+n-j}(y_n)y_j, \quad i = 1, \ldots, n - 1, \tag{G.84}
\]

\[
x_n = y_n. \tag{G.85}
\]

**Proof.** By induction, using the intermediate step that

\[
\gamma_{n-i+1}(x_n) = -\sum_{m=1}^{n-i} \gamma_m(x_n)\lambda_{m+i}(x_n) + \lambda_i'(x_n) \tag{G.86}
\]

for \( i = n - 1, \ldots, 3 \). \[\Box\]

As in Lemma G.1, in the next result we give a closed-form formula for the solutions of the feedback system from Theorem G.6, which will allow us, in Section H.1, to extend the constructive methodology to a class of strict-feedforward systems that are not linearizable.

**Lemma G.3.** Starting from the initial condition \( x \), the solution of the feedback system (G.50)–(G.56), (G.59) at time \( \tau \) is

\[
\xi_i(\tau, x) = e^{-\tau} \left[ \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \sum_{k=0}^{j-i-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right] 
\]

\[
\times \left( x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right) 
\]

\[
+ \sum_{p=i+1}^{n} \lambda_{i+1+n-p} \left( e^{-\tau} \sum_{k=0}^{n-1} \frac{(-\tau)^k}{k!} \sum_{l=n-k}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right) 
\]

\[
\times \left( x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right) 
\]

\[
\times \sum_{j=p}^{n} \binom{n-p}{j-p} (-1)^{j-p} \sum_{k=0}^{j-p-1} \frac{(-\tau)^k}{k!} \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} 
\]

\[
\times \left( x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right) \right], \tag{G.87}
\]

where \( i = 1, \ldots, n \), and the control signal is

\[
u = \tilde{\alpha}_1(\tau, x) = -e^{-\tau} \sum_{i=1}^{n} \binom{n-i}{i-1} \sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{j-i} \sum_{k=0}^{j-i-1} \frac{(-\tau)^k}{k!} 
\]

\[
\times \sum_{l=j-k}^{n} \binom{n-j+k}{l-j+k} \left( x_l - \sum_{m=l+1}^{n} \mu_{l+1+n-m}(x_n)\lambda_m \right). \tag{G.88}
\]
Proof. Analogous to the proof of Lemma G.1, employing also Lemma G.2. □

**Type I and II Systems in Dimensions Two and Three**

We start by pointing out that in dimension two all strict-feedforward systems are simultaneously of Types I and II. This implies that all second-order strict-feedforward systems are linearizable.

**Theorem G.7.** Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_2 + \phi_1(x_2)u, \quad (G.89) \\
\dot{x}_2 &= u, \quad (G.90)
\end{align*}
\]

where \( \phi_1(x_1) \) is continuous and

\[
\phi_1(0) = 0. \quad (G.91)
\]

The control law

\[
u = -x_1 - 2x_2 + \int_0^{x_2} \phi_1(s)\,ds \quad (G.92)
\]

ensures the global asymptotic stability of the origin.

**Proof.** By verification that

\[
\begin{align*}
\dot{z}_1 &= x_2 + u, \quad (G.93) \\
\dot{x}_2 &= u, \quad (G.94)
\end{align*}
\]

where

\[
\begin{align*}
z_1 &= x_1 - \beta_2(x_2), \quad (G.95) \\
\beta_2(x_2) &= -x_2 + \int_0^{x_2} \phi_1(s)\,ds, \quad (G.96)
\end{align*}
\]

and

\[
u = -z_1 - x_2. \quad (G.97)
\]

□

**Example G.2.** Let us now consider an example with

\[
\phi_1(x_2) = -x_2^2. \quad (G.98)
\]

This example was worked out in [196]. In this case the formula (G.92) gives\(^5\)

\[
u = -x_1 - 2x_2 - \frac{x_2^3}{3}. \quad (G.99)
\]

\(^5\) A reader checking the details in [196] will notice that this control law differs from (6.2.12) in [196]. This is due to an extra “\(x_2^2\)” term that has crept into the calculations in [196], in Eq. (6.2.7).
One should recognize that the “−x_1 − 2x_2” portion of the control law (G.99) is responsible for the exponential stabilization of the linearized system. To see that this linear controller is not sufficient for global stabilization, we plug it back into the plant and obtain a closed-loop system, written in the form of a second-order equation, as
\[ \ddot{x}_2 + (2 - x_2^2)\dot{x}_2 + x_2 = 0. \] (G.100)
This is a Van der Pol equation with an unstable limit cycle, which exhibits a finite escape instability. Hence, the nonlinear term “−x_3^3/3,” designed to accommodate the input nonlinearity \( \phi_1(x_2) = -x_2^3 \), is crucial for global stabilization.

The possibilities, as well as the limits, of Type I/II linearizability for strict-feedforward systems are best understood in dimension three. For the following class of systems, which represents a union of all three-dimensional Type I and Type II feedforward systems, a linearizing coordinate change and a stabilizing control law are designed in the next theorem.

