Appendix

We gather here without complete proofs some basic facts about ordinary differential equations and submanifolds of $\mathbf{R}^n$, which we use in the text and may not be included in all standard differential calculus courses.

A.1 Ordinary Differential Equations

We refer here to the book of Hubbard and West [11].

A.1.1 Cauchy Problem

We consider the Cauchy problem for systems of ordinary differential equations (ODE)

$$x'(t) = F(x(t), t),\ x(t_0) = x_0.$$

Here, $\Omega \subset \mathbf{R}_x^n$ is open, $I \subset \mathbf{R}$ is an interval, $m_0 = (x_0, t_0) \in \Omega \times I$, and

$$F : \Omega \times I \to \mathbf{R}_x^n$$

is a C^1 function. A solution is a function $x : I \supset J \to \Omega$ of class C^1, defined on a subinterval $J \ni t_0$ of I, satisfying the equation and the initial condition.

The Cauchy–Lipschitz (local existence) theorem can be stated as follows: Let $m_0 = (x_0, t_0) \in \Omega \times I$, and assume the existence of $a > 0$, $b > 0$, such that $R = \bar{B}(x_0, b) \times [t_0 - a, t_0 + a] \subset \Omega \times I$. Define $M = \max_R |F|$, $\alpha = \min(a, b/M)$.

Theorem A.1 (Cauchy–Lipschitz local existence theorem). *There exists a unique solution $x \in C^1(J), J = [t_0 - \alpha, t_0 + \alpha]$ of the Cauchy problem.*

Proof: The first step in the proof is to write the system in integral form

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)ds.$$

We now look for $x \in C^0(J)$ satisfying this equation, since then $x \in C^1(J)$ and x is a solution of the Cauchy problem:

Step 1. Set $R' = J \times \bar{B}(x_0, b)$. Suppose $y \in C^0(J)$ has its graph in R', and define for $t \in J$

$$x(t) = x_0 + \int_{t_0}^{t} F(y(s), s)ds.$$

Then also x has its graph in R', since $||x(t) - x_0|| \leq M|t - t_0| \leq b$.

Step 2. Using step 1, define a sequence of functions $x^n \in C^0(J)$ by

$$x^0 = x_0, \; x^{n+1}(t) = x_0 + \int_{t_0}^{t} F(x^n(s), s)ds.$$

We claim that, for some constants C_0, C_1, and all n,

$$\delta^n(t) \equiv ||x^{n+1}(t) - x^n(t)|| \leq C_0 C_1^n \frac{|t - t_0|^n}{n!}.$$

This is true for $n = 0$ if C_0 is chosen big enough, which we assume. Suppose the inequality true up to $n - 1$: Subtracting the equations for x^{n+1} and x^n we obtain

$$\delta^n(t) = || \int_{t_0}^{t} [F(x^n(s), s) - F(x^{n-1}(s), s)]ds||.$$

Since F is C^1 on the compact R', there exists a constant C such that, on R',

$$||F(x, t) - F(y, t)|| \leq C||x - y||.$$

Using the induction hypothesis, we obtain for $t \geq t_0$, say,

$$\delta^n(t) \leq C \int_{t_0}^{t} ||x^n(s) - x^{n-1}(s)||ds \leq C C_0 C_1^{n-1} \int_{t_0}^{t} (s - t_0)^{n-1} \frac{ds}{(n-1)!}$$
$$= C C_0 C_1^{n-1} \frac{(t - t_0)^n}{n!}.$$

This is the desired estimate if $C_1 \geq C$. Hence $||x^{n+1} - x^n||_{L^\infty(J)} \leq C_0(C_1\alpha)^n/n!$, which is the general term of a convergent series. Thus $x^{n+1} - x^n$ is a normally converging sequence in $C^0(J)$, and x^n converges uniformly to some $x \in C^0(J)$, which is solution of the integral equation. □

The uniqueness part of the theorem results from the following stronger result.

Theorem A.2 (Global Uniqueness Theorem). *Let x and y be two solutions of the Cauchy problem on some interval $J \subset I$ containing t_0. Then $x \equiv y$.*

It suffices to prove the theorem for J compact. Then, for some C,

$$||x(t) - y(t)|| \leq C \int_{t_0}^{t} ||x(s) - y(s)||ds,$$

and the Gronwall Lemma (Lemma 2.16) implies the result. □

We gave the proof of the Cauchy–Lipschitz theorem in details, since the proof of the existence theorem in Section 2.6 is modeled after it.

