A

Algebraic Varieties

A.1 Basic definitions

Let $k[X_1, X_2, \ldots, X_n]$ be a polynomial algebra over an algebraically closed field k with n indeterminates $X_1, \ldots, X_n$. We sometimes abbreviate it as $k[X] = k[X_1, X_2, \ldots, X_n]$. Let us associate to each polynomial $f(X) \in k[X]$ its zero set

$$V(f) := \{x = (x_1, x_2, \ldots, x_n) \in k^n \mid f(x) = f(x_1, x_2, \ldots, x_n) = 0\}$$

in the n-fold product set k^n of k. For any subset $S \subset k[X]$ we also set $V(S) = \bigcap_{f \in S} V(f)$. Then we have the following properties:

(i) $V(1) = \emptyset$, $V(0) = k^n$.
(ii) $\bigcap_{i \in I} V(S_i) = V(\bigcup_{i \in I} S_i)$.
(iii) $V(S_1) \cup V(S_2) = V(S_1 S_2)$, where $S_1 S_2 := \{fg \mid f \in S_1,\ g \in S_2\}$.

The inclusion $\subset$ of (iii) is clear. We will prove only the inclusion $\supset$. For $x \in V(S_1 S_2) \setminus V(S_2)$ there is an element $g \in S_2$ such that $g(x) \neq 0$. On the other hand, it follows from $x \in V(S_1 S_2)$ that $f(x)g(x) = 0\,({}^\forall f \in S_1)$. Hence $f(x) = 0\,({}^\forall f \in S_1)$ and $x \in V(S_1)$. So the part $\supset$ was also proved.

By (i), (ii), (iii) the set k^n is endowed with the structure of a topological space by taking $\{V(S) \mid S \subset k[X]\}$ to be its closed subsets. We call this topology of k^n the *Zariski topology*. The closed subsets $V(S)$ of k^n with respect to it are called *algebraic sets* in k^n. Note that $V(S) = V(\langle S \rangle)$, where $\langle S \rangle$ denotes the ideal of $k[X]$ generated by S. Hence we may assume from the beginning that S is an ideal of $k[X]$. Conversely, for a subset $W \subset k^n$ the set

$$I(W) := \left\{ f \in k[X] \mid f(x) = 0\ ({}^\forall x \in W) \right\}$$

is an ideal of $k[X]$. When W is a (Zariski) closed subset of k^n, we have clearly $V(I(W)) = W$. Namely, in the diagram

$$\boxed{\text{ideals in } k[X]} \underset{I}{\overset{V}{\rightleftarrows}} \boxed{\text{closed subsets in } k^n}$$

we have $V \circ I = \mathrm{Id}$. However, for an ideal $J \subset k[X]$ the equality $I(V(J)) = J$ does not hold in general. We have only $I(V(J)) \supset J$. The difference will be clarified later by Hilbert's Nullstellensatz.

Let V be a Zariski closed subset of k^n (i.e., an algebraic set in k^n). We regard it as a topological space by the relative topology induced from the Zariski topology of k^n. We denote by $k[V]$ the k-algebra of k-valued functions on V obtained by restricting polynomial functions to V. It is called the *coordinate ring* of V. The restriction map $\rho_V : k[X] \to k[V]$ given by $\rho_V(f) := f|_V$ is a surjective homomorphism of k-algebras with $\mathrm{Ker}\,\rho_V = I(V)$, and hence we have $k[V] \simeq k[X]/I(V)$. For each point $x = (x_1, x_2, \dots, x_n) \in k^n$ of V define a homomorphism $e_x : k[V] \to k$ of k-algebras by $e_x(f) = f(x)$. Then we get a map

$$e : V \longrightarrow \mathrm{Hom}_{k\text{-alg}}(k[V],\ k) \quad (x \longmapsto e_x),$$

where $\mathrm{Hom}_{k\text{-alg}}(k[V],\ k)$ denotes the set of the k-algebra homomorphisms from $k[V]$ to k. Conversely, for a k-algebra homomorphism $\phi : k[V] \to k$ define $x = (x_1, x_2, \dots, x_n) \in k^n$ by $x_i = \phi\rho_V(X_i)$ $(1 \le i \le n)$. Then we have $x \in V$ and $e_x = \phi$. Hence we have an identification $V = \mathrm{Hom}_{k\text{-alg}}(k[V],\ k)$ as a set. Moreover, the closed subsets of V are of the form $V(\rho_V^{-1}(J)) = \{ x \in V \mid e_x(J) = 0 \} \subset V$ for ideals J of $k[V]$. Therefore, the topological space V is recovered from the k-algebra $k[V]$. It indicates the possibility of defining the notion of algebraic sets starting from certain k-algebras without using the embedding into k^n. Note that the coordinate ring $A = k[V]$ is finitely generated over k, and *reduced* (i.e., does not contain non-zero nilpotent elements) because $k[V]$ is a subring of the ring of functions on V with values in the field k.

In this chapter we give an account of the classical theory of "algebraic varieties" based on reduced finitely generated (commutative) algebras over algebraically closed fields (in the modern language of schemes one allows general commutative rings as "coordinate algebras").

The following two theorems are fundamental.

Theorem A.1.1 (A weak form of Hilbert' Nullstellensatz). *Any maximal ideal of the polynomial ring $k[X_1, X_2, \dots, X_n]$ is generated by the elements $X_i - x_i$ $(1 \le i \le n)$ for a point $x = (x_1, x_2, \dots, x_n) \in k^n$.*

Theorem A.1.2 (Hilbert's Nullstellensatz). *We have $I(V(J)) = \sqrt{J}$, where $\sqrt{J}$ is the* **radical** $\{ f \in k[X] \mid f^N \in J$ *for some* $N \gg 0\}$ *of J.*

For the proofs, see, for example, [Mu]. □

For a finitely generated k-algebra A denote by $\mathrm{Specm}\,A$ the set of the maximal ideals of A. For an algebraic set $V \subset k^n$ we have a bijection

$$\mathrm{Hom}_{k\text{-alg}}(k[V],\ k) \simeq \mathrm{Specm}\,k[V] \quad (e \longmapsto \mathrm{Ker}\,e)$$

by Theorem A.1.1. Under this correspondence, the closed subsets in $\mathrm{Specm}\,k[V] \simeq V$ are the sets

$$V(I) = \{ \mathfrak{m} \in \mathrm{Specm}\,k[V] \mid \mathfrak{m} \supset I \},$$

where I ranges through the ideals of $k[V]$. Theorem A.1.2 implies that there is a one-to-one correspondence between the closed subsets in $\operatorname{Specm} k[V]$ and the *radical ideals* I $(I = \sqrt{I})$ of $k[V]$.

A.2 Affine varieties

Motivated by the arguments in the previous section, we start from a finitely generated *reduced* commutative k-algebra A to define an algebraic variety. Namely, we set $V = \operatorname{Specm} A$ and define its topology so that the closed subsets are given by

$$\left\{V(I) = \{\mathfrak{m} \in \operatorname{Specm} A \mid I \subset \mathfrak{m}\} \mid I\text{: ideals of } A\right\}.$$

By Hilbert's Nullstellensatz (its weak form), we get the identification

$$V \simeq \operatorname{Hom}_{k\text{-alg}}(A, k).$$

We sometimes write a point $x \in V$ as $\mathfrak{m}_x \in \operatorname{Specm} A$ or $e_x \in \operatorname{Hom}_{k\text{-alg}}(A, k)$. Under this notation we have

$$f(x) = e_x(f) = (f \mod \mathfrak{m}_x) \in k$$

for $f \in A$. Here, we used the identification $k \simeq A/\mathfrak{m}_x$ obtained by the composite of the morphisms $k \hookrightarrow A \to A/\mathfrak{m}_x$. Hence the ring A is regarded as a k-algebra consisting of certain k-valued functions on V.

Recall that any open subset of V is of the form $D(I) = V \setminus V(I)$, where I is an ideal of A. Since A is a noetherian ring (finitely generated over k), the ideal I is generated by a finite subset $\{f_1, f_2, \dots, f_r\}$ of I. Then we have

$$D(I) = V \setminus \left(\bigcap_{i=1}^{r} V(f_i)\right) = \bigcup_{i=1}^{r} D(f_i),$$

where $D(f) = \{x \in V \mid f(x) \neq 0\} = V \setminus V(f)$ for $f \in A$. We call an open subset of the form $D(f)$ for $f \in A$ a *principal open subset* of V. Principal open subsets form a basis of the open subsets of V. Note that we have the equivalence

$$D(f) \subset D(g) \iff \sqrt{(f)} \subset \sqrt{(g)} \iff f \in \sqrt{(g)}$$

by Hilbert's Nullstellensatz.

Assume that we are given an A-module M. We introduce a sheaf $\widetilde{M}$ on the topological space $V = \operatorname{Specm} A$ as follows. For a multiplicatively closed subset S of A we denote by $S^{-1}M$ the localization of M with respect to S. It consists of the equivalence classes with respect to the equivalence relation $\sim$ on the set of pairs $(s, m) = m/s$ $(s \in S, m \in M)$ given by $m/s \sim m'/s' \iff t(s'm - sm') = 0$ $(\exists t \in S)$. By the ordinary operation rule of fractional numbers $S^{-1}A$ is endowed with a ring structure and $S^{-1}M$ turns out to be an $S^{-1}A$-module. For $f \in A$ we set

$M_f = S_f^{-1}M$, where $S_f := \{1, f, f^2, \dots\}$. Note that for two principal open subsets $D(f) \subset D(g)$ a natural homomorphism $r_f^g : M_g \to M_f$ is defined as follows. We have $f^n = hg$ ($h \in A$, $n \in \mathbb{N}$), and then the element $m/g^l \in M_g$ is mapped to $m/g^l = h^l m/h^l g^l = h^l m/f^{nl} \in M_f$. In the case of $D(f) = D(g)$ we easily see $M_f \simeq M_g$ by considering the inverse.

Theorem A.2.1.
(i) *For an A-module M there exists a unique sheaf $\widetilde{M}$ on $V = \operatorname{Specm} A$ such that for any principal open subset $D(f)$ we have $\widetilde{M}(D(f)) = M_f$, and the restriction homomorphism $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ for $D(f) \subset D(g)$ is given by r_f^g.*
(ii) *The sheaf $\mathcal{O}_V := \widetilde{A}$ is naturally a sheaf of k-algebras on V.*
(iii) *For an A-module M the sheaf $\widetilde{M}$ is naturally a sheaf of $\mathcal{O}_V$-module. The stalk of $\widetilde{M}$ at $x \in V$ is given by*

$$M_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}M = \varinjlim_{f(x)\neq 0} M_f.$$

This is a module over the local ring $\mathcal{O}_{V,x} := A_{\mathfrak{m}_x} = (A \setminus \mathfrak{m}_x)^{-1}A$.

The key point of the proof is the fact that the functor $D(f) \mapsto M_f$ on the category of principal open subsets $\{D(f) \mid f \in A\}$ satisfies the "axioms of sheaves (for a basis of open subsets)," which is assured by the next lemma.

Lemma A.2.2. *Assume that the condition $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is satisfied in the ring A. Then for an A-module M we have an exact sequence*

$$0 \longrightarrow M \stackrel{\alpha}{\longrightarrow} \prod_{i=1}^{r} M_{f_i} \stackrel{\beta}{\longrightarrow} \prod_{i,j} M_{f_i f_j},$$

where the last arrow maps $(s_i)_{i=1}^r$ to $s_i - s_j \in M_{f_i f_j}$ $(1 \le i, j \le r)$.

Proof. We first show that for any $N \in \mathbb{N}$ there exists $g_1, \dots, g_r$ satisfying $\sum_i g_i f_i^N = 1$. Note that our assumption $\langle f_1, f_2, \dots, f_r\rangle \ni 1$ is equivalent to saying that for any $\mathfrak{m} \in \operatorname{Specm} A$ there exists at least one element f_i such that $f_i \notin \mathfrak{m}$ ($\operatorname{Specm} A = \bigcup_{i=1}^r D(f_i)$). If $f_i \notin \mathfrak{m}$, then we have $f_i^N \notin \mathfrak{m}$ for any N since $\mathfrak{m}$ is a prime ideal. Therefore, we get $\langle f_1^N, f_2^N, \dots, f_r^N\rangle \ni 1$, and the assertion is proved.

Let us show the injectivity of α. Assume $m \in \operatorname{Ker} \alpha$. Then there exists $N \gg 0$ such that $f_i^N m = 0$ $(1 \le i \le r)$. Combining it with the equality $\sum_i g_i f_i^N = 1$ we get $m = \sum_i g_i f_i^N m = 0$.

Next assume that $(m_l) \in \prod_l M_{f_l} \in \operatorname{Ker} \beta$. We will show that there is an element $m \in M$ such that $\alpha(m) = (m_l)$. It follows from our assumption that $m_i = m_j$ in $M_{f_i f_j}$ $(1 \le i, j \le r)$. This is equivalent to saying that $(f_i f_j)^N(m_i - m_j) = 0$ $(1 \le i, j \le r)$ for a large number N. That is, $f_j^N f_i^N m_i = f_i^N f_j^N m_j$. Now set $m = \sum_{i=1}^r g_i(f_i^N m_i)$ $(\sum g_i f_i^N = 1)$. Then for any $1 \le l \le r$ we have $f_l^N m = f_l^N \sum_i g_i(f_i^N m_i) = \sum_i g_i f_i^N (f_l^N m_i) = \sum_i g_i f_i^N f_l^N m_l = f_l^N m_l$ and $f_l^N(m - m_l) = 0$. Hence we have $\alpha(m) = (m_l)$. □

For a general open subset $U = \bigcup_{i=1}^{r} D(f_i)$ of U we have

$$\Gamma(U,\ \widetilde{M}) = \left\{(s_i) \in M_{f_i} \mid s_i = s_j \text{ in } M_{f_i f_j}\ (1 \le i, j \le r)\right\}.$$

In particular, for $\mathcal{O}_V = \widetilde{A}$ we have

$$\begin{aligned}\Gamma(U,\ \mathcal{O}_V) = \{f : U \to k \mid &\text{ for each point of } U,\\ &\text{there is an open neighborhood } D(g) \text{ such that } f|_{D(g)} \in A_g.\}\end{aligned}$$

Let $(X,\ \mathcal{O}_X)$ be a pair of a topological space X and a sheaf $\mathcal{O}_X$ of k-algebras on X consisting of certain k-valued functions. We say that the pair $(X, \mathcal{O}_X)$ (or simply X) is an *affine variety* if $(X,\ \mathcal{O}_X)$ is isomorphic to some $(V, \mathcal{O}_V)$ ($V = \operatorname{Specm} A$, $\mathcal{O}_V = \widetilde{A}$) in the sense that there exists a homeomorphism $\phi : X \xrightarrow{\sim} V$ such that the correspondence $f \mapsto f \circ \phi$ gives an isomorphism $\Gamma(\phi(U),\ \mathcal{O}_V) \xrightarrow{\sim} \Gamma(U,\ \mathcal{O}_X)$ for any open subset U of V. In this case we have a natural isomorphism $\phi^\sharp$: $\phi^{-1}\mathcal{O}_V \xrightarrow{\sim} \mathcal{O}_X$ of a sheaf of k-algebras. In particular, we have an isomorphism $\phi_x^\sharp : \mathcal{O}_{V,\phi(x)} \xrightarrow{\sim} \mathcal{O}_{X,x}$ of local rings for any $x \in X$.

A.3 Algebraic varieties

Let $(X,\ \mathcal{O}_X)$ be a pair of a topological space X and a sheaf $\mathcal{O}_X$ of k-algebras consisting of certain k-valued functions. We say that the pair $(X,\ \mathcal{O}_X)$ (or simply X) is called a *prevariety* over k if it is locally an affine variety (i.e., if for any point $x \in X$ there is an open neighborhood $U \ni x$ such that $(U,\ \mathcal{O}_X|_U)$ is isomorphic to an affine variety). In such cases, we call the sheaf $\mathcal{O}_X$ the *structure sheaf* of X and sections of $\mathcal{O}_X$ are called *regular functions*. A morphism $\phi : X \to Y$ between prevarieties X, Y is a continuous map so that for any open subset U of Y and any $f \in \Gamma(U, \mathcal{O}_Y)$ we have $f \circ \phi \in \Gamma(f^{-1}U, \mathcal{O}_X)$.

A prevariety X is called an *algebraic variety* if it is *quasi-compact* and *separated*. Let us explain more precisely these two conditions.

The quasi-compactness is a purely topological condition. We say that a topological space X is quasi-compact if any open covering of X admits a finite subcovering. We say "quasi" because X is not assumed to be Hausdorff. In fact, an algebraic variety is not Hausdorff unless it consists of finitely many points. In particular, an algebraic variety X is covered by finitely many affine varieties.

The condition of separatedness plays an alternative role to that of the Hausdorff condition. To define it we need the notion of products of prevarieties.

We define the product $V_1 \times V_2$ of two affine varieties $V_1,\ V_2$ to be the affine variety associated to the tensor product $A_1 \otimes_k A_2$ ($A_i = \Gamma(V_i,\ \mathcal{O}_{V_i})$) of k-algebras. This operation is possible thanks to the following proposition.

Proposition A.3.1. *For any pair $A_1,\ A_2$ of finitely generated reduced k-algebras the tensor product $A_1 \otimes_k A_2$ is also reduced.*

Proof. Consider the embeddings $V_i \subset \mathbb{A}^{n_i}$ $(i = 1, 2)$ and $V_1 \times V_2 \subset \mathbb{A}^{n_1+n_2}$. Next, define the ring $k[V_1 \times V_2]$ to be the restriction of the polynomial ring $k[\mathbb{A}^{n_1+n_2}]$ to $V_1 \times V_2$. Then the restriction map $\varphi : k[V_1] \otimes_k k[V_2] \to k[V_1 \times V_2]$ is bijective (the right-hand side is obviously reduced). The surjectivity is clear. We can also prove the injectivity, observing that for any linearly independent elements $\{f_i\}$ (resp. $\{g_j\}$) in $k[V_1]$ (resp. $k[V_2]$) over k the elements $\{\varphi(f_i g_j)\}$ are again linearly independent in $k[V_1 \times V_2]$. □

By definition the product $V_1 \times V_2$ of affine varieties V_1, V_2 has a finer topology than the usual product topology.

Now let us give the definition of the product of two prevarieties X, Y. Let $X = \bigcup_i V_i$, $Y = \bigcup_j U_j$ be affine open coverings of X and Y, respectively. Then the product set $X \times Y$ is covered by $\{ V_i \times U_j \}_{(i,j)}$ $(X \times Y = \bigcup_{(i,j)} V_i \times U_j)$. Note that we regard the product sets $V_i \times U_j$ as affine varieties by the above arguments. Namely, the structure sheaf $\mathcal{O}_{V_i \times U_j}$ is associated to the tensor product $\Gamma(V_i, \mathcal{O}_{V_i}) \otimes_k \Gamma(U_j, \mathcal{O}_{U_j})$. Then we can glue $(V_i \times U_j, \mathcal{O}_{V_i \times U_j})$ to get a topology of $X \times Y$ and a sheaf $\mathcal{O}_{X \times Y}$ of k-algebras consisting of certain k-valued functions on $X \times Y$, for which $(X \times Y, \mathcal{O}_{X \times Y})$ is a prevariety. This prevariety is called the product of two prevarieties X and Y. It is the "fiber product" in the category of prevarieties.

Using these definitions, we say that a prevariety X is *separated* if the diagonal set $\Delta = \{(x, x) \in X \times X\}$ is closed in the self-product $X \times X$.

Let us add some remarks.

(i) If $\phi : X \to Y$ is a morphism of algebraic varieties, then its graph $\Gamma_\phi = \{ (x, \phi(x)) \in X \times Y \}$ is a closed subset of $X \times Y$.
(ii) Affine varieties are separated (hence they are algebraic varieties).

A.4 Quasi-coherent sheaves

Let $(X, \mathcal{O}_X)$ be an algebraic variety. We say that a sheaf F of $\mathcal{O}_X$-module (hereafter, we simply call F an $\mathcal{O}_X$-module) is *quasi-coherent* over $\mathcal{O}_X$ if for each point $x \in X$ there exists an affine open neighborhood $V \ni x$ and a module M_V over $A_V = \mathcal{O}_X(V)$ such that $F|_V \simeq \widetilde{M}_V$ as $\mathcal{O}_V$-modules ($\widetilde{M}_V$ is an $\mathcal{O}_V$-module on $V = \operatorname{Specm} A_V$ constructed from the A_V-module M_V by Theorem A.2.1). If, moreover, every M_V is finitely generated over A_V, we say that F is *coherent* over $\mathcal{O}_X$. The next theorem is fundamental.

Theorem A.4.1.

(i) (Chevalley) *The following conditions on an algebraic variety X are equivalent:*
  (a) *X is an affine variety.*
  (b) *For any quasi-coherent $\mathcal{O}_X$-module F we have $H^i(X, F) = 0$ $(i \geq 1)$.*
  (c) *For any quasi-coherent $\mathcal{O}_X$-module F we have $H^1(X, F) = 0$.*

(ii) *Let X be an affine variety and $A = \mathcal{O}_X(X)$ its coordinate ring. Then the functor*

$$\operatorname{Mod}(A) \ni M \longmapsto \widetilde{M} \in \operatorname{Mod}_{qc}(\mathcal{O}_X)$$

from the category $\mathrm{Mod}(A)$ *of* A*-modules to the category* $\mathrm{Mod}_{qc}(\mathcal{O}_X)$ *of quasi-coherent* $\mathcal{O}_X$*-modules induces an equivalence of categories. Namely, any quasi-coherent* $\mathcal{O}_X$*-module is isomorphic to the sheaf* $\widetilde{M}$ *constructed from an* A*-module* M*, and there exists an isomorphism*

$$\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In particular, for a quasi-coherent $\mathcal{O}_X$*-module* F *we have*

$$F \simeq \widetilde{F(X)}.$$

By Theorem A.4.1 (ii) local properties of a quasi-coherent $\mathcal{O}_X$-module F can be deduced from those of the $\mathcal{O}_X(V)$-module $F(V)$ for an affine neighborhood V. For example, F is locally free (resp. coherent) if and only if every point $x \in X$ has an affine open neighborhood V such that $F(V)$ is free (resp. finitely generated) over $\mathcal{O}_X(V)$.

Let us give some examples.

Example A.4.2. *Tangent sheaf* Θ_X and *cotangent sheaf* Ω_X^1 (In this book, Ω_X stands for the sheaf $\Omega_X^n := \bigwedge^n \Omega_X^1$ ($n = \dim X$) of differential forms of top degree). We denote by $\mathcal{E}nd_k\mathcal{O}_X$ the sheaf of k-linear endomorphisms of $\mathcal{O}_X$. We say that a section $\theta \in (\mathcal{E}nd_k\mathcal{O}_X)(X)$ is a vector field on X if for each open subset $U \subset X$ the restriction $\theta(U) := \theta|_U \in (\mathcal{E}nd_k\mathcal{O}_X)(U)$ satisfies the condition

$$\theta(U)(fg) = \theta(U)(f)g + f\theta(U)(g) \quad (f, g \in \mathcal{O}_X(U)).$$

For an open subset U of X, denote the set of the vector fields θ ($\in (\mathcal{E}nd_k\mathcal{O}_U)(U)$) on U by $\Theta(U)$. Then $\Theta(U)$ is an $\mathcal{O}_X(U)$-module, and the presheaf $U \mapsto \Theta(U)$ turns out to be a sheaf (of $\mathcal{O}_X$-modules). We denote this sheaf by Θ_X and call it the *tangent sheaf* of X. When U is affine, we have $\Theta_U \simeq \widetilde{\mathrm{Der}_k(A)}$ for $A = \mathcal{O}_X(U)$, where the right-hand side is the $\mathcal{O}_U$-module associated to the A-module

$$\mathrm{Der}_k(A) := \{\theta \in \mathrm{End}_k\, A \mid \theta(fg) = \theta(f)g + f\theta(g)\ (f, g \in A)\}$$

of the derivations of A over k. It follows from this fact that Θ_X is a coherent $\mathcal{O}_X$-module. Indeed, if $A = k[X]/I$ (here $k[X] = k[X_1, X_2, \dots, X_n]$ is a polynomial ring), then we have

$$\mathrm{Der}_k(k[X]) = \bigoplus_{i=1}^{n} k[X]\partial_i \quad \left(\partial_i := \frac{\partial}{\partial X_i}\right)$$

(free $k[X]$-module of rank n) and

$$\mathrm{Der}_k(A) \simeq \{\theta \in \mathrm{Der}_k(k[X]) \mid \theta(I) \subset I\}.$$

Hence $\mathrm{Der}_k(A)$ is finitely generated over A.

On the other hand we define the *cotangent sheaf* of X by $\Omega_X^1 := \delta^{-1}(\mathcal{J}/\mathcal{J}^2)$, where $\delta : X \to X \times X$ is the diagonal embedding, $\mathcal{J}$ is the ideal sheaf of $\delta(X)$ in $X \times X$ defined by

$$\mathcal{J}(V) = \{f \in \mathcal{O}_{X\times X}(V) \mid f(V \cap \delta(X)) = \{0\}\}$$

for any open subset V of $X \times X$, and δ^{-1} stands for the sheaf-theoretical inverse image functor. Sections of the sheaf Ω_X^1 are called differential forms. By the canonical morphism $\mathcal{O}_X \to \delta^{-1}\mathcal{O}_{X\times X}$ of sheaf of k-algebras Ω_X^1 is naturally an $\mathcal{O}_X$-module. We have a morphism $d : \mathcal{O}_X \to \Omega_X^1$ of $\mathcal{O}_X$-modules defined by $df = f \otimes 1 - 1 \otimes f \mod \delta^{-1}\mathcal{J}^2$. It satisfies $d(fg) = d(f)g + f(dg)$ for any $f, g \in \mathcal{O}_X$. For $\alpha \in \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ we have $\alpha \circ d \in \Theta_X$, which gives an isomorphism $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \simeq \Theta_X$ of $\mathcal{O}_X$-modules.

A.5 Smoothness, dimensions and local coordinate systems

Let x be a point of an algebraic variety X. We say that X is *smooth* (or non-singular) at $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a regular local ring. This condition is satisfied if and only if the cotangent sheaf Ω_X^1 is a free $\mathcal{O}_X$-module on an open neighborhood of x. The smooth points of X form an open subset of X. Let us denote this open subset by X_{reg}. An algebraic variety is called *smooth* (or non-singular) if all of its points are smooth. It is equivalent to saying that Ω_X^1 is a locally free $\mathcal{O}_X$-module. In this case Θ_X is also locally free of the same rank by $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \simeq \Theta_X$. For a smooth point $x \in X$ the *dimension* of X at x is defined by

$$\dim_x X := \operatorname{rank}_{\mathcal{O}_{X,x}} \Theta_{X,x} = \operatorname{rank}_{\mathcal{O}_{X,x}} \Omega_{X,x}^1,$$

where $\Theta_{X,x}$ and $\Omega_{X,x}^1$ are the stalks of Θ_X and Ω_X^1 at x, respectively. It also coincides with the Krull dimension of the regular local ring $\mathcal{O}_{X,x}$. We define the *dimension* of X to be the locally constant function on X_{reg} defined by

$$(\dim X)(x) := \dim_x X.$$

If X is irreducible, the value $\dim_x X$ does not depend on the point $x \in X_{\text{reg}}$.

Theorem A.5.1. *Let X be a smooth algebraic variety of dimension n. Then for each point $p \in X$, there exist an affine open neighborhood V of p, regular functions $x_i \in k[V] = \mathcal{O}_X(V)$, and vector fields $\partial_i \in \Theta_X(V)$ $(1 \leq i \leq n)$ satisfying the conditions*

$$\begin{cases} [\partial_i, \partial_j] = 0, \quad \partial_i(x_j) = \delta_{ij} \ (1 \leq i, \ j \leq n) \\ \Theta_V = \bigoplus_{i=1}^n \mathcal{O}_V \, \partial_i. \end{cases}$$

Moreover, we can choose the functions $x_1, x_2, \ldots, x_n$ so that they generate the maximal ideal $\mathfrak{m}_p$ of the local ring $\mathcal{O}_{X,p}$ at p.

Proof. By the theory of regular local rings there exist $n(= \dim_x X)$ functions $x_1, \dots, x_n \in \mathfrak{m}_p$ generating the ideal $\mathfrak{m}_p$. Then $dx_1, \dots, dx_n$ is a basis of the free $\mathcal{O}_{X,p}$-module $\Omega^1_{X,p}$. Hence we can take an affine open neighborhood V of p such that $\Omega^1_X(V)$ is a free module with basis $dx_1, \dots, dx_n$ over $\mathcal{O}_X(V)$. Taking the dual basis $\partial_1, \dots, \partial_n \in \Theta_X(V) \simeq \mathrm{Hom}_{\mathcal{O}_X(V)}(\Omega^1_X(V), \mathcal{O}_X(V))$ we get $\partial_i(x_j) = \delta_{ij}$. Write $[\partial_i, \partial_j]$ as $[\partial_i, \partial_j] = \sum_{l=1}^n g^l_{ij} \partial_l$ ($g^l_{ij} \in \mathcal{O}_X(V)$). Then we have $g^l_{ij} = [\partial_i, \partial_j] x_l = \partial_i \partial_j x_l - \partial_j \partial_i x_l = 0$. Hence $[\partial_i, \partial_j] = 0$. □

Definition A.5.2. The set $\{ x_i, \partial_i \mid 1 \le i \le n \}$ defined over an affine open neighborhood of p satisfying the conditions of Theorem A.5.1 is called a *local coordinate system* at p.

It is clear that this notion is a counterpart of the local coordinate system of a complex manifold. Note that the local coordinate system $\{x_i\}$ defined on an affine open subset V of a smooth algebraic variety does not necessarily separate the points in V. We only have an *étale morphism* $V \to k^n$ given by $q \mapsto (x_1(q), \dots, x_n(q))$.

We have the following relative version of Theorem A.5.1.

Theorem A.5.3. *Let Y be a smooth subvariety of a smooth algebraic variety X. Assume that* $\dim_p Y = m$, $\dim_p X = n$ *at $p \in Y$. Then we can take an affine open neighborhood V of p in X and a local coordinate system $\{x_i, \partial_i \mid 1 \le i \le n\}$ such that $Y \cap V = \{q \in V \mid x_i(q) = 0 \ (m < i \le n)\}$ (hence $k[Y \cap V] = k[V] / \sum_{i>m} k[V] x_i$) and $\{x_i, \partial_i \mid 1 \le i \le m\}$ is a local coordinate system of $Y \cap V$. Here we regard ∂_i $(1 \le i \le m)$ as derivations on $k[Y \cap V]$ by using the relation $\partial_i x_j = 0$ $(j > m)$.*

Proof. The result follows from the fact that smooth ⇒ locally complete intersection. □

B

Derived Categories and Derived Functors

In this appendix, we give a brief account of the theory of derived categories without proofs. The basic references are Hartshorne [Ha1], Verdier [V2], Borel et al. [Bor3, Chapter 1], Gelfand–Manin [GeM], Kashiwara–Schapira [KS2], [KS4]. We especially recommend the reader to consult Kashiwara–Schapira [KS4] for details on this subject.