**Theorem G.8.** Consider the class of systems
\[
\begin{align*}
\dot{x}_1 &= x_2 + \pi_2(x_2)x_3 + \left( \frac{x_3\phi_2(x_3) - \int_0^{x_3} \phi_2(s)ds}{x_3^2} \right) u, \\
\dot{x}_2 &= x_3 + \phi_2(x_3)u, \\
\dot{x}_3 &= u,
\end{align*}
\] (G.101) (G.102) (G.103)
where \( \pi_2(\cdot), \pi_3(\cdot) \in C^0 \), and \( \phi_2(\cdot) \in C^1 \) vanish at the origin and
\[ \pi_2(x_2)\phi_2(x_3) \equiv 0. \] (G.104)
Then the control law
\[ u = -y_1 - 3y_2 - 3y_3, \] (G.105)
where
\[
\begin{align*}
y_1 &= x_1 - \int_0^{x_2} \pi_2(s)ds - \mu_3(x_3)x_2 - \int_0^{x_3} \pi_3(s)ds \\
&\quad + \frac{1}{2} x_3 (\mu_3(x_3))^2 + \frac{1}{2} \int_0^{x_3} (\mu_3(s))^2 ds, \\
y_2 &= x_2 - \int_0^{x_3} \phi_2(s)ds, \\
y_3 &= x_3,
\end{align*}
\] (G.106) (G.107) (G.108)
and
\[ \mu_3(x_3) = \frac{\int_0^{x_3} \phi_2(s)ds}{x_3^3}, \] (G.109)
achieves global asymptotic stability of the origin.
Proof. One can verify that

\[ \ddot{y}_1 + 3 \dot{y}_1 + 3 \dot{y}_1 + y_1 = 0 \]  \hspace{1cm} (G.110)

and that

\[ \dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & -1 & -1 \end{bmatrix} z. \]  \hspace{1cm} (G.111)

\[ \square \]

A Type II example of a system from this class is

\[ \dot{x}_1 = x_2 + \left( \frac{1}{2} x_2 + x_3 \sin x_3 \right) u, \]  \hspace{1cm} (G.112)

\[ \dot{x}_2 = x_3 + x_3 u, \]  \hspace{1cm} (G.113)

\[ \dot{x}_3 = u, \]  \hspace{1cm} (G.114)

which is stabilized (and feedback-linearized) using

\[ u = -x_1 - 3x_2 - 3x_3 + \frac{x_2 x_3}{2} + \frac{3}{2} x_3^2 - \frac{1}{6} x_3^3 \]

\[ + x_3 \sin x_3 + \cos x_3 - 1. \]  \hspace{1cm} (G.115)

We point out that the key restriction in this example is the boldfaced 1/2. If this value were anything else (say, 1 or 0), this system would not be linearizable. It would, however, be stabilizable using the procedure we present in Section H.1.
Appendix H
Strict-Feedforward Systems: Not Linearizable

In this appendix we review three major extensions to the algorithms for linearizable strict-feedforward systems in Appendix H. The first extension is for certain strict-feedforward systems that are not linearizable. The second extension is for a class of systems referred to as the “block-feedforward” systems. The third extension combines forwarding and backstepping for an interlaced feedforward-feedback class of nonlinear systems.

H.1 Algorithms for Nonlinearizable Feedforward Systems

In this section we expand upon the Type I and II feedforward systems, to develop algorithms for feedforward systems that are not linearizable. Two classes of systems that we consider consist of a linearizable subsystem \([x_1, \ldots, x_n]^T\) and a scalar equation \(x_0\) that is (possibly) not linearizable. This, structure belongs to the class of nonflat Liouvillian systems of defect equal to one; see Chelouah [24] (especially Example 2).

Consider the following extension of the Type I strict-feedforward systems:

\[
\begin{align*}
\dot{x}_0 &= x_1 + \psi_0(x) + \phi_0(x)u, \\
\dot{x}_1 &= x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \\
\dot{x}_i &= x_{i+1}, \quad i = 2, \ldots, n - 1, \\
\dot{x}_n &= u,
\end{align*}
\]

\[(H.1) \quad (H.2) \quad (H.3) \quad (H.4)\]

where \(x\) denotes \([x_1, \ldots, x_n]^T\) (i.e., \(x_0\) is not included in \(x\)),

\[
\psi_0(0) = \phi_0(0) = \pi_j(0) = 0, \quad j = 2, \ldots, n,
\]

\[(H.5)\]
and
\[ \frac{\partial \psi_0(0)}{\partial x_i} = 0, \quad i = 1, \ldots, n. \] (H.6)

The subsystem (H.2)–(H.4) is linearizable. This makes it possible to develop a closed-form formula for a globally stabilizing SJK-type control law.