Using these theorems, it is easy to establish the following result.

Theorem A.3 (Maximal Interval Theorem). *There exists a unique maximal solution $x \in C^1(J)$ of the Cauchy problem defined on an open interval $J =]T_*, T^*[$. If F is defined on $\mathbf{R}^n \times]a, b[$ and $T^* < b$, then*

$$||x(t)|| \to +\infty,\ t \to T^*,\ t < T^*.$$

In this statement, "maximal solution" means that there exists no solution y defined on $K \supset J$ and (strictly) extending x. A simple illustration of this theorem is the *scalar* Cauchy problem

$$x'(t) = F(x(t)),\ x(0) = x_0,\ F \in C^1(\mathbf{R}),\ F > 0.$$

Let $G(x) = \int_0^x \frac{ds}{F(s)}$ be a primitive of $1/F$. For any solution $x \in C^1(I)$,

$$\frac{d}{dt}[G(x(t))] = \frac{1}{F}(x(t)) \times x'(t) = 1,$$

hence $G(x(t)) = G(x_0) + t$, and x will exist as long as $G(x_0) + t$ is in the range of G. Suppose for instance

$$\int_{-\infty}^{0} \frac{ds}{F(s)} = \infty,\ \alpha = \int_0^{+\infty} \frac{ds}{F(s)} < \infty.$$

Then G is a strictly increasing function from $-\infty$ to α, and the maximal interval is $]-\infty, T^*[$ with $G(x_0) + T^* = \alpha$, that is $T^* = \int_{x_0}^{+\infty} \frac{ds}{F(s)}$. The other three cases are handled similarly.

It remains for us to understand how the maximal interval depends on the initial value x_0. Though this is a difficult problem, one can easily obtain the following theorem.

Theorem A.4. *Let $\bar{x} \in C^1(]T_*, T^*[)$ be the maximal solution of the Cauchy problem*

$$x'(t) = F(x(t), t),\ x(t_0) = \bar{x}_0.$$

Fix a and b such that $T_* < a < t_0 < b < T^*$. Then there exists $\epsilon > 0$ such that all solutions with initial data $x(t_0) = x_0$ satisfying $||x_0 - \bar{x}_0|| \leq \epsilon$ are defined on a maximal interval containing $[a, b]$.

For instance, the solution of $x'(t) = x^2(t),\ x(0) = x_0$ is $x(t) = x_0/(1 - tx_0)$. If we take $\bar{x}_0 = 0$, the solution $\bar{x}$ is global; the solution with data x_0 will be defined on $[-M, M]$ as soon as $|x_0| < 1/M$.

A.1.2 Flows

In the special case when F does not depend on t, we call the system "**autonomous**." It is enough then to consider the Cauchy problem

$$x'(t) = F(x(t)),\ x(0) = x_0.$$

The solution is denoted by $\Phi(t, x_0)$, and called the flow of F. The point of this notation is to emphasize the dependence of the solution on its initial value x_0, and this is very convenient, as we shall see in applications (See Chapters 1–3). By definition, for each x_0, the function $\Phi(t, x_0)$ is defined on the maximal interval $]T_*(x_0), T^*(x_0)[$; hence Φ is defined on $U \subset \mathbf{R} \times \mathbf{R}^n$

$$U = \{(t, x), x \in \Omega, T_*(x) < t < T^*(x)\}.$$

The important result about Φ is the following.

Theorem A.5 (Flow Theorem). *Let Φ be the flow of F. Then U is open and $\Phi \in C^1(U)$.*

We do not prove this theorem (though it can be obtained as an application of the implicit function theorem), but explain why U is open. Let $m_0 = (t_0 > 0, x_0) \in U$: This implies $t_0 + \eta < T^*(x_0)$ for some $\eta > 0$, hence $[0, t_0 + \eta]$ is contained in $]T_*(x_0), T^*(x_0)[$. Using the above theorem about the maximal interval, we obtain that for some $\epsilon > 0$, $T^*(x) > t_0 + \eta$ if $||x - x_0|| \leq \epsilon$. Hence $B(x_0, \epsilon) \times]0, t_0 + \eta[$ is an open set contained in U and containing m_0.

A.1.3 Lower and Upper Fences

Consider a *scalar* equation $x'(t) = F(x(t), t)$.

