Appendix

A.1 A Basic Fact from Convex Analysis

Here, we recall the following property of convex closed sets.

Proposition A.1.1 *Let $K \subset \mathbb{R}^p$ be a closed convex set. Then for every $x \in \mathbb{R}^p$, there is a unique point y_x in K closest to x. Furthermore, for any $z \in K$, we have $|y_x - z| \leq |x - z|$, where the equality holds if and only if $x \in K$.*

Proof Existence of a point y_x in K closest to x follows from the fact that K is a closed set. If there were two distinct points $z_1, z_2 \in K$ closest to x, then the height $[x, u]$ of the isosceles triangle $\triangle z_1 x z_2$ would be shorter than the distance from x to z_1 and z_2. Since $u \in [z_1, z_2]$, we have $u \in K$ contradicting the choice of z_1 and z_2. Thus, y_x is unique.

If $x \in K$, then $y_x = x$ and we have $|y_x - z| = |x - z|$. Assume that $x \notin K$. If $z = y_x$, then $|y_x - z| = 0 < |x - z|$. Assume that $z \neq y_x$. If points x, y_x, and z are collinear, then y_x is strictly between x and z, which implies that $|y_x - z| < |x - z|$. Then we also assume that x, y_x, and z are noncollinear. Let $[x, z^*]$ be the height of the triangle $\triangle x y_x z$. If $z^* \in (y_x, z]$, then $z^* \in K$ and z^* is closer to x than y_x. If $z \in (y_x, z^*]$, then z is closer to x than y_x (and $z \in K$). Both cases contradict the choice of y_x. Thus, $y_x \in (z, z^*]$. Then the interior angle at the vertex y_x of the triangle $\triangle x y_x z$ is at least $90°$, which makes the side $[x, z]$ of $\triangle x y_x z$ the longest. Consequently, $|y_x - z| < |x - z|$. This argument also shows that the inequality $|y_x - z| \leq |x - z|$ becomes equality if and only if $x \in K$. □

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A.2 Certain Properties of Sequences

Summation by parts formula. We start by presenting the well-known Abel's lemma, which is discrete analogue of the integration by parts formula.

Lemma A.2.1 *Let* $\{a_k\}_{k=1}^{n+1}$ *and* $\{b_k\}_{k=1}^{n+1}$ *be two arbitrary sequences of numbers. Then*

$$\sum_{k=1}^{n} a_k(b_{k+1} - b_k) = a_{n+1}b_{n+1} - a_1 b_1 - \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k).$$

Proof We have

$$\begin{aligned}
&\sum_{k=1}^{n} a_k(b_{k+1} - b_k) + \sum_{k=1}^{n} b_{k+1}(a_{k+1} - a_k) \\
&= \sum_{k=1}^{n} a_k b_{k+1} - \sum_{k=1}^{n} a_k b_k + \sum_{k=2}^{n+1} a_k b_k - \sum_{k=1}^{n} a_k b_{k+1} \\
&= a_{n+1}b_{n+1} - a_1 b_1
\end{aligned}$$

and the assertion of the lemma follows. □

Quasi-monotone sequences. Let $\{a_n\}_{n=1}^{\infty} \subset [-\infty, \infty]$ be a sequence which does not assume infinite values of both signs. The sequence $\{a_n\}_{n=1}^{\infty}$ is called *quasi-monotone increasing* if

$$(n + m)a_{n+m} \geq na_n + ma_m, \quad n, m \in \mathbb{N}.$$

Similarly, the sequence $\{a_n\}_{n=1}^{\infty}$ is called *quasi-monotone decreasing* if

$$(n + m)a_{n+m} \leq na_n + ma_m, \quad n, m \in \mathbb{N}.$$

If a sequence is monotone, it is clearly quasi-monotone. Every quasi-monotone sequence has a limit in $[-\infty, \infty]$ as the following result asserts (see, e.g., [135]).