B.1 Motivation

The notion of derived categories is indispensable if one wants to fully understand the theory of D-modules. Many operations of D-modules make sense only in derived categories, and the Riemann–Hilbert correspondence, which is the main subject of Part I, cannot be formulated without this notion. Derived categories were introduced by A. Grothendieck [Ha1], [V2]. We hear that M. Sato arrived at the same notion independently in his way of creating algebraic analysis. In this section we explain the motivation of the theory of derived categories and give an outline of the theory.

Let us first recall the classical definition of right derived functors. Let $\mathcal{C}$, $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ a left exact functor. Assume that the category $\mathcal{C}$ has *enough injectives*, i.e., for any object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a monomorphism $X \to I$ into an injective object I. Then for any $X \in \mathrm{Ob}(\mathcal{C})$ there exists an exact sequence

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

such that I^k is an injective object for any $k \in \mathbb{Z}$. Such an exact sequence is called an *injective resolution* of X. Next consider the complex

$$I^{\cdot} = \left[0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow I^3 \cdots\right]$$

in $\mathcal{C}$ and apply to it the functor F. Then we obtain a complex

$$F(I^{\cdot}) = \left[0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow F(I^2) \longrightarrow F(I^3) \cdots\right]$$

in $\mathcal{C}'$. As is well known in homological algebra the nth cohomology group of $F(I^{\cdot})$:

$$H^n F(I^{\cdot}) = \mathrm{Ker}[F(I^n) \longrightarrow F(I^{n+1})]/\,\mathrm{Im}[F(I^{n-1}) \longrightarrow F(I^n)]$$

does not depend on the choice of injective resolutions, and is uniquely determined up to isomorphisms. Set $R^n F(X) = H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$. Then $R^n F$ defines a functor $R^n F : \mathcal{C} \to \mathcal{C}'$. We call $R^n F$ the nth *derived functor* of F. For $n < 0$ we have $R^n F = 0$ and $R^0 F = F$. Similar construction can be applied also to complexes in $\mathcal{C}$ which are bounded below. Indeed, consider a complex

$$X^{\cdot} = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^k \longrightarrow X^{k+1} \longrightarrow X^{k+2} \longrightarrow \cdots\right]$$

in $\mathcal{C}$ such that $X^i = 0$ for any $i < k$. Then there exists a complex

$$I^{\cdot} = \left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow I^k \longrightarrow I^{k+1} \longrightarrow I^{k+2} \longrightarrow \cdots\right]$$

of injective objects in $\mathcal{C}$ and a *quasi-isomorphism* $f : X^{\cdot} \to I^{\cdot}$, i.e., a morphism of complexes $f : X^{\cdot} \to I^{\cdot}$ which induces an isomorphism $H^i(X^{\cdot}) \simeq H^i(I^{\cdot})$ for any $i \in \mathbb{Z}$. We call $I^{\cdot}$ an injective resolution of $X^{\cdot}$. Since the injective resolution $I^{\cdot}$ and the complex $F(I^{\cdot})$ are uniquely determined up to homotopy equivalences, the nth cohomology group $H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$ of $F(I^{\cdot})$ is uniquely determined up to isomorphisms. Set $R^n F(X^{\cdot}) = H^n F(I^{\cdot}) \in \mathrm{Ob}(\mathcal{C}')$. If we introduce the homotopy category $K^+(\mathcal{C})$ of complexes in $\mathcal{C}$ which are bounded below (for the definition see Section B.2 below), this gives a functor $R^n F : K^+(\mathcal{C}) \to \mathcal{C}'$. Such derived functors for "complexes" are frequently used in algebraic geometry.

However, this classical construction of derived functors has some defects. Since we treat only cohomology groups $\{H^n F(I^{\cdot})\}_{n\in\mathbb{Z}}$ of $F(I^{\cdot})$, we lose various important information of the complex $F(I^{\cdot})$ itself. Moreover, the above construction of derived functors is not convenient for the composition of functors. For example, let $G : \mathcal{C}' \longrightarrow \mathcal{C}''$ be another left exact functor. Then, for $X \in \mathrm{Ob}(\mathcal{C})$ the equality $R^{i+j}(G \circ F)(X) = R^i G(R^j F(X))$ cannot be expected in general. The theory of spectral sequences was invented as a remedy for such problems, but the best way is to treat everything at the level of complexes without taking cohomology groups. Namely, we want to introduce certain categories of complexes and define a lifting RF (which will be also called a derived functor of F) of $R^n F$'s between such categories of complexes. This is the theory of derived categories. Indeed, the language of derived categories allows one to formulate complicated relations among various functors in a very beautiful and efficient way.

Now let us briefly explain the construction of derived categories. Let $C(\mathcal{C})$ be the category of complexes in $\mathcal{C}$. Since the injective resolutions $f : X^{\cdot} \to I^{\cdot}$ of $X^{\cdot}$ are just quasi-isomorphisms in $C(\mathcal{C})$, we should change the family of morphisms of $C(\mathcal{C})$ so that quasi-isomorphisms are isomorphisms in the new category. For this purpose we use a general theory of localizations of categories (see Section B.4). However, this localization cannot be applied directly to the category $C(\mathcal{C})$. So we first define the homotopy category $K(\mathcal{C})$ by making homotopy equivalences in $C(\mathcal{C})$ invertible, and then apply the localization. The derived category $D(\mathcal{C})$ thus obtained is an additive category and not an abelian category any more. Therefore, we cannot consider short exact sequences $0 \to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to 0$ of complexes in $D(\mathcal{C})$ as in $C(\mathcal{C})$.

Nevertheless, we can define the notion of distinguished triangles in $D(\mathcal{C})$ which is a substitute for that of short exact sequences of complexes. In other words, the derived category $D(\mathcal{C})$ is a triangulated category in the sense of Definition B.3.6. As in the case of short exact sequences in $C(\mathcal{C})$, from a distinguished triangle in $D(\mathcal{C})$ we can deduce a cohomology long exact sequence in $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and assume that $\mathcal{C}$ has enough injectives. Denote by $D^+(\mathcal{C})$ (resp. $D^+(\mathcal{C}')$) the full subcategory of $D(\mathcal{C})$ (resp. $D(\mathcal{C})$) consisting of complexes which are bounded below. Then we can construct a (right) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ of F, which sends distinguished triangles to distinguished triangles. If we identify an object X of $\mathcal{C}$ with a complex

$$\left[\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots\right]$$

concentrated in degree 0 and hence with an object of $D^+(\mathcal{C})$, we have an isomorphism $H^n(RF(X)) \simeq R^nF(X)$ in $\mathcal{C}'$. From this we see that the new derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ extends classical ones. Moreover, this new construction of derived functors turns out to be very useful for the compositions of various functors. Let $G : \mathcal{C}' \to \mathcal{C}''$ be another left exact functor and assume that $\mathcal{C}'$ has enough injectives. Then also the derived functors RG and $R(G \circ F)$ exist and (under a sufficiently weak hypothesis) we have a beautiful composition rule $RG \circ RF = R(G \circ F)$. Since in the theory of D-modules we frequently use the compositions of various derived functors, such a nice property is very important.

B.2 Categories of complexes

Let $\mathcal{C}$ be an abelian category, e.g., the category of R-modules over a ring R, the category $\mathrm{Sh}(T_0)$ of sheaves on a topological space T_0. Denote by $C(\mathcal{C})$ the category of complexes in $\mathcal{C}$. More precisely, an object $X^\cdot$ of $C(\mathcal{C})$ consists of a family of objects $\{X^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and that of morphisms $\{d^n_{X^\cdot} : X^n \longrightarrow X^{n+1}\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying $d^{n+1}_{X^\cdot} \circ d^n_{X^\cdot} = 0$ for any $n \in \mathbb{Z}$. A morphism $f : X^\cdot \longrightarrow Y^\cdot$ in $C(\mathcal{C})$ is a family of morphisms $\{f^n : X^n \longrightarrow Y^n\}_{n\in\mathbb{Z}}$ in $\mathcal{C}$ satisfying the condition $d^n_{Y^\cdot} \circ f^n = f^{n+1} \circ d^n_{X^\cdot}$ for any $n \in \mathbb{Z}$. Namely, a morphism in $C(\mathcal{C})$ is just a chain map between two complexes in $\mathcal{C}$. For an object $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ of $C(\mathcal{C})$ we say that $X^\cdot$ is bounded below (resp. bounded above, resp. bounded) if it satisfies the condition $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$, resp. $|i| \gg 0$). We denote by $C^+(\mathcal{C})$ (resp. $C^-(\mathcal{C})$, resp. $C^b(\mathcal{C})$) the full subcategory of $C(\mathcal{C})$ consisting of objects which are bounded below (resp. bounded above, resp. bounded). These are naturally abelian categories. Moreover, we identify $\mathcal{C}$ with the full subcategory of $C(\mathcal{C})$ consisting of complexes concentrated in degree 0.

Definition B.2.1. We say that a morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$ is a *quasi-isomorphism* if it induces an isomorphism $H^n(X^\cdot) \simeq H^n(Y^\cdot)$ between cohomology groups for any $n \in \mathbb{Z}$.

Definition B.2.2.

(i) For a complex $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ with differentials $d^n_{X^\cdot} : X^n \to X^{n+1}$ ($n \in \mathbb{Z}$) and an integer $k \in \mathbb{Z}$, we define the *shifted complex* $X^\cdot[k]$ by

$$\begin{cases} X^n[k] = X^{n+k}, \\ d^n_{X^\cdot[k]} = (-1)^k d^{n+k}_{X^\cdot} : X^n[k] = X^{n+k} \longrightarrow X^{n+1}[k] = X^{n+k+1}. \end{cases}$$

(ii) For a morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$, we define the *mapping cone* $M_f{}^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ by

$$\begin{cases} M^n_f = X^{n+1} \oplus Y^n, \\ d^n_{M_f{}^\cdot} : M^n_f = X^{n+1} \oplus Y^n \longrightarrow M^{n+1}_f = X^{n+2} \oplus Y^{n+1} \\ \qquad\qquad\qquad \cup\!\!\!\shortmid \qquad\qquad\qquad\qquad\qquad \cup\!\!\!\shortmid \\ \qquad\qquad (x^{n+1}, y^n) \longmapsto (-d^{n+1}_{X^\cdot}(x^{n+1}), f^{n+1}(x^{n+1}) + d^n_{Y^\cdot}(y^n)). \end{cases}$$

There exists a natural short exact sequence

$$0 \longrightarrow Y^\cdot \xrightarrow{\alpha(f)} M_f{}^\cdot \xrightarrow{\beta(f)} X^\cdot[1] \longrightarrow 0$$

in $C(\mathcal{C})$, from which we obtain the cohomology long exact sequence

$$\cdots \longrightarrow H^{n-1}(M_f{}^\cdot) \longrightarrow H^n(X^\cdot) \longrightarrow H^n(Y^\cdot) \longrightarrow H^n(M_f{}^\cdot) \longrightarrow \cdots$$

in $\mathcal{C}$. Since the connecting homomorphisms $H^n(X^\cdot) \longrightarrow H^n(Y^\cdot)$ in this long exact sequence coincide with $H^n(f) : H^n(X^\cdot) \longrightarrow H^n(Y^\cdot)$ induced by $f : X^\cdot \to Y^\cdot$, we obtain the following useful result.

Lemma B.2.3. *A morphism $f : X^\cdot \to Y^\cdot$ in $C(\mathcal{C})$ is a quasi-isomorphism if and only if $H^n(M_f{}^\cdot) = 0$ for any $n \in \mathbb{Z}$.*

Definition B.2.4. For a complex $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ and an integer $k \in \mathbb{Z}$ we define the *truncated complexes* by

$$\tau^{\leqslant k} X^\cdot = \tau^{<k+1} X^\cdot := [\cdots \to X^{k-1} \to Z^k = \mathrm{Ker}\, d^k_{X^\cdot} \to 0 \to 0 \to \cdots],$$
$$\tau^{\geqslant k+1} X^\cdot = \tau^{>k} X^\cdot := [\cdots \to 0 \to 0 \to B^{k+1} = \mathrm{Im}\, d^k_{X^\cdot} \to X^{k+1} \to \cdots].$$

For $X^\cdot \in \mathrm{Ob}(C(\mathcal{C}))$ there exists a short exact sequence

$$0 \longrightarrow \tau^{\leqslant k} X^\cdot \longrightarrow X^\cdot \longrightarrow \tau^{>k} X^\cdot \longrightarrow 0$$

in $C(\mathcal{C})$ for each $k \in \mathbb{Z}$. Note that the complexes $\tau^{\leqslant k} X^\cdot$ and $\tau^{>k} X^\cdot$ satisfy the following conditions, which explain the reason why we call them "truncated" complexes:

$$H^j(\tau^{\leqslant k} X^\cdot) \simeq \begin{cases} H^j(X^\cdot) & j \leq k \\ 0 & j > k, \end{cases}$$

$$H^j(\tau^{>k} X^\cdot) \simeq \begin{cases} H^j(X^\cdot) & j > k \\ 0 & j \leq k. \end{cases}$$

B.3 Homotopy categories

In this section, before constructing derived categories, we define homotopy categories. Derived categories are obtained by applying a localization of categories to homotopy categories. In order to apply the localization, we need a family of morphisms called a multiplicative system (see Definition B.4.2 below). But the quasi-isomorphisms in $C(\mathcal{C})$ do not form a multiplicative system. Therefore, for the preparation of the localization, we define the *homotopy categories* $K^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) of an abelian category $\mathcal{C}$ as follows. First recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) is *homotopic to* 0 (we write $f \sim 0$ for short) if there exists a family $\{s_n : X^n \to Y^{n-1}\}_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{C}$ such that $f^n = s^{n+1} \circ d^n_{X^{\cdot}} + d^{n-1}_{Y^{\cdot}} \circ s^n$ for any $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X^{n-1} & \xrightarrow{d^{n-1}_{X^{\cdot}}} & X^n & \xrightarrow{d^n_{X^{\cdot}}} & X^{n+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle f^{n-1}}\downarrow & \swarrow{\scriptstyle s^n} & {\scriptstyle f^n}\downarrow & \swarrow{\scriptstyle s^{n+1}} & \downarrow{\scriptstyle f^{n+1}} & & \\
\cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d^{n-1}_{Y^{\cdot}}} & Y^n & \xrightarrow{d^n_{Y^{\cdot}}} & Y^{n+1} & \longrightarrow & \cdots .
\end{array}$$

We say also that two morphisms $f, g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ in $C^{\#}(\mathcal{C})$ are *homotopic* (we write $f \sim g$ for short) if the difference $f - g \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ is homotopic to 0.

Definition B.3.1. For $\# = \emptyset, +, -, b$ we define the homotopy category $K^{\#}(\mathcal{C})$ of $\mathcal{C}$ by

$$\begin{cases} \mathrm{Ob}(K^{\#}(\mathcal{C})) = \mathrm{Ob}(C^{\#}(\mathcal{C})), \\ \mathrm{Hom}_{K^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \big/ \mathrm{Ht}(X^{\cdot}, Y^{\cdot}), \end{cases}$$

where $\mathrm{Ht}(X^{\cdot}, Y^{\cdot})$ is a subgroup of $\mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot})$ defined by $\mathrm{Ht}(X^{\cdot}, Y^{\cdot}) = \big\{ f \in \mathrm{Hom}_{C^{\#}(\mathcal{C})}(X^{\cdot}, Y^{\cdot}) \mid f \sim 0 \big\}$.

The homotopy categories $K^{\#}(\mathcal{C})$ are not abelian, but they are still additive categories. We may regard the categories $K^{\#}(\mathcal{C})$ ($\# = +, -, b$) as full subcategories of $K(\mathcal{C})$. Moreover, $\mathcal{C}$ is naturally identified with the full subcategories of these homotopy categories consisting of complexes concentrated in degree 0. Since morphisms which are homotopic to 0 induce zero maps in cohomology groups, the additive functors $H^n : K^{\#}(\mathcal{C}) \to \mathcal{C}$ ($X^{\cdot} \longmapsto H^n(X^{\cdot})$) are well defined. We say that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $K^{\#}(\mathcal{C})$ is a quasi-isomorphism if it induces an isomorphism $H^n(X^{\cdot}) \simeq H^n(Y^{\cdot})$ for any $n \in \mathbb{Z}$. Recall that a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $C^{\#}(\mathcal{C})$ is called a *homotopy equivalence* if there exists a morphism $g : Y^{\cdot} \to X^{\cdot}$ in $C^{\#}(\mathcal{C})$ such that $g \circ f \sim \mathrm{id}_{X^{\cdot}}$ and $f \circ g \sim \mathrm{id}_{Y^{\cdot}}$. Homotopy equivalences in $C^{\#}(\mathcal{C})$ are isomorphisms in $K^{\#}(\mathcal{C})$ and hence quasi-isomorphisms. As in the case of the categories $C^{\#}(\mathcal{C})$, we can also define truncation functors $\tau^{\geqslant k} : K(\mathcal{C}) \to K^{+}(\mathcal{C})$ and $\tau^{\leqslant k} : K(\mathcal{C}) \to K^{-}(\mathcal{C})$.

Since the homotopy category $K^{\#}(\mathcal{C})$ is not abelian, we cannot consider short exact sequences in it any more. So we introduce the notion of distinguished triangles in

$K^{\#}(\mathcal{C})$ which will be a substitute for that of short exact sequences in the derived category $D^{\#}(\mathcal{C})$.

Definition B.3.2.

(i) A sequence $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ of morphisms in $K^{\#}(\mathcal{C})$ is called a *triangle*.

(ii) A *morphism of triangles* between two triangles $X_1^{\cdot} \longrightarrow Y_1^{\cdot} \longrightarrow Z_1^{\cdot} \longrightarrow X_1^{\cdot}[1]$ and $X_2^{\cdot} \longrightarrow Y_2^{\cdot} \longrightarrow Z_2^{\cdot} \longrightarrow X_2^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a commutative diagram

$$\begin{array}{ccccccc} X_1^{\cdot} & \longrightarrow & Y_1^{\cdot} & \longrightarrow & Z_1^{\cdot} & \longrightarrow & X_1^{\cdot}[1] \\ {\scriptstyle h}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h[1]} \\ X_2^{\cdot} & \longrightarrow & Y_2^{\cdot} & \longrightarrow & Z_2^{\cdot} & \longrightarrow & X_2^{\cdot}[1] \end{array}$$

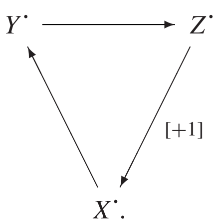
in $K^{\#}(\mathcal{C})$.

(iii) We say that a triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in $K^{\#}(\mathcal{C})$ is a *distinguished triangle* if it is isomorphic to a mapping cone triangle $X_0^{\cdot} \xrightarrow{f} Y_0^{\cdot} \xrightarrow{\alpha(f)} M_f^{\cdot} \xrightarrow{\beta(f)} X_0^{\cdot}[1]$ associated to a morphism $f : X_0^{\cdot} \longrightarrow Y_0^{\cdot}$ in $C^{\#}(\mathcal{C})$, i.e., there exists a commutative diagram

$$\begin{array}{ccccccc} X_0^{\cdot} & \xrightarrow{f} & Y_0^{\cdot} & \xrightarrow{\alpha(f)} & M_f^{\cdot} & \xrightarrow{\beta(f)} & X_0^{\cdot}[1] \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ X^{\cdot} & \longrightarrow & Y^{\cdot} & \longrightarrow & Z^{\cdot} & \longrightarrow & X^{\cdot}[1] \end{array}$$

in which all vertical arrows are isomorphisms in $K^{\#}(\mathcal{C})$.

A distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ is sometimes denoted by $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \xrightarrow{+1}$ or by



Proposition B.3.3. *The family of distinguished triangles in $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ satisfies the following properties* (TR0) $\sim$ (TR5)*:*

(TR0) *A triangle which is isomorphic to a distinguished triangle is also distinguished.*

(TR1) *For any $X^{\cdot} \in \mathrm{Ob}(\mathcal{C}_0)$, $X^{\cdot} \xrightarrow{\mathrm{id}_{X^{\cdot}}} X^{\cdot} \longrightarrow 0 \longrightarrow X^{\cdot}[1]$ is a distinguished triangle.*

(TR2) *Any morphism* $f : X^\bullet \longrightarrow Y^\bullet$ *in* $\mathcal{C}_0$ *can be embedded into a distinguished triangle* $X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$.

(TR3) *A triangle* $X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1]$ *in* $\mathcal{C}_0$ *is distinguished if and only if* $Y^\bullet \xrightarrow{g} Z^\bullet \xrightarrow{h} X^\bullet[1] \xrightarrow{-f[1]} Y^\bullet[1]$ *is distinguished.*

(TR4) *Given two distinguished triangles* $X_1{}^\bullet \xrightarrow{f_1} Y_1{}^\bullet \longrightarrow Z_1{}^\bullet \longrightarrow X_1{}^\bullet[1]$ *and* $X_2{}^\bullet \xrightarrow{f_2} Y_2{}^\bullet \longrightarrow Z_2{}^\bullet \longrightarrow X_2{}^\bullet[1]$ *and a commutative diagram*

$$\begin{array}{ccc} X_1{}^\bullet & \xrightarrow{f_1} & Y_1{}^\bullet \\ {\scriptstyle h}\downarrow & & \downarrow \\ X_2{}^\bullet & \xrightarrow{f_2} & Y_2{}^\bullet \end{array}$$

in $\mathcal{C}_0$*, then we can embed them into a morphism of triangles, i.e., into a commutative diagram in* $\mathcal{C}_0$*:*

$$\begin{array}{ccccccc} X_1{}^\bullet & \longrightarrow & Y_1{}^\bullet & \longrightarrow & Z_1{}^\bullet & \longrightarrow & X_1{}^\bullet[1] \\ {\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle h[1]} \\ X_2{}^\bullet & \longrightarrow & Y_2{}^\bullet & \longrightarrow & Z_2{}^\bullet & \longrightarrow & X_2{}^\bullet[1]. \end{array}$$

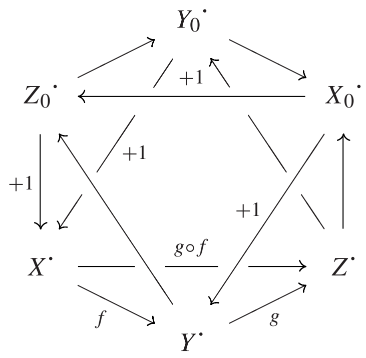
(TR5) *Let*

$$\begin{cases} X^\bullet \xrightarrow{f} Y^\bullet \longrightarrow Z_0{}^\bullet \longrightarrow X^\bullet[1] \\ Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow X_0{}^\bullet \longrightarrow Y^\bullet[1] \\ X^\bullet \xrightarrow{g\circ f} Z^\bullet \longrightarrow Y_0{}^\bullet \longrightarrow X^\bullet[1], \end{cases}$$

be three distinguished triangles. Then there exists a distinguished triangle $Z_0{}^\bullet \longrightarrow Y_0{}^\bullet \longrightarrow X_0{}^\bullet \longrightarrow Z_0{}^\bullet[1]$ *which can be embedded into the commutative diagram*

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \longrightarrow & Z_0{}^\bullet & \longrightarrow & X^\bullet[1] \\ {\scriptstyle \mathrm{id}}\| & & {\scriptstyle g}\downarrow & & \downarrow & & \|{\scriptstyle \mathrm{id}} \\ X^\bullet & \xrightarrow{g\circ f} & Z^\bullet & \longrightarrow & Y_0{}^\bullet & \longrightarrow & X^\bullet[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle \mathrm{id}}\| & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ Y^\bullet & \xrightarrow{g} & Z^\bullet & \longrightarrow & X_0{}^\bullet & \longrightarrow & Y^\bullet[1] \\ \downarrow & & \downarrow & & {\scriptstyle \mathrm{id}}\| & & \downarrow \\ Z_0{}^\bullet & \longrightarrow & Y_0{}^\bullet & \longrightarrow & X_0{}^\bullet & \longrightarrow & Z_0{}^\bullet[1]. \end{array}$$

For the proof, see [KS2, Proposition 1.4.4]. The property (TR5) is called the *octahedral axiom*, because it can be visualized by the following figure:



Corollary B.3.4. *Set* $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ *and let* $X^{\bullet} \xrightarrow{f} Y^{\bullet} \xrightarrow{g} Z^{\bullet} \xrightarrow{h} X^{\bullet}[1]$ *be a distinguished triangle in* $\mathcal{C}_0$*.*

(i) *For any* $n \in \mathbb{Z}$ *the sequence* $H^n(X^{\bullet}) \xrightarrow{H^n(f)} H^n(Y^{\bullet}) \xrightarrow{H^n(g)} H^n(Z^{\bullet})$ *in* $\mathcal{C}$ *is exact.*
(ii) *The composite* $g \circ f$ *is zero.*
(iii) *For any* $W^{\bullet} \in \mathrm{Ob}(\mathcal{C}_0)$*, the sequences*

$$\begin{cases} \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, X^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, Y^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(W^{\bullet}, Z^{\bullet}) \\ \mathrm{Hom}_{\mathcal{C}_0}(Z^{\bullet}, W^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(Y^{\bullet}, W^{\bullet}) \longrightarrow \mathrm{Hom}_{\mathcal{C}_0}(X^{\bullet}, W^{\bullet}) \end{cases}$$

associated to the distinguished triangle $X^{\bullet} \xrightarrow{f} Y^{\bullet} \xrightarrow{g} Z^{\bullet} \xrightarrow{h} X^{\bullet}[1]$ *are exact in the abelian category* $\mathcal{A}b$ *of abelian groups.*

Corollary B.3.5. *Set* $\mathcal{C}_0 = K^{\#}(\mathcal{C})$*.*

(i) *Let*

$$\begin{array}{ccccccc} X_1^{\bullet} & \longrightarrow & Y_1^{\bullet} & \longrightarrow & Z_1^{\bullet} & \longrightarrow & X_1^{\bullet}[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ X_2^{\bullet} & \longrightarrow & Y_2^{\bullet} & \longrightarrow & Z_2^{\bullet} & \longrightarrow & X_2^{\bullet}[1]. \end{array}$$

be a morphism of distinguished triangles in $\mathcal{C}_0$*. Assume that* f *and* g *are isomorphisms. Then* h *is also an isomorphism.*
(ii) *Let* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z^{\bullet} \longrightarrow X^{\bullet}[1]$ *and* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z'^{\bullet} \longrightarrow X^{\bullet}[1]$ *be two distinguished triangles in* $\mathcal{C}_0$*. Then* $Z^{\bullet} \simeq Z'^{\bullet}$*.*
(iii) *Let* $X^{\bullet} \xrightarrow{u} Y^{\bullet} \longrightarrow Z^{\bullet} \longrightarrow X^{\bullet}[1]$ *be a distinguished triangle in* $\mathcal{C}_0$*. Then* u *is an isomorphism if and only if* $Z^{\bullet} \simeq 0$*.*

By abstracting the properties of the homotopy categories $K^{\#}(\mathcal{C})$ let us introduce the notion of triangulated categories as follows. In the case when $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ for an abelian category $\mathcal{C}$, the automorphism $T : \mathcal{C}_0 \longrightarrow \mathcal{C}_0$ in the definition below is the degree shift functor $(\bullet)[1] : K^{\#}(\mathcal{C}) \to K^{\#}(\mathcal{C})$.

Definition B.3.6. Let $\mathcal{C}_0$ be an additive category and $T : \mathcal{C}_0 \to \mathcal{C}_0$ an automorphism of $\mathcal{C}_0$.

(i) A sequence of morphisms $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ is called a triangle in $\mathcal{C}_0$.
(ii) Consider a family $\mathcal{T}$ of triangles in $\mathcal{C}_0$, called distinguished triangles. We say that the pair $(\mathcal{C}_0, \mathcal{T})$ is a *triangulated category* if the family $\mathcal{T}$ of distinguished triangles satisfies the axioms obtained from (TR0) $\sim$ (TR5) in Proposition B.3.3 by replacing $(\bullet)[1]$'s with $T(\bullet)$'s everywhere.

It is clear that Corollary B.3.4 (ii), (iii) and Corollary B.3.5 are true for any triangulated category $(\mathcal{C}_0, \mathcal{T})$. Derived categories that we introduce in the next section are also triangulated categories. Note also that the morphism ψ in (TR4) is not unique in general, which is the source of some difficulties in using triangulated categories.

Definition B.3.7. Let $(\mathcal{C}_0, \mathcal{T})$, $(\mathcal{C}'_0, \mathcal{T}')$ be two triangulated categories and $T : \mathcal{C}_0 \to \mathcal{C}_0$, $T' : \mathcal{C}'_0 \to \mathcal{C}'_0$ the corresponding automorphisms. Then we say that an additive functor $F : \mathcal{C}_0 \to \mathcal{C}'_0$ is a *functor of triangulated categories* (or a ∂*-functor*) if $F \circ T = T' \circ F$ and F sends distinguished triangles in $\mathcal{C}_0$ to those in $\mathcal{C}'_0$.

Definition B.3.8. An additive functor $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ from a triangulated category $(\mathcal{C}_0, \mathcal{T})$ into an abelian category $\mathcal{A}$ is called a *cohomological functor* if for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$, the associated sequence $F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ in $\mathcal{A}$ is exact.

The assertions (i) and (iii) of Corollary B.3.4 imply that the functors $H^n : K^{\#}(\mathcal{C}) \longrightarrow \mathcal{C}$ and $\mathrm{Hom}_{K^{\#}(\mathcal{C})}(W^{\cdot}, \bullet) : K^{\#}(\mathcal{C}) \longrightarrow \mathcal{A}b$ are cohomological functors, respectively. Let $F : \mathcal{C}_0 \longrightarrow \mathcal{A}$ be a cohomological functor. Then by using the axiom (TR3) repeatedly, from a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ in $\mathcal{C}_0$ we obtain a long exact sequence

$$\begin{aligned} \cdots \longrightarrow F(T^{-1}Z) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(TX) \\ \longrightarrow F(TY) \longrightarrow \cdots \end{aligned}$$

in the abelian category $\mathcal{A}$.

B.4 Derived categories

In this section, we shall construct derived categories $D^{\#}(\mathcal{C})$ from homotopy categories $K^{\#}(\mathcal{C})$ by adding morphisms so that quasi-isomorphisms are invertible in $D^{\#}(\mathcal{C})$. For this purpose, we need the general theory of localizations of categories. Now let $\mathcal{C}_0$ be a category and S a family of morphisms in $\mathcal{C}_0$. In what follows, for two functors $F_1, F_2 : \mathcal{A}_1 \to \mathcal{A}_2$ $(\in Fun(\mathcal{A}_1, \mathcal{A}_2))$ we denote by $\mathrm{Hom}_{Fun(\mathcal{A}_1,\mathcal{A}_2)}(F_1, F_2)$ the set of *natural transformations* (i.e., morphisms of functors) from F_1 to F_2.