Definition A.6. *A real function $y \in C^1([a, b[)$ is called a lower fence (resp., an upper fence) for the equation $x'(t) = F(x(t), t)$ if $y'(t) \leq F(y(t), t)$ (resp., $y'(t) \geq F(y(t), t)$).*

The point of this definition lies in the following theorem.

Theorem A.7 (Fence Theorem). *Suppose we are given a solution $x \in C^1([a, b[)$ of the scalar equation $x'(t) = F(x(t), t)$. If $y \in C^1([a, b[)$ is a lower fence (resp., an upper fence) with $y(a) \leq x(a)$ (resp., $y(a) \geq x(a)$), then for all $t \in [a, b[$, $y(t) \leq x(t)$ (resp., $y(t) \geq x(t)$).*

Since $F \in C^1$, the proof is very simple, and we give it for a lower fence: Let $\delta(t) = x(t) - y(t), \delta(a) \geq 0$; then

$$\delta'(t) \geq F(x(t), t) - F(y(t), t) = \alpha(t)\delta(t),$$
$$\alpha(t) = \int_0^1 (\partial_x F)(sx(t) + (1 - s)y(t), t) ds,$$

and the function α is continuous. Setting $A(t) = \int_a^t \alpha(s) ds$ and $z(t) = \delta(t)e^{-A(t)}$, we obtain

$$z'(t) = e^{-A(t)}(\delta'(t) - \alpha(t)\delta(t)) \geq 0, \; z(a) \geq 0.$$

Hence $z(t) \geq 0$ in $[a, b[$, which implies $\delta(t) \geq 0$. □

A.2 Submanifolds

We refer here to the book of M. Spivak [22].

A.2.1 First Definitions

Definition A.8. *A set $S \subset \mathbf{R}^n_x$ is a submanifold of dimension d if, for all $x_0 \in S$, there exists a C^1-diffeomorphism ϕ from a neighborhood U of x_0*

onto a neighborhood V *of the origin in* $\mathbf{R}^n_y$ *such that*

$$\phi(S \cap U) = V \cap \Pi_d,$$

where $\Pi_d = \{y \in \mathbf{R}^n, y_{d+1} = \cdots = y_n = 0\}$.

In other words, looking at S through the "glasses" ϕ, we see only a piece of d-plane. Obviously, if we split the coordinates in $\mathbf{R}^n$ as

$$x = (y, z),\ y = (x_1, \ldots, x_d),\ z = (x_{d+1}, \ldots, x_n),$$

the set S defined by $S = \{x, z = f(y)\}$ for some $f \in C^1$ (the graph of f) is a submanifold of dimension d.

Definition A.9. *Suppose* $x_0 \in S \subset \mathbf{R}^n$ *is a submanifold of dimension* d *and there exists a curve* $x \in C^1(]-\eta, \eta[)$ *in* $\mathbf{R}^n$ *with* $x(t) \in S$, $x(0) = x_0$. *Then* $x'(0)$ *is called a tangent vector to* S *at* x_0.

From these definitions, we obtain easily the following theorem.

Theorem A.10. *The set of all tangent vectors to* S *at* x_0 *is a subspace of dimension* d *denoted by* $T_{x_0}S$, *called "tangent plane to* S *at* x_0."

In practice, submanifolds turn out to be defined in two different ways: By a set of equations, or as parametrized surfaces.

A.2.2 Submanifolds Defined by Equations

The simplest case is this:

Theorem A.11. *Let* $f \in C^1(\mathbf{R}^n)$ *be a real function with* $\nabla f \neq 0$. *Then*

$$S = \{x \in \mathbf{R}^n, f(x) = 0\}$$

is a submanifold of dimension $n-1$, *whose tangent plane* $T_m S$ *is orthogonal to* $\nabla f(m)$.

More generally, let $f_1, \ldots, f_q \in C^1(\mathbf{R}^n)$ be q given real functions:

Theorem A.12. *If all* f_i *vanish at* m *and the differentials* $D_m f_1, \ldots, D_m f_q$ *are independent, the set* $S = \{x \in \mathbf{R}^n, f_1(x) = \cdots = f_q(x) = 0\}$ *is a submanifold of dimension* $n-q$ *in a neighborhood of* m. *The tangent plane* $T_m S$ *is the intersection of the kernels of the* $D_m f_i$.