Theorem A.2.2 *Every quasi-monotone increasing sequence* $\{a_n\}_{n=1}^{\infty}$ *contained in* $(-\infty, +\infty]$ *satisfies*

$$\lim_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} a_n. \tag{A.2.1}$$

Every quasi-monotone decreasing sequence $\{a_n\}_{n=1}^{\infty} \subset [-\infty, +\infty)$ *satisfies*

$$\lim_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} a_n. \tag{A.2.2}$$

Remark A.2.3 If a quasi-monotone increasing (decreasing) sequence $\{a_n\}_{n=1}^{\infty}$ assumes the value $-\infty$ ($+\infty$) for some infinite subsequence of indices n, which does not coincide with $\mathbb{N}$, then $\lim\limits_{n\to\infty} a_n$ does not exist in $[-\infty, \infty]$.

Proof Assume first that $\{a_n\}_{n=1}^{\infty}$ is quasi-monotone increasing. If $a_n = +\infty$ for some $n \in \mathbb{N}$, then $a_k = +\infty$ for any $k \geq n$ and (A.2.1) follows trivially.

Assume that $a_n < +\infty$ for every $n \geq 1$. Then the sequence $\{a_n\}_{n=0}^{\infty}$, where $a_0 = 0$ will remain quasi-monotone increasing. Let m be a fixed positive integer. For every $n \in \mathbb{N}$, there exist nonnegative integers $l = l(n)$ and $r = r(n)$ such that $r(n) < m$ and $n = lm + r$. Applying the quasi-monotonicity of $\{a_n\}_{n=0}^{\infty}$ inductively, we obtain

$$na_n \geq lm \cdot a_{lm} + ra_r \geq lm \cdot a_m + ra_r = (n-r)a_m + ra_r = na_m + r(a_r - a_m).$$

Let $t_m := \min\{a_0, a_1, \ldots, a_{m-1}\}$. Then

$$a_n \geq a_m + \frac{r(n)}{n}(t_m - a_m), \quad n \in \mathbb{N}.$$

Since m is fixed, the sequence $\{r(n)\}_{n=1}^{\infty}$ is bounded, and we have

$$\liminf_{n\to\infty} a_n \geq a_m.$$

In view of arbitrariness of m, we have

$$\liminf_{n\to\infty} a_n \geq \sup_{m\in\mathbb{N}} a_m.$$

Since clearly, $\limsup_{n\to\infty} a_n \leq \sup\limits_{m\in\mathbb{N}} a_m$, relation (A.2.1) follows.

If $\{a_n\}_{n=1}^{\infty}$ is quasi-monotone decreasing, the sequence $b_n = -a_n$ is quasi-monotone increasing. Applying relation (A.2.1) to the sequence $\{b_n\}_{n=1}^{\infty}$, we obtain relation (A.2.2) for the sequence $\{a_n\}_{n=1}^{\infty}$. □

A.3 Certain Properties of the Möbius Function

In this section, we recall properties of a multiplicative function that plays an important role in Number Theory and Combinatorics. The Möbius function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is defined in the following way: if $n \in \mathbb{N}$ has factorization $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$ into prime factors, then

$$\mu(n) := \begin{cases} 0, & \text{if } m_i \geq 2 \text{ for some } 1 \leq i \leq k, \\ 1, & \text{if } m_i = 1 \text{ for all } 1 \leq i \leq k \text{ and } k \text{ is even}, \\ -1, & \text{if } m_i = 1 \text{ for all } 1 \leq i \leq k \text{ and } k \text{ is odd}. \end{cases}$$

Define also

$$\epsilon(n) := \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases}$$

We first present an identity which is crucial in the proof of the Möbius inversion formula. Recall that the notation $r|n$ means "r is a divisor of n".

Lemma A.3.1 *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{r \in \mathbb{N} \\ r|n}} \mu(r) = \epsilon(n).$$

Proof If $n = 1$, we have $\mu(1) = 1$ and the assertion of the lemma holds trivially. Assume that $n \geq 2$ and let $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$ be the prime factorization of n. If $r \in \mathbb{N}$ is a divisor of n, then $\mu(r) \neq 0$ if and only if the prime factorization of r has only simple factors from some subset of the set $\{p_1, \ldots, p_k\}$. Then using the binomial formula, we have

$$\sum_{\substack{r \in \mathbb{N} \\ r|n}} \mu(r) = \sum_{i=0}^{k} \sum_{\substack{A \subseteq \{1,\ldots,k\} \\ \#A = i}} \mu\left(\prod_{j \in A} p_j\right) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = 0 = \epsilon(n).$$

□

The following version of the Möbius inversion formula is established next.