We propose the following design algorithm. Start by computing the expressions in Lemma G.1. Then calculate
\[ \beta_1(x) = -\int_0^\infty [\xi_1(\tau, x) + \psi_0(\xi(\tau, x)) + \phi_0(\xi(\tau, x)) \tilde{\alpha}_1(\tau, x)] \, d\tau, \] (H.7)
\[ w_0(x) = \phi_0(x) - \frac{\partial \beta_1(x)}{\partial x_1} \pi_n(x_n) - \frac{\partial \beta_1(x)}{\partial x_n}, \] (H.8)

and
\[ u = \alpha_0(x_0, x) = -w_0(x)(x_0 - \beta_1(x)) - \sum_{i=1}^{n} \left( \frac{n}{i - 1} \right) x_i + \sum_{i=2}^{n} \int_0^{x_i} \pi_i(s) \, ds. \] (H.9)

**Theorem H.1.** The feedback system (H.1)–(H.4), (H.9) is globally asymptotically stable at the origin.

**Proof.** Lengthy calculations verify that
\[ \frac{d}{dt} \sum_{i=0}^{n} z_i^2 = -w_0^2 z_0^2 - \sum_{i=1}^{n} z_i^2 - \left( w_0 z_0 + \sum_{i=1}^{n} z_i \right)^2, \] (H.10)

where \( w_0(0) = 1 \) and
\[ z_0 = x_0 - \beta_1, \] (H.11)
\[ z_i = \sum_{j=1}^{n} \left( \frac{n-i}{j-i} \right) x_j - \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) \, ds \] (H.12)

for \( i = 1, \ldots, n. \)

Next, consider the following extension of the Type II strict-feedforward systems:
\[ \dot{x}_0 = x_1 + \psi_0(x) + \phi_0(x)u, \] (H.13)
\[ \dot{x}_1 = x_2 + \phi_i(x_{i+1})u, \quad i = 1, \ldots, n-2, \] (H.14)
\[ \dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u, \] (H.15)
\[ \dot{x}_n = u, \] (H.16)

where the \( \phi_i \)’s satisfy the conditions of Theorem G.6.

We propose the following design algorithm. Start by computing the expressions in Theorem G.3. Then calculate
\[ \beta_1(x) = -\int_0^\infty [\xi_1(\tau, x) + \psi_0(\xi(\tau, x)) + \phi_0(\xi(\tau, x)) \tilde{\alpha}_1(\tau, x)] \, d\tau, \] (H.17)
\[ w_0(x) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial \beta_1(x)}{\partial x_i} \phi_i(x_{i+1}) - \frac{\partial \beta_1(x)}{\partial x_n}, \quad (H.18) \]

and

\[ u = \alpha_0(x_0, x) = -w_0(x)(x_0 - \beta_1(x)) - x_1 - \sum_{m=2}^{n} x_m \left[ \binom{n}{m-1} - \sum_{j=1}^{m} \binom{n}{j-1} \mu_{j+1+n-m}(x_n) \right]. \quad (H.19) \]

**Theorem H.2.** The feedback system (H.13)–(H.16), (H.19) is globally asymptotically stable at the origin.

**Proof.** The same as the proof of Theorem H.1, except that

\[ z_i = x_i + \sum_{m=i+1}^{n} x_m \left[ \binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right] \quad (H.20) \]

for \( i = 1, \ldots, n-1 \). \( \square \)

The restriction (F.7) can be lifted in some cases, to expand the class of strict-feedback systems that are not linearizable but for which an explicit feedback law can be developed.

Consider the example

\[
\begin{align*}
\dot{x}_1 &= x_2 + x_3^2, \\
\dot{x}_2 &= \sinh x_3 + x_3 u, \\
\dot{x}_3 &= u,
\end{align*}
\]

which, although only a slight variation from (12.154)–(12.156), is not represented in the class (H.1)–(H.4). The difference in (H.22) is easily accommodated by the coordinate/prefeedback change

\[
\begin{align*}
X_3 &= \sinh x_3, \\
v &= \sqrt{1 + (\sinh x_3)^2} u,
\end{align*}
\]

which converts (H.21) into

\[
\begin{align*}
\dot{x}_1 &= x_2 + (\sinh^{-1}(X_3))^2, \\
\dot{x}_2 &= X_3 + \frac{\sinh^{-1}(X_3)}{\sqrt{1 + X_3^2}} v, \\
\dot{X}_3 &= v.
\end{align*}
\]

This system fits the forms in Section H.1.
However, the system
\[
\dot{x}_i = \sin(x_{i+1}), \quad i = 1, \ldots, n-1, \\
\dot{x}_n = u,
\]
(H.29, H.30)
suggested to us by Teel, (very) remotely motivated by the ball-and-beam problem [217], cannot be brought into those forms, except in the case \( n = 2 \), where the resulting control law is
\[
u = -x_2 - \frac{\sin x_2}{x_2} \left( x_1 - \int_0^{x_2} \frac{\sin \xi}{\xi} d\xi \right).
\]
(H.31)

### H.2 Block-Forwarding

In this section we extend the class of systems to which the SJK forwarding procedure is applicable. Then we present our explicit controller formulas for this class of systems.