Definition B.4.1. A *localization* of the category $\mathcal{C}_0$ by S is a pair $((\mathcal{C}_0)_S, Q)$ of a category $(\mathcal{C}_0)_S$ and a functor $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ which satisfies the following universal properties:

(i) $Q(s)$ is an isomorphism for any $s \in S$.
(ii) For any functor $F : \mathcal{C}_0 \to \mathcal{C}_1$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and an isomorphism $F \simeq F_S \circ Q$ of functors:

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{Q} & (\mathcal{C}_0)_S \\ {\scriptstyle F}\downarrow & \swarrow {\scriptstyle \exists F_S} & \\ \mathcal{C}_1. & & \end{array}$$

(iii) Let $G_1, G_2 : (\mathcal{C}_0)_S \to \mathcal{C}_1$ be two functors. Then the natural morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S,\mathcal{C}_1)}(G_1, G_2) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(G_1 \circ Q, G_2 \circ Q)$$

is a bijection.

By the property (iii), F_S in (ii) is unique up to isomorphisms. Moreover, since the localization $((\mathcal{C}_0)_S, Q)$ is characterized by universal properties (if it exists) it is unique up to equivalences of categories. We call this operation "a localization of categories" because it is similar to the more familiar localization of (non-commutative) rings. As we need the so-called "Ore conditions" for the construction of localizations of rings, we have to impose some conditions on S to ensure the existence of the localization $((\mathcal{C}_0)_S, Q)$.

Definition B.4.2. Let $\mathcal{C}_0$ be a category and S a family of morphisms. We call the family S a *multiplicative system* if it satisfies the following axioms:

(M1) $\mathrm{id}_X \in S$ for any $X \in \mathrm{Ob}(\mathcal{C}_0)$.
(M2) If $f, g \in S$ and their composite $g \circ f$ exists, then $g \circ f \in S$.
(M3) Any diagram

$$\begin{array}{ccc} & & Y' \\ & & {\scriptstyle s}\downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{C}_0$ with $s \in S$ fits into a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{C}_0$ with $t \in S$. We impose also the condition obtained by reversing all arrows.
(M4) For $f, g \in \mathrm{Hom}_{\mathcal{C}_0}(X, Y)$ the following two conditions are equivalent:

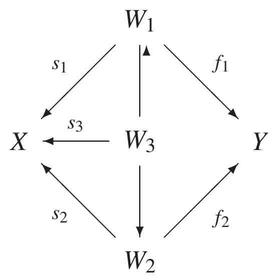
(i) $^{\exists}s : Y \longrightarrow Y'$, $s \in S$ such that $s \circ f = s \circ g$.
(ii) $^{\exists}t : X' \longrightarrow X$, $t \in S$ such that $f \circ t = g \circ t$.

Let $\mathcal{C}_0$ be a category and S a multiplicative system in it. Then we can define a category $(\mathcal{C}_0)_S$ by

(objects): $\mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)_S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) = \left\{(X \overset{s}{\longleftarrow} W \overset{f}{\longrightarrow} Y) \mid s \in S\right\} / \sim$$

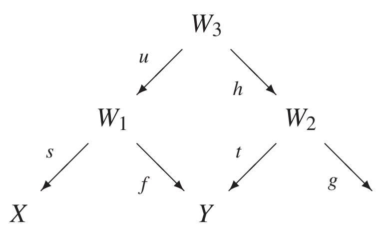
where two diagrams $(X \overset{s_1}{\longleftarrow} W_1 \overset{f_1}{\longrightarrow} Y)$ $(s_1 \in S)$ and $(X \overset{s_2}{\longleftarrow} W_2 \overset{f_2}{\longrightarrow} Y)$ $(s_2 \in S)$ are equivalent ($\sim$) if and only if they fit into a commutative diagram



with $s_3 \in S$. We omit the details here. Let us just explain how we compose morphisms in the category $(\mathcal{C}_0)_S$. Assume that we are given two morphisms

$$\begin{cases} \left[(X \overset{s}{\longleftarrow} W_1 \overset{f}{\longrightarrow} Y)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Y) \\ \left[(Y \overset{t}{\longleftarrow} W_2 \overset{g}{\longrightarrow} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(Y, Z) \end{cases}$$

$(s, t \in S)$ in $(\mathcal{C}_0)_S$. Then by the axiom (M3) of multiplicative systems we can construct a commutative diagram



with $u \in S$ and the composite of these two morphisms in $(\mathcal{C}_0)_S$ is given by

$$\left[(X \overset{s \circ u}{\longleftarrow} W_3 \overset{g \circ h}{\longrightarrow} Z)\right] \in \mathrm{Hom}_{(\mathcal{C}_0)_S}(X, Z).$$

Moreover, there exists a natural functor $Q : \mathcal{C}_0 \longrightarrow (\mathcal{C}_0)_S$ defined by

$$\begin{cases} Q(X) = X \quad \text{for } X \in \mathrm{Ob}(\mathcal{C}_0), \\ \mathrm{Hom}_{\mathcal{C}_0}(X, Y) \longrightarrow \mathrm{Hom}_{(\mathcal{C}_0)_S}(Q(X), Q(Y)) \\ \quad \Psi \qquad\qquad\qquad \Psi \\ \quad f \qquad \longmapsto \quad \left[(X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for any } X, Y \in \mathrm{Ob}(\mathcal{C}_0). \end{cases}$$

We can easily check that the pair $((\mathcal{C}_0)_S, Q)$ satisfies the conditions of the localization of $\mathcal{C}_0$ by S. Since morphisms in S are invertible in the localized category $(\mathcal{C}_0)_S$, a morphism $[(X \xleftarrow{s} W \xrightarrow{f} Y)]$ in $(\mathcal{C}_0)_S$ can be written also as $Q(f) \circ Q(s)^{-1}$. If, moreover, $\mathcal{C}_0$ is an additive category, then we can show that $(\mathcal{C}_0)_S$ is also an additive category and $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ is an additive functor. Since we defined the localization $(\mathcal{C}_0)_S$ of $\mathcal{C}_0$ by universal properties, also the following category $(\mathcal{C}_0)^S$ satisfies the conditions of the localization:

(objects): $\mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$.
(morphisms): For $X, Y \in \mathrm{Ob}((\mathcal{C}_0)^S) = \mathrm{Ob}(\mathcal{C}_0)$, we set

$$\mathrm{Hom}_{(\mathcal{C}_0)^S}(X, Y) = \left\{(X \xrightarrow{f} W \xleftarrow{s} Y) \mid s \in S\right\} / \sim$$

where we define the equivalence $\sim$ of diagrams similarly. Namely, a morphism in the localization $(\mathcal{C}_0)_S$ can be written also as $Q(s)^{-1} \circ Q(f)$ for $s \in S$. The following elementary lemma will be effectively used in the next section.

Lemma B.4.3. *Let $\mathcal{C}_0$ be a category and S a multiplicative system in it. Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by T the family of morphisms in $\mathcal{J}_0$ which belong to S. Assume, moreover, that for any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in S such that $J \in \mathrm{Ob}(\mathcal{J}_0)$. Then T is a multiplicative system in $\mathcal{J}_0$, and the natural functor $(\mathcal{J}_0)_T \to (\mathcal{C}_0)_S$ gives an equivalence of categories.*

Now let us return to the original situation and consider a homotopy category $\mathcal{C}_0 = K^{\#}(\mathcal{C})$ ($\# = \emptyset, +, -, b$) of an abelian category $\mathcal{C}$. Denote by S the family of quasi-isomorphisms in it. Then we can prove that S is a multiplicative system.

Definition B.4.4. We set $D^{\#}(\mathcal{C}) = (K^{\#}(\mathcal{C}))_S$ and call it a *derived category* of $\mathcal{C}$. The canonical functor $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is called the *localization functor*.

By construction, quasi-isomorphisms are isomorphisms in the derived category $D^{\#}(\mathcal{C})$. Moreover, if we define distinguished triangles in $D^{\#}(\mathcal{C})$ to be the triangles isomorphic to the images of distinguished triangles in $K^{\#}(\mathcal{C})$ by Q, then $D^{\#}(\mathcal{C})$ is a triangulated category and $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ is a functor of triangulated categories. We can also prove that the canonical morphisms $\mathcal{C} \to D(\mathcal{C})$ and $D^{\#}(\mathcal{C}) \to D(\mathcal{C})$ ($\# = +, -, b$) are fully faithful. Namely, the categories $\mathcal{C}$ and $D^{\#}(\mathcal{C})$ ($\# = +, -, b$) can be identified with full subcategories of $D(\mathcal{C})$. By the results in the previous section, to a distinguished triangle $X^{\cdot} \longrightarrow Y^{\cdot} \longrightarrow Z^{\cdot} \longrightarrow X^{\cdot}[1]$ in the derived category $D^{\#}(\mathcal{C})$ we can associate a cohomology long exact sequence

$$\cdots \longrightarrow H^{-1}(Z^\cdot) \longrightarrow H^0(X^\cdot) \longrightarrow H^0(Y^\cdot) \longrightarrow H^0(Z^\cdot) \longrightarrow H^1(X^\cdot) \longrightarrow \cdots$$

in $\mathcal{C}$. The following lemma is very useful to construct examples of distinguished triangles in $D^\#(\mathcal{C})$.

Lemma B.4.5. *Any short exact sequence* $0 \longrightarrow X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow 0$ *in* $C^\#(\mathcal{C})$ *can be embedded into a distinguished triangle* $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \longrightarrow X^\cdot[1]$ *in* $D^\#(\mathcal{C})$.

Proof. Consider the short exact sequence

$$0 \longrightarrow M_{\mathrm{id}_{X^\cdot}}{}^\cdot \xrightarrow[\begin{pmatrix} \mathrm{id}_{X^\cdot} & 0 \\ 0 & f \end{pmatrix}]{} M_f{}^\cdot \xrightarrow[(0,\, g)]{\varphi} Z^\cdot \longrightarrow 0$$

in $C^\#(\mathcal{C})$. Since the mapping cone $M_{\mathrm{id}_{X^\cdot}}{}^\cdot$ is quasi-isomorphic to 0 by Lemma B.2.3, we obtain an isomorphism $\varphi : M_f{}^\cdot \simeq Z^\cdot$ in $D^\#(\mathcal{C})$. Hence there exists a commutative diagram

$$\begin{array}{ccccccc} X^\cdot & \xrightarrow{f} & Y^\cdot & \xrightarrow{\alpha(f)} & M_f{}^\cdot & \xrightarrow{\beta(f)} & X^\cdot[1] \\ \mathrm{id}\,\Big\| & & \mathrm{id}\,\Big\| & & \varphi\wr\Big\downarrow & & \mathrm{id}\,\Big\| \\ X^\cdot & \xrightarrow{f} & Y^\cdot & \xrightarrow{g} & Z^\cdot & \xrightarrow{\beta(f)\circ\varphi^{-1}} & X^\cdot[1] \end{array}$$

in $D^\#(\mathcal{C})$, which shows that $X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{\beta(f)\circ\varphi^{-1}} X^\cdot[1]$ is a distinguished triangle. □

Definition B.4.6. An abelian subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *thick subcategory* if for any exact sequence $X_1 \to X_2 \to X_3 \to X_4 \to X_5$ in $\mathcal{C}$ with $X_i \in \mathrm{Ob}(\mathcal{C}')$ $(i = 1, 2, 4, 5)$, X_3 belongs to $\mathcal{C}'$.

Proposition B.4.7. *Let* $\mathcal{C}'$ *be a thick abelian subcategory of an abelian category* $\mathcal{C}$ *and* $\mathrm{D}^\#_{\mathcal{C}'}(\mathcal{C})$ *the full subcategory of* $\mathrm{D}^\#(\mathcal{C})$ *consisting of objects* $X^\cdot$ *such that* $H^n(X^\cdot) \in \mathrm{Ob}(\mathcal{C}')$ *for any* $n \in \mathbb{Z}$. *Then* $\mathrm{D}^\#_{\mathcal{C}'}(\mathcal{C})$ *is a triangulated category.*

B.5 Derived functors

Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ an additive functor. Let us consider the problem of constructing a ∂-functor $\tilde{F} : D^\#(\mathcal{C}) \to D^\#(\mathcal{C}')$ between their derived categories which is naturally associated to $F : \mathcal{C} \to \mathcal{C}'$. This problem can be easily solved if $\# = +$ or $-$ and F is an exact functor. Indeed, let $Q : K^\#(\mathcal{C}) \to D^\#(\mathcal{C})$, $Q' : K^\#(\mathcal{C}') \to D^\#(\mathcal{C}')$ be the localization functors and consider the functor $K^\# F : K^\#(\mathcal{C}) \to K^\#(\mathcal{C}')$ defined by $X^\cdot \mapsto F(X^\cdot)$. Then by Lemma B.2.3 the functor $K^\# F$ sends quasi-isomorphisms in $K^\#(\mathcal{C})$ to those in $K^\#(\mathcal{C}')$. Hence it follows from the universal properties of the localization $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ that there exists a unique functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ which makes the following diagram commutative:

$$\begin{array}{ccc} K^{\#}(\mathcal{C}) & \xrightarrow{K^{\#}F} & K^{\#}(\mathcal{C}') \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\ D^{\#}(\mathcal{C}) & \xrightarrow[\tilde{F}]{} & D^{\#}(\mathcal{C}'). \end{array}$$

In this situation, we call the functor $\tilde{F} : D^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ a "localization" of $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$. However, many important additive functors that we encounter in sheaf theory or homological algebra are not exact. They are only left exact or right exact. So in such cases the functor $Q' \circ K^{\#}F : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C}')$ does not factorize through $Q : K^{\#}(\mathcal{C}) \to D^{\#}(\mathcal{C})$ in general. In other words, there is no localization of the functor $Q' \circ K^{\#}F$. As a remedy for this problem we will introduce the following notion of right (or left) localizations. In what follows, let $\mathcal{C}_0$ be a general category, S a multiplicative system in $\mathcal{C}_0$, $F : \mathcal{C}_0 \to \mathcal{C}_1$ a functor. As before we denote by $Q : \mathcal{C}_0 \to (\mathcal{C}_0)_S$ the canonical functor.

Definition B.5.1. A *right localization* of F is a pair (F_S, τ) of a functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ and a morphism of functors $\tau : F \to F_S \circ Q$ such that for any functor $G : (\mathcal{C}_0)_S \to \mathcal{C}_1$ the morphism

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S,\mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(F, G \circ Q)$$

is bijective. Here the morphism above is obtained by the composition of

$$\mathrm{Hom}_{Fun((\mathcal{C}_0)_S,\mathcal{C}_1)}(F_S, G) \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(F_S \circ Q, G \circ Q)$$

$$\xrightarrow{\tau} \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(F, G \circ Q).$$

We say that F is *right localizable* if it has a right localization.

The notion of left localizations can be defined similarly. Note that by definition the functor F_S is a representative of the functor

$$G \longrightarrow \mathrm{Hom}_{Fun(\mathcal{C}_0,\mathcal{C}_1)}(F, G \circ Q).$$

Therefore, if a right localization (F_S, τ) of F exists, it is unique up to isomorphisms. Let us give a useful criterion for the existence of the right localization of $F : \mathcal{C}_0 \to \mathcal{C}_1$.

Proposition B.5.2. *Let $\mathcal{J}_0$ be a full subcategory of $\mathcal{C}_0$ and denote by T the family of morphisms in $\mathcal{J}_0$ which belong to S. Assume the following conditions:*

(i) *For any $X \in \mathrm{Ob}(\mathcal{C}_0)$ there exists a morphism $s : X \to J$ in S such that $J \in \mathrm{Ob}(\mathcal{J}_0)$.*
(ii) *$F(t)$ is an isomorphism for any $t \in T$.*

Then F is localizable.

A very precise proof of this proposition can be found in Kashiwara–Schapira [KS4, Proposition 7.3.2]. Here we just explain how the functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined. First, by Lemma B.4.3 there exists an equivalence of categories $\Phi : (\mathcal{J}_0)_T \xrightarrow{\sim} (\mathcal{C}_0)_S$. Let $\iota : \mathcal{J}_0 \to \mathcal{C}_0$ be the inclusion. Then by the condition (ii) above the functor $F \circ \iota : \mathcal{J}_0 \to \mathcal{C}_1$ factorizes through the localization functor $\mathcal{J}_0 \to (\mathcal{J}_0)_T$ and we obtain a functor $F_T : (\mathcal{J}_0)_T \to \mathcal{C}_1$. The functor $F_S : (\mathcal{C}_0)_S \to \mathcal{C}_1$ is defined by $F_S = F_T \circ \Phi^{-1}$:

$$\begin{array}{ccc} (\mathcal{J}_0)_T & \xrightarrow{F_T} & \mathcal{C}_1 \\ \Phi \wr \downarrow & \nearrow_{F_S} & \\ (\mathcal{C}_0)_S . & & \end{array}$$

Now let us return to the original situation and assume that $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a left exact functor. In this situation, by Proposition B.5.2 we can give a criterion for the existence of a right localization of the functor $Q' \circ K^+F : K^+(\mathcal{C}) \to D^+(\mathcal{C}')$ as follows.

Definition B.5.3. A *right derived functor* of F is a pair (RF, τ) of a ∂-functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ and a morphism of functors $\tau : Q' \circ K^+(F) \to RF \circ Q$

$$\begin{array}{ccc} K^+(\mathcal{C}) & \xrightarrow{K^+(F)} & K^+(\mathcal{C}') \\ Q \downarrow & \swarrow_{\tau} & \downarrow Q' \\ D^+(\mathcal{C}) & \xrightarrow{RF} & D^+(\mathcal{C}') \end{array}$$

such that for any functor $\tilde{G} : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ the morphism

$$\mathrm{Hom}_{Fun(D^+(\mathcal{C}),D^+(\mathcal{C}'))}(RF, \tilde{G}) \longrightarrow \mathrm{Hom}_{Fun(K^+(\mathcal{C}),D^+(\mathcal{C}'))}(Q' \circ K^+(F), \tilde{G} \circ Q)$$

induced by τ is an isomorphism. We say that F is *right derivable* if it admits a right derived functor.

Similarly, for right exact functors F we can define the notion of *left derived functors* $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$. By definition, if a right derived functor (RF, τ) of a left exact functor F exists, it is unique up to isomorphisms. Moreover, for an exact functor F the natural functor $D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ defined simply by $X^\cdot \mapsto F(X^\cdot)$ gives a right derived functor. In other words, any exact functor is right (and left) derivable.

Definition B.5.4. Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor between abelian categories. We say that a full additive subcategory $\mathcal{J}$ of $\mathcal{C}$ is *F-injective* if the following conditions are satisfied:

(i) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an object $I \in \mathrm{Ob}(\mathcal{J})$ and an exact sequence $0 \to X \to I$.

(ii) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{C}$ and $X', X \in \mathrm{Ob}(\mathcal{J})$, then $X'' \in \mathrm{Ob}(\mathcal{J})$.

(iii) For any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ such that $X', X, X'' \in \mathrm{Ob}(\mathcal{J})$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ in $\mathcal{C}'$ is also exact.

Similarly we define *F-projective* subcategories of $\mathcal{C}$ by reversing all arrows in the conditions above.

Example B.5.5.

(i) Assume that the abelian category $\mathcal{C}$ has enough injectives. Denote by $\mathcal{I}$ the full additive subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$. Then $\mathcal{I}$ is F-injective for any additive functor $F : \mathcal{C} \to \mathcal{C}'$ (use the fact that any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathrm{Ob}(\mathcal{I})$ splits).

(ii) Let T_0 be a topological space and set $\mathcal{C} = \mathrm{Sh}(T_0)$. Let $F = \Gamma(T_0, \bullet) : \mathrm{Sh}(T_0) \to \mathcal{A}b$ be the global section functor. Then $F = \Gamma(T_0, \bullet)$ is left exact and

$$\mathcal{J} = \{\text{flabby sheaves on } T_0\} \subset \mathrm{Sh}(T_0)$$

is an F-injective subcategory of $\mathcal{C}$.

(iii) Let T_0 be a topological space and $\mathcal{R}$ a sheaf of rings on T_0. Denote by $\mathrm{Mod}(\mathcal{R})$ the abelian category of sheaves of left $\mathcal{R}$-modules on T_0 and let $\mathcal{P}$ be the full subcategory of $\mathrm{Mod}(\mathcal{R})$ consisting of flat $\mathcal{R}$-modules, i.e., objects $M \in \mathrm{Mod}(\mathcal{R})$ such that the stalk M_x at x is a flat $\mathcal{R}_x$-module for any $x \in T_0$. For a right $\mathcal{R}$-module N, consider the functor $F_N = N \otimes_{\mathcal{R}} (\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(T_0)$. Then the category $\mathcal{P}$ is F_N-projective. For the details see Section C.1.

Assume that for the given left exact functor $F : \mathcal{C} \to \mathcal{C}'$ there exists an F-injective subcategory $\mathcal{J}$ of $\mathcal{C}$. Then it is well known that for any $X^\cdot \in \mathrm{Ob}(K^+(\mathcal{C}))$ we can construct a quasi-isomorphism $X^\cdot \to J^\cdot$ into an object $J^\cdot$ of $K^+(\mathcal{J})$. Such an object $J^\cdot$ is called an *F-injective resolution* of $X^\cdot$. Moreover, by Lemma B.2.3 we see that the functor $Q' \circ K^+F : K^+(\mathcal{J}) \to D^+(\mathcal{C}')$ sends quasi-isomorphisms in $K^+(\mathcal{J})$ to isomorphisms in $D^+(\mathcal{C}')$. Therefore, applying Proposition B.5.2 to the special case when $\mathcal{C}_0 = K^+(\mathcal{C})$, $\mathcal{C}_1 = D^+(\mathcal{C}')$, $\mathcal{J}_0 = K^+(\mathcal{J})$ and S is the family of quasi-isomorphisms in $K^+(\mathcal{C})$, we obtain the fundamental important result.

Theorem B.5.6. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left (resp. right) exact functor and assume that there exists an F-injective (resp. F-projective) subcategory of $\mathcal{C}$. Then the right (resp. left) derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ (resp. $LF : D^-(\mathcal{C}) \to D^-(\mathcal{C}')$) of F exists.*

Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and $\mathcal{J}$ an F-injective subcategory of $\mathcal{C}$. Then the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ is constructed as follows. Denote by S (resp. T) the family of quasi-isomorphisms in $K^+(\mathcal{C})$ (resp. $K^+(\mathcal{J})$). Then there exist an equivalence of categories $\Phi : K^+(\mathcal{J})_T \xrightarrow{\sim} K^+(\mathcal{C})_S = D^+(\mathcal{C})$ and a natural functor $\Psi : K^+(\mathcal{J})_T \to D^+(\mathcal{C}')$ induced by $K^+F : K^+(\mathcal{C}) \to K^+(\mathcal{C}')$ such that $RF = \Psi \circ \Phi^{-1}$. Consequently the right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ sends $X^\cdot \in \mathrm{Ob}(D^+(\mathcal{C}))$ to $F(J^\cdot) \in \mathrm{Ob}(D^+(\mathcal{C}'))$, where $J^\cdot \in \mathrm{Ob}(K^+(\mathcal{J}))$

is an F-injective resolution of $X^{\cdot}$. If $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects in $\mathcal{C}$, then we have, moreover, an equivalence of categories $K^{+}(\mathcal{I}) \simeq D^{+}(\mathcal{C})$. This follows from the basic fact that quasi-isomorphisms in $K^{+}(\mathcal{I})$ are homotopy equivalences. Using this explicit description of derived functors, we obtain the following useful composition rule.

Proposition B.5.7. *Let $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ be abelian categories and $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ left exact functors. Assume that $\mathcal{C}$ (resp. $\mathcal{C}'$) has an F-injective (resp. G-injective) subcategory $\mathcal{J}$ (resp. $\mathcal{J}'$) and $F(X) \in \mathrm{Ob}(\mathcal{J}')$ for any $X \in \mathrm{Ob}(\mathcal{J})$. Then $\mathcal{J}$ is $(G \circ F)$-injective and*

$$R(G \circ F) = RG \circ RF : D^{+}(\mathcal{C}) \longrightarrow D^{+}(\mathcal{C}'').$$

Definition B.5.8. Assume that a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ is right derivable. Then for each $n \in \mathbb{Z}$ we set

$$R^{n}F = H^{n} \circ RF : D^{+}(\mathcal{C}) \to \mathcal{C}'.$$

Since $RF : D^{+}(\mathcal{C}) \to D^{+}(\mathcal{C}')$ sends distinguished triangles to distinguished triangles, the functors $R^{n}F : D^{+}(\mathcal{C}) \to \mathcal{C}'$ defined above are cohomological functors. Now let us identify $\mathcal{C}$ with the full subcategory of $D^{+}(\mathcal{C})$ consisting of complexes concentrated in degree 0. Then we find that for $X \in \mathrm{Ob}(\mathcal{C})$, $R^{n}F(X) \in \mathrm{Ob}(\mathcal{C}')$ coincides with the nth derived functor of F in the classical literature.

B.6 Bifunctors in derived categories

In this section we introduce some important bifunctors in derived categories which will be frequently used throughout this book. First, let us explain the bifunctor $\mathrm{RHom}(\bullet, \bullet)$. Let $\mathcal{C}$ be an abelian category. For two complexes $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}))$ in $\mathcal{C}$ define a new complex $\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot}) \in \mathrm{Ob}(C(\mathcal{A}b))$ by

$$\begin{cases} \mathrm{Hom}^{n}(X^{\cdot}, Y^{\cdot}) = \displaystyle\prod_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^{i}, Y^{j}) \\ d^{n} = d^{n}_{\mathrm{Hom}^{\cdot}(X^{\cdot},Y^{\cdot})} : \displaystyle\prod_{j-i=n} \mathrm{Hom}_{\mathcal{C}}(X^{i}, Y^{j}) \longrightarrow \displaystyle\prod_{j-i=n+1} \mathrm{Hom}_{\mathcal{C}}(X^{i}, Y^{j}) \\ \qquad\qquad\qquad \psi \qquad\qquad\qquad\qquad\qquad \psi \\ \qquad\qquad \{f_{ij}\} \longmapsto \left\{g_{ij} = (-1)^{n+1} f_{i+1,j} \circ d^{i}_{X^{\cdot}} + d^{j-1}_{Y^{\cdot}} \circ f_{i,j-1}\right\}. \end{cases}$$

This is the simple complex associated to the double complex $\mathrm{Hom}(X^{\cdot}, Y^{\cdot})$, which satisfies the conditions

$$\begin{cases} \mathrm{Ker}\, d^{n} \simeq \mathrm{Hom}_{C(\mathcal{C})}(X^{\cdot}, Y^{\cdot}[n]) \\ \mathrm{Im}\, d^{n-1} \simeq \mathrm{Ht}(X^{\cdot}, Y^{\cdot}[n]) \\ H^{n}[\mathrm{Hom}^{\cdot}(X^{\cdot}, Y^{\cdot})] \simeq \mathrm{Hom}_{K(\mathcal{C})}(X^{\cdot}, Y^{\cdot}[n]). \end{cases}$$

for any $n \in \mathbb{Z}$. We thus defined a bifunctor

$$\mathrm{Hom}^\cdot(\bullet, \bullet) : C(\mathcal{C})^{\mathrm{op}} \times C(\mathcal{C}) \longrightarrow C(\mathcal{A}b),$$

where $(\bullet)^{\mathrm{op}}$ denotes the *opposite category*. It is easy to check that it induces also a bifunctor

$$\mathrm{Hom}^\cdot(\bullet, \bullet) : K(\mathcal{C})^{\mathrm{op}} \times K(\mathcal{C}) \longrightarrow K(\mathcal{A}b)$$

in homotopy categories. Similarly we also obtain

$$\mathrm{Hom}^\cdot(\bullet, \bullet) : K^-(\mathcal{C})^{\mathrm{op}} \times K^+(\mathcal{C}) \longrightarrow K^+(\mathcal{A}b),$$

taking boundedness into account. From now on, assume that the category $\mathcal{C}$ has enough injectives and denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of injective objects. The following lemma is elementary.

Lemma B.6.1. *Let* $X^\cdot \in \mathrm{Ob}(K(\mathcal{C}))$ *and* $I^\cdot \in \mathrm{Ob}(K^+(\mathcal{I}))$. *Assume that* $X^\cdot$ *or* $I^\cdot$ *is quasi-isomorphic to* 0. *Then the complex* $\mathrm{Hom}^\cdot(X^\cdot, Y^\cdot) \in \mathrm{Ob}(K(\mathcal{A}b))$ *is also quasi-isomorphic to* 0.

Let $X^\cdot \in \mathrm{Ob}(K(\mathcal{C}))$ and consider the functor

$$\mathrm{Hom}^\cdot(X^\cdot, \bullet) : K^+(\mathcal{C}) \longrightarrow K(\mathcal{A}b).$$

Then by Lemmas B.2.3 and B.6.1 and Proposition B.5.2, we see that this functor induces a functor

$$R_{II}\,\mathrm{Hom}^\cdot(X^\cdot, \bullet) : D^+(\mathcal{C}) \longrightarrow D(\mathcal{A}b)$$

between derived categories. Here we write "R_{II}" to indicate that we localize the bifunctor $\mathrm{Hom}^\cdot(\bullet, \bullet)$ with respect to the second factor. Since this construction is functorial with respect to $X^\cdot$, we obtain a bifunctor

$$R_{II}\,\mathrm{Hom}^\cdot(\bullet, \bullet) : K(\mathcal{C})^{\mathrm{op}} \times D^+(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

By the universal properties of the localization $Q : K(\mathcal{C}) \to D(\mathcal{C})$, this bifunctor factorizes through Q and we obtain a bifunctor

$$R_I R_{II}\,\mathrm{Hom}^\cdot(\bullet, \bullet) : D(\mathcal{C})^{\mathrm{op}} \times D^+(\mathcal{C}) \longrightarrow D(\mathcal{A}b).$$

We set $\mathrm{RHom}_{\mathcal{C}}(\bullet, \bullet) = R_I R_{II}\,\mathrm{Hom}^\cdot(\bullet, \bullet)$ and call it the functor RHom. Similarly taking boundedness into account, we also obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{C}}(\bullet, \bullet) : D^-(\mathcal{C})^{\mathrm{op}} \times D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{A}b).$$

These are bifunctors of triangulated categories. The following proposition is very useful to construct canonical morphisms in derived categories.

Proposition B.6.2. *For* $Z^\cdot \in \mathrm{Ob}(K^+(\mathcal{C}))$ *and* $I^\cdot \in \mathrm{Ob}(K^+(\mathcal{I}))$ *the natural morphism*

$$Q : \mathrm{Hom}_{K^+(\mathcal{C})}(Z^\cdot, I^\cdot) \longrightarrow \mathrm{Hom}_{D^+(\mathcal{C})}(Z^\cdot, I^\cdot)$$

is an isomorphism. In particular for any $X^\cdot, Y^\cdot \in \mathrm{Ob}(D^+(\mathcal{C}))$ *and* $n \in \mathbb{Z}$ *there exists a natural isomorphism*

$$H^n \mathrm{RHom}_{\mathcal{C}}(X^\cdot, Y^\cdot) = H^n\,\mathrm{Hom}^\cdot(X^\cdot, I^\cdot) \xrightarrow{\sim} \mathrm{Hom}_{D^+(\mathcal{C})}(X^\cdot, Y^\cdot[n]),$$

where $I^\cdot$ *is an injective resolution of* $Y^\cdot$.