Theorem A.3.2 *Let $\psi : \mathbb{N} \to \mathbb{R}$ be a function which vanishes for every n sufficiently large. Then*

$$\sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \mu(r)\psi(mr) = \psi(1).$$

Proof Since $\psi(n) = 0$ for every n greater than some n_0 using Lemma A.3.1, we have

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \mu(r)\psi(mr) &= \sum_{n=1}^{n_0} \psi(n) \sum_{\substack{(r,m) \in \mathbb{N}^2 \\ mr = n}} \mu(r) \\ &= \sum_{n=1}^{n_0} \psi(n) \sum_{\substack{r \in \mathbb{N} \\ r|n}} \mu(r) = \sum_{n=1}^{n_0} \psi(n)\epsilon(n) = \psi(1). \end{aligned}$$

□

We next present the well-known identity called the Euler product. Here, we use the notation $\mathbb{P}$ for the set of prime numbers.

Theorem A.3.3 *For every $s > 1$, we have*

$$\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}.$$

Proof Let p_j be the jth smallest prime number. For every $m, l \in \mathbb{N}$, we have

$$\prod_{j=1}^{l}\left(\sum_{i=0}^{m}p_j^{-si}\right)\leq\sum_{n=1}^{M}\frac{1}{n^s}<\zeta(s),$$

where $M = p_1^m \cdot \ldots \cdot p_l^m$. Letting $m \to \infty$ we have

$$\prod_{j=1}^{l}\frac{1}{1-p_j^{-s}}=\prod_{j=1}^{l}\left(\sum_{i=0}^{\infty}p_j^{-si}\right)\leq\zeta(s).$$

Letting $l \to \infty$ we obtain that

$$\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}\leq\zeta(s). \tag{A.3.1}$$

On the other hand, for every $q \in \mathbb{N}$, we have

$$\sum_{n=1}^{q}\frac{1}{n^s}\leq\prod_{j=1}^{q}\left(\sum_{i=0}^{q}p_j^{-si}\right)\leq\prod_{j=1}^{q}\frac{1}{1-p_j^{-s}}.$$

Letting $q \to \infty$ we obtain that

$$\zeta(s)\leq\prod_{p\in\mathbb{P}}\frac{1}{1-p^{-s}}.$$

This together with (A.3.1) implies that $\prod_{p\in\mathbb{P}}(1-p^{-s})^{-1}=\zeta(s)$ and the assertion of the theorem follows. □

The following formula relates the Möbius function and the Riemann zeta function.

Theorem A.3.4 *For every $s > 1$, we have*

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}.$$

Proof For every $l \in \mathbb{N}$, we have

$$\prod_{j=1}^{l}(1-p_j^{-s}) = \sum_{A\subset\{1,\ldots,l\}}(-1)^{\#A}\prod_{i\in A}p_i^{-s}$$
$$= \sum_{A\subset\{1,\ldots,l\}}\mu\left(\prod_{i\in A}p_i\right)\prod_{i\in A}p_i^{-s} = \sum_{n\in B_l}\frac{\mu(n)}{n^s},$$

where $B_l := \{p_1^{k_1}\cdot\ldots\cdot p_l^{k_l} : k_1,\ldots,k_l \in \mathbb{N}\cup\{0\}\}$. Letting $l\to\infty$ and taking into account Theorem A.3.3 and the fact that the series $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ converges absolutely, we obtain

$$\frac{1}{\zeta(s)} = \lim_{l\to\infty}\prod_{j=1}^{l}(1-p_j^{-s}) = \lim_{l\to\infty}\sum_{n\in B_l}\frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}. \qquad \square$$

We conclude this section by stating the multiplicative property of the Möbius function mentioned in the beginning of this section. The proof is straightforward and we omit it here.

Theorem A.3.5 *For every $m,n\in\mathbb{N}$ coprime, we have $\mu(mn)=\mu(m)\mu(n)$.*

A.4 Integral Representation of Completely Monotone Functions

Completely monotone potential functions (see Definition 2.2.4) that appear in several parts of this book, give a rise to positive-definite kernels and are involved in the definition of universal optimality of point configurations. Here, we present the proof of the well-known Hausdorff–Bernstein–Widder theorem (see Theorem A.4.5), which provides an integral representation for functions from this class.