Consider the class of block-strict-feedforward systems
\[
\dot{x}_i = x_{i+1} + \psi_i \left( x_{i+1}, q_{i+1} \right) + \phi_i \left( x_{i+1}, q_{i+1} \right) u, \\
\dot{q}_i = A_i q_i + \omega_i \left( x_i, q_{i+1} \right),
\]
(H.35, H.36)
where \( i = 1, 2, \ldots, n \), each \( x_i \) is scalar-valued, each \( q_i \) is \( r_i \)-vector-valued,
\[
x_i = \left[ x_i, x_{i+1}, \ldots, x_n \right]^T, \\
q_i = \left[ q_i^T, q_{i+1}^T, \ldots, q_n^T \right]^T,
\]
(H.37, H.38)
\( A_i \) is a Hurwitz matrix for all \( i = 1, 2, \ldots, n \),
\[
x_{n+1} = u, \\
q_{n+1} = 0, \\
\phi_n = 0,
\]
(H.39, H.40, H.41)

---

1 The blocks considered here are less general than those in [219, 145, 74]. We can generalize the idea we are presenting (even somewhat beyond the classes considered [219, 145, 74]) to include blocks \( q_i \) that are merely input-to-state stable with respect to \( \left( x_i, q_{i+1} \right) \), rather than being linear in \( q_i \). A simple example is the system
\[
\dot{q} = -q^3 + x_2, \\
\dot{x}_1 = x_2 + qu, \\
\dot{x}_2 = u,
\]
(H.32, H.33, H.34)
This generalization would, however, preclude closed-form solvability of the problem; the result would only be an extension of [195].
and

\[
\frac{\partial \psi_i(0)}{\partial x_j} = 0, \tag{H.42}
\]

\[
\phi_i(0) = 0, \tag{H.43}
\]

\[
\omega_i(0) = 0 \tag{H.44}
\]

for \(i = 1, 2, \ldots, n-1, j = i+1, \ldots, n\). This class of systems should be understood as a dual of the block-strict-feedback systems in Section 4.5.2 of [112].

The control law for this class of systems is designed as follows. Let

\[
\begin{align*}
\beta_{n+1} &= 0, \tag{H.45} \\
\alpha_{n+1} &= 0 \tag{H.46}
\end{align*}
\]

For \(i = n, n-1, \ldots, 2, 1\),

\[
\begin{align*}
 z_i &= x_i - \beta_{i+1} \tag{H.47} \\
 w_i(x_{i+1}, q_{i+1}) &= \phi_i - \sum_{j=i+1}^{n-1} \frac{\partial \beta_{i+1}}{\partial x_j} \phi_j - \frac{\partial \beta_{i+1}}{\partial x_n}, \tag{H.48} \\
 \alpha_i(x_i, q_{i+1}) &= \alpha_{i+1} - w_i \tau, \tag{H.49} \\
 \beta_i(x_i, q_i) &= -\int_0^\infty \left[ \xi_i^{[i]}(\tau, x_i, q_i) + \psi_i(\xi_i^{[i]}(\tau, x_i, q_i), \eta_i^{[i]}(\tau, x_i, q_i)) \\
 &\quad + \phi_i(\xi_i^{[i]}(\tau, x_i, q_i), \eta_i^{[i]}(\tau, x_i, q_i)) \\
 &\quad \times \alpha_i(\xi_i^{[i]}(\tau, x_i, q_i), \eta_i^{[i]}(\tau, x_i, q_i)) \right] d\tau, \tag{H.50}
\end{align*}
\]

where the notation in the integrand of (H.50) refers to the solutions of the (sub)system(s)

\[
\frac{d}{d\tau} \xi_j^{[i]} = \xi_j^{[i+1]} + \psi_j(\xi_j^{[i+1]}), \tag{H.51}
\]

\[
\frac{d}{d\tau} \eta_j^{[i]} = A_j \eta_j^{[i]} + \omega_j(\xi_j^{[i]}, \eta_j^{[i+1]}), \tag{H.52}
\]

for \(j = i-1, i, \ldots, n\), at time \(\tau\), starting from the initial condition \((x_i, q_i)\). The control law is

\[
u = \alpha_1. \tag{H.53}
\]

**Theorem H.3.** The feedback system (H.35), (H.36), (H.53) is globally asymptotically stable at the origin.
Proof. As in the proof of Theorem F.1, the Lyapunov function

\[ V = \frac{1}{2} \sum_{i=1}^{n} z_i^2 \]  \hspace{1cm} (H.54)

has a negative-definite derivative:

\[ \dot{V} = -\frac{1}{2} \sum_{i=1}^{n} w_i^2 z_i^2 - \frac{1}{2} \left( \sum_{i=2}^{n} z_i w_i \right)^2. \] \hspace{1cm} (H.55)

This implies that \( x_n(t) \) converges to zero. Since \( \omega_n(0) = 0, \) we have that \( \omega_n(x_n(t)) \) converges to zero. Because \( A_n \) is Hurwitz, \( q_n(t) \) converges to zero. One can show recursively that \( w_i(0) = 1 \) and \( \beta_i(0) = 0. \) It then follows that \( w_{n-1}(x_n(t), q_n(t)) \) converges to one. Since (H.55) guarantees that \( w_{n-1}z_{n-1} \) goes to zero, \( z_{n-1}(t) \) also goes to zero. Hence,

\[ x_{n-1}(t) = z_{n-1}(t) + \beta_n(x_n(t), q_n(t)) \] \hspace{1cm} (H.56)

converges to zero. Continuing in the same fashion, one shows that \( x(t), q(t) \to 0 \) as \( t \to \infty. \) This establishes that the equilibrium \( x = 0, q = 0 \) is (uniformly) attractive. Global stability is argued in a similar, recursive fashion, using (H.55) and the fact that the subsystems (H.36) are input-to-state stable. In conclusion, the origin is globally asymptotically stable. \( \square \)

As in Section F.2, the solution \((\xi^*[i](\tau, s_i, q_*), \eta^*[i](\tau, s_i, q_*))\), needed in the integral (H.50), is impossible to obtain analytically in general. For this reason, we consider two classes of block-feedforward systems, inspired by feedforward systems of Types I and II, for which a closed-form controller can be obtained.