Proof: To see this, let us complete the free system of the q vectors $\nabla f_1(m), \ldots, \nabla f_q(m)$ by vectors $a_{q+1}, \ldots, a_n$ into a basis of $\mathbf{R}^n$. Set now $g_i(x) = a_i \cdot (x - m)$, $i = q+1, \ldots, n$. Define the map $\psi : \mathbf{R}^n_x \to \mathbf{R}^n_y$ by

$$x \mapsto y = \psi(x) = (f_1(x), \ldots, f_q(x), g_{q+1}(x), \ldots, g_n(x)),\ \psi(m) = 0.$$

The differential $D_m\psi$ is represented by a matrix whose lines form a basis of $\mathbf{R}^n$, hence it is invertible. By the impicit function theorem, ψ is a local diffeomorphism from a neighborhood U of m onto a neighborhood V of 0. The image of $S \cap U$ by ψ is the piece in V of $n-q$ plane $\{y_1 = \cdots = y_q = 0\}$. Hence S is a submanifold of dimension $n-q$.

If $x \in C^1(]-\eta,\eta[)$ is a curve on S, $f_i(x(t)) = 0$ for all i, hence, by differentiation, $x'(0)$ belongs to the kernel of $D_m f_i$. Since this happens for all i, $x'(0)$ belongs to the intersection E of these kernels. Thus, $T_m S$ has dimension $n-d$ and is included in E which also has dimension $n-d$, and this implies $T_m S = E$. □

The condition about the differentials of the defining functions f_i is easy to understand. Suppose $q = 2$: each equation $f_i = 0$ defines a submanifold S_i of codimension 1, and $S = S_1 \cap S_2$. The condition that ∇f_1 and ∇f_2 be independent just means that S_1 and S_2 are not tangent at m, which is a very reasonable requirement.

A.2.3 Parametrized Surfaces

Let $f : \mathbf{R}^p_u \supset \Omega \to \mathbf{R}^n, f(u) = (f_1(u), \ldots, f_n(u))$ be a C^1 function, and set

$$S = \{x \in \mathbf{R}^n, \exists\, u \in \Omega,\ x = f(u)\}.$$

Intuitively, S, a set of points depending on the p parameters $(u_1, \ldots, u_p)$, should be a submanifold of dimension p.

Theorem A.13. *Assume $m_0 = f(u_0)$ and $D_{u_0} f$ injective. Then there exists a neighborhood U of u_0 such that $f(U) \subset S$ is a submanifold of dimension p. The tangent space $T_{m_0}[f(U)]$ is spanned by the vectors $(\partial_1 f(u_0), \ldots, \partial_p f(u_0))$.*

The simplest example is a curve $p = 1$, for which the condition of the theorem is just $f'(u_0) \neq 0$, defining the tangent to the curve. In general, consider the $(n \times p)$-matrix representing $D_{u_0} f$: Its columns are the vectors $\partial_i f(u_0)$, which are independent since the differential is injective. Hence there is a $p \times p$ block B, say the first p lines, which is invertible. This block B is the differential at u_0 of the map

$$\phi : \Omega \to \mathbf{R}^p,\ \phi(u) = (f_1(u), \ldots, f_p(u)).$$

Let p_0 be the projection of m_0 on the subspace generated by the first p vectors. Since B is invertible, ϕ is a C^1 diffeomorphism from a neighborhood U of u_0 onto a neighborhood V of the projection p_0. Then $f(U)$ is the graph of $f(\phi^{-1})$ over V, hence a submanifold of dimension p.

To visualize $T_{m_0}[f(U)]$, consider the "coordinate curve"

$$u_i \mapsto ((u_0)_1, \dots, (u_0)_{i-1}, u_i, (u_0)_{i+1}, \dots, (u_0)_p).$$

This is the parallel to the u_i-axis through u_0. The image of this curve by f is a C^1 curve on S with tangent, by definition, $\partial_i f(u_0)$. Therefore these vectors are tangent vectors and span a subspace of $T_{m_0}[f(U)]$ of dimension p, that is, the whole of the tangent space. □

A.2.4 Graphs

Let the coordinates in $\mathbf{R}^n_x$ be split as $x = (y, z), y \in \mathbf{R}^p, z \in \mathbf{R}^q$, $p + q = n$. The subspace $\mathbf{R}^p_y$ is thought of as "horizontal," the subspace $\mathbf{R}^q_z$ as "vertical." Let

$$f : \mathbf{R}^p_y \supset \omega \to \mathbf{R}^{n-p},\ f(y) = (f_1(y), \dots, f_{n-p}(y))$$

be a C^1 function, and

$$S = \{x = (y, z) \in \mathbf{R}^n,\ y \in \omega,\ z = f(y)\}.$$

We call S the graph of f over ω. Then the tangent space to S at $m_0 = (y_0, z_0)$ does not contain any vertical vector $(0, V)$. Conversely, we have the following theorem.