Given a sequence $\mathbf{x}=\{x_n\}_{n=0}^{\infty}$, denote $\Delta^0 x_n = x_n$, $\Delta x_n := x_{n+1}-x_n$, and let $\Delta^k x_n := \Delta(\Delta^{k-1})x_n$, $k=1,2,3,\ldots$. A sequence $\mathbf{x}$ is called *completely monotone* if $(-1)^k\Delta^k x_n \geq 0$, $n,k\geq 0$ and *strictly completely monotone* if $(-1)^k\Delta^k x_n > 0$, $n,k\geq 0$. We start by proving the following auxiliary statement.

Lemma A.4.1 *If $f:[0,\infty)\to\mathbb{R}$ is completely monotone, then the sequence $\mathbf{y}:=\{f(a+nh)\}_{n=0}^{\infty}$ is completely monotone for every $a\geq 0$ and $h>0$. If f is strictly completely monotone, then so is $\mathbf{y}$.*

Proof We must show the positivity of the sequence $(-1)^k \Delta^k \mathbf{y}$ for every $k \geq 0$. Let $p(t)$ be the polynomial of degree at most k such that $p(a+hi) = f(a+hi)$, $i = n, n+1, \dots, n+k$, and let $g(t) = f(t) - p(t)$. Since the function g vanishes at $k+1$ distinct points of the interval $[a+hn, a+h(n+k)]$, multiple application of the Rolle's theorem shows that there is a point ξ inside this interval such that $g^{(k)}(\xi) = 0$. Then $f^{(k)}(\xi) = p^{(k)}(\xi) = k! b_k$, where b_k is the coefficient t^k of the polynomial p.

Let $b := a + hn$. Recall the following well-known representation (Lagrange interpolation formula):

$$p(t) = \sum_{i=0}^{k} f(b+hi) \prod_{\substack{j=0\\ j\neq i}}^{k} \frac{t-b-hj}{h(i-j)} = \sum_{i=0}^{k} \frac{f(b+hi)(-1)^{k-i}}{h^k i!(k-i)!} \prod_{\substack{j=0\\ j\neq i}}^{k} (t-b-hj).$$

Then with $y_n = f(a+hn)$ the coefficient b_k of p becomes

$$b_k = \sum_{i=0}^{k} \frac{f(b+hi)(-1)^{k-i}}{h^k i!(k-i)!} = \frac{1}{h^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(b+hi)$$

$$= \frac{1}{h^k k!} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} y_{n+i} = \frac{1}{h^k k!} \Delta^k y_n.$$

Since f is completely monotone, we finally obtain

$$(-1)^k \Delta^k y_n = (-1)^k h^k k! b_k = (-1)^k h^k f^{(k)}(\xi) \geq 0,$$

where the inequality is strict if f is strictly completely monotone. □

We also need the result of the following calculations.

Lemma A.4.2 *Let n be a fixed positive integer and let*

$$g_k(t) := \prod_{i=0}^{n-1} \frac{kt-i}{k-i}, \quad k \geq n.$$

Then $\lim_{k\to\infty} g_k(t) = t^n$, where the limit is uniform over $t \in [0,1]$.

Proof We first notice that for every fixed $i = 0, 1, \dots, n-1$, we have

$$\left| \frac{kt-i}{k-i} - t \right| \leq \frac{n-1}{k-n+1}, \quad t \in [0,1], \quad k \geq n.$$

Then the sequence of functions $g_{k,i}(t) := \frac{kt-i}{k-i}$ converges uniformly to t on $[0,1]$ for every $i = 0, 1, \ldots, n-1$. Since the function $q(t) = t$ is bounded on $[0,1]$, the product $g_k = g_{k,0} \cdot g_{k,1} \cdot \ldots \cdot g_{k,n-1}$ converges uniformly to t^n on $[0,1]$. □

Next, we recall the Hausdorff theorem, which gives the solution to the Hausdorff moment problem (for positive measures).