Consider the class of systems we refer to as Type I block-feedforward systems:

\[
\begin{align*}
\dot{x}_0 &= x_1 + \psi_0(x, q) + \phi_0(x, q)u, \\
\dot{q}_0 &= A_0 q_0 + \omega_0(x_0, x, q), \\
\dot{x}_1 &= x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \\
\dot{q}_1 &= A_1 q_1 + \omega_1(x, q_2), \\
\dot{x}_i &= x_{i+1}, \quad i = 2, \ldots, n - 1, \\
\dot{q}_i &= A_i q_i + \omega_i(x, q_{i+1}), \\
\dot{x}_n &= u, \\
\dot{q}_n &= A_n q_n + \omega_n(x_n),
\end{align*}
\]  \hspace{1cm} (H.57)-(H.64)

where \( x \) denotes \([x_1, \ldots, x_n]^T\), \( q \) denotes \([q_1^T, \ldots, q_n^T]^T\) (i.e., it does not include \( q_0 \)).

\[ \psi_0(0) = \phi_0(0) = \omega_0(0) = \omega_1(0) = \omega_j(0) = \pi_j(0) = 0, \quad j = 2, \ldots, n \]  \hspace{1cm} (H.65)
and
\[
\frac{\partial \psi_0(0)}{\partial x_i} = 0, \quad i = 1, \ldots, n. \tag{H.66}
\]

The subsystem \((x_1, \ldots, x_n)\) is linearizable, which makes it possible to develop a closed-form formula. The first step in the design algorithm is to compute the expressions in Lemma G.1. It is worth noting that \(\xi(\tau, x)\) and \(\bar{\alpha}_1(\tau, x)\) are both independent of \(q\). Then, for \(i = n, n - 1, \ldots, 2\), we calculate
\[
\eta_i(\tau, q_i, x) = e^{A_i\tau} q_i + \int_0^\tau e^{A_i(\tau - \sigma)} \omega_i \left( \xi_i(\sigma, x), \eta_{i+1} \left( \sigma, q_{i+1}, x \right) \right) d\sigma,
\]
followed by
\[
\beta_1(x, q) = -\int_0^\infty [\xi_1(\tau, x) + \psi_0(\xi(\tau, x), \eta(\tau, q, x))] \\
+ \phi_0(\xi(\tau, x), \eta(\tau, q, x)) \bar{\alpha}_1(\tau, x) d\tau, \tag{H.68}
\]
\[
w_0(x, q) = \phi_0(x) - \frac{\partial \beta_1(x, q)}{\partial x_1} \pi_n(x_n) - \frac{\partial \beta_1(x, q)}{\partial x_n}, \tag{H.69}
\]
and
\[
u = \omega_0(x_0, x, q) = -w_0(x, q)(x_0 - \beta_1(x, q)) \\
- \sum_{i=1}^n \left( \binom{n}{i-1} x_i + \sum_{j=2}^n \int_0^{x_i} \pi_i(s) ds \right). \tag{H.70}
\]

**Theorem H.4.** The feedback system (H.57)–(H.64), (H.70) is globally asymptotically stable at the origin.

**Proof.** Lengthy calculations verify that the same expressions hold as in the proof of Theorem H.1. In the present proof, however, \(z_0\) depends not only on \(x_0, x\) but also on \(q_n, q_{n-1}, \ldots, q_1\). Thus, convergence to the origin is proved in the following order: \(x_n, x_{n-1}, \ldots, x_1, q_n, q_{n-1}, \ldots, q_1, x_0, q_0\). Global stability is argued similarly. Hence, the equilibrium \(x_0 = q_0 = 0, x = 0, q = 0\) is globally asymptotically stable. \(\square\)

Finally, consider the class of systems we refer to as *Type II block-feedforward* systems:
\[
\begin{align*}
\dot{x}_0 &= x_1 + \psi_0(x, q) + \phi_0(x, q)u, \tag{H.71} \\
\dot{q}_0 &= A_0 q_0 + \omega_0(x_0, x, q), \tag{H.72} \\
\dot{x}_i &= x_{i+1} + \phi_i(x_{i+1})u, \quad i = 1, \ldots, n - 1, \tag{H.73} \\
\dot{q}_i &= A_i q_i + \omega_k(x_i, q_{i+1}), \tag{H.74} \\
\dot{x}_n &= u, \tag{H.75} \\
\dot{q}_n &= A_n q_n + \omega_n(x_n), \tag{H.76}
\end{align*}
\]
where the $\phi_i$’s satisfy the conditions of Theorem G.6. With $\xi(\tau, x)$ and $\tilde{\alpha}_1(\tau, x)$ calculated as in Theorem G.3, and the $\eta_i$’s and $\tilde{\beta}_1$ calculated as in (H.67), (H.68), respectively, the algorithm’s final step is to calculate