In the classical literature we denote $H^n\mathrm{RHom}_{\mathcal{C}}(X^{\cdot}, Y^{\cdot})$ by $Ext^n_{\mathcal{C}}(X^{\cdot}, Y^{\cdot})$ and call it the nth *hyperextension group* of $X^{\cdot}$ and $Y^{\cdot}$.

Next we shall explain the bifunctor $\bullet \otimes^L \bullet$. Let T_0 be a topological space and $\mathcal{R}$ a sheaf of rings on T_0. Denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on T_0. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. Set $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$. Then there exists a bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \mathrm{Sh}(T_0).$$

For two complexes $X^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}_1))$ and $Y^{\cdot} \in \mathrm{Ob}(C(\mathcal{C}_2))$ we define a new complex $(X^{\cdot} \otimes_{\mathcal{R}} Y^{\cdot})^{\cdot} \in \mathrm{Ob}(C(\mathrm{Sh}(T_0)))$ by

$$\begin{cases} (X^{\cdot} \otimes_{\mathcal{R}} Y^{\cdot})^n = \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \\ d^n = d^n_{(X^{\cdot} \otimes_{\mathcal{R}} Y^{\cdot})^{\cdot}} : \quad \displaystyle\prod_{i+j=n} X^i \otimes_{\mathcal{R}} Y^j \longrightarrow \displaystyle\prod_{i+j=n+1} X^i \otimes_{\mathcal{R}} Y^j \\ \qquad\qquad\qquad\qquad \cup\!\!\!\!\!\!\quad \qquad\qquad\qquad\qquad \cup \\ \qquad\qquad \{x_i \otimes y_j\} \longmapsto \left\{ d^i_{X^{\cdot}}(x_i) \times y_j + (-1)^i x_i \otimes d^j_{Y^{\cdot}}(y_j) \right\}. \end{cases}$$

This is the simple complex associated to the double complex $X^{\cdot} \otimes_{\mathcal{R}} Y^{\cdot}$. We thus defined a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^{\cdot} : C(\mathcal{C}_1) \times C(\mathcal{C}_2) \longrightarrow C(\mathrm{Sh}(T_0))$$

which also induces a bifunctor

$$(\bullet \otimes_{\mathcal{R}} \bullet)^{\cdot} : K(\mathcal{C}_1) \times K(\mathcal{C}_2) \longrightarrow K(\mathrm{Sh}(T_0))$$

in homotopy categories. Similarly we also obtain

$$(\bullet \otimes_{\mathcal{R}} \bullet)^{\cdot} : K^-(\mathcal{C}_1) \times K^-(\mathcal{C}_2) \longrightarrow K^-(\mathrm{Sh}(T_0)),$$

taking boundedness into account. Note that in general the abelian categories $\mathcal{C}_1 = \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $\mathcal{C}_2 = \mathrm{Mod}(\mathcal{R})$ do not have enough projectives (unless the topological space T_0 consists of a point). So we cannot use projective objects in these categories to derive the above bifunctor. However, it is well known that for any $Y \in \mathrm{Ob}(\mathcal{C}_2)$ there exist a flat $\mathcal{R}$-module $P \in \mathrm{Ob}(\mathcal{C}_2)$ and an epimorphism $P \to Y$. Therefore, we can use the full subcategory $\mathcal{P}$ of $\mathcal{C}_2$ consisting of flat $\mathcal{R}$-modules.

Lemma B.6.3. *Let $X^{\cdot} \in \mathrm{Ob}(K(\mathcal{C}_1))$ and $P^{\cdot} \in \mathrm{Ob}(K^-(\mathcal{P}))$. Assume that $X^{\cdot}$ or $P^{\cdot}$ is quasi-isomorphic to* 0. *Then the complex $(X^{\cdot} \otimes_{\mathcal{R}} P^{\cdot})^{\cdot} \in \mathrm{Ob}(K(\mathrm{Sh}(T_0)))$ is also quasi-isomorphic to* 0.

By this lemma and previous arguments we obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D(\mathrm{Sh}(T_0))$$

in derived categories. Taking boundedness into account, we also obtain a bifunctor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{C}_1) \times D^-(\mathcal{C}_2) \longrightarrow D^-(\mathrm{Sh}(T_0)).$$

These are bifunctors of triangulated categories. In the classical literature we denote $H^{-n}(X^{\cdot} \otimes^L_{\mathcal{R}} Y^{\cdot})$ by $Tor^{\mathcal{R}}_n(X^{\cdot}, Y^{\cdot})$ and call it the nth *hypertorsion group* of $X^{\cdot}$ and $Y^{\cdot}$.

C

Sheaves and Functors in Derived Categories

In this appendix, assuming only few prerequisites for sheaf theory, we introduce basic operations of sheaves in derived categories and their main properties without proofs. For the details we refer to Hartshorne [Ha1], Iversen [Iv], Kashiwara–Schapira [KS2], [KS4]. We also give a proof of Kashiwara's non-characteristic deformation lemma.

C.1 Sheaves and functors

In this section we quickly recall basic operations in sheaf theory. For a topological space X we denote by $\mathrm{Sh}(X)$ the abelian category of sheaves on X. The abelian group of sections of $F \in \mathrm{Sh}(X)$ on an open subset $U \subset X$ is denoted by $F(U)$ or $\Gamma(U, F)$, and the subgroup of $\Gamma(U, F)$ consisting of sections with compact supports is denoted by $\Gamma_c(U, F)$. We thus obtain left exact functors $\Gamma(U, \bullet), \Gamma_c(U, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$ for each open subset $U \subset X$, where $\mathcal{A}b$ denotes the abelian category of abelian groups. If $\mathcal{R}$ is a sheaf of rings on X, we denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$) the abelian category of sheaves of left (resp. right) $\mathcal{R}$-modules on X. Here $\mathcal{R}^{\mathrm{op}}$ denotes the opposite ring of $\mathcal{R}$. For example, in the case where $\mathcal{R}$ is the constant sheaf $\mathbb{Z}_X$ with germs $\mathbb{Z}$ the category $\mathrm{Mod}(\mathcal{R})$ is $\mathrm{Sh}(X)$. For $F, G \in \mathrm{Sh}(X)$ (resp. $M, N \in \mathrm{Mod}(\mathcal{R})$) we denote by $\mathrm{Hom}(F, G)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, N)$) the abelian group of sheaf homomorphisms (resp. sheaf homomorphisms commuting with the actions of $\mathcal{R}$) on X from F to G (resp. from M to N). We thus obtain left exact bifunctors

$$\begin{cases} \mathrm{Hom}(\bullet, \bullet) : \mathrm{Sh}(X)^{\mathrm{op}} \times \mathrm{Sh}(X) \longrightarrow \mathcal{A}b, \\ \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b. \end{cases}$$

For a subset $Z \subset X$, we denote by $i_Z : Z \to X$ the inclusion map.

Definition C.1.1. Let $f : X \to Y$ be a morphism of topological spaces, $F \in \mathrm{Sh}(X)$ and $G \in \mathrm{Sh}(Y)$.

(i) The *direct image* $f_*F \in \mathrm{Sh}(Y)$ of F by f is defined by $f_*F(V) = F(f^{-1}(V))$ for each open subset $V \subset Y$. This gives a left exact functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If Y is the space pt consisting of one point, the functor f_* is the global section functor $\Gamma(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(ii) The *proper direct image* $f_!F \in \mathrm{Sh}(Y)$ of F by f is defined by $f_!F(V) = \{s \in F(f^{-1}(V)) | f|_{\mathrm{supp}\, s} : \mathrm{supp}\, s \to V \quad \text{is proper}\}$ for each open subset $V \subset Y$. This gives a left exact functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. If Y is pt, the functor $f_!$ is the global section functor with compact supports $\Gamma_c(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The *inverse image* $f^{-1}G \in \mathrm{Sh}(X)$ of G by f is the sheaf associated to the presheaf $(f^{-1}G)'$ defined by $(f^{-1}G)'(U) = \varinjlim G(V)$ for each open subset $U \subset X$, where V ranges through the family of open subsets of Y containing $f(U)$. Since we have an isomorphism $(f^{-1}G)_x \simeq G_{f(x)}$ for any $x \in X$, we obtain an exact functor $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

When we treat proper direct images $f_!$ in this book, all topological spaces are assumed to be locally compact and Hausdorff. For two morphisms $f : X \to Y$, $g : Y \to Z$ of topological spaces, we have obvious relations $g_* \circ f_* = (g \circ f)_*$, $g_! \circ f_! = (g \circ f)_!$ and $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$. For $F \in \mathrm{Sh}(X)$ and a subset $Z \subset X$ the inverse image $i_Z^{-1}F \in \mathrm{Sh}(Z)$ of F by the inclusion map $i_Z : Z \to X$ is sometimes denoted by $F|_Z$. If Z is a locally closed subset of X (i.e., a subset of X which is written as an intersection of an open subset and a closed subset), then it is well known that the functor $(i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ is exact.

Proposition C.1.2. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a **cartesian square** *of topological spaces, i.e., X' is homeomorphic to the fiber product $X \times_Y Y'$. Then there exists an isomorphism of functors $g^{-1} \circ f_! \simeq f'_! \circ g'^{-1}$: $\mathrm{Sh}(X) \to \mathrm{Sh}(Y')$.*

Definition C.1.3. Let X be a topological space, $Z \subset X$ a locally closed subset and $F \in \mathrm{Sh}(X)$.

(i) Set $F_Z = (i_Z)_!(i_Z)^{-1}F \in \mathrm{Sh}(X)$. Since we have $(F_Z)_x \simeq F_x$ (resp. $(F_Z)_x \simeq 0$) for any $x \in Z$ (resp. $x \in X \setminus Z$), we obtain an exact functor $(\bullet)_Z : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$.

(ii) Take an open subset W of X containing Z as a closed subset of W. Since the abelian group $\mathrm{Ker}[F(W) \to F(W \setminus Z)]$ does not depend on the choice of W, we denote it by $\Gamma_Z(X, F)$. This gives a left exact functor $\Gamma_Z(X, \bullet) : \mathrm{Sh}(X) \to \mathcal{A}b$.

(iii) The subsheaf $\Gamma_Z(F)$ of F is defined by $\Gamma_Z(F)(U) = \Gamma_{Z \cap U}(U, F|_U)$ for each open subset $U \subset X$. This gives a left exact functor $\Gamma_Z(\bullet) : \mathrm{Sh}(X) \to \mathrm{Sh}(X)$. By construction we have an isomorphism of functors $\Gamma(X, \bullet) \circ \Gamma_Z(\bullet) = \Gamma_Z(X, \bullet)$.

Note that if U is an open subset of X and $j = i_U : U \to X$, then there exists an isomorphism of functors $j_* \circ j^{-1} \simeq \Gamma_U(\bullet)$.

Lemma C.1.4. *Let X be a topological space, Z a locally closed subset of X and Z' a closed subset of Z. Also let Z_1, Z_2 (resp. U_1, U_2) be closed (resp. open) subsets of X and $F \in \mathrm{Sh}(X)$.*

(i) *There exists a natural exact sequence*

$$0 \longrightarrow F_{Z\setminus Z'} \longrightarrow F_Z \longrightarrow F_{Z'} \longrightarrow 0 \tag{C.1.1}$$

in $\mathrm{Sh}(X)$.

(ii) *There exist natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F), \tag{C.1.2}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F), \tag{C.1.3}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F). \tag{C.1.4}$$

in $\mathrm{Sh}(X)$.

Recall that a sheaf $F \in \mathrm{Sh}(X)$ on X is called *flabby* if the restriction morphism $F(X) \to F(U)$ is surjective for any open subset $U \subset X$.

Lemma C.1.5.

(i) *Let Z be a locally closed subset of X and $F \in \mathrm{Sh}(X)$ a flabby sheaf. Then the sheaf $\Gamma_Z(F)$ is flabby. Moreover, for any morphism $f : X \to Y$ of topological spaces the direct image f_*F is flabby.*

(ii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in* $\mathrm{Sh}(X)$ *and Z a locally closed subset of X. Assume that F' is flabby. Then the sequences* $0 \to \Gamma_Z(X, F') \to \Gamma_Z(X, F) \to \Gamma_Z(X, F'') \to 0$ *and* $0 \to \Gamma_Z(F') \to \Gamma_Z(F) \to \Gamma_Z(F'') \to 0$ *are exact.*

(iii) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in* $\mathrm{Sh}(X)$ *and assume that F' and F are flabby. Then F'' is also flabby.*

(iv) *In the situation of Lemma* C.1.4*, assume, moreover, that F is flabby. Then there are natural exact sequences*

$$0 \longrightarrow \Gamma_{Z'}(F) \longrightarrow \Gamma_Z(F) \longrightarrow \Gamma_{Z\setminus Z'}(F) \longrightarrow 0, \tag{C.1.5}$$

$$0 \longrightarrow \Gamma_{Z_1\cap Z_2}(F) \longrightarrow \Gamma_{Z_1}(F) \oplus \Gamma_{Z_2}(F) \longrightarrow \Gamma_{Z_1\cup Z_2}(F) \longrightarrow 0, \tag{C.1.6}$$

$$0 \longrightarrow \Gamma_{U_1\cup U_2}(F) \longrightarrow \Gamma_{U_1}(F) \oplus \Gamma_{U_2}(F) \longrightarrow \Gamma_{U_1\cap U_2}(F) \longrightarrow 0. \tag{C.1.7}$$

in $\mathrm{Sh}(X)$.

Definition C.1.6. Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X.

(i) For $M, N \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(M, N) \in \mathrm{Sh}(X)$ of $\mathcal{R}$-linear homomorphisms from M to N is defined by $\mathcal{H}om_{\mathcal{R}}(M, N)(U) = \mathrm{Hom}_{\mathcal{R}|_U}(M|_U, N|_U)$ for each open subset $U \subset X$. This gives a left exact bifunctor $\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : \mathrm{Mod}(\mathcal{R})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$. By definition we have $\Gamma(X; \mathcal{H}om_{\mathcal{R}}(M, N)) = \mathrm{Hom}_{\mathcal{R}}(M, N)$.

(ii) For $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$, $N \in \mathrm{Mod}(\mathcal{R})$ the tensor product $M \otimes_{\mathcal{R}} N \in \mathrm{Sh}(X)$ of M and N is the sheaf associated to the presheaf $(M \otimes_{\mathcal{R}} N)'$ defined by $(M \otimes_{\mathcal{R}} N)'(U) = M(U) \otimes_{\mathcal{R}(U)} N(U)$ for each open subset $U \subset X$. Since by definition we have an isomorphism $(M \otimes_{\mathcal{R}} N)_x \simeq M_x \otimes_{\mathcal{R}_x} N_x$ for any $x \in X$, we obtain a right exact bifunctor $\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \to \mathrm{Sh}(X)$.

Note that for any $M \in \mathrm{Mod}(\mathcal{R})$ the sheaf $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M)$ is a left $\mathcal{R}$-module by the right multiplication of $\mathcal{R}$ on $\mathcal{R}$ itself, and there exists an isomorphism $\mathcal{H}om_{\mathcal{R}}(\mathcal{R}, M) \simeq M$ of left $\mathcal{R}$-modules. Now recall that $M \in \mathrm{Mod}(\mathcal{R})$ is an injective (resp. a projective) object of $\mathrm{Mod}(\mathcal{R})$ if the functor $\mathrm{Hom}_{\mathcal{R}}(\bullet, M)$ (resp. $\mathrm{Hom}_{\mathcal{R}}(M, \bullet)$) is exact.

Proposition C.1.7. *Let $\mathcal{R}$ be a sheaf of rings on X. Then the abelian category* $\mathrm{Mod}(\mathcal{R})$ *has* **enough injectives**.

An injective object in $\mathrm{Mod}(\mathcal{R})$ is sometimes called an *injective sheaf* or an injective $\mathcal{R}$-module.

Definition C.1.8. We say that $M \in \mathrm{Mod}(\mathcal{R})$ is *flat* (or a flat $\mathcal{R}$-module) if the functor $\bullet \otimes_{\mathcal{R}} M : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \to \mathrm{Sh}(X)$ is exact.

By the definition of tensor products, $M \in \mathrm{Mod}(\mathcal{R})$ is flat if and only if the stalk M_x is a flat $\mathcal{R}_x$-module for any $x \in X$. Although in general the category $\mathrm{Mod}(\mathcal{R})$ does not have enough projectives (unless X is the space pt consisting of one point), we have the following useful result.

Proposition C.1.9. *Let $\mathcal{R}$ be a sheaf of rings on X. Then for any $M \in \mathrm{Mod}(\mathcal{R})$ there exist a flat $\mathcal{R}$-module P and an epimorphism $P \to M$.*

Lemma C.1.10. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in* $\mathrm{Mod}(\mathcal{R})$.

(i) *Assume that M' and M are injective. Then M'' is also injective.*
(ii) *Assume that M and M'' are flat. Then M' is also flat.*

Proposition C.1.11. *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on X.*

(i) *Let $M_1 \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ and $M_2 \in \mathrm{Mod}(\mathcal{R})$. Then there exists an isomorphism*

$$f^{-1}M_1 \otimes_{f^{-1}\mathcal{R}} f^{-1}M_2 \simeq f^{-1}(M_1 \otimes_{\mathcal{R}} M_2) \tag{C.1.8}$$

in $\mathrm{Sh}(Y)$.

(ii) *Let $M \in \mathrm{Mod}(\mathcal{R})$ and $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$\mathcal{H}om_{\mathcal{R}}(M, f_*N) \simeq f_*\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M, N) \tag{C.1.9}$$

in $\mathrm{Sh}(X)$*, where f_*N is a left $\mathcal{R}$-module by the natural ring homomorphism $\mathcal{R} \to f_*f^{-1}\mathcal{R}$. In particular we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{R}}(M, f_*N) \simeq \mathrm{Hom}_{f^{-1}\mathcal{R}}(f^{-1}M, N). \tag{C.1.10}$$

Namely, the functor f_* is a right adjoint of f^{-1}.

(iii) *Let* $M \in \mathrm{Mod}(\mathcal{R}^{\mathrm{op}})$ *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$*. Then there exists a natural morphism*

$$M \otimes_{\mathcal{R}} f_! N \longrightarrow f_!(f^{-1}M \otimes_{f^{-1}\mathcal{R}} N) \tag{C.1.11}$$

in $\mathrm{Sh}(X)$*. Moreover, this morphism is an isomorphism if* M *is a flat* $\mathcal{R}^{\mathrm{op}}$*-module.*

Corollary C.1.12. *Let* $f : Y \to X$ *be a morphism of topological spaces,* $\mathcal{R}$ *a sheaf of rings on* X *and* $N \in \mathrm{Mod}(f^{-1}\mathcal{R})$ *an injective* $f^{-1}\mathcal{R}$*-module. Then the direct image* f_*N *is an injective* $\mathcal{R}$*-module.*

Lemma C.1.13. *Let* Z *be a locally closed subset of* X*,* $\mathcal{R}$ *a sheaf of rings on* X *and* $M, N \in \mathrm{Mod}(\mathcal{R})$*. Then we have natural isomorphisms*

$$\Gamma_Z \mathcal{H}om_{\mathcal{R}}(M, N) \simeq \mathcal{H}om_{\mathcal{R}}(M, \Gamma_Z N) \simeq \mathcal{H}om_{\mathcal{R}}(M_Z, N). \tag{C.1.12}$$

Corollary C.1.14. *Let* $\mathcal{R}$ *be a sheaf of rings on* X*,* Z *a locally closed subset of* X *and* $M, N \in \mathrm{Mod}(\mathcal{R})$*. Assume that* N *is an injective* $\mathcal{R}$*-module. Then the sheaf* $\mathcal{H}om_{\mathcal{R}}(M, N)$ (*resp.* $\Gamma_Z N$) *is flabby* (*resp. an injective* $\mathcal{R}$*-module*)*. In particular, any injective* $\mathcal{R}$*-module is flabby.*

C.2 Functors in derived categories of sheaves

Applying the results in Appendix B to functors of operations of sheaves, we can introduce various functors in derived categories of sheaves as follows.

Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X. Since the category $\mathrm{Mod}(\mathcal{R})$ is abelian, we obtain a derived category $D(\mathrm{Mod}(\mathcal{R}))$ of complexes in $\mathrm{Mod}(\mathcal{R})$ and its full subcategories $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ ($\# = +, -, b$). In this book for $\# = \emptyset, +, -, b$ we sometimes denote $D^{\#}(\mathrm{Mod}(\mathcal{R}))$ by $D^{\#}(\mathcal{R})$ for the sake of simplicity. For example, we set $D^+(\mathbb{Z}_X) = D^+(\mathrm{Sh}(X))$. Now let Z be a locally closed subset of X, and $f : Y \to X$ a morphism of topological spaces. Consider the following left exact functors:

$$\begin{cases} \Gamma(X, \bullet), \Gamma_c(X, \bullet), \Gamma_Z(X, \bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathcal{A}b, \\ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ f_*, f_! : \mathrm{Mod}(f^{-1}\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}). \end{cases} \tag{C.2.1}$$

Since the categories $\mathrm{Mod}(\mathcal{R})$ and $\mathrm{Mod}(f^{-1}\mathcal{R})$ have enough injectives we obtain their derived functors

$$\begin{cases} \mathrm{R}\Gamma(X, \bullet), \mathrm{R}\Gamma_c(X, \bullet), \mathrm{R}\Gamma_Z(X, \bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b), \\ \mathrm{R}\Gamma_Z(\bullet) : D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{R}), \\ Rf_*, Rf_! : D^+(f^{-1}\mathcal{R}) \longrightarrow D^+(\mathcal{R}). \end{cases} \tag{C.2.2}$$

For example, for $M^{\cdot} \in D^+(\mathcal{R})$ the object $\mathrm{R}\Gamma(X, F^{\cdot}) \in D^+(\mathcal{A}b)$ is calculated as follows. First take a quasi-isomorphism $M^{\cdot} \xrightarrow{\sim} I^{\cdot}$ in $C^+(\mathcal{R})$ such that I^k is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Then we have $\mathrm{R}\Gamma(X, F^{\cdot}) \simeq \Gamma(X, I^{\cdot})$.

Since by Lemma C.1.5 the full subcategory $\mathcal{J}$ of $\mathrm{Mod}(\mathcal{R})$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective in the sense of Definition B.5.4, we can also take a quasi-isomorphism $M^{\cdot} \xrightarrow{\sim} J^{\cdot}$ in $C^{+}(\mathcal{R})$ such that $J^{k} \in \mathcal{J}$ for any $k \in \mathbb{Z}$ and show that $\mathrm{R}\Gamma(X, F^{\cdot}) \simeq \Gamma(X, J^{\cdot})$. Let us apply Proposition B.5.7 to the identity $\Gamma_Z(X, \bullet) = \Gamma(X, \bullet) \circ \Gamma_Z(\bullet) : \mathrm{Mod}(\mathcal{R}) \to \mathcal{A}b$. Then by the fact that the functor $\Gamma_Z(\bullet) \simeq \mathcal{H}om_{\mathcal{R}}(\mathcal{R}_Z, \bullet)$ sends injective sheaves to injective sheaves (Corollary C.1.14), we obtain an isomorphism

$$\mathrm{R}\Gamma(X, \mathrm{R}\Gamma_Z(M^{\cdot})) \simeq \mathrm{R}\Gamma_Z(X, M^{\cdot}) \tag{C.2.3}$$

in $D^{+}(\mathcal{A}b)$ for any $M^{\cdot} \in D^{+}(\mathcal{R})$. Similarly by Corollary C.1.12 we obtain an isomorphism

$$\mathrm{R}\Gamma(X, Rf_*(N^{\cdot})) \simeq \mathrm{R}\Gamma(Y, N^{\cdot}) \tag{C.2.4}$$

in $D^{+}(\mathcal{A}b)$ for any $N^{\cdot} \in D^{+}(f^{-1}\mathcal{R})$ (also the similar formula $\mathrm{R}\Gamma_c(X, Rf_!(N^{\cdot})) \simeq \mathrm{R}\Gamma_c(Y, N^{\cdot})$ can be proved). For $M^{\cdot} \in D^{+}(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{R}\Gamma(X, M^{\cdot})$, $H^i\mathrm{R}\Gamma_Z(X, M^{\cdot})$, $H^i\mathrm{R}\Gamma_Z(M^{\cdot})$ simply by $H^i(X, M^{\cdot})$, $H^i_Z(X, M^{\cdot})$, $H^i_Z(M^{\cdot})$, respectively. Now let us consider the functors

$$\begin{cases} f^{-1} : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(f^{-1}\mathcal{R}), \\ (\bullet)_Z : \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Mod}(\mathcal{R}), \\ (i_Z)_! : \mathrm{Sh}(Z) \to \mathrm{Sh}(X). \end{cases} \tag{C.2.5}$$

Since these functors are exact, they extend naturally to the following functors in derived categories:

$$\begin{cases} f^{-1} : D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(f^{-1}\mathcal{R}), \\ (\bullet)_Z : D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(\mathcal{R}), \\ (i_Z)_! : D^{\#}(\mathrm{Sh}(Z)) \to D^{\#}(\mathrm{Sh}(X)) \end{cases} \tag{C.2.6}$$

for $\# = \emptyset, +, -, b$. Let Z' be a closed subset of Z and Z_1, Z_2 (resp. U_1, U_2) closed (resp. open) subsets of X. Then by Lemma C.1.5 and Lemma B.4.5, for $M^{\cdot} \in D^{+}(\mathcal{R})$ we obtain the following distinguished triangles in $D^{+}(\mathcal{R})$:

$$M_{Z\setminus Z'}{}^{\cdot} \longrightarrow M_Z{}^{\cdot} \longrightarrow M_{Z'}{}^{\cdot} \xrightarrow{+1}, \tag{C.2.7}$$

$$\mathrm{R}\Gamma_{Z'}(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_Z(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{Z\setminus Z'}(M^{\cdot}) \xrightarrow{+1}, \tag{C.2.8}$$

$$\mathrm{R}\Gamma_{Z_1\cap Z_2}(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{Z_1}(M^{\cdot}) \oplus \mathrm{R}\Gamma_{Z_2}(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{Z_1\cup Z_2}(M^{\cdot}) \xrightarrow{+1}, \tag{C.2.9}$$

$$\mathrm{R}\Gamma_{U_1\cup U_2}(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{U_1}(M^{\cdot}) \oplus \mathrm{R}\Gamma_{U_2}(M^{\cdot}) \longrightarrow \mathrm{R}\Gamma_{U_1\cap U_2}(M^{\cdot}) \xrightarrow{+1}. \tag{C.2.10}$$

The following result is also well known.

Proposition C.2.1. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian square of topological spaces. Then there exists an isomorphism of functors $g^{-1} \circ Rf_! \simeq Rf'_! \circ g'^{-1} : D^+(\mathrm{Sh}(X)) \to D^+(\mathrm{Sh}(Y'))$.

For the proof see [KS2, Proposition 2.6.7]. From now on, let us introduce bifunctors in derived categories of sheaves. Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Then by applying the construction in Section B.6 to $\mathcal{C} = \mathrm{Mod}(\mathcal{R})$ we obtain a bifunctor

$$\mathrm{RHom}_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathcal{A}b). \tag{C.2.11}$$

Similarly we obtain a bifunctor

$$R\mathcal{H}om_{\mathcal{R}}(\bullet, \bullet) : D^-(\mathcal{R})^{\mathrm{op}} \times D^+(\mathcal{R}) \longrightarrow D^+(\mathrm{Sh}(X)). \tag{C.2.12}$$

For $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$ the objects $\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \in D^+(\mathcal{A}b)$ and $R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \in D^+(\mathrm{Sh}(X))$ are more explicitly calculated as follows. Take a quasi-isomorphism $N^\cdot \xrightarrow{\sim} I^\cdot$ such that I^k is an injective $\mathcal{R}$-module for any $k \in \mathbb{Z}$ and consider the simple complex $\mathrm{Hom}^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot) \in C^+(\mathcal{A}b)$ (resp. $\mathcal{H}om^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot) \in C^+(\mathrm{Sh}(X))$) associated to the double complex $\mathrm{Hom}_{\mathcal{R}}(M^\cdot, I^\cdot)$ (resp. $\mathcal{H}om_{\mathcal{R}}(M^\cdot, I^\cdot)$) as in Section B.6. Then we have isomorphisms $\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq \mathrm{Hom}^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot)$ and $R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq \mathcal{H}om^\cdot_{\mathcal{R}}(M^\cdot, I^\cdot)$. For $M^\cdot \in D^-(\mathcal{R})$, $N^\cdot \in D^+(\mathcal{R})$ and $i \in \mathbb{Z}$ we sometimes denote $H^i\mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot)$, $H^i R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot)$ simply by $Ext_{\mathcal{R}}(M^\cdot, N^\cdot)$, $\mathcal{E}xt_{\mathcal{R}}(M^\cdot, N^\cdot)$, respectively. Since the full subcategory of $\mathrm{Sh}(X)$ consisting of flabby sheaves is $\Gamma(X, \bullet)$-injective, from Corollary C.1.14 and the obvious identity $\Gamma(X, \mathcal{H}om_{\mathcal{R}}(\bullet, \bullet)) = \mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet)$, we obtain an isomorphism

$$\mathrm{R}\Gamma(X, R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot)) \simeq \mathrm{RHom}_{\mathcal{R}}(M^\cdot, N^\cdot) \tag{C.2.13}$$

in $D^+(\mathcal{A}b)$ for any $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$. Let us apply the same argument to the identities in Lemma C.1.13. Then by Lemma C.1.5 and Corollary C.1.14 we obtain the following.

Proposition C.2.2. *Let Z be a locally closed subset of X, $\mathcal{R}$ a sheaf of rings on X, $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(\mathcal{R})$. Then we have isomorphisms*

$$\mathrm{R}\Gamma_Z R\mathcal{H}om_{\mathcal{R}}(M^\cdot, N^\cdot) \simeq R\mathcal{H}om_{\mathcal{R}}(M^\cdot, \mathrm{R}\Gamma_Z N^\cdot) \simeq R\mathcal{H}om_{\mathcal{R}}(M_Z{}^\cdot, N^\cdot). \tag{C.2.14}$$

Similarly, from Proposition C.1.11 (ii) we obtain the following.

Proposition C.2.3 (Adjunction formula). *Let $f : Y \to X$ be a morphism of topological spaces, $\mathcal{R}$ a sheaf of rings on X. Let $M^\cdot \in D^-(\mathcal{R})$ and $N^\cdot \in D^+(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$R\mathcal{H}om_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq Rf_*R\mathcal{H}om_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.15}$$

in $D^+(\mathrm{Sh}(X))$. *Moreover, we have an isomorphism*

$$\mathrm{RHom}_{\mathcal{R}}(M^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{RHom}_{f^{-1}\mathcal{R}}(f^{-1}M^{\cdot}, N^{\cdot}) \tag{C.2.16}$$

in $D^+(\mathcal{A}b)$.

By Proposition B.6.2 and the same argument as above, we also obtain the following.