Theorem A.4.3 *A sequence* $\mathbf{x} = \{x_n\}_{n=0}^{\infty}$ *is completely monotone if and only if there is a finite and positive Borel measure* μ *supported on* $[0,1]$ *such that*

$$x_n = \int_0^1 t^n d\mu(t), \quad n \geq 0. \tag{A.4.1}$$

Here we agree that $t^0 \equiv 1$, $t \in [0,1]$.

Proof Assuming (A.4.1), we will have

$$(-1)^k \Delta^k x_n = (-1)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} x_{n+i} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \int_0^1 t^{n+i} d\mu(t)$$

$$= \int_0^1 t^n \left(\sum_{i=0}^{k} \binom{k}{i} (-t)^i \right) d\mu(t) = \int_0^1 t^n (1-t)^k d\mu(t) \geq 0, \quad k, n \geq 0,$$

which proves the "if" part.

To establish the "only if" part, we let

$$\lambda_{k,m} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} x_m, \quad 0 \leq m \leq k, \quad k \geq 0.$$

Define the measure μ_k, $k \geq 0$, supported at points m/k, $0 \leq m \leq k$, by $\mu_k(\{m/k\}) = \lambda_{k,m}$. We fix $n \geq 0$ and let g_k be as in Lemma A.4.2 (when $n = 0$ we let $g_k(t) \equiv 1$). Notice that

$$g_k\left(\frac{m}{k}\right) = \prod_{i=0}^{n-1} \frac{m-i}{k-i} = \frac{m!(k-n)!}{(m-n)!k!} = \frac{\binom{m}{n}}{\binom{k}{n}}, \quad n \geq 0.$$

Then if $n \geq 1$, we have

$$\begin{aligned}
\sum_{m=n}^{k} g_k\left(\frac{m}{k}\right)\lambda_{k,m} &= \sum_{m=n}^{k} \frac{\binom{m}{n}}{\binom{k}{n}} \binom{k}{m} (-1)^{k-m} \sum_{i=0}^{k-m} \binom{k-m}{i} (-1)^{k-m-i} x_{m+i} \\
&= \sum_{m=n}^{k} \sum_{i=0}^{k-m} \binom{k-n}{k-m-i} \binom{i+m-n}{i} (-1)^i x_{m+i} \\
&= \sum_{j=n}^{k} \sum_{m=n}^{j} \binom{k-n}{k-j} \binom{j-n}{j-m} (-1)^{j-m} x_j \\
&= \sum_{j=n}^{k} \binom{k-n}{k-j} x_j \sum_{m=n}^{j} \binom{j-n}{j-m} (-1)^{j-m} \\
&= \sum_{j=n}^{k} \binom{k-n}{k-j} \left(\sum_{\ell=0}^{j-n} \binom{j-n}{\ell} (-1)^{\ell}\right) x_j = x_n.
\end{aligned}$$

In the case $n = 0$, a similar argument also proves that

$$\int_0^1 d\mu_k(t) = \sum_{m=0}^{k} \lambda_{k,m} = x_0, \quad k \geq 0,$$

which implies that the total mass of each measure μ_k, $k \geq 0$, is x_0. Choose arbitrary $\epsilon > 0$. In the case $n \geq 1$, let K_ϵ be a positive integer such that

$$|g_k(t) - t^n| \leq \epsilon, \quad t \in [0,1], \quad \text{and} \quad \left(\frac{n}{k}\right)^n \leq \epsilon, \quad k > K_\epsilon,$$

Then for every $k > K_\epsilon$, since $\lambda_{k,m} \geq 0$, we will have

$$\begin{aligned}
\left| x_n - \int_0^1 t^n d\mu_k(t) \right| &= \left| \sum_{m=n}^{k} g_k\left(\frac{m}{k}\right)\lambda_{k,m} - \sum_{m=0}^{k} \left(\frac{m}{k}\right)^n \lambda_{k,m} \right| \\
&\leq \sum_{m=n}^{k} \left| g_k\left(\frac{m}{k}\right) - \left(\frac{m}{k}\right)^n \right| \lambda_{k,m} + \sum_{m=0}^{n-1} \left(\frac{m}{k}\right)^n \lambda_{k,m} \\
&\leq \epsilon \sum_{m=n}^{k} \lambda_{k,m} + \sum_{m=0}^{n-1} \left(\frac{n}{k}\right)^n \lambda_{k,m} \leq \epsilon \sum_{m=0}^{k} \lambda_{k,m} = \epsilon x_0.
\end{aligned}$$