$$w_0(x, q) = \phi_0(x) - \sum_{i=1}^{n-1} \frac{\partial \tilde{\beta}_1(x, q)}{\partial x_i} \phi_i(x_{i+1}) - \frac{\partial \tilde{\beta}_1(x, q)}{\partial x_n} \tag{H.77}$$

and

$$u = \alpha_0(x_0, x, q)$$

$$= -w_0(x, q) (x_0 - \beta_1(x, q)) - x_1$$

$$- \sum_{m=2}^{n} x_m \left( \binom{n}{m-1} - \sum_{j=1}^{m} \binom{n}{j-1} \mu_{j+1+n-m}(x_n) \right). \tag{H.78}$$

**Theorem H.5.** The feedback system (H.71)–(H.76), (H.78) is globally asymptotically stable at the origin.

**Proof.** Analogous to the proof of Theorem H.4. □

### H.3 Interlaced Feedforward-Feedback Systems

#### General Design

The ability to stabilize systems that are neither in the strict-feedback form nor in the strict-feedforward form is nicely illustrated in [196]. In this section we present designs for two classes of systems obtained by interlacing strict-feedback systems [112] with feedforward systems of Types I and II.

First, consider the class of interlaced systems of Type I:

$$\dot{x}_1 = x_2 + \sum_{j=2}^{n-1} \pi_j(x_j)x_{j+1} + \pi_n(x_n)u, \tag{H.79}$$

$$\dot{x}_i = x_{i+1}, \quad i = 2, \ldots, n, \tag{H.80}$$

$$\dot{x}_{n+1} = x_{n+2} + f_1(\bar{x}_1, x_{n+1}), \tag{H.81}$$

$$\dot{x}_{n+j} = x_{n+j+1} + f_j(x_1, \bar{x}_{n+j}), \quad j = 2, \ldots, N, \tag{H.82}$$

where $x_{n+N+1} = u$. In this system $\bar{x}_{n+j}$ denotes $[x_{n+1}, \ldots, x_{n+j}]^T$, and, as before, $\bar{x}_j$ denotes $[x_j, x_{j+1}, \ldots, x_n]^T$ (which means, in particular, that $\bar{x}_1 = [x_1, \ldots, x_n]^T$).

It is clear from the above notation that the overall system order is $n + N$, where the feedforward part (top) is of order $n$ and the feedback part (bottom) is of order $N$. We assume that $\pi_i(0) = 0, i = 2, \ldots, n$, and $f_i(0) = 0, i = 1, \ldots, N$. The control synthesis for this system is given in the following theorem.
Theorem H.6. The control law given by

\[ z_i = x_i + \sum_{j=i+1}^{n} \left( n - i \right) x_j - \delta_{i,1} \sum_{j=2}^{n} \int_0^{x_j} \pi_j(s) ds, \]  

(H.83)

\[ \alpha_i(\bar{z}_i) = -\sum_{i=1}^{n} z_i \]  

(H.84)

for \( i = 1, \ldots, n, \)

\[ z_{n+1} = x_{n+1} - \alpha_1, \]  

(H.85)

\[ \alpha_{n+1}(\bar{z}_1, z_{n+1}) = -(n + 1)z_{n+1} + \sum_{l=1}^{n} (n - l)z_l - f_1(\bar{z}_1, x_{n+1}), \]  

(H.86)

\[ z_{n+j} = x_{n+j} - \alpha_{n+j-1}(\bar{z}_1, z_{n+j-1}), \]  

(H.87)

\[ \alpha_{n+j} = -z_{n+j-1} - z_{n+j} - f_j(\bar{z}_1, \bar{x}_{n+j}) \]  

\[ + \sum_{l=1}^{n} \frac{\partial \alpha_{n+j-1}}{\partial z_l} \left( -\sum_{k=1}^{i} z_k + z_{n+1} \right) + \frac{\partial \alpha_{n+j-1}}{\partial z_{n+1}} \left( -\sum_{k=1}^{n} z_k + z_{n+2} \right) \]  

\[ + \sum_{l=2}^{j-1} \frac{\partial \alpha_{n+j-1}}{\partial z_{n+l}} (-z_{n+l} + z_{n+l+1}) \]  

(H.88)

for \( j = 2, \ldots, n, \) and

\[ u = \alpha_{n+N} \]  

(H.89)

globally asymptotically stabilizes the system (H.79)–(H.82) at the origin.