Theorem A.14. *Let S be a submanifold of dimension p such that $T_m S$ does not contain any vertical vector. Then S is the graph of some C^1 function in a neighborhood of m.*

Proof: Let S be defined by independent equations $g_1 = \dots = g_q = 0$, and define

$$g : \mathbf{R}^n \to \mathbf{R}^q,\ g(x) = (g_1(x), \dots, g_q(x)),\ g(m) = 0.$$

The last $n - p$ columns of the $(n - p \times n)$-matrix representing $D_m g$ form a square block B. The assumption about $T_m S$ means that no (nonzero) vector of the form $(0, V)$ is in the kernel of $D_m g$, and since $D_m g(0, V) = BV$, this means that B is invertible. Now we can use the implicit function theorem at m to solve the equation $g(y, z) = 0$ for z, since $\partial_z g(m) = B$. This yields a C^1 function $f : \mathbf{R}^p_y \to \mathbf{R}^q_z$ defined near the projection p of m, for which

$$S = \{x = (y, z),\ z = f(y)\}.$$

□

A.2.5 Weaving

Consider, in $\mathbf{R}^n_x$, a p-submanifold Σ containing m $(p < n)$, and a function $F : \mathbf{R}^n \to \mathbf{R}^n$ of class C^1 in a neighborhood of m. The flow of F, defined on an open set U, is denoted by Φ.

Theorem A.15 (Weaving). *Assume that $F(m)$ is not tangent to Σ at m. Then there exists a neighborhood V of $(m, 0)$ in U such that*

$$S = \{x = \Phi(t, y),\ y \in \Sigma,\ (y, t) \in V\}$$

is a $(p + 1) - submanifold$.

Proof: Geometrically, S is the union of the trajectories of the system $x'(t) = F(x(t))$ starting from points of Σ. Let Σ be properly parametrized by $u \in \mathbf{R}^p$, that is, assume that there exists $f : \mathbf{R}^p_u \supset \omega \to \mathbf{R}^n_x$, $0 \in \omega$, $f(0) = m$, such that $\Sigma = f(\omega)$. As in Theorem A.13, assume that the vectors $\partial_{u_i} f(0)$ are independent. In this case, S is naturally parametrized by

$$(t, u) \mapsto \psi(t, u) = \Phi(t, f(u))$$

for (t, u) close to $(0, 0)$. According to Theorem A.13, we need only prove that the vectors $(\partial_t \psi, \partial_{u_1} \psi, \ldots, \partial_{u_p} \psi)$ are independent. But

$$\partial_t \psi(0, 0) = F(m),\ \partial_{u_i} \psi(0, 0) = \partial_{u_i} f(0),$$

and since F is not tangent to Σ, these vectors are independent. □

A.2.6 Stokes Formula

We do not give this formula in full generality, but mention only two very useful special cases.

Formula A.16 (Green-Riemann formula). *In the plane $\mathbf{R}^2_{x,y}$, let D be a compact domain such that its boundary ∂D is piecewise C^1 and can be oriented clockwise. Then for $P, Q \in C^1(D)$,*

$$\int_D (\partial_x Q - \partial_y P) dx dy = \int_{\partial D} P dx + Q dy.$$

The meaning of the integral on the right is

$$\int_a^b [P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t)] dt$$

for a C^1 parametrization $[a, b] \ni t \mapsto (x(t), y(t))$ of ∂D.

In $\mathbf{R}_x^3$, a slightly different formulation is customary.

Formula A.17 (Stokes formula). *Let* $D \subset \mathbf{R}_x^3$ *be a compact domain with (piecewise)* C^1 *boundary* ∂D*. Let*

$$X : D \to \mathbf{R}_x^3,\ X(x) = (X_1(x), X_2(x), X_3(x))$$

be a C^1 *function.*

Then

$$\int_D [\Sigma \partial_i (X_i(x))] dx = \int_{\partial D} [\Sigma X_i(x) N_i(x)] d\sigma.$$

Here, $N = (N_1, N_2, N_3)$ is the unit outward normal to ∂D, and $d\sigma$ is its surface element. We say that the integral in D of the divergence of X equals the outgoing flux of X through ∂D.

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