Thus,

$$\lim_{k\to\infty} \int_0^1 t^n d\mu_k(t) = x_n, \quad n \geq 0.$$

By Helly's selection theorem (see Theorem 1.6.7) and the positivity of the measures μ_k, there is a subsequence $\{\mu_{k_l}\}_{l=1}^{\infty}$ that converges weak* to a positive and finite Borel measure μ supported on $[0, 1]$. Then

$$x_n = \lim_{l \to \infty} \int_0^1 t^n d\mu_{k_l}(t) = \int_0^1 t^n d\mu(t), \quad n \geq 0,$$

which completes the proof of (A.4.1). □

The Hausdorff theorem helps us in constructing the measure for the "weighted average" representation of a completely monotone function.

Lemma A.4.4 *If a function* $f : [0, \infty) \to \mathbb{R}$ *is completely monotone, then there is a finite and positive Borel measure* ν *supported on* $[0, \infty)$ *such that*

$$f(r) = \int_0^\infty e^{-rt} d\nu(t), \quad r \geq 0. \tag{A.4.2}$$

Proof In view of Lemma A.4.1, the sequence $\{f(n)\}_{n=0}^\infty$ is completely monotone. By Theorem A.4.3, there is a finite and positive Borel measure μ supported on $[0, 1]$ such that $f(n) = \int_0^1 t^n d\mu(t)$, $n \geq 0$. For arbitrary $m \in \mathbb{N}$, the sequence $\{f\left(\frac{n}{m}\right)\}_{n=0}^\infty$ is also completely monotone by Lemma A.4.1. By Theorem A.4.3, there is ν_m, a finite and positive Borel measure supported on $[0, 1]$ such that $f(n/m) = \int_0^1 t^n d\nu_m(t)$, $n \in \mathbb{N} \cup \{0\}$. Let μ_m be the image of the measure ν_m with respect to the mapping $u = t^m$. Then

$$\int_0^1 u^n d\mu(u) = f(n) = \int_0^1 t^{nm} d\nu_m(t) = \int_0^1 u^n d\mu_m(u).$$

Then every polynomial has the same integral with respect to the measures μ_m and μ, and, in view of the Weierstrass approximation theorem, every function continuous on $[0, 1]$ (in particular, $v(t) = t^{n/m}$) will have the same integral with respect to these measures. Consequently,

$$f\left(\frac{n}{m}\right) = \int_0^1 t^n d\nu_m(t) = \int_0^1 u^{n/m} d\mu_m(u) = \int_0^1 u^{n/m} d\mu(u), \quad m \in \mathbb{N}, \quad n \geq 0.$$

Since the functions $f(x)$ and $g(x) := \int_0^1 t^x d\mu(t)$ are continuous on $(0, \infty)$ and coincide at all positive rationals, we have $f = g$ on $(0, \infty)$. Since also $f(0) = g(0)$ and f is continuous at $t = 0$ by assumption, we have

$$\mu([0,1]) = f(0) = \lim_{x\to 0+} f(x) = \lim_{x\to 0+} \int_{[0,1]} t^x \, d\mu(t)$$
$$= \lim_{x\to 0+} \int_{(0,1]} t^x \, d\mu(t) \le \mu((0,1]),$$

which implies that $\mu((0,1]) = \mu([0,1])$ and, hence, $\mu(\{0\}) = 0$. If ν is the image of the measure μ with respect to the mapping $u = -\log t$, $t \in (0,1]$, we finally obtain

$$f(x) = \int_{(0,1]} t^x d\mu(t) = \int_{[0,\infty)} e^{-xu} d\nu(u), \quad x \ge 0,$$

which proves (A.4.2). □

Now, we are ready to state and prove the Hausdorff–Bernstein–Widder theorem (see, e.g., [288, p. 161]).