Proof. It can be verified that the closed-loop system in the \( z \)-coordinates is

\[ \dot{z}_i = -\sum_{k=1}^{i} z_k + z_{n+1}, \quad i = 1, \ldots, n, \]  

(H.90)

\[ \dot{z}_{n+1} = -\sum_{k=1}^{n} z_k - z_{n+1} + z_{n+2}, \]  

(H.91)

\[ \dot{z}_{n+j} = -z_{n+j} + z_{n+j+1}, \quad j = 2, \ldots, N, \]  

(H.92)

where \( z_{n+N+1} = 0 \). The Lyapunov function

\[ V = \sum_{i=1}^{n+N} z_i^2 \]  

(H.93)

satisfies

\[ \dot{V} = -\sum_{i=1}^{n+N} z_i^2 - \sum_{i=n+2}^{n+N} (z_i - z_{i-1})^2 - \left( \sum_{i=1}^{n} z_i \right)^2, \]  

(H.94)

which proves the result. \( \square \)
Next, consider the class of *interlaced systems of Type II*:

\[
\dot{x}_1 = x_2 + \phi_1(x_{i+1})u, \quad i = 1, \ldots, n-2, \\
\dot{x}_{n-1} = x_n + \phi_{n-1}(x_n)u, \\
\dot{x}_n = x_{n+1}, \\
\dot{x}_{n+1} = x_{n+2} + f_1(x_1, x_{n+1}), \\
\dot{x}_{n+j} = x_{n+j+1} + f_j(x_1, x_{n+j}), \quad j = 2, \ldots, N, \\
\dot{x}_{n+N+1} = u.
\]

where \( x_{n+N+1} = u \). We assume that \( \phi_i(0) = f_j(0) = 0 \), and the \( \phi_i \)’s satisfy the conditions of Theorem G.6.

**Theorem H.7.** The control law given by

\[
\begin{align*}
\dot{z}_i &= x_i + \sum_{m=i+1}^{n} x_m \left[ \binom{n-i}{m-i} - \sum_{j=i}^{m} \binom{n-i}{j-i} \mu_{j+1+n-m}(x_n) \right], \\
\dot{z}_n &= x_n,
\end{align*}
\]

and (H.84)–(H.89) globally asymptotically stabilizes the system (H.95)–(H.99) at the origin.

**Proof.** The same as Theorem H.6. \( \Box \)

Since the interlaced systems of both Types I and II are feedback linearizable, one does not have to necessarily commit to the integrator forwarding + integrator backstepping design procedure. It suffices to define an output with respect to which one has a relative degree equal to the order of the system, with which one can pursue full-state feedback linearization by conversion to the Brunovsky canonical form. This is spelled out in the next theorem.

**Theorem H.8.** The systems (H.79)–(H.82) and (H.95)–(H.99) are of relative degree \( n+N \) from \( u \) to the respective outputs

\[
y_1 = x_1 - \sum_{j=2}^{n} \int_{0}^{x_j} \pi_j(s) ds
\]

and

\[
y_1 = x_1 - \sum_{j=2}^{n} \mu_{2+n-j}(x_n)x_j.
\]

\[\text{H } \text{Strict-Feedforward Systems: Not Linearizable}\]
Example: Combining Block-Backstepping and Block-Forwarding

In this section we show that block-backstepping and block-forwarding can be combined in a similar manner on an example that is outside the forms considered in Section H.3 (and also outside those in [196]):

\[
\dot{q} = -2q + x_2^2, \quad (H.105)
\]
\[
\dot{x}_1 = x_2 + qx_3, \quad (H.106)
\]
\[
\dot{x}_2 = x_3 + q, \quad (H.107)
\]
\[
\dot{x}_3 = u + qx_1. \quad (H.108)
\]

This system is neither in the block-strict-feedforward form (because of \(qx_1\) in the \(x_3\)-equation) nor in the block-strict-feedback form (because of \(qx_3\) in the \(x_1\)-equation). However, the \(x_1,x_2,q\)-subsystem is block-strict-feedforward if one views \(x_3\) as control, and the \(x_2,x_3,q\)-subsystem is block-strict-feedback with \(u\) as control. Hence, we will derive a controller for this system using one step of forwarding followed by one step of backstepping.

Following the design from Section H.2, we first calculate

\[
\xi_2^{[2]}(\tau,x_2) = x_2 e^{-\tau} \quad (H.109)
\]

and

\[
\eta_2^{[2]}(\tau,x_2,q) = (q + \tau x_2^2) e^{-\tau}. \quad (H.110)
\]

Then we derive

\[
\beta_2(x_2,q) = -x_2 + \frac{qx_2}{3} + \frac{q x_2^2}{8} + \frac{q^2}{4} + \frac{x_3^2}{9} + \frac{x_3^4}{32}, \quad (H.111)
\]
\[
w_1(x_2,q) = 1 + \frac{2}{3}q - \frac{q x_2}{3} - \frac{x_2^2}{3} - \frac{x_3^2}{8}. \quad (H.112)
\]

The system is converted from the \(x_1,x_2,x_3\)-coordinates into \(z_1,x_2,z_3\) (note that \(x_2\) is unaltered), where

\[
z_1 = x_1 - \beta_1, \quad (H.113)
\]
\[
z_3 = x_3 + q + w_1 z_1 + x_2. \quad (H.114)
\]