Proposition C.2.4. *In the situation of Proposition* C.2.3, *for any* $L^{\cdot} \in D^+(\mathcal{R})$ *and* $N^{\cdot} \in D^+(f^{-1}\mathcal{R})$ *there exists an isomorphism*

$$\mathrm{Hom}_{D^+(\mathcal{R})}(L^{\cdot}, Rf_*N^{\cdot}) \simeq \mathrm{Hom}_{D^+(f^{-1}\mathcal{R})}(f^{-1}L^{\cdot}, N^{\cdot}). \tag{C.2.17}$$

Namely, the functor $f^{-1} : D^+(\mathcal{R}) \to D^+(f^{-1}\mathcal{R})$ *is a left adjoint of* $Rf_* : D^+(f^{-1}\mathcal{R}) \to D^+(\mathcal{R})$.

Next we shall introduce the derived functor of the bifunctor of tensor products. Let X be a topological space and $\mathcal{R}$ a sheaf of rings on X. Then by the results in Section B.6, there exists a right exact bifunctor of tensor products

$$\bullet \otimes_{\mathcal{R}} \bullet : \mathrm{Mod}(\mathcal{R}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{R}) \longrightarrow \mathrm{Sh}(X), \tag{C.2.18}$$

and its derived functor

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^-(\mathcal{R}^{\mathrm{op}}) \times D^-(\mathcal{R}) \longrightarrow D^-(\mathrm{Sh}(X)). \tag{C.2.19}$$

From now on, let us assume, moreover, that $\mathcal{R}$ has *finite weak global dimension*, i.e., there exists an integer $d > 0$ such that the weak global dimension of the ring $\mathcal{R}_x$ is less than or equal to d for any $x \in X$. Then for any $M^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) we can construct a quasi-isomorphism $P^{\cdot} \to M^{\cdot}$ for $P^{\cdot} \in C^+(\mathrm{Mod}(\mathcal{R}))$ (resp. $C^b(\mathrm{Mod}(\mathcal{R}))$) such that P^k is a flat $\mathcal{R}$-module for any $k \in \mathbb{Z}$. Hence we obtain also bifunctors

$$\bullet \otimes^L_{\mathcal{R}} \bullet : D^{\#}(\mathcal{R}^{\mathrm{op}}) \times D^{\#}(\mathcal{R}) \longrightarrow D^{\#}(\mathrm{Sh}(X)) \tag{C.2.20}$$

for $\# = +, b$. By definition, we immediately obtain the following.

Proposition C.2.5. *Let* $f : Y \to X$ *be a morphism of topological spaces and* $\mathcal{R}$ *a sheaf of rings on* X. *Let* $M^{\cdot} \in D^-(\mathcal{R}^{\mathrm{op}})$ *and* $N^{\cdot} \in D^-(\mathcal{R})$. *Then there exists an isomorphism*

$$f^{-1}M^{\cdot} \otimes^L_{f^{-1}\mathcal{R}} f^{-1}N^{\cdot} \simeq f^{-1}(M^{\cdot} \otimes^L_{\mathcal{R}} N^{\cdot}) \tag{C.2.21}$$

in $D^-(\mathrm{Sh}(Y))$.

The following result is also well known.

Proposition C.2.6 (Projection formula). *Let $f : Y \to X$ be a morphism of topological spaces and $\mathcal{R}$ a sheaf of rings on X. Assume that $\mathcal{R}$ has finite weak global dimension. Let $M^\bullet \in D^+(\mathcal{R}^{\mathrm{op}})$ and $N^\bullet \in D^+(f^{-1}\mathcal{R})$. Then there exists an isomorphism*

$$M^\bullet \otimes^L_{\mathcal{R}} Rf_! N^\bullet \xrightarrow{\sim} Rf_!(f^{-1}M^\bullet \otimes^L_{f^{-1}\mathcal{R}} N^\bullet) \tag{C.2.22}$$

in $D^+(\mathrm{Sh}(X))$.

For the proof see [KS2, Proposition 2.6.6].

Finally, let us explain the *Poincaré–Verdier duality*. Now let $f : X \to Y$ be a continuous map of locally compact and Hausdorff topological spaces. Let A be a commutative ring with finite global dimension, e.g., a field k. In what follows, we always assume the following condition for f.

Definition C.2.7. We say that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ has *finite cohomological dimension* if there exists an integer $d > 0$ such that for any sheaf F on X we have $H^k Rf_!(F) = 0$ for any $k > d$.

Theorem C.2.8 (Poincaré–Verdier duality theorem). *In the situation as above, there exists a functor of triangulated categories $f^! : D^+(A_Y) \to D^+(A_X)$ such that for any $M^\bullet \in D^b(A_X)$ and $N^\bullet \in D^+(A_Y)$ we have isomorphisms*

$$Rf_* R\mathcal{H}om_{A_X}(M^\bullet, f^! N^\bullet) \simeq R\mathcal{H}om_{A_Y}(Rf_! M^\bullet, N^\bullet), \tag{C.2.23}$$

$$\mathrm{RHom}_{A_X}(M^\bullet, f^! N^\bullet) \simeq \mathrm{RHom}_{A_Y}(Rf_! M^\bullet, N^\bullet) \tag{C.2.24}$$

in $D^+(A_Y)$ and $D^+(\mathrm{Mod}(A))$, respectively.

We call the functor $f^! : D^+(A_Y) \to D^+(A_X)$ the *twisted inverse image* functor by f. Since the construction of this functor $f^!(\bullet)$ is a little bit complicated, we do not explain it here. For the details see Kashiwara–Schapira [KS2, Chapter III]. Let us give basic properties of twisted inverse images. First, for a morphism $g : Y \to Z$ of topological spaces satisfying the same assumption as f, we have an isomorphism $(g \circ f)^! \simeq f^! \circ g^!$ of functors.

Theorem C.2.9. *Let $f : X \to Y$ be as above. Then for any $M^\bullet \in D^+(A_X)$ and $N^\bullet \in D^+(A_Y)$ we have an isomorphism*

$$\mathrm{Hom}_{D^+(A_X)}(M^\bullet, f^! N^\bullet) \simeq \mathrm{Hom}_{D^+(A_Y)}(Rf_! M^\bullet, N^\bullet).$$

Namely, the functor $f^! : D^+(A_Y) \to D^+(A_X)$ is a right adjoint of $Rf_! : D^+(A_X) \to D^+(A_Y)$.

Proposition C.2.10. *Let $f : X \to Y$ be as above. Then for any $N_1{}^\bullet \in D^b(A_Y)$ and $N_2{}^\bullet \in D^+(A_Y)$ we have an isomorphism*

$$f^! R\mathcal{H}om_{A_Y}(N_1{}^\bullet, N_2{}^\bullet) \simeq R\mathcal{H}om_{A_X}(f^{-1}N_1{}^\bullet, f^! N_2{}^\bullet).$$

Proposition C.2.11. *Assume that X is a locally closed subset of Y and let $f = i_X : X \hookrightarrow Y$ be the embedding. Then we have an isomorphism*

$$f^!(N^\cdot) \simeq f^{-1}(\mathrm{R}\Gamma_{f(X)}(N^\cdot)) = (\mathrm{R}\Gamma_{f(X)}(N^\cdot))|_X \tag{C.2.25}$$

in $D^+(A_X)$ for any $N^\cdot \in D^+(A_Y)$.

Proposition C.2.12. *Assume that X and Y are real C^1-manifolds and $f : X \to Y$ is a C^1-submersion. Set $d = \dim X - \dim Y$. Then*

(i) *$H^j(f^!(A_Y)) = 0$ for any $j \neq -d$ and $H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ is a locally constant sheaf of rank one over A_X.*
(ii) *For any $N^\cdot \in D^+(A_Y)$ there exists an isomorphism*

$$f^!(A_Y) \otimes^L_{A_X} f^{-1}(N^\cdot) \xrightarrow{\sim} f^!(N^\cdot). \tag{C.2.26}$$

Definition C.2.13. In the situation of Proposition C.2.12 we set $\mathrm{or}_{X/Y} = H^{-d}(f^!(A_Y)) \in \mathrm{Mod}(A_X)$ and call it the *relative orientation sheaf* of $f : X \to Y$. If, moreover, Y is the space $\{\mathrm{pt}\}$ consisting of one point, we set $\mathrm{or}_X = \mathrm{or}_{X/Y} \in \mathrm{Mod}(A_X)$ and call it the *orientation sheaf* of X.

In the situation of Proposition C.2.12 above we thus have an isomorphism $f^!(A_Y) \simeq \mathrm{or}_{X/Y}[\dim X - \dim Y]$ and for any $N^\cdot \in D^+(A_Y)$ there exists an isomorphism

$$f^!(N^\cdot) \simeq \mathrm{or}_{X/Y} \otimes_{A_X} f^{-1}(N^\cdot)[\dim X - \dim Y]. \tag{C.2.27}$$

Note that in the above isomorphism we wrote $\otimes_{A_X}$ instead of $\otimes^L_{A_X}$ because $\mathrm{or}_{X/Y}$ is flat over A_X.

Definition C.2.14. Let $f : X \to Y$ be as above. Assume, moreover, that Y is the space $\{\mathrm{pt}\}$ consisting of one point and the morphism f is $X \to \{\mathrm{pt}\}$. Then we set $\omega_X{}^\cdot = f^!(A_{\{\mathrm{pt}\}}) \in D^+(A_X)$ and call it the *dualizing complex* of X. We sometimes denote $\omega_X{}^\cdot$ simply by ω_X.

To define the dualizing complex $\omega_X{}^\cdot \in D^+(A_X)$ of X, we assumed that the functor $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(\{\mathrm{pt}\}) = \mathcal{A}b$ for $f : X \to \{\mathrm{pt}\}$ has finite cohomological dimension. This assumption is satisfied if X is a topological manifold or a real analytic space. In what follows we assume that all topological spaces that we treat satisfy this assumption.

Definition C.2.15. For $M^\cdot \in D^b(A_X)$ we set

$$\mathbf{D}_X(M^\cdot) = R\mathcal{H}om_{A_X}(M^\cdot, \omega_X{}^\cdot) \in D^+(A_X)$$

and call it the *Verdier dual* of $M^\cdot$.

Since for the morphism $f : X \to Y$ of topological spaces we have $f^!\omega_Y{}^\cdot \simeq \omega_X{}^\cdot$, from Proposition C.2.10 we obtain an isomorphism

$$(f^! \circ \mathbf{D}_Y)(N^\cdot) \simeq (\mathbf{D}_X \circ f^{-1})(N^\cdot) \tag{C.2.28}$$

for any $N^\cdot \in D^b(A_Y)$. Similarly, from Theorem C.2.8 we obtain an isomorphism

$$(Rf_* \circ \mathbf{D}_X)(M^\cdot) \simeq (\mathbf{D}_Y \circ Rf_!)(M^\cdot) \tag{C.2.29}$$

for any $M^\cdot \in D^b(A_X)$.

Example C.2.16. In the situation as above, assume, moreover, that A is a field k, X is an orientable C^1-manifold of dimension n, and $Y = \{\mathrm{pt}\}$. In this case there exist isomorphisms $\omega_X^\cdot \simeq \mathrm{or}_X[n] \simeq k_X[n]$. Let $M^\cdot \in D^b(k_X)$ and set $\mathbf{D}'_X(M^\cdot) = R\mathcal{H}om_{k_X}(M^\cdot, k_X)$. Then by the isomorphism (C.2.29) we obtain an isomorphism $H^{n-i}(X, \mathbf{D}'_X(M^\cdot)) \simeq [H^i_c(X, M^\cdot)]^*$ for any $i \in \mathbb{Z}$, where we set $H^i_c(X, \bullet) = H^i\mathrm{R}\Gamma_c(X, \bullet)$. In the very special case where $M^\cdot = k_X$ we thus obtain the famous Poincaré duality theorem: $H^{n-i}(X, k_X) \simeq [H^i_c(X, k_X)]^*$.

C.3 Non-characteristic deformation lemma

In this section, we prove the *non-characteristic deformation lemma* (due to Kashiwara), which plays a powerful role in deriving results on global cohomology groups of complexes of sheaves from their local properties. First, we introduce some basic results on projective systems of abelian groups. Recall that a pair $M = (M_n, \rho_{n,m})$ of a family of abelian groups M_n ($n \in \mathbb{N}$) and that of group homomorphisms $\rho_{n,m} : M_m \to M_n$ ($m \geq n$) is called a *projective system* of abelian groups (indexed by $\mathbb{N}$) if it satisfies the conditions: $\rho_{n,n} = \mathrm{id}_{M_n}$ for any $n \in \mathbb{Z}$ and $\rho_{n,m} \circ \rho_{m,l} = \rho_{n,l}$ for any $n \leq m \leq l$. If $M = (M_n, \rho_{n,m})$ is a projective system of abelian groups, we denote its projective limit by $\varprojlim M$ for short. We define morphisms of projective systems of abelian groups in an obvious way. Then the category of projective systems of abelian groups is abelian. However, the functor $\varprojlim(*)$ from this category to that of abelian groups is not exact. It is only left exact. As a remedy for this problem we introduce the following notion.

Definition C.3.1. Let $M = (M_n, \rho_{n,m})$ be a projective system of abelian groups. We say that M satisfies the *Mittag-Leffler condition* (or M-L condition) if for any $n \in \mathbb{N}$ decreasing subgroups $\rho_{n,m}(M_m)$ ($m \geq n$) of M_n is stationary.

Let us state basic results on projective systems satisfying the M-L condition. Since the proofs of the following lemmas are straightforward, we leave them to the reader.

Lemma C.3.2. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of projective systems of abelian groups.

(i) *Assume L and N satisfy the M-L condition. Then M satisfies the M-L condition.*
(ii) *Assume M satisfies the M-L condition. Then N satisfies the M-L condition.*

Lemma C.3.3. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of projective systems of abelian groups. Assume that L satisfies the M-L condition. Then the sequence

$$0 \longrightarrow \varprojlim L \longrightarrow \varprojlim M \longrightarrow \varprojlim N \longrightarrow 0$$

is exact.

Now let X be a topological space and $F^{\cdot} \in D^b(\mathbb{Z}_X)$. Namely, $F^{\cdot}$ is a bounded complex of sheaves of abelian groups on X.

Proposition C.3.4. *Let $\{U_n\}_{n\in\mathbb{N}}$ be an increasing sequence of open subsets of X and set $U = \cup_{n\in\mathbb{N}} U_n$. Then*

(i) *The natural morphism $\phi_i : H^i(U, F^{\cdot}) \to \varprojlim_n H^i(U_n, F^{\cdot})$ is surjective for any $i \in \mathbb{Z}$.*
(ii) *Assume that for an integer $i \in \mathbb{Z}$ the projective system $\{H^{i-1}(U_n, F^{\cdot})\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then $\phi_i : H^i(U, F^{\cdot}) \to \varprojlim_n H^i(U_n, F^{\cdot})$ is bijective.*

Proof. We may assume that each term F^i of $F^{\cdot}$ is a flabby sheaf. Then we have $H^i(U, F^{\cdot}) = H^i(F^{\cdot}(U)) = H^i(\varprojlim_n F^{\cdot}(U_n))$. Hence the morphism ϕ_i is

$$H^i(\varprojlim_n F^{\cdot}(U_n)) \to \varprojlim_n H^i(F^{\cdot}(U_n)).$$

Note that for any $i \in \mathbb{Z}$ the projective system $\{F^i(U_n)\}_{n\in\mathbb{N}}$ satisfies the M-L condition by the flabbiness of F^i. Set $Z_n^i = \mathrm{Ker}[F^i(U_n) \to F^{i+1}(U_n)]$ and $B_n^i = \mathrm{Im}[F^{i-1}(U_n) \to F^i(U_n)]$. Then we have exact sequences

$$0 \longrightarrow Z_n^i \longrightarrow F^i(U_n) \longrightarrow B_n^{i+1} \longrightarrow 0 \tag{C.3.1}$$

and by Lemma C.3.2 (ii) the projective systems $\{B_n^i\}_{n\in\mathbb{N}}$ satisfy the M-L condition. Therefore, applying Lemma C.3.3 to the exact sequences

$$0 \longrightarrow B_n^i \longrightarrow Z_n^i \longrightarrow H^i(U_n, F^{\cdot}) \longrightarrow 0 \tag{C.3.2}$$

we get an exact sequence

$$0 \longrightarrow \varprojlim_n B_n^i \longrightarrow \varprojlim_n Z_n^i \longrightarrow \varprojlim_n H^i(U_n, F^{\cdot}) \longrightarrow 0. \tag{C.3.3}$$

Since the functor $\varprojlim(*)$ is left exact, we also have isomorphisms

$$\varprojlim_n Z_n^i \simeq \mathrm{Ker}[\varprojlim_n F^i(U_n) \to \varprojlim_n F^{i+1}(U_n)] = \mathrm{Ker}[F^i(U) \to F^{i+1}(U)].$$

Now let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} & F^{i-1}(U) & \longrightarrow & \mathrm{Ker}[F^i(U) \to F^{i+1}(U)] & \longrightarrow & H^i(U; F^\cdot) & \longrightarrow 0 \\ & \downarrow & & \wr\downarrow & & \downarrow \phi_i & \\ 0 \longrightarrow & \varprojlim_n B_n^i & \longrightarrow & \varprojlim_n Z_n^i & \longrightarrow & \varprojlim_n H^i(U_n, F^\cdot) & \longrightarrow 0 \end{array}$$

Then we see that ϕ_i is surjective. The assertion (i) was proved. Let us prove (ii). Assume that the projective system $\{H^{i-1}(U_n, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Then applying Lemma C.3.2 (i) to the exact sequence (C.3.2) we see that the projective system $\{Z_n^{i-1}\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Lemma C.3.3 and (C.3.1) we get an exact sequence

$$0 \longrightarrow \varprojlim_n Z_n^{i-1} \longrightarrow F^{i-1}(U) \longrightarrow \varprojlim_n B_n^i \longrightarrow 0,$$

which shows that the left vertical arrow in the above diagram is surjective. Hence ϕ_i is bijective. This completes the proof. □

We also require the following.

Lemma C.3.5. *Let $\{M_t, \rho_{t,s}\}$ be a projective system of abelian groups indexed by $\mathbb{R}$. Assume that for any $t \in \mathbb{R}$ the natural morphisms*

$$\begin{cases} \alpha_t : M_t \longrightarrow \varprojlim_{s<t} M_s, \\ \beta_t : \varinjlim_{s>t} M_s \longrightarrow M_t \end{cases}$$

are injective (resp. surjective). Then for any pair $t_1 \leq t_2$ the morphism $\rho_{t_1,t_2} : M_{t_2} \to M_{t_1}$ is injective (resp. surjective).

Proof. Since the proof of injectivity is easy, we only prove surjectivity. Let $t_1 \leq t_2$ and $m_1 \in M_{t_1}$. Denote by S the set of all pairs (t, m) of $t_1 \leq t \leq t_2$ and $m \in M_t$ satisfying $\rho_{t_1,t}(m) = m_1$. Let us order this set S in the following way: $(t, m) \leq (t', m') \iff t \leq t'$ and $\rho_{t,t'}(m') = m$. Then by the surjectivity of α_s for any s we can easily prove that S is an inductively ordered set. Therefore, by *Zorn's lemma* there exists a maximal element (t, m) of S. If $t = t_2$, then $\rho_{t_1,t_2}(m) = m_1$. If $t < t_2$ then by the surjectivity of β_s's for any s, there exist t_3 with $t < t_3 \leq t_2$ and $m_3 \in M_{t_3}$ such that $\rho_{t,t_3}(m_3) = m$. This contradicts the maximality of the element (t, m). □

Now let us introduce the non-characteristic deformation lemma (due to Kashiwara). This result is very useful to derive global results from the local properties of $F^\cdot \in D^b(\mathbb{C}_X)$. Here we introduce only its weak form, which is enough for the applications in this book (see also [KS2, Proposition 2.7.2]).

Theorem C.3.6 (Non-characteristic deformation lemma). *Let X be a C^∞-manifold, $\{\Omega_t\}_{t\in\mathbb{R}}$ a family of relatively compact open subsets of X, and $F^\cdot \in D^b(\mathbb{C}_X)$. Assume the following conditions:*

(i) *For any pair* $s < t$ *of real numbers,* $\Omega_s \subset \Omega_t$.
(ii) *For any* $t \in \mathbb{R}$, $\Omega_t = \bigcup_{s<t} \Omega_s$.
(iii) *For* $^\forall t \in \mathbb{R}$, $\bigcap_{s>t}(\Omega_s \setminus \Omega_t) = \partial\Omega_t$ *and for* $^\forall x \in \partial\Omega_t$*, we have*

$$\left[\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)\right]_x \simeq 0.$$

Then we have an isomorphism

$$\mathrm{R}\Gamma\left(\bigcup_{s\in\mathbb{R}} \Omega_s, F^\cdot\right) \xrightarrow{\sim} \mathrm{R}\Gamma(\Omega_t, F^\cdot)$$

for any $t \in \mathbb{R}$.

Proof. We prove the theorem by using Lemma C.3.5. First let us prove that for any $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ the canonical morphism

$$\varinjlim_{s>t} H^i(\Omega_s, F^\cdot) \longrightarrow H^i(\Omega_t, F^\cdot)$$

is an isomorphism. Since we have $\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)|_{\partial\Omega_t} \simeq 0$ by the assumption (iii), we obtain

$$\mathrm{R}\Gamma(\overline{\Omega_t}, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)) \simeq \mathrm{R}\Gamma(\partial\Omega_t, \mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot)) \simeq 0.$$

Then by the distinguished triangle

$$\mathrm{R}\Gamma_{X\setminus\Omega_t}(F^\cdot) \longrightarrow F^\cdot \longrightarrow \mathrm{R}\Gamma_{\Omega_t}(F^\cdot) \xrightarrow{+1}$$

we get an isomorphism

$$\mathrm{R}\Gamma(\overline{\Omega_t}, F^\cdot) \simeq \mathrm{R}\Gamma(\Omega_t, F^\cdot).$$

Taking the cohomology groups of both sides we finally obtain the desired isomorphisms

$$\varinjlim_{s>t} H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot). \tag{C.3.4}$$

Now consider the following assertions:

$$(A)_i^t : \varprojlim_{s<t} H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot)$$

for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. Assume that for an integer j the assertion $(A)_i^t$ is proved for any $i < j$ and $t \in \mathbb{R}$. Then by Lemma C.3.5 we get an isomorphism $H^i(\Omega_s, F^\cdot) \simeq H^i(\Omega_t, F^\cdot)$ for any $i < j$ and any pair $s > t$. This implies that for each $t \in \mathbb{R}$ the projective system $\{H^{j-1}(\Omega_{t-\frac{1}{n}}, F^\cdot)\}_{n\in\mathbb{N}}$ satisfies the M-L condition. Hence by Proposition C.3.4 the assertion $(A)_j^t$ is proved for any $t \in \mathbb{R}$. Repeating this argument, we can finally prove $(A)_i^t$ for all $i \in \mathbb{Z}$ and all $t \in \mathbb{R}$. Together with the isomorphisms (C.3.4), we obtain by Lemma C.3.5 an isomorphism $\mathrm{R}\Gamma(\Omega_s, F^\cdot) \simeq \mathrm{R}\Gamma(\Omega_t, F^\cdot)$ for any pair $s > t$. This completes the proof. □

D

Filtered Rings

D.1 Good filtration

Let A be a ring. Assume that we are given a family $F = \{F_l A\}_{l \in \mathbb{Z}}$ of additive subgroups of A satisfying

(a) $F_l A = 0$ for $l < 0$,
(b) $1 \in F_0 A$,
(c) $F_l A \subset F_{l+1} A$,
(d) $(F_l A)(F_m A) \subset F_{l+m} A$,
(e) $A = \bigcup_{l \in \mathbb{Z}} F_l A$.

Then we call (A, F) a *filtered ring*. For a filtered ring (A, F) we set

$$\operatorname{gr}^F A = \bigoplus_{l \in \mathbb{Z}} \operatorname{gr}_l^F A, \qquad \operatorname{gr}_l^F A = F_l A / F_{l-1} A.$$

The canonical map $F_l A \to \operatorname{gr}_l^F A$ is denoted by σ_l. The additive group $\operatorname{gr}^F A$ is endowed with a structure of a ring by

$$\sigma_l(a)\sigma_m(b) = \sigma_{l+m}(ab).$$

We call the ring $\operatorname{gr}^F A$ the *associated graded ring*.

Let (A, F) be a filtered ring. Let M be a (left) A-module M, and assume that we are given a family $F = \{F_p M\}_{p \in \mathbb{Z}}$ of additive subgroups of M satisfying

(a) $F_p M = 0$ for $p \ll 0$,
(b) $F_p M \subset F_{p+1} M$,
(c) $(F_l A)(F_p M) \subset F_{l+p} M$,
(d) $M = \bigcup_{p \in \mathbb{Z}} F_p M$.

Then F is called a filtration of M and (M, F) is called a filtered (left) A-module. For a filtered A-module (M, F) we set

$$\mathrm{gr}^F M = \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}_p^F M, \qquad \mathrm{gr}_p^F M = F_pM/F_{p-1}M.$$

Denote the canonical map $F_pM \to \mathrm{gr}_p^F M$ by τ_p. The additive group $\mathrm{gr}^F M$ is endowed with a structure of a $\mathrm{gr}^F A$-module by

$$\sigma_l(a)\tau_p(m) = \tau_{l+p}(am).$$

We call the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$ the *associated graded module*.

We can also define the notion of a filtration of a right A-module and the associated graded module of a right filtered A-module. We will only deal with left A-modules in the following; however, parallel facts also hold for right modules.

Proposition D.1.1. *Let M be an A-module.*

(i) *Let F be a filtration of M such that $\mathrm{gr}^F M$ is finitely generated over $\mathrm{gr}^F A$. Then there exist finitely many integers p_k $(k = 1, \dots, r)$ and $m_k \in F_{p_k}M$ such that for any p we have $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. In particular, the A-module M is generated by finitely many elements $m_1, \dots, m_k$.*

(ii) *Let M an A-module generated by finitely many elements $m_1, \dots, m_k$. For $p_k \in \mathbb{Z}$ $(k = 1, \dots, r)$ set $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. Then F is a filtration of M such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

Proof. (i) We take integers p_k $(k = 1, \dots, r)$ and $m_k \in F_{p_k}M$ so that $\{\tau_{p_k}(m_k)\}_{1\leq k\leq r}$ generates the $\mathrm{gr}^F A$-module $\mathrm{gr}^F M$. Then we can show $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$ by induction on p. (ii) is obvious. □

Corollary D.1.2. *The following conditions on an A-module M are equivalent:*

(i) *M is a finitely generated A-module,*

(ii) *there exists a filtration F of M such that $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module.*

Let (M, F) be a filtered A-module. If $\mathrm{gr}^F M$ is a finitely generated $\mathrm{gr}^F A$-module, then F is called a *good filtration* of M, and (M, F) is called a good filtered A-module.

Proposition D.1.3. *Let M be a finitely generated A-module and let F, G be filtrations of M. If F is good, then there exist an integers a such that for any $p \in \mathbb{Z}$ we have*

$$F_pM \subset G_{p+a}M.$$

In particular, if G is also good, then for $a \gg 0$ we have

$$F_{p-a}M \subset G_pM \subset F_{p+a}M \qquad (\forall p).$$

Proof. By Proposition D.1.1 we can take elements m_k $(1 \leq k \leq r)$ of M and integers p_k $(1 \leq k \leq r)$ such that $F_pM = \sum_{p\geq p_k}(F_{p-p_k}A)m_k$. Take $q_k \in \mathbb{Z}$ such that $m_k \in G_{q_k}M$ and denote the maximal value of $q_k - p_k$ by a. Then we have

$$\begin{aligned} F_pM &= \sum_{p \geq p_k} (F_{p-p_k}A)m_k \subset \sum_{p \geq p_k} (F_{p-p_k}A)G_{q_k}M \\ &\subset \sum_{p \geq p_k} G_{p+(q_k-p_k)}M \subset G_{p+a}M. \end{aligned}$$

The proof is complete. □

Let (M, F) be a filtered A-module, and let

$$0 \to L \to M \to N \to 0$$

be an exact sequence of A-modules. Then we have the induced filtrations of L and N defined by

$$F_pL = F_pM \cap L, \qquad F_pN = \mathrm{Im}(F_pM \to N),$$

for which we have the exact sequence

$$0 \to \mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N \to 0.$$

Hence, if (M, F) is a good filtered A-module, then so is (N, F). If, moreover, $\mathrm{gr}^F A$ is a left noetherian ring, then (L, F) is also a good filtered A-module.

Proposition D.1.4. *Let (A, F) be a filtered ring. If $\mathrm{gr}^F A$ is a left (or right) noetherian ring, then so is A.*

Proof. In order to show that A is a left noetherian ring it is sufficient to show that any left ideal I of A is finitely generated. Define a filtration F of a left A-module I by $F_pI = I \cap F_pA$. Then $\mathrm{gr}^F I$ is a left ideal of $\mathrm{gr}^F A$. Since $\mathrm{gr}^F A$ is a left noetherian ring, $\mathrm{gr}^F I$ is finitely generated over $\mathrm{gr}^F A$. Hence I is finitely generated by Corollary D.1.2. The statement for right noetherian rings is proved similarly. □

D.2 Global dimensions

Let (M, F), (N, F) be filtered A-modules. An A-homomorphism $f : M \to N$ such that $f(F_pM) \subset F_pN$ for any p is called a *filtered* ***A****-homomorphism*. In this case we write $f : (M, F) \to (N, F)$. A filtered A-homomorphism $f : (M, F) \to (N, F)$ induces a homomorphism $\mathrm{gr}\, f : \mathrm{gr}^F M \to \mathrm{gr}^F N$ of $\mathrm{gr}^F A$-modules. A filtered A-homomorphism $f : (M, F) \to (N, F)$ is called *strict* if it satisfies $f(F_pM) = \mathrm{Im}\, f \cap F_pN$. The following fact is easily proved.

Lemma D.2.1. *Let $f : (L, F) \to (M, F)$, $g : (M, F) \to (N, F)$ be strict filtered A-homomorphisms such that $L \to M \to N$ is exact. Then $\mathrm{gr}^F L \to \mathrm{gr}^F M \to \mathrm{gr}^F N$ is exact.*

Let W be a free A-module of rank $r < \infty$ with basis $\{w_k\}_{1 \leqq k \leqq r}$. For integers p_k $(1 \leqq k \leqq r)$ we can define a filtration F of W by $F^pW = \sum_k (F_{p-p_k}A)w_k$. This type of filtered A-module (W, F) is called a filtered free A-module of rank r. We can easily show the following.

Lemma D.2.2. *Assume that A is left noetherian. For a good filtered A-module (M, F) we can take filtered free A-modules (W_i, F) $(i \in \mathbb{N})$ of finite ranks and strict filtered A-homomorphisms $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ such that*

$$\cdots \to W_1 \to W_0 \to M \to 0$$

is an exact sequence of A-modules.