Theorem A.4.5 *A function $f : (0,\infty) \to \mathbb{R}$ is completely monotone if and only if there is a positive Borel measure ν supported on $[0,\infty)$ such that*

$$f(r) = \int_0^\infty e^{-rt} d\nu(t), \quad r > 0. \tag{A.4.3}$$

Proof The "if" part can be established by verifying directly the relation

$$(-1)^k f^{(k)}(r) = \int_0^\infty e^{-rt} t^k d\nu(t) \ge 0, \quad r > 0, \quad k \ge 0. \tag{A.4.4}$$

To prove the "only if" part we use Lemma A.4.4. For $\delta \in (0,1)$, the function $F_\delta(t) := f(t+\delta)$ is completely monotone on the interval $[0,\infty)$. Hence, there is a finite and positive Borel measure ν_δ supported on $[0,\infty)$ such that

$$F_\delta(x) = \int_0^\infty e^{-xt} d\nu_\delta(t) = \int_{(0,1]} u^x d\mu_\delta(u),$$

where μ_δ is the image of the measure ν_δ under the mapping $u = e^{-t}$. For every $n \in \mathbb{N}$, the integral in

$$f(n) = F_\delta(n-\delta) = \int_{(0,1]} u^n u^{-\delta} d\mu_\delta(u), \quad 0 < \delta < 1,$$

does not depend on δ. Consequently, the integral of any polynomial p such that $p(0) = 0$ over the measure $d\eta_\delta(u) := u^{-\delta} d\mu_\delta(u)$ is independent of δ as well. Then by the Weierstrass approximation theorem, so is the integral with respect to η_δ of any function $v \in C[0,1]$ such that $v(0) = 0$ (in particular,

$v(u) = u^x$, $x > 0$). Consequently, for every $x > 0$, taking a positive number δ to be less than $\min\{1, x\}$, we will have

$$f(x) = F_\delta(x-\delta) = \int_{(0,1]} u^x d\eta_\delta(u) = \int_{(0,1]} u^x d\eta_{1/2}(u) = \int_0^\infty e^{-xt} d\nu(t),$$

where ν is the image of the measure $\eta_{1/2}$ under the mapping $t = -\log u$. $\square$

The second part of this section discusses the analyticity of absolutely monotone functions. Let

$$S_f^a(x) := \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

denote the formal Taylor series of a function f at a point a. The following result is sometimes called the Little Bernstein Theorem (see e.g. [111, Section I.5]).

Theorem A.4.6 *Let f be an absolutely monotone function on an interval $I \subset \mathbb{R}$ and $-\infty < a < b < \infty$.*

(i) *If $I = [a, b)$, then S_f^a is the analytic continuation of f into the open disk $D_{a,b} := \{|z - a| < b - a\}$. If $I = [a, \infty)$, then S_f^a is the analytic continuation of f into the whole complex plane.*
(ii) *If $I = (-\infty, \infty)$, then S_f^0 is the analytic continuation of f into the whole complex plane.*
(iii) *If $I = (-\infty, b)$, then f has an analytic continuation into the half-plane $\{\mathrm{Re}\, z < b\}$.*

We first establish the following auxiliary statement.

Lemma A.4.7 *Let f be absolutely monotone on an interval $[\alpha, \beta)$. Then S_f^α exists on $D_{\alpha,\beta}$. Let g be absolutely monotone on $[\alpha, \beta)$ with $f^{(n)}(\alpha) = g^{(n)}(\alpha)$, $n \geq 0$. Then $f(t) = g(t)$, $t \in [\alpha, (\alpha+\beta)/2)$.*

Proof Choose arbitrary $\alpha \leq c < (\alpha+\beta)/2$ and let

$$S_f^c(t; N) := \sum_{n=0}^N \frac{f^{(n)}(c)}{n!}(t-c)^n.$$

Let $x \in [c, \beta)$. Since f and all its derivatives are nonnegative, the sequence $\{S_f^c(x; N)\}_{N=0}^\infty$ is increasing. By Taylor's formula, there is a number $\xi \in (c, x)$ such that

$$f(x) - S_f^c(x, N) = \frac{f^{(N+1)}(\xi)}{(N+1)!}(x-c)^{N+1}.$$

Observe that $S_f^c(x; N) \leq f(x)$, which implies the convergence of the series

$$S_f^c(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{A.4.5}$$

for every $x \in [c, \beta)$. Then the series in (A.4.5) converges for every $z \in D_{c,\beta}$, in particular, for $z = \alpha$, which belongs to $D_{c,\beta}$ by assumption. Furthermore, the function S_f^α is well defined on $D_{\alpha,\beta}$.