Note that (H.113) corresponds to one step of forwarding, resulting in a “virtual control” \(-q - w_1 z_1 - x_2\) for \(x_3\) as a control input, whereas (H.114) corresponds to one step of backstepping. The control law

\[
u = -z_3 - x_2 - w_1 z_1 - x_1 q + 2q - x_2^2 - w_1^2 (x_3 + q + x_2)
\]
\[
- (x_3 + q) + z_1 \left[ \left( \frac{x_2}{4} - \frac{2}{3} \right) (-2q + x_2^2) \right.
\]
\[
+ \left( \frac{q}{4} - \frac{2}{3} x_2 + \frac{3}{8} x_2^2 \right) (x_3 + q) \right]. \quad (H.115)
\]
results in the system being transformed into

\[
\begin{align*}
\dot{z}_1 &= -w_1^2 z_1 + w_1 z_3, \\
\dot{x}_2 &= -w_1 z_1 - x_2 + z_3, \\
\dot{z}_3 &= -w_1 z_1 - x_2 - z_3.
\end{align*}
\] (H.116, H.117, H.118)

The stability of this system follows from the Lyapunov function

\[
V(x, q) = z_1(x_1, x_2, q)^2 + x_2^2 + z_3(x_1, x_2, x_3, q)^2
\] (H.119)

because

\[
\dot{V} = -w_1^2 z_1^2 - x_2^2 - (w_1 z_1 + x_2)^2 - 2z_3^2.
\] (H.120)

The convergence to zero can be seen in the following order: \(x_2\) [from (H.120)], \(q\) [from (H.105)], \(x_1\) [from (H.113) and (H.111)], \(x_3\) [from (H.114)].
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### Index

$\ell_\infty$ stability, 179, 185, 187

$L_p$ stability, 179

adaptive control, 109, 119, 149
adaptive state estimator of transport PDE, 122
advection-diffusion PDE, 267
advection-reaction PDE, 250
Agmon’s inequality, 394
antistable wave equation, 363, 385
backstepping, 3
backstepping transformation, 19, 172, 175, 177, 231, 275
Barbalat’s lemma, 404
Bessel functions, 413
block-forwarding, 444

Cauchy–Schwartz inequality, 393
chain of integrators, 426, 429
class-$K$ function, 403
class-$KL$ function, 403
cross-term approach, 215

Datko’s counterexample, 83, 383
delay mismatch, 65, 73
delay systems, 1
delay-heat cascade, 331
delay-time function, 87
delay-wave cascade, 357
differential game, 61
diffusion coefficient, 262
disturbance attenuation, 61

energy functional—wave equation, 278
explicit closed-loop solutions for linearizable
strict-feedforward systems, 224
explicit feedback law for linearizable
strict-feedforward systems, 222
exponential stability, 404
feedforward systems, 214
finite spectrum assignment, 18, 22, 39
first-order hyperbolic PDEs, 235
forward completeness, 411
forward-complete systems, 171, 181
Goursat-type PDE, 257
Gronwall’s lemma, 402
Hölder’s inequality, 397
heat equation, 254
heat equation transfer function, 255
heat-wave cascade, 385
hyperbolic PDEs, 235
input delay, 2
input-to-state stability, 61, 183, 406
integrator forwarding, 198, 218, 422
interlaced feedforward-feedback systems, 448
inverse backstepping transformation, 25, 156, 159, 172, 175, 177, 231, 276
inverse optimality, 54, 59
KdV-like PDE, 243
kernel PDE, 237, 249
Krasovskii’s theorem, 7

linearizability of strict-feedforward systems, 218, 427
linearizable strict-feedforward systems, 174, 196, 217, 425
logarithmic Lyapunov function, 113
Lyapunov–Krasovskii functional, 8, 24, 50, 92, 113
nested saturations, 211, 214
nonlinear predictor, 155, 158, 172, 199, 221
nonlinear small-gain theorem, 214
nonlinear systems with time-varying input
delay, 212
nonlinearizable feedforward systems, 441
observer, 42, 306, 316
observer for time-varying sensor delay, 97
phase margin, 59
plant-predictor, 175
Poincaré’s inequality, 393
prediction time function, 87
predictor feedback for time-varying delay, 87
projection operator, 111, 122, 139, 417
reaction-diffusion equation, 388
reaction-diffusion PDE, 267, 334
reduced-order observer, 46
reduction approach, 18, 22
reference trajectory, 137
Riccati equation, 60
robustness to delay mismatch, 66, 75
sensor delay, 41
SJK algorithm, 422
Smith predictor, 23, 49, 168
stability, 403
state delay, 2
strict-feedback systems, 173
strict-feedforward systems, 174, 191, 197, 421
target system, 19, 172, 175, 256, 271, 291
target system for time-varying delay, 90
target-predictor, 175
time-varying input delay, 85
tracking error, 138
trajectory tracking, 135
transport PDE, 19, 73, 107, 178
truncated $L_p$ norm, 397
Type I linearizable strict-feedforward systems, 430, 437
Type II linearizable strict-feedforward systems, 433, 437
uncompensated nonlinear controller, 165
uniform asymptotic stability, 404
update law, 111, 122, 138
wave equation, 269
wave propagation speed, 283
wave-heat cascade, 388
Wirtinger’s inequality, 394
Young’s convolution theorem, 397
Young’s inequality, 393