For filtered A-modules (M, F), (N, F) and $p \in \mathbb{Z}$ set

$$F^p \operatorname{Hom}_A(M, N) = \{f \in \operatorname{Hom}_A(M, N) \mid f(F_q M) \subset F_{q+p} N \ (\forall q \in \mathbb{Z})\}.$$

This defines an increasing filtration of the abelian group $\operatorname{Hom}_A(M, N)$. Set

$$\operatorname{gr}^F_p \operatorname{Hom}_A(M, N) = F_p \operatorname{Hom}_A(M, N)/F_{p-1} \operatorname{Hom}_A(M, N),$$
$$\operatorname{gr}^F \operatorname{Hom}_A(M, N) = \bigoplus_p \operatorname{gr}^F_p \operatorname{Hom}_A(M, N).$$

Then we have a canonical homomorphism

$$\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$$

of abelian groups. The following is easily proved.

Lemma D.2.3. *Let (M, F) be a good filtered A-module and (N, F) a filtered A-module.*

(i) $\operatorname{Hom}_A(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Hom}_A(M, N)$.
(ii) $F_p \operatorname{Hom}_A(M, N) = 0$ *for* $p \ll 0$.
(iii) *The canonical homomorphism* $\operatorname{gr}^F \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$ *is injective. Moreover, it is surjective if (M, F) is a filtered free A-module of finite rank.*

Lemma D.2.4. *Let (A, F) be a filtered ring such that $\operatorname{gr}^F A$ is left noetherian. Let (M, F) be a good filtered A-module and (N, F) a filtered A-module. Then there exists an increasing filtration F of the abelian group $\operatorname{Ext}^i_A(M, N)$ such that*

(i) $\operatorname{Ext}^i_A(M, N) = \bigcup_{p \in \mathbb{Z}} F_p \operatorname{Ext}^i_A(M, N)$,
(ii) $F_p \operatorname{Ext}^i_A(M, N) = 0$ *for* $p \ll 0$,
(iii) $\operatorname{gr}^F \operatorname{Ext}^i_A(M, N)$ *is isomorphic to a subquotient of* $\operatorname{Ext}^i_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F N)$.

Proof. Take (W_i, F) $(i \in \mathbb{N})$, $(W_{i+1}, F) \to (W_i, F)$ $(i \in \mathbb{N})$ and $(W_0, F) \to (M, F)$ as in Lemma D.2.2. Then we have $\operatorname{Ext}^i_A(M, N) = H^i(K^\cdot)$ with $K^\cdot = \operatorname{Hom}_A(W_\cdot, N)$. Note that each term $K^i = \operatorname{Hom}_A(W_i, N)$ of $K^\cdot$ is equipped with increasing filtration F satisfying $d^i(F_p K^i) \subset F_p K^{i+1}$, where $d^i : K^i \to K^{i+1}$ denotes the boundary homomorphism.

Consider the complex $\operatorname{gr}^F K^\cdot$ with ith term $\operatorname{gr}^F K^i$. We have $\operatorname{gr}^F K^\cdot \simeq \operatorname{Hom}_{\operatorname{gr}^F A}(\operatorname{gr}^F W_\cdot, \operatorname{gr}^F N)$ by Lemma D.2.3. Hence by the exact sequence

$$\cdots \to \mathrm{gr}^F W_1 \to \mathrm{gr}^F W_0 \to \mathrm{gr}^F M \to 0,$$

(see Lemma D.2.1) we obtain

$$\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = H^i(\mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F W_., \mathrm{gr}^F N) \simeq H^i(\mathrm{gr}^F K^\cdot).$$

Now define an increasing filtration of $H^i(K^\cdot) = \mathrm{Ext}^i_A(M, N)$ by

$$F_p H^i(K^\cdot) = \mathrm{Im}(H^i(F^p K^\cdot) \to H^i(K^\cdot)).$$

For each $i \in \mathbb{N}$ we have

$$K^i = \bigcup_p F_p K^i, \qquad F_p K^i = 0 \ (p \ll 0).$$

by Lemma D.2.3. From this we easily see that

$$H^i(K^\cdot) = \bigcup_p F_p H^i(K^\cdot), \qquad F_p H^i(K^\cdot) = 0 \ (p \ll 0).$$

It remains to show that $\mathrm{gr}^F H^i(K^\cdot)$ is a subquotient of $H^i(\mathrm{gr}^F K^\cdot)$. By definition we have

$$\mathrm{gr}^F_p H^i(K^\cdot) = (F_p K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1})/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + \mathrm{Im}\, d^{i-1}),$$
$$H^i(\mathrm{gr}^F_p K^\cdot) = \mathrm{Ker}(F_p K^i \to \mathrm{gr}^F_p K^{i+1})/(F_{p-1} K^i + d^{i-1}(F_p K^{i-1})).$$

Set $L = F_p K^i \cap \mathrm{Ker}\, d^i/(F_{p-1} K^i \cap \mathrm{Ker}\, d^i + d^{i-1}(F_p K^{i-1}))$. Then we can easily check that L is isomorphic to a submodule of $H^i(\mathrm{gr}^F_p K^\cdot)$ and that $\mathrm{gr}^F_p H^i(K^\cdot)$ is a quotient of L. □

Let us consider the situation where $N = A$ (with canonical filtration F) in Lemma D.2.3 and Lemma D.2.4. Let (A, F) be a filtered ring and let (M, F) be a good filtered A-module. We easily see that the filtration F of the right A-module $\mathrm{Hom}_A(M, A)$ is a good filtration and the canonical homomorphism $\mathrm{gr}^F \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N)$ preserves the $\mathrm{gr}^F A$-modules structure. Hence (the proof of) Lemma D.2.4 implies the following.

Lemma D.2.5. *Let (A, F) be a filtered ring such that $\mathrm{gr}^F A$ is left noetherian, and let (M, F) be a good filtered A-module. Then there exists a good filtration F of the right A-module $\mathrm{Ext}^i_A(M, A)$ such that $\mathrm{gr}^F \mathrm{Ext}^i_A(M, A)$ is isomorphic to a subquotient of $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$ as a right $\mathrm{gr}^F A$-module.*

Theorem D.2.6. *Let (A, F) be a filtered ring such that $\mathrm{gr}^F A$ is left (resp. right) noetherian. Then the left (resp. right) global dimension of the ring A is smaller than or equal to that of $\mathrm{gr}^F A$.*

Proof. We will only show the statement for left global dimensions. Denote the left global dimension of $\mathrm{gr}^F A$ by n. If $n = \infty$, there is nothing to prove. Assume that $n < \infty$. We need to show $\mathrm{Ext}^i_A(M, N) = 0$ $(i > n)$ for arbitrary A-modules M, N. Since A is left noetherian, we may assume that M is finitely generated. Choose a good filtration F of M and a filtration F of N. Then we have $\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F N) = 0$ $(i > n)$. Hence the assertion follows from Lemma D.2.4. □

D.3 Singular supports

Let R be a commutative noetherian ring and let M be a finitely generated R-module. We denote by $\mathrm{supp}(M)$ the set of prime ideals $\mathfrak{p}$ of R satisfying $M_{\mathfrak{p}} \neq 0$, and by $\mathrm{supp}_0(M)$ the set of minimal elements of $\mathrm{supp}(M)$. We have $\mathfrak{p} \in \mathrm{supp}(M)$ if and only if $\mathfrak{p}$ contains the annihilating ideal

$$\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}.$$

In fact, we have

$$\sqrt{\mathrm{Ann}_R(M)} = \bigcap_{\mathfrak{p} \in \mathrm{supp}(M)} \mathfrak{p}.$$

For $\mathfrak{p} \in \mathrm{supp}_0(M)$ we denote the *length* of the artinian $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ by $\ell_{\mathfrak{p}}(M)$. We set $\ell_{\mathfrak{q}}(M) = 0$ for a prime ideal $\mathfrak{q} \notin \mathrm{supp}(M)$. For an exact sequence

$$0 \to L \to M \to N \to 0$$

of finitely generated R-modules we have

$$\mathrm{supp}(M) = \mathrm{supp}(L) \cup \mathrm{supp}(N).$$

Moreover, for $\mathfrak{p} \in \mathrm{supp}_0(M)$ we have

$$\ell_{\mathfrak{p}}(M) = \ell_{\mathfrak{p}}(L) + \ell_{\mathfrak{p}}(N).$$

In the rest of this section (A, F) denotes a filtered ring such that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let M be a finitely generated A-module. By choosing a good filtration F we can consider $\mathrm{supp}(\mathrm{gr}^F M)$ and $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ for $\mathfrak{p} \in \mathrm{supp}_0(M)$.

Lemma D.3.1. $\mathrm{supp}(\mathrm{gr}^F M)$ *and* $\ell_{\mathfrak{p}}(\mathrm{gr}^F M)$ *for* $\mathfrak{p} \in \mathrm{supp}_0(M)$ *do not depend on the choice of a good filtration* F.

Proof. We say two good filtrations F and G are "*adjacent*" if they satisfy the condition

$$F_i M \subset G_i M \subset F_{i+1} M \quad ({}^{\forall} i \in \mathbb{Z}).$$

We first show the assertion in this case. Consider the natural homomorphism $\varphi_i : F_i M / F_{i-1} M \to G_i M / G_{i-1} M$. Then we have $\mathrm{Ker}\, \varphi_i \simeq G_{i-1} M / F_{i-1} M \simeq \mathrm{Coker}\, \varphi_{i-1}$. Therefore, the morphism $\varphi : \mathrm{gr}^F M \to \mathrm{gr}^G M$ entails an isomorphism $\mathrm{Ker}\, \varphi \simeq \mathrm{Coker}\, \varphi$. Consider the exact sequence

$$0 \to \mathrm{Ker}\, \varphi \to \mathrm{gr}^F M \xrightarrow{\varphi} \mathrm{gr}^G M \to \mathrm{Coker}\, \varphi \to 0$$

of finitely generated $\mathrm{gr}^F A$-modules. From this we obtain

$$\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ker}\, \varphi) \cup \mathrm{supp}(\mathrm{Im}\, \varphi),$$
$$\mathrm{supp}(\mathrm{gr}^G M) = \mathrm{supp}(\mathrm{Im}\, \varphi) \cup \mathrm{supp}(\mathrm{Coker}\, \varphi).$$

Hence $\operatorname{Ker}\varphi \simeq \operatorname{Coker}\varphi$ implies $\operatorname{supp}(\operatorname{gr}^F M) = \operatorname{supp}(\operatorname{gr}^G M)$. Moreover, for $\mathfrak{p} \in \operatorname{supp}_0(\operatorname{gr}^F M) = \operatorname{supp}_0(\operatorname{gr}^G M)$ we have

$$\ell_{\mathfrak{p}}(\operatorname{gr}^F M) = \ell_{\mathfrak{p}}(\operatorname{Ker}\varphi) + \ell_{\mathfrak{p}}(\operatorname{Im}\varphi) = \ell_{\mathfrak{p}}(\operatorname{gr}^G M).$$

The assertion is proved for adjacent good filtrations.

Let us consider the general case. Namely, assume that F and G are arbitrary good filtrations of M. For $k \in \mathbb{Z}$ set

$$F_i^{(k)} M = F_i M + G_{i+k} M \quad (i \in \mathbb{Z}).$$

By Proposition D.1.3 $F^{(k)}$ is a good filtration of M satisfying the conditions

$$\begin{cases} F^{(k)} = F \quad (k \ll 0), \\ F^{(k)} = G[k] \quad (k \gg 0), \\ F^{(k)} \text{ and } F^{(k+1)} \text{ are adjacent,} \end{cases}$$

where $G[k]$ is a filtration obtained form G by the degree shift $[k]$. Therefore, our assertion follows form the adjacent case. □

Definition D.3.2. For a finitely generated A-module M we set

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{supp}(\operatorname{gr}^F M), \\ \operatorname{SS}_0(M) &= \operatorname{supp}_0(\operatorname{gr}^F M), \\ J_M &= \sqrt{\operatorname{Ann}_{\operatorname{gr}^F A}(\operatorname{gr}^F M)} = \bigcap_{\mathfrak{p} \in \operatorname{SS}_0(M)} \mathfrak{p}, \\ d(M) &= \operatorname{Krull\,dim}\left(\operatorname{gr}^F A / J_M\right), \\ m_{\mathfrak{p}}(M) &= \ell_{\mathfrak{p}}(\operatorname{gr}^F M) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M) \text{ or } \mathfrak{p} \notin \operatorname{SS}(M)), \end{aligned}$$

where F is a good filtration of M. $\operatorname{SS}(M)$ and J_M are called the *singular support* and the *characteristic ideal* of M, respectively.

Lemma D.3.3. *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

of finitely generated A-modules we have

$$\begin{aligned} \operatorname{SS}(M) &= \operatorname{SS}(L) \cup \operatorname{SS}(N), \\ d(M) &= \max\{d(L), d(N)\}, \\ m_{\mathfrak{p}}(M) &= m_{\mathfrak{p}}(L) + m_{\mathfrak{p}}(N) \qquad (\mathfrak{p} \in \operatorname{SS}_0(M)). \end{aligned}$$

Proof. Take a good filtration F of M. With respect to the induced filtrations of L and N we have a short exact sequence

$$0 \longrightarrow \operatorname{gr}^F M \longrightarrow \operatorname{gr}^F N \longrightarrow \operatorname{gr}^F L \longrightarrow 0.$$

Hence the assertions for SS and $\ell_{\mathfrak{p}}$ are obvious. The assertion for d follows from the one for SS. □

Since $\mathrm{gr}^F A$ is commutative, we have $[F_pA, F_qA] \subset F_{p+q-1}A$. Here, for $a, b \in A$ we set $[a, b] = ab - ba$. Hence we obtain a bi-additive product

$$\{\ ,\ \} : \mathrm{gr}^F_p A \times \mathrm{gr}^F_q A \to \mathrm{gr}^F_{p+q-1} A, \quad (\{\sigma_p(a), \sigma_q(b)\} = \sigma_{p+q-1}([a, b])).$$

Its bi-additive extension

$$\{\ ,\ \} : \mathrm{gr}^F A \times \mathrm{gr}^F A \to \mathrm{gr}^F A$$

is called the *Poisson bracket*. It satisfies the following properties:

(i) $\{a, b\} + \{b, a\} = 0$,
(ii) $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$,
(iii) $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

We say that an ideal I of $\mathrm{gr}^F A$ is *involutive* if it satisfies $\{I, I\} \subset I$.

We state the following deep result of Gabber [Ga] without proof.

Theorem D.3.4. *Assume that (A, F) is a filtered ring such that the center of A contains a subring isomorphic to $\mathbb{Q}$ and that $\mathrm{gr}^F A$ is a commutative noetherian ring. Let M be a finitely generated A-module. Then any $\mathfrak{p} \in \mathrm{SS}_0(M)$ is involutive. In particular, J_M is involutive.*

D.4 Duality

In this section (A, F) is a filtered ring such that $\mathrm{gr}^F A$ is a regular commutative ring of pure dimension m (a commutative ring R is called a regular ring of pure dimension m if its localization at any maximal ideal is a regular local ring of dimension m). In particular, $\mathrm{gr}^F A$ is a noetherian ring whose global dimension and Krull dimension are equal to m. Hence A is a left and right noetherian ring with global dimension $\leqq m$ by Proposition D.1.4 and Theorem D.2.6. We will consider properties of the Ext-groups $\mathrm{Ext}^i_A(M, A)$ for finitely generated A-modules M.

Note first that for any (left) A-module M the Ext-groups $\mathrm{Ext}^i_A(M, A)$ are endowed with a right A-module structure (i.e., a left A^{op}-module structure, where A^{op} denotes the opposite ring) by the right multiplication of A on A. Since A has global dimension $\leqq m$, we have $\mathrm{Ext}^i_A(M, A) = 0$ for $i > m$. Moreover, if M is finitely generated, then $\mathrm{Ext}^i_A(M, A)$ are also finitely generated since A is left noetherian.

Let us give a formulation in terms of the derived category. Let $\mathrm{Mod}(A)$ and $\mathrm{Mod}_f(A)$ denote the category of (left) A-modules and its full subcategory consisting of finitely generated A-modules, respectively. Denote by $D^b(A)$ and $D^b_f(A)$ the bounded derived category of $\mathrm{Mod}(A)$ and its full subcategory consisting of complexes whose cohomology groups belong to $\mathrm{Mod}_f(A)$. Our objectives are the functors

$$\mathbb{D} = R\,\mathrm{Hom}_A(\bullet, A) : D^b_f(A) \to D^b_f(A^{\mathrm{op}})^{\mathrm{op}},$$
$$\mathbb{D}' = R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}}) : D^b_f(A^{\mathrm{op}}) \to D^b_f(A)^{\mathrm{op}},$$

where $D^b_f(A^{\mathrm{op}})$ is defined similarly.

Proposition D.4.1. *We have* $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$ *and* $\mathbb{D} \circ \mathbb{D}' \simeq \mathrm{Id}$.

Proof. By symmetry we have only to show $\mathbb{D}' \circ \mathbb{D} \simeq \mathrm{Id}$. We first construct a canonical morphism $M^{\cdot} \to \mathbb{D}'\mathbb{D}M^{\cdot}$ for $M^{\cdot} \in D^b_f(A)$. Set $H^{\cdot} = \mathbb{D}M^{\cdot} = R\operatorname{Hom}_A(M^{\cdot}, A)$. By

$$R\operatorname{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot}, A) \simeq R\operatorname{Hom}_A(M^{\cdot}, R\operatorname{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}}))$$

we have

$$\operatorname{Hom}_{A\otimes_{\mathbb{Z}}A^{\mathrm{op}}}(M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot}, A) \simeq \operatorname{Hom}_A(M^{\cdot}, R\operatorname{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}}))$$

Hence the canonical morphism $M^{\cdot} \otimes_{\mathbb{Z}} H^{\cdot} (= M^{\cdot} \otimes_{\mathbb{Z}} R\operatorname{Hom}_A(M^{\cdot}, A)) \to A$ in $D^b(A \otimes_{\mathbb{Z}} A^{\mathrm{op}})$ gives rise to a canonical morphism

$$M^{\cdot} \to R\operatorname{Hom}_{A^{\mathrm{op}}}(H^{\cdot}, A^{\mathrm{op}})(= \mathbb{D}'\mathbb{D}M)$$

in $D^b(A)$. It remains to show that $M^{\cdot} \to \mathbb{D}'\mathbb{D}M^{\cdot}$ is an isomorphism. By taking a free resolution of $M^{\cdot}$ we may replace $M^{\cdot}$ with A. In this case the assertion is clear. □

For $M^{\cdot} \in D^b_f(A)$ (or $D^b_f(A^{\mathrm{op}})$) we set

$$\mathrm{SS}(M^{\cdot}) = \bigcup_i \mathrm{SS}(H^i(M^{\cdot})).$$

We easily see by Lemma D.3.3 that for a distinguished triangle

$$L^{\cdot} \longrightarrow M^{\cdot} \longrightarrow N^{\cdot} \xrightarrow{+1}$$

we have $\mathrm{SS}(M^{\cdot}) \subset \mathrm{SS}(L^{\cdot}) \cup \mathrm{SS}(N^{\cdot})$.

Proposition D.4.2. *For* $M^{\cdot} \in D^b_f(A)$ (*resp.* $D^b_f(A^{\mathrm{op}})$) *we have* $\mathrm{SS}(\mathbb{D}M^{\cdot}) = \mathrm{SS}(M^{\cdot})$ (*resp.* $\mathrm{SS}(\mathbb{D}'M^{\cdot}) = \mathrm{SS}(M^{\cdot})$).

Proof. By Proposition D.4.1 and symmetry we have only to show $\mathrm{SS}(\mathbb{D}M^{\cdot}) \subset \mathrm{SS}(M^{\cdot})$ for $M^{\cdot} \in D^b_f(A)$. We use induction on the cohomological length of $M^{\cdot}$. We first consider the case where $M^{\cdot} = M \in \mathrm{Mod}_f(A)$. Take a good filtration F of M and consider a good filtration F of $\operatorname{Ext}^i_A(M, A)$ as in Lemma D.2.5. By Lemma D.2.5 we have

$$\begin{aligned}\mathrm{SS}(\operatorname{Ext}^i_A(M, A)) &= \operatorname{supp}(\operatorname{gr}^F \operatorname{Ext}^i_A(M, A)) \subset \operatorname{supp}(\operatorname{Ext}^i_{\operatorname{gr}^F A}(\operatorname{gr}^F M, \operatorname{gr}^F A)) \\ &\subset \operatorname{supp}(\operatorname{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

The assertion is proved in the case where $M^{\cdot} = M \in \mathrm{Mod}_f(A)$. Now we consider the general case. Set $k = \min\{i \mid H^i(M^{\cdot}) \neq 0\}$. Then we have a distinguished triangle

$$H^k(M^{\cdot})[-k] \longrightarrow M^{\cdot} \longrightarrow N^{\cdot} \xrightarrow{+1},$$

where $N^{\cdot} = \tau^{\geqslant k+1}M^{\cdot}$. By

$$H^i(N^\cdot) = \begin{cases} H^i(M^\cdot) & (i \neq k) \\ 0 & (i = k) \end{cases}$$

we obtain $\mathrm{SS}(M^\cdot) = \mathrm{SS}(N^\cdot) \cup \mathrm{SS}(H^k(M^\cdot))$. Moreover, by the hypothesis of induction we have $\mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(N^\cdot)$ and $\mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \subset \mathrm{SS}(H^k(M^\cdot))$. Hence by the distinguished triangle

$$\mathbb{D}N^\cdot \longrightarrow \mathbb{D}M^\cdot \longrightarrow (\mathbb{D}H^k(M^\cdot))[k] \xrightarrow{+1}$$

we obtain

$$\mathrm{SS}(\mathbb{D}M^\cdot) \subset \mathrm{SS}(\mathbb{D}H^k(M^\cdot)) \cup \mathrm{SS}(\mathbb{D}N^\cdot) \subset \mathrm{SS}(H^k(M^\cdot)) \cup \mathrm{SS}(N^\cdot) = \mathrm{SS}(M^\cdot).$$

The proof is complete. □

For a finitely generated A-module M set

$$j(M) := \min\{i \mid \mathrm{Ext}^i_A(M, A) \neq 0\}.$$

Theorem D.4.3. *Let M be a finitely generated A-module.*

(i) $j(M) + d(M) = m$,
(ii) $d(\mathrm{Ext}^i_A(M, A)) \leq m - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(M)}_A(M, A)) = d(M)$.

(*Recall that m denotes the global dimension of* $\mathrm{gr}^F A$.)

The following corresponding fact for regular commutative rings is well known (see [Ser2], [Bj1]).

Theorem D.4.4. *Let R be a regular commutative ring of dimension m'. For a finitely generated R-module N we set $d(N) :=$ Krull $\dim(R/\mathrm{Ann}_R N)$ and $j(N) :=$ $\min\{\, i \mid \mathrm{Ext}^i_R(N, R) \neq 0\}$.*

(i) $d(N) + j(N) = m'$,
(ii) $d(\mathrm{Ext}^i_R(N, R)) \leq m' - i \quad (i \in \mathbb{Z})$,
(iii) $d(\mathrm{Ext}^{j(N)}_R(N, R)) = d(N)$.

Proof of Theorem D.4.3. We apply Theorem D.4.4 to the case $R = \mathrm{gr}^F A$. Fix a good filtration F of M. By Lemma D.2.4 (iii) we have $\mathrm{SS}(\mathrm{Ext}^i_A(M, A)) \subset \mathrm{supp}(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A))$ and hence

$$d(\mathrm{Ext}^i_A(M, A)) \leqq d(\mathrm{Ext}^i_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

Thus (ii) follows from the corresponding fact for $\mathrm{gr}^F A$. Moreover, we have $\mathrm{Ext}^i_A(M, A) = 0$ for $i < j(\mathrm{gr}^F M)$. Hence in order to show (i) and (iii) it is sufficient to verify

$$d(\mathrm{Ext}_A^{j(\mathrm{gr}^F M)}(M, A)) = d(M).$$

By Proposition D.4.2 we have

$$d(M) = \max_{i \geqq j(\mathrm{gr}^F M)} d(\mathrm{Ext}_A^i(M, A)).$$

For $i > j(\mathrm{gr}^F M)$ we have

$$d(\mathrm{Ext}_A^i(M, A)) \leqq m - i < m - j(\mathrm{gr}^F M) = d(\mathrm{gr}^F M) = d(M),$$

and hence we must have $d(\mathrm{Ext}_A^{j(\mathrm{gr}^F M)}(M, A)) = d(M)$. □

Corollary D.4.5. *For an exact sequence*

$$0 \to L \to M \to N \to 0$$

of finitely generated A-modules we have

$$j(M) = \min\{j(L), j(N)\}.$$

D.5 Codimension filtration

In this section (A, F) is a filtered ring such that $\mathrm{gr}^F A$ is a regular commutative ring of pure dimension m. For a finitely generated A-module M and $s \geq 0$ we denote by $C^s(M)$ the sum of all submodules N of M satisfying $j(N) \geq s$. Since $C^s(M)$ is finitely generated, we have $j(C^s(M)) \geq s$ by Corollary D.4.5 and hence $C^s(M)$ is the largest submodule N of M satisfying $j(N) \geq s$. By definition we have a decreasing filtration

$$0 = C^{m+1}(M) \subset C^m(M) \subset \cdots \subset C^1(M) \subset C^0(M) = M.$$

We say that a finitely generated A-module M is purely s-codimensional if $C^s(M) = M$ and $C^{s+1}(M) = 0$.

Lemma D.5.1. *For any finitely generated A-module M $C^s(M)/C^{s+1}(M)$ is purely s-codimensional.*

Proof. Set $N = C^s(M)/C^{s+1}(M)$. Then we have $j(N) \geq j(C^s(M)) \geq s$ and hence $C^s(N) = N$. Set $K = \mathrm{Ker}\,(C^s(M) \to N/C^{s+1}(N))$. Then by the exact sequence

$$0 \to C^{s+1}(M) \to K \to C^{s+1}(N) \to 0$$

we have $j(K) = \min\{j(C^{s+1}(M)), j(C^{s+1}(N))\} \geq s + 1$. By the maximality of $C^{s+1}(M)$ we obtain $K = C^{s+1}(M)$, i.e., $C^{s+1}(N) = 0$. □

We will give a cohomological interpretation of this filtration.

For a finitely generated A-module M and $s \geq 0$ we set

$$T^s(M) = \mathrm{Ext}^0_{A^{\mathrm{op}}}(\tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A), A^{\mathrm{op}}).$$

By Proposition D.4.1 we have $T^0(M) = M$. By $j(\mathrm{Ext}^s(M, A)) \geq s$ we have $\mathrm{Ext}^{s-1}_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}}) = 0$. Hence by applying $R\,\mathrm{Hom}_{A^{\mathrm{op}}}(\bullet, A^{\mathrm{op}})$ to the distinguished triangle

$$\mathrm{Ext}^s_A(M, A)[-s] \longrightarrow \tau^{\geqslant s} R\,\mathrm{Hom}_A(M, A) \longrightarrow \tau^{\geqslant s+1} R\,\mathrm{Hom}_A(M, A) \xrightarrow{+1}$$

and taking the cohomology groups we obtain an exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}).$$

Hence we obtain a decreasing filtration

$$0 = T^{m+1}(M) \subset T^m(M) \subset \cdots \subset T^1(M) \subset T^0(M) = M.$$

Proposition D.5.2. *For any s we have $C^s(M) = T^s(M)$.*

Proof. By $j(\mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s(M, A), A^{\mathrm{op}})) \geq s$ we see from the exact sequence

$$0 \to T^{s+1}(M) \to T^s(M) \to \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}})$$

using the backward induction on s that $j(T^s(M)) \geq s$. Hence $T^s(M) \subset C^s(M)$. It remains to show the opposite inclusion. Set $N = C^s(M)$. By $j(N) \geq s$ we have

$$\tau^{\geqslant s} R\,\mathrm{Hom}_A(N, A) = R\,\mathrm{Hom}_A(N, A),$$

and hence $N = T^s(N)$. By the functoriality of T^s we have a commutative diagram

$$\begin{array}{ccc} T^s(N) & = & N \\ \downarrow & & \downarrow \\ T^s(M) & \longrightarrow & M, \end{array}$$

which implies $N \subset T^s(M)$. □

Theorem D.5.3. *Let M be a finitely generated A-module which is purely s-codimensional. Then for any $\mathfrak{p} \in \mathrm{SS}_0(M)$ we have*

$$\text{Krull}\dim((\mathrm{gr}^F A)/\mathfrak{p}) = m - s.$$

Proof. The assertion being trivial for $M = 0$ we assume that $M \neq 0$. In this case we have $j(M) = s$. Let F be a good filtration of M. Then there exists a good filtration F of $N = \mathrm{Ext}^s_A(M, A)$ such that $\mathrm{gr}^F N$ is a subquotient of $\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)$. Hence

$$\begin{aligned}\mathrm{SS}(N) = \mathrm{supp}(\mathrm{gr}^F N) &\subset \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)) \\ &\subset \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{SS}(M).\end{aligned}$$

On the other hand since M is purely s-codimensional, we have

$$M = T^s(M)/T^{s+1}(M) \subset \mathrm{Ext}^s_{A^{\mathrm{op}}}(\mathrm{Ext}^s_A(M, A), A^{\mathrm{op}}) = \mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})$$

and hence $\mathrm{SS}(M) \subset \mathrm{SS}(\mathrm{Ext}^s_{A^{\mathrm{op}}}(N, A^{\mathrm{op}})) \subset \mathrm{SS}(N)$. Therefore, we have

$$\mathrm{SS}(M) = \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ext}^s_{\mathrm{gr}^F A}(\mathrm{gr}^F M, \mathrm{gr}^F A)).$$

By $j(\mathrm{gr}^F M) = j(M) = s$ we have $j((\mathrm{gr}^F A)/\mathfrak{p}) \geq s$ for any $\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M)$. Set $\Lambda = \{\mathfrak{p} \in \mathrm{supp}_0(\mathrm{gr}^F M) \mid j((\mathrm{gr}^F A)/\mathfrak{p}) = s)\}$. By a well-known fact in commutative algebra there exists a submodule L of $\mathrm{gr}^F M$ such that $j(L) > s$ and $\mathrm{supp}_0(\mathrm{gr}^F M/L) = \Lambda$. We need to show $\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{gr}^F M/L)$. We have obviously $\mathrm{supp}(\mathrm{gr}^F M) \supset \mathrm{supp}(\mathrm{gr}^F M/L)$. On the other hand by $\mathrm{Ext}^s(L, \mathrm{gr}^F A) = 0$ we have an injection $\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A) \to \mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A)$, and hence

$$\begin{aligned}\mathrm{supp}(\mathrm{gr}^F M) = \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M, \mathrm{gr}^F A)) &\subset \mathrm{supp}(\mathrm{Ext}^s(\mathrm{gr}^F M/L, \mathrm{gr}^F A)) \\ &\subset \mathrm{supp}(\mathrm{gr}^F M/L).\end{aligned}$$

The proof is complete. □

E

Symplectic Geometry

In this chapter we first present basic results in symplectic geometry laying special emphasis on cotangent bundles of complex manifolds. Most of the results are well known and we refer the reader to Abraham–Mardsen [AM] and Duistermaat [Dui] for details. Next we will precisely study conic Lagrangian analytic subsets in the cotangent bundles of complex manifolds. We prove that such a Lagrangian subset is contained in the union of the conormal bundles of strata in a Whitney stratification of the base manifold (Kashiwara's theorem in [Kas3], [Kas8]).

E.1 Symplectic vector spaces

Let V be a finite-dimensional vector space over a field k. A *symplectic form* σ on V is a non-degenerate anti-symmetric bilinear form on V. If a vector space V is endowed with a symplectic form σ, we call the pair (V, σ) a *symplectic vector space*. The dimension of a symplectic vector space is even. Let (V, σ) be a symplectic vector space. Denote by V^* the dual of V. Then for any $\theta \in V^*$ there exists a unique $H_\theta \in V$ such that

$$< \theta, v >= \sigma(v, H_\theta) \quad (v \in V)$$

by the non-degeneracy of σ. The correspondence $\theta \mapsto H_\theta$ defines the *Hamiltonian isomorphism* $H : V^* \simeq V$. For a linear subspace W of V consider its orthogonal complement $W^\perp = \{v \in V | \sigma(v, W) = 0\}$ with respect to σ. Then again by the non-degeneracy of σ we obtain $\dim W + \dim W^\perp = \dim V$. Now let us introduce the following important linear subspaces of V.

Definition E.1.1. A linear subspace W of V is called *isotropic* (resp. *Lagrangian*, resp. *involutive*) if it satisfies $W \subset W^\perp$ (resp. $W = W^\perp$, resp. $W \supset W^\perp$).

Note that if a linear subspace $W \subset V$ is isotropic (resp. Lagrangian, resp. involutive) then $\dim W \leq \frac{1}{2} \dim V$ (resp. $\dim W = \frac{1}{2} \dim V$, resp. $\dim W \geq \frac{1}{2} \dim V$). Moreover, a one-dimensional subspace (resp. a hyperplane) of V is always isotropic (resp. involutive).

Example E.1.2. Let W be a finite-dimensional vector space and W^* its dual. Set $V = W \oplus W^*$ and define a bilinear form σ on V by

$$\sigma((x, \xi), (x', \xi')) =< x', \xi > - < x, \xi' > \qquad ((x, \xi), (x', \xi') \in V = W \oplus W^*).$$

Then (V, σ) is a symplectic vector space. Moreover, W and W^* are Lagrangian subspaces of V.

E.2 Symplectic structures on cotangent bundles

A complex manifold X is called a (holomorphic) *symplectic manifold* if there exists a holomorphic 2-form σ globally defined on X which induces a symplectic form on the tangent space $T_x X$ of X at each $x \in X$. The dimension of a symplectic manifold is necessarily even. As one of the most important examples of symplectic manifolds, we treat here cotangent bundles of complex manifolds.

Now let X be a complex manifold and TX (resp. T^*X) its tangent (resp. cotangent) bundle. We denote by $\pi : T^*X \to X$ the canonical projection. By differentiating π we obtain the tangent map $\pi' : T(T^*X) \to (T^*X) \times_X (TX)$ and its dual $\rho_\pi : (T^*X) \times_X (T^*X) \to T^*(T^*X)$. If we restrict ρ_π to the diagonal T^*X of $(T^*X) \times_X (T^*X)$ then we get a map $T^*X \to T^*(T^*X)$. Since this map is a holomorphic section of the bundle $T^*(T^*X) \to T^*X$, it corresponds to a (globally defined) holomorphic 1-form α_X on T^*X. We call α_X the *canonical* **1**-*form*. If we take a local coordinate $(x_1, x_2, \dots, x_n)$ of X on an open subset $U \subset X$, then any point p of $T^*U \subset T^*X$ can be written uniquely as $p = (x_1, x_2, \dots, x_n; \xi_1 dx_1 + \xi_2 dx_2 + \cdot + \xi_n dx_n)$ where $\xi_i \in \mathbb{C}$. We call $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ the local coordinate system of T^*X associated to $(x_1, x_2, \dots, x_n)$. In this local coordinate of T^*X the canonical 1-form α_X is written as $\alpha_X = \sum_{i=1}^n \xi_i dx_i$. Set $\sigma_X = d\alpha_X = \sum_{i=1}^n d\xi_i \wedge dx_i$. Then we see that the holomorphic 2-form σ_X defines a symplectic structure on $T_p(T^*X)$ at each point $p \in T^*X$. Namely, the cotangent bundle T^*X is endowed with a structure of a symplectic manifold by σ_X. We call σ_X the (canonical) *symplectic form* of T^*X. Since there exists the Hamiltonian isomorphism $H : T_p^*(T^*X) \simeq T_p(T^*X)$ at each $p \in T^*X$, we obtain the global isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. For a holomorphic function f on T^*X we define a holomorphic vector field H_f to be the image of the 1-form df by $H : T^*(T^*X) \simeq T(T^*X)$. The vector field H_f is called the *Hamiltonian vector field* of f. Define the *Poisson bracket* of two holomorphic functions f, g on T^*X by $\{f, g\} = H_f(g) = \sigma_X(H_f, H_g)$. In the local coordinate $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ of T^*X we have the explicit formula

$$H_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} \right).$$

We can also easily verify the following:

$$\begin{cases} \{f, g\} = -\{g, f\}, \\ \{f, hg\} = h\{f, g\} + g\{f, h\}, \\ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0. \end{cases}$$

Moreover, we have $[H_f, H_g] = H_{\{f,g\}}$, where $[H_f, H_g]$ is the Lie bracket of H_f and H_g.

Definition E.2.1. An analytic subset V of T^*X is called isotropic (resp. Lagrangian, resp. involutive) if for any smooth point $p \in V_{\mathrm{reg}}$ of V the tangent space T_pV at p is a isotropic (resp. a Lagrangian, resp. an involutive) subspace in $T_p(T^*X)$.

By definition the dimension of a Lagrangian analytic subset of T^*X is equal to $\dim X$.

Example E.2.2.
(i) Let $Y \subset X$ be a complex submanifold of X. Then the *conormal bundle* $T^*_Y X$ of Y in X is a Lagrangian submanifold of T^*X.
(ii) Let f be a holomorphic function on X. Set $\Lambda_f = \{(x, \operatorname{grad} f(x) | x \in X\}$. Then Λ_f is a Lagrangian submanifold of T^*X.

For an analytic subset V of T^*X denote by $\mathcal{I}_V$ the subsheaf of $\mathcal{O}_{T^*X}$ consisting of holomorphic functions vanishing on V.

Lemma E.2.3. *For an analytic subset V of T^*X the following conditions are equivalent:*

(i) *V is involutive.*
(ii) $\{\mathcal{I}_V, \mathcal{I}_V\} \subset \mathcal{I}_V$.

Proof. By the definition of Hamiltonian isomorphisms, for each smooth point $p \in V_{\mathrm{reg}}$ of V the orthogonal complement $(T_pV)^\perp$ of T_pV in the symplectic vector space $T_p(T^*X)$ is spanned by the Hamiltonian vector fields H_f of $f \in \mathcal{I}_V$. Assume that V is involutive. If $f, g \in \mathcal{I}_V$ then the Hamiltonian vector field H_f is tangent to V_{reg} and hence $\{f, g\} = H_f(g) = 0$ on V_{reg}. Since $\{f, g\}$ is holomorphic and V_{reg} is dense in V, $\{f, g\} = 0$ on the whole V, i.e., $\{f, g\} \in \mathcal{I}_V$. The part (i) $\Longrightarrow$ (ii) was proved. The converse can be proved more easily. □

E.3 Lagrangian subsets of cotangent bundles

Let X be a complex manifold of dimension n. Since the fibers of the cotangent bundle T^*X are complex vector spaces, there exists a natural action of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ on T^*X. We say that an analytic subset V of T^*X is *conic* if V is stable by this action of $\mathbb{C}^\times$. In this subsection we focus our attention on conic Lagrangian analytic subsets of T^*X.

First let us examine the image $H(\alpha_X)$ of the canonical 1-form α_X by the Hamiltonian isomorphism $H : T^*(T^*X) \simeq T(T^*X)$. In a local coordinate $(x_1, x_2, \dots, x_n; \xi_1, \xi_2, \dots, \xi_n)$ of T^*X the holomorphic vector field $H(\alpha_X)$ thus obtained has the form

$$H(\alpha_X) = -\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}.$$

It follows that $-H(\alpha_X)$ is the infinitesimal generator of the action of $\mathbb{C}^\times$ on T^*X. We call this vector field the *Euler vector field*.

Lemma E.3.1. *Let V be a conic complex submanifold of T^*X. Then V is isotropic if and only if the pull-back $\alpha_X|_V$ of α_X to V is identically zero.*

Proof. Assume that $\alpha_X|_V$ is identically zero on V. Then also the pull-back of the symplectic 2-form $\sigma_X = d\alpha_X$ to V vanishes. Hence V is isotropic. Let us prove the converse. Assume that V is isotropic. Then for any local section δ of the tangent bundle $TV \to V$ we have

$$\langle \alpha_X, \delta \rangle = \sigma_X(\delta, H(\alpha_X)) = 0$$

on V because the Euler vector field $-H(\alpha_X)$ is tangent to V by the conicness of V. This means that $\alpha_X|_V$ is identically zero on V. □

Corollary E.3.2. *Let Λ be a conic Lagrangian analytic subset of T^*X. Then the pull-back of α_X to the regular part Λ_{reg} of Λ is identically zero.*

Let Z be an analytic subset of the base space X and denote by $\overline{T^*_{Z_{\mathrm{reg}}}X}$ the closure of the conormal bundle $T^*_{Z_{\mathrm{reg}}}X$ of Z_{reg} in T^*X. Since the closure is taken with respect to the classical topology of T^*X, it is not clear if $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X or not. In Proposition E.3.5 below, we will prove the analyticity of $\overline{T^*_{Z_{\mathrm{reg}}}X}$. For this purpose, recall the following definitions.

Definition E.3.3. Let S be an analytic space. A locally finite partition $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S by locally closed complex manifolds S_α's is called a *stratification* of S if for each S_α the closure $\overline{S_\alpha}$ and the boundary $\partial S_\alpha = \overline{S_\alpha} \setminus S_\alpha$ are analytic and unions of S_β's. A complex manifold S_α in it is called a *stratum* of the stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$.

Definition E.3.4. Let S be an analytic space. Then we say that a subset S' of S is *constructible* if there exists stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S such that S' is a union of some strata in it.

For an analytic space S the family of constructible subsets of S is closed under various set-theoretical operations. Note also that by definition the closure of a constructible subset is analytic. Moreover, if $f : S \to S'$ is a morphism (resp. proper morphism) of analytic spaces, then the inverse (resp. direct) image of a constructible subset of S' (resp. S) by f is again constructible. Now we are ready to prove the following.

Proposition E.3.5. *$\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X.*

Proof. We may assume that Z is irreducible. By Hironaka's theorem there exists a proper holomorphic map $f : Y \to X$ from a complex manifold Y and an analytic subset $Z' \neq Z$ of Z such that $f(Y) = Z$, $Z_0 := Z \setminus Z'$ is smooth, and the restriction of f to $Y_0 := f^{-1}(Z_0)$ induces a biholomorphic map $f|_{Y_0} : Y_0 \simeq Z_0$. From $f : Y \to X$ we obtain the canonical morphisms

$$T^*Y \xleftarrow{\rho_f} Y \times_X T^*X \xrightarrow{\varpi_f} T^*X.$$

We easily see that $T^*_{Z_0}X = \varpi_f \rho_f^{-1}(T^*_{Y_0}Y_0)$, where $T^*_{Y_0}Y_0 \simeq Y_0$ is the zero-section of $T^*Y_0 \subset T^*Y$. Since $T^*_{Y_0}Y_0$ is a constructible subset of T^*Y and ϖ_f is proper, $T^*_{Z_0}X$ is a constructible subset of T^*X. Hence the closure $\overline{T^*_{Z_0}X} = \overline{T^*_{Z_{\mathrm{reg}}}X}$ is an analytic subset of T^*X. □

By Example E.2.2 and Proposition E.3.5, for an irreducible analytic subset Z of X we conclude that the closure $\overline{T^*_{Z_{\mathrm{reg}}}X}$ is an irreducible conic Lagrangian analytic subset of T^*X. The following result, which was first proved by Kashiwara [Kas3], [Kas8], shows that any irreducible conic Lagrangian analytic subset of T^*X is obtained in this way.

Theorem E.3.6 (Kashiwara [Kas3], [Kas8]). *Let Λ be a conic Lagrangian analytic subset of T^*X. Assume that Λ is irreducible. Then $Z = \pi(\Lambda)$ is an irreducible analytic subset of X and $\Lambda = \overline{T^*_{Z_{\mathrm{reg}}}X}$.*

Proof. Since Λ is conic, $Z = \pi(\Lambda) = (T^*_X X) \cap \Lambda$ is an analytic subset of X. Moreover, by definition we easily see that Z is irreducible. Denote by Λ_0 the open subset of $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda_{\mathrm{reg}}$ consisting of points where the map $\pi|_{\Lambda_{\mathrm{reg}}}$ has the maximal rank. Then Λ_0 is open dense in $\pi^{-1}(Z_{\mathrm{reg}}) \cap \Lambda$ and the maximal rank is equal to $\dim Z$. Now let p be a point in Λ_0. Taking a local coordinate $(x_1, x_2, \dots, x_n)$ of X around the point $\pi(p) \in Z_{\mathrm{reg}}$, we may assume that $Z = \{x_1 = x_2 = \cdots = x_d = 0\}$ where $d = n - \dim Z$. Let us choose a local section $s : Z_{\mathrm{reg}} \hookrightarrow \Lambda_{\mathrm{reg}}$ of $\pi|_{\Lambda_{\mathrm{reg}}}$ such that $s(\pi(p)) = p$. Let $i_{\Lambda_{\mathrm{reg}}} : \Lambda_{\mathrm{reg}} \hookrightarrow T^*X$ be the embedding. Then by Corollary E.3.2 the pull-back of the canonical 1-form α_X to Z_{reg} by $i_{\Lambda_{\mathrm{reg}}} \circ s$ is zero. On the other hand, this 1-form on Z_{reg} has the form $\xi_{d+1}(x')dx_{d+1} + \cdots + \xi_n(x')dx_n$, where we set $x' = (x_{d+1}, \dots, x_n)$. Therefore, the point p should be contained in $\{\xi_{d+1} = \cdots = \xi_n = 0\}$. We proved that $\Lambda_0 \subset T^*_{Z_{\mathrm{reg}}}X$. Since $\dim \Lambda_0 = \dim T^*_{Z_{\mathrm{reg}}}X = n$ we obtain $\overline{T^*_{Z_{\mathrm{reg}}}X} = \overline{\Lambda_0} \subset \Lambda$. Then the result follows from the irreducibility of Λ. □

To treat general conic Lagrangian analytic subsets of T^*X let us briefly explain Whitney stratifications.

Definition E.3.7. Let S be an analytic subset of a complex manifold M. A stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ of S is called a *Whitney stratification* if it satisfies the following Whitney conditions (a) and (b):

(a) Assume that a sequence $x_i \in S_\alpha$ of points converges to a point $y \in S_\beta$ $(\alpha \neq \beta)$ and the limit T of the tangent spaces $T_{x_i}S_\alpha$ exists. Then we have $T_y S_\beta \subset T$.
(b) Let $x_i \in S_\alpha$ and $y_i \in S_\beta$ be two sequences of points which converge to the same point $y \in S_\beta$ $(\alpha \neq \beta)$. Assume further that the limit l (resp. T) of the lines l_i jointing x_i and y_i (resp. of the tangent spaces $T_{x_i}S_\alpha$) exists. Then we have $l \subset T$.

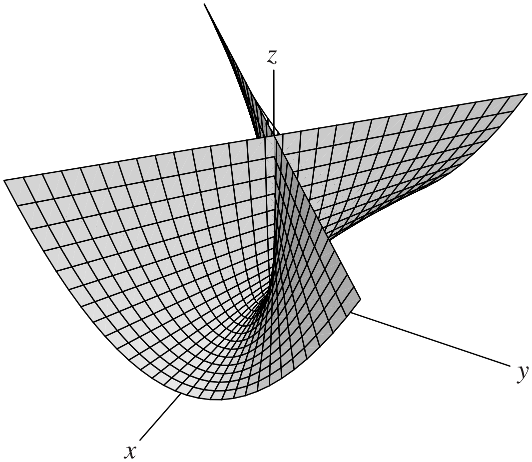
It is well known that any stratification of an analytic set can be refined to satisfy the Whitney conditions. Intuitively, the Whitney conditions means that the geometrical normal structure of the stratification $S = \bigsqcup_{\alpha \in A} S_\alpha$ is locally constant along each stratum S_α as is illustrated in the example below.

Example E.3.8 (Whitney's umbrella). Consider the analytic set $S = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 = zx^2\}$ in $\mathbb{C}^3$ and the following two stratifications $S = \bigsqcup_{i=1}^{2} S'_i$ and $S = \bigsqcup_{i=1}^{3} S_i$ of S:

$$\begin{cases} S'_1 = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \\ S'_2 = S \setminus S'_1 \end{cases}$$

$$\begin{cases} S_1 = \{0\} \\ S_2 = \{(x, y, z) \in \mathbb{C}^3 \mid x = y = 0\} \setminus S_1 \\ S_3 = S \setminus (S_1 \sqcup S_2) \end{cases}$$

Then the stratification $S = \bigsqcup_{i=1}^{3} S_i$ satisfies the Whitney conditions (a) and (b), but the stratification $S = \bigsqcup_{i=1}^{2} S'_i$ does not.



We see that along each stratum S_i $(i = 1, 2, 3)$, the geometrical normal structure of $S = \bigsqcup_{i=1}^{3} S_i$ is constant.

Now consider a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of a complex manifold X. Then it is a good exercise to prove that for each point $x \in X$ there exists a sufficiently small sphere centered at x which is transversal to all the strata X_α's. This result follows easily from the Whitney condition (b). For the details see [Kas8], [Schu]. Moreover, by the Whitney conditions we can prove easily that the union $\bigsqcup_{\alpha \in A} T^*_{X_\alpha} X$ of the conormal bundles $T^*_{X_\alpha} X$ is a closed analytic subset of T^*X (the analyticity follows from Proposition E.3.5). The following theorem was proved by Kashiwara [Kas3], [Kas8] and plays a crucial role in proving the constructibility of the solutions to holonomic D-modules.

Theorem E.3.9. *Let X be a complex manifold and Λ a conic Lagrangian analytic subset of T^*X. Then there exists a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that*

$$\Lambda \subset \bigsqcup_{\alpha \in A} T^*_{X_\alpha} X.$$

Proof. Let $\Lambda = \cup_{i \in I} \Lambda_i$ be the irreducible decomposition of Λ and set $Z_i = \pi(\Lambda_i)$. Then we can take a Whitney stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X such that Z_i is a union of strata in it for any $i \in I$. Note that for each $i \in I$ there exists a (unique) stratum $X_{\alpha_i} \subset (Z_i)_{\mathrm{reg}}$ which is open dense in Z_i. Hence we have $\Lambda_i = \overline{T^*_{(Z_i)_{\mathrm{reg}}} X} = \overline{T^*_{X_{\alpha_i}} X}$ by Theorem E.3.6 and $\Lambda = \cup_{i \in I} \Lambda_i \subset \sqcup_{\alpha \in A} T^*_{X_\alpha} X$. □

References

[AM] R. Abraham and J. E. Mardsen, *Foundation of Mechanics*, Benjamin/Cummings, Reading, MA, 1978.

[AJS] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of Quantum Groups at a pth Root of Unity and of Semisimple Groups in Characteristic p: Independence of p*, Astérisque, Vol. 220, Société Mathématique de France, Paris, 1994.

[An1] E. Andronikof, A microlocal version of the Riemann–Hilbert correspondence, *Topol. Methods Nonlinear Anal.*, **4**-2 (1994), 417–425.

[An2] E. Andronikof, *Microlocalisation tempérée*, Mémoires de la Société Mathématique de France (N.S.), Vol. 57, Société Mathématique de France, Paris, 1994.

[AB] D. V. Anosov and A. A. Bolibruch, *The Riemann–Hilbert Problem*, Aspects of Mathematics, Vol. E22, Vieweg, Braunschweig, Germany, 1994.

[AK] K. Aomoto and M. Kita, *The Theory of Hypergeometric Functions*, Springer-Verlag, Tokyo, 1994 (in Japanese).

[Ber1] J. Bernstein, Modules over a ring of differential operators: An investigation of the fundamental solutions of equations with constant coefficients, *Functional Anal. Appl.*, **5** (1971), 89–101.

[Ber2] J. Bernstein, Analytic continuation of generalized functions with respect to a parameter, *Functional Anal. Appl.*, **6** (1972), 273–285.

[Ber3] J. Bernstein, *Algebraic Theory of D-Modules*, unpublished notes.

[BL] J. Bernstein and V. Lunts, *Equivariant Sheaves and Functors*, Lecture Notes in Mathematics, Vol. 1578, Springer-Verlag, Berlin, 1994.

[Bei] A. A. Beilinson, On the derived category of perverse sheaves, in *K-Theory, Arithmetic and Geometry* (*Moscow*, 1984–1986), Lecture Notes in Mathematics, Vol. 1289, Springer-Verlag, Berlin, 1987, 27–41.

[BB] A. A. Beilinson and J. Bernstein, Localisation de $\mathfrak{g}$-modules, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **292**-1 (1981), 15–18.

[BB2] A. A. Beilinson and J. Bernstein, A proof of Jantzen conjectures, *Adv. Soviet Math.*, **16** (1993) (I. M. Gel′fand Seminar), 1–50.

[BBD] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces* I (*Luminy*, 1981), Astérisque, Vol. 100, Société Mathématique de France, Paris, 1982, 5–171.

[Bj1] J.-E. Björk, *Rings of Differential Operators*, North-Holland Mathematical Library, Vol. 21, North-Holland, Amsterdam, 1979.

[Bj2] J.-E. Björk, *Analytic D-Modules and Applications*, Mathematics and Its Applications, Vol. 247, Kluwer Academic Publishers, Dordrecht, Netherlands, 1993.

[BS] J.-M. Bony and P. Schapira, Existence et prolongement des solutions holomorphes des équations aux dérivées partielles, *Invent. Math.*, **17** (1972), 95–105.

[Bor1] A. Borel, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Math., Vol. 126, Springer-Verlag, New York, 1991.

[Bor2] A. Borel (ed.), *Intersection Cohomology* (*Bern*, 1983), Progress in Mathematics, Vol. 50, Birkhäuser Boston, Cambridge, MA, 1984.

[Bor3] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-Modules*, Perspectives in Mathematics, Vol. 2, Academic Press, Boston, 1987.

[BM] W. Borho and R. MacPherson, Partial resolutions of nilpotent varieties, in *Analysis and Topology on Singular Spaces* II, III (*Luminy*, 1981), Astérisque, Vol. 101, Société Mathématique de France, Paris, 1983, 23–74.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris, 1972, Chapters 4–6.

[BMM] J. Briançon, P. Maisonobe, and M. Merle, Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom, *Invent. Math.*, **117**-3 (1994), 531–550.

[Br] J.-L. Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques, in *Géométrie et analyse microlocales*, Astérisque, Vols. 140 and 141, Société Mathématique de France, Paris, 1986, 3–134.

[BK] J.-L. Brylinski and M. Kashiwara, Kazhdan–Lusztig conjecture and holonomic systems, *Invent. Math.*, **64**-3 (1981), 387–410.

[BMV] J.-L. Brylinski, B. Malgrange, and J.-L. Verdier, Transformation de Fourier géométrique I, II, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **297** (1983), 55–58 and **303** (1986), 193–198.

[Ca1] L. Casian, Kazhdan–Lusztig multiplicity formulas for Kac–Moody algebras, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **310**-6 (1990), 333–337.

[Ca2] L. Casian, Proof of the Kazhdan–Lusztig conjecture for Kac–Moody algebras (the characters ch $L_\omega\rho - \rho$), *Adv. Math.*, **119**-2 (1996), 207–281.

[Ch] C. Chevalley, *Séminaire sur la classification des groupes de Lie algébriques*, École Normale Supérieure, Paris, 1956–1958.

[CG] N. Chriss and V. Ginzburg, *Representation Theory and Complex Geometry*, Birkhäuser Boston, Cambridge, MA, 1997.

[Co] V. Colin, Specialization of formal cohomology and asymptotic expansions, *Publ. Res. Inst. Math. Sci.*, **37**-1 (2001), 37–69.

[D] A. D'Agnolo, Radon transform and the Cavalieri condition: A cohomological approach, *Duke Math. J.*, **93**-3 (1998), 597–632.

[DS1] A. D'Agnolo and P. Schapira, An inverse image theorem for sheaves with applications to the Cauchy problem, *Duke Math. J.*, **64**-3 (1991), 451–472.

[DS2] A. D'Agnolo and P. Schapira, Radon–Penrose transform for *D*-modules, *J. Functional Anal.*, **139**-2 (1996), 349–382.

[De1] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970.

[De2] P. Deligne, Théorie de Hodge I, Actes du Congrés International des Mathématiciens (Nice 1970) 425–430 II, III, *Publ. Math. IHES*, **40** (1971), 5–57 and **44** (1974), 5–77.

[De3] P. Deligne, La conjecture de Weil I, II, *Publ. Math. IHES*, **43** (1974), 273–307 and **52** (1980), 137–252.

[Di] A. Dimca, *Sheaves in Topology*, Universitext, Springer-Verlag, Berlin, 2004.

[Dr] V. Drinfeld, On a conjecture of Kashiwara, *Math. Res. Lett.*, **8**-5–6 (2001), 713–728.

[Du] A. S. Dubson, Formule pour l'indice des complexes constructibles et des Modules holonomes, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **298**-6 (1984), 113–116.

[Dui] J. J. Duistermaat, *Fourier Integral Operators*, Lecture Notes, Courant Institute, New York, 1973.

[Dy] E. B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *Amer. Math. Soc. Transl.*, **6** (1957), 111–244.

[F] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 2, Springer-Verlag, Berlin, 1984.

[Ga] O. Gabber, The integrability of the characteristic variety, *Amer. J. Math.*, **103**-3 (1981), 445–468.

[Gai] D. Gaitsgory, *On de Jong's Conjecture*, preprint, math.AG/0402184.

[GGM] A. Galligo, M. Granger, and P. Maisonobe, D-modules et faisceaux pervers dont le support singulier est un croisement normal, *Ann. Inst. Fourier* (*Grenoble*), **35**-1 (1985), 1–48.

[GKZ] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Generalized Euler integrals and A-hypergeometric functions, *Adv. Math.*, **84**-2 (1990), 255–271.

[GeM] S. I. Gelfand and Y. Manin, *Methods of Homological Algebra* I: *Introduction to Cohomology Theory and Derived Categories*, Springer-Verlag, Berlin, 1991.

[GMV] S. I. Gelfand, R. MacPherson, and K. Vilonen, Perverse sheaves and quivers, *Duke Math. J.*, **83**-3 (1996), 621–643.

[GMV2] S. I. Gelfand, R. MacPherson, and K. Vilonen, *Microlocal Perverse Sheaves*, preprint, math.AG/0509440.

[Gi] V. Ginsburg, Characteristic varieties and vanishing cycles, *Invent. Math.*, **84**-2 (1986), 327–402.

[Gon] A. B. Goncharov, Differential equations and integral geometry, *Adv. Math.*, **131**-2 (1997), 279–343.

[GM1] M. Goresky and R. MacPherson, Intersection homology II, *Invent. Math.*, **72**-1 (1983), 77–129.

[GM2] M. Goresky and R. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 14, Springer-Verlag, Berlin, 1988.

[Go] L. Göttsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, *Math. Ann.*, **286**-1–3 (1990), 193–207.

[Gri] P. A. Griffiths, Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems, *Bull. Amer. Math. Soc.*, **76** (1970), 228–296.

[GS] P. A. Griffiths and W. Schmid, Recent developments in Hodge theory: A discussion of techniques and results, in *Discrete Subgroups of Lie Groups and Applicatons to Moduli* (*International Colloquium, Bombay*, 1973), Oxford University Press, Bombay, 1975, 31–127.

[Gr] I. Grojnowski, Instantons and affine algebras I: The Hilbert scheme and vertex operators, *Math. Res. Lett.*, **3**-2 (1996), 275–291.

[Gro1] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tôhoku Math. J.* (2), **9** (1957), 119–221.

[Gro2] A. Grothendieck, Eléments de géométrie algébrique, EGA (avec J. Dieudonné), *Publ. Math. IHES*, **4** (1960), **8**, **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[Gro3] A. Grothendieck, L'enterrement (1): Ou la robe de l'empereur de Chine, in *Recoltes et semailles: Réflexions et témoignage sur un passé de mathématicien*, Université des Sciences et Techniques du Languedoc, Montpellier, France, 1985.

[G1] S. Guillermou, *Introduction aux faisceaux pervers*, personal note.

[Gui] S. Guillermou, Index of transversally elliptic D-modules, *Ann. Sci. École Norm. Sup.* (4) **34**-2 (2001), 223–265.

[Ha1] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics, Vol. 20, Springer-Verlag, Berlin, 1966 (lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64; with an appendix by P. Deligne).

[Ha2] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York, 1977.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, *Ann. Math.* (2), **79** (1964), 109–203 **79** (1964), 205–326.

[Hr] E. Horikawa, *Introduction to Complex Algebraic Geometory*, Iwanami Shoten, Tokyo, 1990 (in Japanese).

[Ho1] R. Hotta, Holonomic D-modules in representation theory, in *The Arcata Conference on Representations of Finite Groups* (*Arcata, CA*, 1986), Proceedings of Symposia in Pure Mathematics, Vol. 47, American Mathematical Society, Providence, RI, 1987, 87–102.

[Ho2] R. Hotta, *Introduction to Algebra: Groups and Modules*, Shōkabō, Tokyo, 1987 (in Japanese).

[Ho3] R. Hotta, *Introduction to D-Modules*, Lecture Notes Series, Vol. 1, Institute of Mathematical Science, Madras, India, 1987.

[Ho4] R. Hotta, *Equivariant D-Modules*, Travaux en Cours, Hermann, Paris, 1995.

[HK1] R. Hotta and M. Kashiwara, The invariant holonomic system on a semisimple Lie algebra, *Invent. Math.*, **75**-2 (1984), 327–358.

[HK2] R. Hotta and M. Kashiwara, Quotients of the Harish-Chandra system by primitive ideals, in *Geometry Today* (*Rome*, 1984), Progress in Mathematics, Vol. 60, Birkhäuser Boston, Cambridge, MA, 1985, 185–205.

[Hu1] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Vol. 9, Springer-Verlag, New York, 1972.

[Hu2] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, Vol. 21, Springer-Verlag, New York, 1975.

[Is] R. Ishimura, The Cauchy–Kowalevski theorem for E-modules, *J. Math. Pures Appl.* (9), **77**-7 (1998), 647–654.

[Iv] B. Iversen, *Cohomology of Sheaves*, Springer-Verlag, Berlin, 1987.

[J] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, Vol. 750, Springer-Verlag, Berlin, 1979.

[Kas1] M. Kashiwara, *Algebraic Study of Partial Differential Equations*, Master's thesis, University of Tokyo, Tokyo, 1970 (in Japanese).

[Kas2] M. Kashiwara, Index theorem for a maximally overdetermined system of linear differential equations, *Proc. Japan Acad.*, **49** (1973), 803–804.

[Kas3] M. Kashiwara, On the maximally overdetermined system of linear differential equations I, *Publ. Res. Inst. Math. Sci.*, **10** (1974–1975), 563–579.

[Kas4] M. Kashiwara, B-functions and holonomic systems: Rationality of roots of B-functions, *Invent. Math.*, **38**-1 (1976–1977), 33–53.

[Kas5] M. Kashiwara, On the holonomic systems of linear differential equations II, *Invent. Math.*, **49**-2 (1978), 121–135.

[Kas6] M. Kashiwara, Faisceaux constructibles et systèmes holonômes d'équations aux dérivées partielles linéaires à points singuliers réguliers, in *Séminaire Goulaouic-Schwartz*, Exp. 19, 1979–1980, École Polytechnique, Palaiseau, France, 1980 (in French).

[Kas7] M. Kashiwara, Holonomic systems of linear differential equations with regular singularities and related topics in topology, in *Algebraic Varieties and Analytic Varieties* (*Tokyo*, 1981), Advanced Studies in Pure Mathematics, Vol. 1, Elsevier, Amsterdam, 1983, 49–54.

[Kas8] M. Kashiwara, *Systems of Microdifferential Equations*, Progress in Mathematics, Vol. 34, Birkhäuser Boston, Cambridge, MA, 1983 (based on lecture notes by Teresa Monteiro Fernandes; translated from the French).

[Kas9] M. Kashiwara, Vanishing cycle sheaves and holonomic systems of differential equations, in *Algebraic Geometry* (*Tokyo/Kyoto*, 1982), Lecture Notes in Mathematics, Vol. 1016, Springer-Verlag, Berlin, 1983, 134–142.

[Kas10] M. Kashiwara, The Riemann–Hilbert problem for holonomic systems, *Publ. Res. Inst. Math. Sci.*, **20**-2 (1984), 319–365.

[Kas11] M. Kashiwara, Index theorem for constructible sheaves, in *Differential Systems and Singularities* (*Luminy*, 1983), Astérisque, Vol. 130, Société Mathématique de France, Paris, 1985, 193–209.

[Kas12] M. Kashiwara, *Introduction to Microlocal Analysis*, Monographies de L'Enseignement Mathématique, Vol. 32, Série des Conférences de l'Union Mathématique Internationale, Vol. 7, L'Enseignement Mathématique, Geneva, 1986.

[Kas13] M. Kashiwara, A study of variation of mixed Hodge structure, *Publ. Res. Inst. Math. Sci.*, **22**-5 (1986), 991–1024.

[Kas14] M. Kashiwara, Representation theory and D-modules on flag varieties, in *Orbites unipotentes et représentations* III, Astérisque, Vols. 173 and 174, Société Mathématique de France, Paris, 1989, 55–109.

[Kas15] M. Kashiwara, Kazhdan–Lusztig conjecture for a symmetrizable Kac–Moody Lie algebra, in *The Grothendieck Festschrift*, Vol. II, Progress in Mathematics, Vol. 87, Birkhäuser Boston, Cambridge, MA, 1990, 407–433.

[Kas16] M. Kashiwara, *Algebraic Study of Systems of Partial Differential Equations*, Master's thesis, University of Tokyo, Tokyo, 1970; Mémoires de la Société Mathématique de France (N.S.), Vol. 63, Société Mathématique de France, Paris, 1995 (translated by A. D'Agnolo and J.-P. Schneiders).

[Kas17] M. Kashiwara, Semisimple holonomic D-modules, in *Topological Field Theory, Primitive Forms and Related Topics* (*Kyoto*, 1996), Progress in Mathematics, Vol. 160, Birkhäuser Boston, Cambridge, MA, 1998, 267–271.

[Kas18] M. Kashiwara, *D-Modules and Microlocal Calculus*, Iwanami Series in Modern Mathematics, Iwanami Shoten, Tokyo, 2000 (in Japanese); Translations of Mathematical Monographs, Vol. 217, American Mathematical Society, Providence, RI, 2003 (in English; translated by Mutsumi Saito).

[Kas19] M. Kashiwara, t-structures on the derived categories of D-modules and O-modules, *Moscow Math. J.*, **4**-4 (2004), 847–868.

[KK1] M. Kashiwara and T. Kawai, On the characteristic variety of a holonomic system with regular singularities, *Adv. Math.*, **34**-2 (1979), 163–184.

[KK2] M. Kashiwara and T. Kawai, Second-microlocalization and asymptotic expansions, in *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory* (*Proceedings of the International Colloquium, Centre Physique, Les Houches*, 1979), Lecture Notes in Physics, Vol. 126, Springer-Verlag, Berlin, 1980, 21–76.

[KK3] M. Kashiwara and T. Kawai, On holonomic systems of microdifferential equations III: Systems with regular singularities, *Publ. Res. Inst. Math. Sci.*, **17**-3 (1981), 813–979.

[KK4] M. Kashiwara and T. Kawai, Microlocal analysis, *Publ. Res. Inst. Math. Sci.*, **19**-3 (1983), 1003–1032.

[KKK] M. Kashiwara, T. Kawai, and T. Kimura, *Foundations of Algebraic Analysis*, Princeton Mathematical Series, Vol. 37, Princeton University Press, Princeton, NJ, 1986 (translated from the Japanese by Goro Kato).

[six] M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, Eigenfunctions of invariant differential operators on a symmetric space, *Ann. Math.* (2), **107**-1 (1978), 1–39.

[KFS] M. Kashiwara, T. Monteiro Fernandes, and P. Schapira, Truncated microsupport and holomorphic solutions of D-modules, *Ann. Sci. École Norm. Sup.* (4), **36**-4 (2003), 583–599.

[KO] M. Kashiwara and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, *Ann. Math.* (2), **106**-1 (1977), 145–200.

[KS1] M. Kashiwara and P. Schapira, *Microlocal Study of Sheaves*, Astérisque, Vol. 128, Société Mathématique de France, Paris, 1985.

[KS2] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, Vol. 292, Springer-Verlag, Berlin, 1990.

[KS3] M. Kashiwara and P. Schapira, *Moderate and Formal Cohomology Associated with Constructible Sheaves*, Mémoires de la Société Mathématique de France (N.S.), Vol. 64, Société Mathématique de France, Paris, 1996.

[KS4] M. Kashiwara and P. Schapira, *Categories and Sheaves*, Grundlehren der Mathematischen Wissenschaften, Vol. 332, Springer-Verlag, Berlin, 2006.

[KT1] M. Kashiwara and T. Tanisaki, The characteristic cycles of holonomic systems on a flag manifold related to the Weyl group algebra, *Invent. Math.*, **77**-1 (1984), 185–198.

[KT2] M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for symmetrizable Kac–Moody Lie algebra II: Intersection cohomologies of Schubert varieties, in *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory* (*Paris*, 1989), Progress in Mathematics, Vol. 92, Birkhäuser Boston, Cambridge, MA, 1990, 159–195.

[KT3] M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for affine Lie algebras with negative level, *Duke Math. J.*, **77**-1 (1995), 21–62.

[KT4] M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for affine Lie algebras with negative level II: Nonintegral case, *Duke Math. J.*, **84**-3 (1996), 771–813.

[KT5] M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for symmetrizable Kac–Moody Lie algebras III: Positive rational case, *Asian J. Math.*, **2**-4 (1998), 779–832.

[KT6] M. Kashiwara and T. Tanisaki, On Kazhdan–Lusztig conjectures, *Sūgaku*, **47**-3 (1995), 269–285; *Sugaku Expositions*, **11**-2 (1998), 177–195.

[KT7] M. Kashiwara and T. Tanisaki, Characters of irreducible modules with non-critical highest weights over affine Lie algebras, in *Representations and Quantizations* (*Shanghai*, 1998), China Higher Education Press, Beijing, 2000, 275–296.

[Kat] N. M. Katz, An overview of Deligne's work on Hilbert's twenty-first problem, in *Mathematical Developments Arising from Hilbert Problems* (*Proceedings of Symposia in Pure Mathematics, Vol.* XXVIII, *Northern Illinois University, De Kalb, IL*, 1974), American Mathematical Society, Providence, RI, 1976, 537–557.

[KL1] D. Kazhdan and G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.*, **53**-2 (1979), 165–184.

[KL2] D. Kazhdan and G. Lusztig, Schubert varieties and Poincaré duality, in *Geometry of the Laplace Operator* (*University of Hawaii, Honolulu, HI*, 1979), Proceedings of Symposia in Pure Mathematics, Vol. XXXVI, American Mathematical Society, Providence, RI, 1980, 185–203.

[KL3] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I–IV, *J. Amer. Math. Soc.*, **6** (1993), 905–947, 949–1011 and **7** (1994), 335–381, 383–453.

[Ki] F. Kirwan, *An Introduction to Intersection Homology Theory*, Pitman Research Notes in Mathematics Series, Vol. 187, Longman, Harlow, UK, 1988.

[Kos] B. Kostant, Lie group representations on polynomial rings, *Amer. J. Math.*, **85** (1963), 327–404.

[LM] D. T. Lê and Z. Mebkhout, Introduction to linear differential systems, in *Singularities, Part* 2 (*Arcata, CA*, 1981), Proceedings of Symposia in Pure Mathematics, Vol. 40, American Mathematical Society, Providence, RI, 1983, 31–63.

[L1] G. Lusztig, Green polynomials and singularities of unipotent classes, *Adv. Math.*, **42**-2 (1981), 169–178.

[L2] G. Lusztig, Intersection cohomology methods in representation theory, in I. Satake, ed., *Proceedings of the International Congress of Mathematicians* (*Kyoto* 1990), Springer-Verlag, 1991, 155–174.

[Mac] R. MacPherson, Chern classes for singular algebraic varieties, *Ann. Math.* (2), **100** (1974), 423–432.

[MS] P. Maisonobe and C. Sabbah, *D-modules cohérents et holonomes* (*Les cours du CIMPA*), Travaux en Cours, Vols. 45 and 46, Hermann, Paris, 1993.

[Ma1] B. Malgrange, *Intégrales asymptotiques et monodromie*, *Ann. Sci. École Norm. Sup.* (4), **7** (1974), 405–430 (1975).

[Ma2] B. Malgrange, On I. N. Bernšteĭn's polynomials, in *Séminaire Goulaouic-Schwartz* 1973–1974: *Equations aux dériveés partielles et analyse fonctionelle*, Exp. 20, Centre de Mathématiques, École Polytechnique, Paris, 1974 (in French); *Uspehi Mat. Nauk*, **29**-4 (1974) (volume II of a collection of articles dedicated to the memory of Ivan Georgievič Petrovskiĭ), 81–88 (translated by A. G. Kušnirenko).

[Ma3] B. Malgrange, Sur les points singuliers des équations différentielles, *Enseign. Math.* (2), **20** (1974), 147–176.

[Ma4] B. Malgrange, Le polynôme de Bernstein d'une singularité isolée, in *Fourier Integral Operators and Partial Differential Equations* (*International Colloquium, University of Nice, Nice*, 1974), Lecture Notes in Mathematics, Vol. 459, Springer-Verlag, Berlin, 1975, 98–119.

[Ma5] B. Malgrange, L'involutivité des caractéristiques des systèmes différentiels et microdifférentiels, in *Séminaire Bourbaki*, Exp. 522 (1977–1978), Lecture Notes in Mathematics, Vol. 710, Springer-Verlag, Berlin, 1979, 277–289.

[Ma6] B. Malgrange, La classification des connexions irrégulières à une variable, in *Mathematics and Physics* (*Paris*, 1979/1982), Progress in Mathematics, Vol. 37, Birkhäuser Boston, Cambridge, MA, 1983, 381–399.

[Ma7] B. Malgrange, Polynômes de Bernstein–Sato et cohomologie évanescente, in *Analysis and Topology on Singular Spaces* II, III (*Luminy*, 1981), Astérisque, Vol. 101, Société Mathématique de France, Paris, 1983, 243–267.

[Ma8] B. Malgrange, *Équations différentielles à coefficients polynomiaux*, Progress in Mathematics, Vol. 96, Birkhäuser Boston, Cambridge, MA, 1991.

[Mar] C. Marastoni, Grassmann duality for *D*-modules, *Ann. Sci. École Norm. Sup.* (4), **31**-4 (1998), 459–491.

[MT] C. Marastoni and T. Tanisaki, Radon transforms for quasi-equivariant *D*-modules on generalized flag manifolds, *Differential Geom. Appl.*, **18**-2 (2003), 147–176.

[Mas1] D. B. Massey, The Sebastiani–Thom isomorphism in the derived category, *Compositio Math.*, **125**-3 (2001), 353–362.

[Mas2] D. B. Massey, Numerical control over complex analytic singularities, *Mem. Amer. Math. Soc.*, **163**-778 (2003), no. 778.

[Mat] T. Matsuki, The orbits of affine symmetric spaces under the action of minimal parabolic subgroups, *J. Math. Soc. Japan*, **31**-2 (1979), 331–357.

[Matm] H. Matsumura, *Commutative Algebra*, 2nd ed., Mathematics Lecture Note Series, Vol. 56, Benjamin/Cummings, Reading, MA, 1980.

[Me1] Z. Mebkhout, Sur le problème de Hilbert–Riemann, in *Complex Analysis, Microlocal Calculus and Relativistic Quantum Theory* (*Proceedings of the International Colloquium, Centre Physique, Les Houches*, 1979), Lecture Notes in Physics, Vol. 126, Springer-Verlag, Berlin, 1980, 90–110.

[Me2] Z. Mebkhout, Théorèmes de bidualité locale pour les D_X-modules holonomes, *Ark. Mat.*, **20**-1 (1982), 111–124.

[Me3] Z. Mebkhout, Théorèmes de dualité globale pour les D_X-modules cohérents, *Math. Scand.*, **50**-1 (1982), 25–43.

[Me4] Z. Mebkhout, Une équivalence de catégories, *Compositio Math.*, **51**-1 (1984), 51–62; Une autre équivalence de catégories, *Compositio Math.*, **51**-1 (1984), 63–88.

[Me5] Z. Mebkhout, *Le formalisme des six opérations de Grothendieck pour les D_X-modules cohérents*, Travaux en Cours, Vol. 35, Hermann, Paris, 1989.

[Me6] Z. Mebkhout, Le théorème de comparaison entre cohomologies de de Rham d'une variété algébrique complexe et le théorème d'existence de Riemann, *Publ. Math. IHES*, **69** (1989), 47–89.

[MV] I. Mirković and K. Vilonen, Perverse sheaves on affine Grassmannians and Langlands duality, *Math. Res. Lett.*, **7**-1 (2000), 13–24.

[Mn] M. Miyanishi, *Algebraic Geometry*, Shōkabō, Tokyo, 1990 (in Japanese).

[My] T. Miyata and A. H., eds., *Proceedings of the* 5*th Seminar on Algebra* II (*Hyogo*, 1982), 1983 (in Japanese).

[Mo] T. Mochizuki, *Asymptotic Behaviour of Tame Harmonic Bundles and an Application to Pure Twistor D-Modules*, preprint, math.DG/0312230.

[Mu] D. Mumford, *The Red Book of Varieties and Schemes*, Lecture Notes in Mathematics, Vol. 1358, Springer-Verlag, Berlin, 1988.

[Na1] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, *Ann. Math.* (2), **145**-2 (1997), 379–388.

[Na2] H. Nakajima, *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, Vol. 18, American Mathematical Society, Providence, RI, 1999.

[NT] P. Nang and K. Takeuchi, Characteristic cycles of perverse sheaves and Milnor fibers, *Math. Z.*, **249**-3 (2005), 493–511; addendum, *Math. Z.*, **250**-3 (2005), 729.

[N] M. Noumi, Regular holonomic systems and their minimal extensions I, in *Group Representations and Systems of Differential Equations* (*Tokyo*, 1982), Advanced Studies in Pure Mathematics, Vol. 4, Elsevier, Amsterdam, 1984, 209–221.

[Oa1] T. Oaku, *Gröbner Bases and Linear Partial Differetial Equations* (*Introduction to Computational Algebraic Analysis*), Lecture Notes in Mathematics, Vol. 38, Sophia University, Tokyo, 1994 (in Japanese).

[Oa2] T. Oaku, An algorithm of computing b-functions, *Duke Math. J.*, **87**-1 (1997), 115–132.

[O] T. Oda, Introduction to algebraic analysis on complex manifolds, in *Algebraic Varieties and Analytic Varieties* (*Tokyo*, 1981), Advanced Studies in Pure Mathematics, Vol. 1, North-Holland, Amsterdam, 1983, 29–48.

[P] F. Pham, *Singularités des systèmes différentiels de Gauss–Manin*, Progress in Mathematics, Vol. 2, Birkhäuser Boston, Cambridge, MA, 1979.

[Ri] R. W. Richardson, Jr., Conjugacy classes in Lie algebras and algebraic groups, *Ann. Math.* (2), **86** (1967), 1–15.

[R] J.-E. Roos, Détermination de la dimension homologique globale des algèbres de Weyl, *C. R. Acad. Sci. Paris Sér.* A–B, **274** (1972), A23–A26.

[Sab1] C. Sabbah, Quelques remarques sur la géométrie des espaces conormaux, in *Differential Systems and Singularities* (*Luminy*, 1983), Astérisque, Vol. 130, Société Mathématique de France, Paris, 1985, 161–192.

[Sab2] C. Sabbah, *Polarizable Twistor D-Modules*, Astérisque, Vol. 300, Société Mathématique de France, Paris, 2005.

[Sa1] M. Saito, Modules de Hodge polarisables, *Publ. Res. Inst. Math. Sci.*, **24**-6 (1988), 849–995.

[Sa2] M. Saito, Induced D-modules and differential complexes, *Bull. Soc. Math. France*, **117**-3 (1989), 361–387.

[Sa3] M. Saito, Mixed Hodge modules, *Publ. Res. Inst. Math. Sci.*, **26**-2 (1990), 221–333.

[SST] M. Saito, B. Sturmfels, and N. Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Algorithms and Computation in Mathematics, Vol. 6, Springer-Verlag, Berlin, 2000.

[SKKO] M. Sato, M. Kashiwara, T. Kimura, and T. Oshima, Microlocal analysis of prehomogeneous vector spaces, *Invent. Math.*, **62**-1 (1980–1981), 117–179.

[SKK] M. Sato, T. Kawai, and M. Kashiwara, Microfunctions and pseudo-differential equations, in *Hyperfunctions and Pseudo-Differential Equations* (*Katata*, 1971) (dedicated to the memory of André Martineau), Lecture Notes in Mathematics, Vol. 287, Springer-Verlag, Berlin, 1973, 265–529.

[Sc] P. Schapira, *Microdifferential Systems in the Complex Domain*, Grundlehren der Mathematischen Wissenschaften, Vol. 269, Springer-Verlag, Berlin, 1985.

[SS] P. Schapira and J.-P. Schneiders, *Elliptic Pairs* I: *Relative Finiteness and Duality*, Astérisque, Vol. 224, Société Mathématique de France, Paris, 1994, 5–60; *Elliptic Pairs* II: *Euler Class and Relative Index Theorem*, Astérisque, Vol. 224, Société Mathématique de France, Paris, 1994, 61–98.

[SV] W. Schmid and K. Vilonen, Characteristic cycles of constructible sheaves, *Invent. Math.*, **124**-1–3 (1996), 451–502.

[Sch] J.-P. Schneiders, Un théorème de dualité relative pour les modules différentiels, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **303**-3 (1986), 235–238.

[Schu] J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) (Mathematics Institute of the Polish Academy of Sciences Mathematical Monographs (New Series)), Vol. 63, Birkhäuser, Basel, Switzerland, 2003.

[Sek1] J. Sekiguchi, Regular holonomic systems and their minimal extensions II: Application to the multiplicity formula for Verma modules, in *Group Representations and Systems of Differential Equations* (*Tokyo*, 1982), Advanced Studies in Pure Mathematics, Vol. 4, North-Holland, Amsterdam, 1984, 243–253.

[Sek2] J. Sekiguchi, *Applications of Partial Differential Equations to Representation Theory*, Lecture Notes in Mathematics, Vol. 27, Sophia University, Tokyo, 1988 (in Japanese).

[Ser1] J.-P. Serre, Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* (*Grenoble*), **6** (1955–1956), 1–42.

[Ser2] J.-P. Serre, *Algèbre locale: Multiplicités* (Cours au Collège de France, 1957–1958; rédigé par Pierre Gabriel), 2nd ed., Lecture Notes in Mathematics, Vol. 11, Springer-Verlag, Berlin, 1965.

[Sp1] T. A. Springer, *Linear Algebraic Groups*, Progress in Mathematics, Vol. 9, Birkhäuser Boston, Cambridge, MA, 1981.

[Sp2] T. A. Springer, Quelques applications de la cohomologie d'intersection, in *Séminaire Bourbaki*, Exp. 589, Astérisque, Vol. 92–93, Société Mathématique de France, Paris, 1982, 410–431.

[Ta] N. Takayama, Software for rings and modules, *Sūgaku*, **45**-3 (1993), 256–263 (in Japanese).

[Tk1] K. Takeuchi, Dimension formulas for the hyperfunction solutions to holonomic D-modules, *Adv. Math.*, **180**-1 (2003), 134–145.

[Tk2] K. Takeuchi, Perverse sheaves and Milnor fibers over singular varieties, in J.-P. Brasselet and T. Suwa, eds., *Singularities in Geometry and Topology* 2004, Advanced Studies in Pure Mathematics, Vol. 46, Kinokuniya, Tokyo and American Mathematical Society, Providence, RI, 2007.

[T1] T. Tanisaki, Holonomic systems on a flag variety associated to Harish-Chandra modules and representations of a Weyl group, in *Algebraic Groups and Related Topics* (*Kyoto/Nagoya*, 1983), Advanced Studies in Pure Mathematics, Vol. 6, American Mathematical Society, Providence, RI, 1985, 139–154.

[T2] T. Tanisaki, Hodge modules, equivariant K-theory and Hecke algebras, *Publ. Res. Inst. Math. Sci.*, **23**-5 (1987), 841–879.

[Th] R. Thom, Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.*, **75** (1969), 240–284.

[V1] J.-L. Verdier, Classe d'homologie associée à un cycle, in *Séminaire de géométrie analytique* (*École Normale Supérieure, Paris*, 1974–75), Astérisque, Vol. 36–37, Société Mathématique de France, Paris, 1976, 101–151.

[V2] J.-L. Verdier, Catégories dérivées, in *Cohomologie etale* (*SGA* $4\frac{1}{2}$), Lecture Notes in Mathematics, Vol. 569, Springer-Verlag, Berlin, 1977, 262–311.

[V3] J.-L. Verdier, Géométrie microlocale, in *Algebraic Geometry* (*Tokyo/Kyoto*, 1982), Lecture Notes in Mathematics, Vol. 1016, Springer-Verlag, Berlin, 1983, 127–133.

[V4] J.-L. Verdier, Dualité dans la cohomologie des espaces localement compacts, in *Séminaire Bourbaki*, Vol. 9, Exp. 300, Société Mathématique de France, Paris, 1995, 337–349.

[Was] I. Waschkies, The stack of microlocal perverse sheaves, *Bull. Soc. Math. France*, **132**-3 (2004), 397–462.

[Z] M. Zerner, Domaine d'holomorphie des fonctions vérifiant une équation aux dérivées partielles, *C. R. Acad. Sci. Paris Sér.* I *Math.*, **272** (1971), 1646–1648.

List of Notation

- $\mathbb{N} = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$
- $\mathbb{N}^+ = \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$

Chapter 1

Chapter 2

Chapter 3

Chapter 4

Chapter 5

Chapter 6

Chapter 7

Chapter 8

Chapter 9

Chapter 10

Chapter 11

Chapter 12

Chapter 13

Appendix A

Appendix B

Appendix C

Appendix D

Appendix E

Index

Progress in Mathematics

Edited by:

Progress in Mathematics is a series of books intended for professional mathematicians and scientists, encompassing all areas of pure mathematics. This distinguished series, which began in 1979, includes authored monographs and edited collections of papers on important research developments as well as expositions of particular subject areas.

We encourage preparation of manuscripts in some form of TEX for delivery in camera-ready copy which leads to rapid publication, or in electronic form for interfacing with laser printers or typesetters.

Proposals should be sent directly to the editors or to: Birkhäuser Boston, 675 Massachusetts Avenue, Cambridge, MA 02139, USA or to Birkhauser Verlag, 40-44 Viadukstrasse, CH-4051 Basel, Switzerland.

150 SHIOTA. Geometry of Subanalytic and Semialgebraic Sets
151 HUMMEL. Gromov's Compactness Theorem for Pseudo-holomorphic Curves
152 GROMOV. Metric Structures for Riemannian and Non-Riemannian Spaces. Sean Bates (translator)
153 BUESCU. Exotic Attractors: From Liapunov Stability to Riddled Basins
154 BÖTTCHER/KARLOVICH. Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators
155 DRAGOMIR/ORNEA. Locally Conformal Kähler Geometry
156 GUIVARC'H/JI/TAYLOR. Compactifications of Symmetric Spaces
157 MURTY/MURTY. Non-vanishing of L-functions and Applications
158 TIRAO/VOGAN/WOLF (eds). Geometry and Representation Theory of Real and p-adic Groups
159 THANGAVELU. Harmonic Analysis on the Heisenberg Group
160 KASHIWARA/MATSUO/SAITO/SATAKE (eds). Topological Field Theory, Primitive Forms and Related Topics
161 SAGAN/STANLEY (eds). Mathematical Essays in Honor of Gian-Carlo Rota
162 ARNOLD/GREUEL/STEENBRINK. Singularities: The Brieskorn Anniversary Volume
163 BERNDT/SCHMIDT. Elements of the Representaiton Theory of the Jacobi Group
164 ROUSSARIE. Bifurcations of Planar Vector Fields and Hilbert's Sixteenth Problem
165 MIGLIORE. Introduction to Liaison Theory and Deficiency Modules
166 ELIAS/GIRAL/MIRO-ROIG/ZARZUELA (eds). Six Lectures on Commutative Algebra
167 FACCHINI. Module Theory
168 BALOG/KATONA/SZA'SZ/RECSKI (eds). European Congress of Mathematics, Budapest, July 22-26, 1996. Vol. I
169 BALOG/KATONA/SZA'SZ/RECSKI (eds). European Congress of Mathematics, Budapest, July 22-26, 1996. Vol. II
168/169 Sets Vols I, II
170 PATERSON. Groupoids, Inverse Semigroups, and their Operator Algebras
171 REZNIKOV/SCHAPPACHER (eds). Regulators in Analysis, Geometry and Number Theory
172 BRYLINSKI/BRYLINSKI/NISTOR/TSYGAN/XU (eds). Advances in Geometry

173 DRÄXLER/MICHLER/RINGEL (eds). Computational Methods for Representation of Groups and Algebras: Euroconference in Essen
174 GOERSS/JARDINE. Simplicial Homotopy Theory
175 BAÑUELOS/MOORE. Probabilistic Behavior of Harmonic Functions
176 BASS/LUBOTZKY. Tree Lattices
177 BIRKENHAKE/LANGE. Complex Tori
178 PUIG. On the Local Structure of Morita and Rickard Equivalence Between Brauer Blocks
179 MORALES RUIZ. Differential Galois Theory and Non-Integrability of Hamiltonian Systems
180 PATERNAIN. Geodesic Flows
181 HAUSER/LIPMAN/OORT/QUIRÓS (eds). Resolution of Singularities: In Tribute to Oscar Zariski
182 BILLEY/LAKSHMIBAI. Singular Loci of Schubert Varieties
183 KAPOVICH. Hyperbolic Manifolds and Discrete Groups
184 du SAUTOY/SEGAL/SHALEV (eds). New Horizons in pro-*p* Groups
185 FARAUT/KANEYUKI/KORÁNYI/LU/ROOS. Analysis and Geometry on Complex Homogeneous Domains
186 KLEINERT. Units in Skew Fields
187 BOST/LOESER/RAYNAUD. Courbes Semi-stables et Groupe Fondamental en Géométrie Algébrique
188 DOLBEAULT/IORDAN/HENKIN/SKODA/TRÉPREAU (eds). Complex Analysis and Geometry: International Conference in Honor of Pierre Lelong
189 ANDRÉ/BALDASSARI. DeRham Cohomology of Differential Modules on Algebraic Varieties
190 van den ESSEN. Polynomial Automorphisms and the Jacobian Conjecture.
191 KASHIWARA/MIWA (eds). Physical Combinatorics
192 DEHORNOY. Braids and Self-Distributivity
193 KNUDSON. Homology of Linear Groups
194 JUHL. Cohomological Theory of Dynamical Zeta Functions
195 FABER/van der GEER/OORT (eds). Moduli of Abelian Varieties
196 AGUADÉ/BROTO/CASACUBERTA (eds). Cohomological Methods in Homotopy Theory
197 CHERIX/COWLING/JOLISSAINT/JULG/VALETTE (eds). Groups with the Haagerup property
198 LANDSMAN/PFLAUM/SCHLICHENMAIER (eds). Quantization of Singular Symplectic Quotients
199 PEYRE/TSCHINKEL. Rational Points on Algebraic Varieties
200 GOLUBITSKY/STEWART. The Symmetry Perspective
201 CASACUBERTA/MIRÓ-ROIG/VERDERA/XAMBÓ-DESCAMPS (eds). European Congress of Mathematics, Volume 1
202 CASACUBERTA/MIRÓ-ROIG/VERDERA/XAMBÓ-DESCAMPS (eds). European Congress of Mathematics, Volume 2
203 BLAIR. Riemannian Geometry of Contact and Symplectic Manifolds
204 KUMAR. Kac–Moody Groups, their Flag Varieties, and Representation Theory
205 BOUWKNEGT/WU. Geometric Analysis and Applications to Quantum Field Theory
206 SNAITH. Algebraic *K*-groups as Galois Modules
207 LONG. Index Theory for Symplectic Paths with Applications
208 TURAEV. Torsions of 3-dimensional Manifolds
209 UNTERBERGER. Automorphic Pseudodifferential Analysis and Higher Level Weyl Calculi
210 JOSEPH/MELNIKOV/RENTSCHLER (eds). Studies in Memory of Issai Schur
211 HERNÁNDEZ-LERMA/LASSERE. Markov Chains and Invariant Probabilities
212 LUBOTZKY/SEGAL. Subgroup Growth
213 DUVAL/GUIEU/OVSIENKO (eds). The Orbit Method in Geometry and Physics. In Honor of A.A. Kirillov
214 DUNGEY/TER ELST/ROBINSON. Analysis on Lie Groups with Polynomial Growth
215 ARONE/HUBBUCK/LEVI/WEISS. Algebraic Topology.
216 BOROVIK/GELFAND/WHITE. Coxeter Matroids.
217 THANGAVELU. An Introduction to the Uncertainty Principle: Hardy's Theorem on Lie Groups.