For every $t \in [\alpha, c]$, there is a number $\eta \in [\alpha, t]$ such that

$$\left|f(t) - S_f^\alpha(t; N)\right| = \frac{f^{(N+1)}(\eta)}{(N+1)!}(t-\alpha)^{N+1} \le \frac{f^{(N+1)}(c)}{(N+1)!}(c-\alpha)^{N+1}.$$

Since the series in (A.4.5) converges for $z = \alpha$, we have $\frac{f^{(N+1)}(c)}{(N+1)!}(c-\alpha)^{N+1} \to 0$, $N \to \infty$. Then $f(t) = S_f^\alpha(t)$, $t \in [\alpha, c]$. Similarly, we show that $g(t) = S_g^\alpha(t)$, $t \in [\alpha, c]$. By assumption, $S_f^\alpha = S_g^\alpha$, which yields the equality $f = g$ on $[\alpha, c]$ and, hence, on $[\alpha, (\alpha+\beta)/2)$. □

Proof of Theorem A.4.6. In the case $I = [a, b)$ pick a number $q \in (1/2, 1)$ and let $c_0 := a$ and $c_n := qc_{n-1} + (1-q)b$, $n \in \mathbb{N}$. Then $c_n \to b$, $n \to \infty$. If $f^{(k)}(c_n) = (S_f^a)^{(k)}(c_n)$, $k \ge 0$, (this is clearly true when $n = 0$), taking into account Lemma A.4.7 and absolute monotonicity of S_f^a on $[a, b)$, we have $f = S_f^a$ on $[c_n, (c_n + b)/2)$ (which contains c_{n+1}). Consequently, $f^{(k)}(c_{n+1}) = (S_f^a)^{(k)}(c_{n+1})$. Then by induction, $f = S_f^a$ on $\cup_{n=1}^{\infty}[c_n, c_{n+1}] = [a, b)$.

If $I = [a, \infty)$, for every $x \in I$, applying Lemma A.4.7 with β sufficiently large, we obtain that $S_f^a(x)$ exists and equals $f(x)$.

If $I = (-\infty, \infty)$, then $f = S_f^0$ on $[0, \infty)$. For any $c < 0$, $f = S_f^c$ on $[c, \infty)$ and S_f^c is the Taylor expansion of S_f^0 at c. Then $f = S_f^0$ on $[c, \infty)$ and hence on the whole interval I.

If $I = (-\infty, b)$, in view of (i), f has analytic continuation into the disk $D_{a,b}$ for any $-\infty < a < b$. The union of such disks is the half-plane $\{\operatorname{Re} z < b\}$. □

The following sufficient condition for the pointwise convergence of the Gegenbauer expansion (see Section 5.1) of an absolutely monotone function holds.

Theorem A.4.8 *Let f be a function absolutely monotone on the interval $[-1, 1+\epsilon)$ for some $\epsilon > 0$. Then the Gegenbauer expansion of f converges to f at every point of $[-1, 1]$ for every $\lambda > -1/2$.*

Theorem A.4.8 is a consequence of Theorem A.4.6 and of the following known result by Szegő [267, Theorem 9.1.1].

Theorem A.4.9 *Let f be a function analytic on the closed interval $[-1, 1]$. Then the Gegenbauer expansion of f converges to f pointwise on $[-1, 1]$ for every $\lambda > -1/2$.*

A.5 Certain Properties of Orthogonal Polynomials

In this section, we present certain basic properties of orthogonal polynomials that are needed in Chapter 5 and Section 5.7. An extensive review of properties of orthogonal polynomials can be found, for example, in the classical book by Szegő [267] or in the book [10, Chapters 5–7]. While the proofs are essentially the same, here we consider orthogonality with respect to a signed measure satisfying some additional assumptions (rather than a positive measure). Specifically, we consider signed measures as given in the next definition:

Definition A.5.1 Let ν be a signed Borel measure on $\mathbb{R}$ with respect to which all polynomials are integrable. We say that ν is *positive definite up to degree* N if for all polynomials p such that $\deg(p) \le N$, we have

$$\|p\|_\nu^2 := \int (p(t))^2 \, d\nu(t) \geq 0$$

with equality only if $p \equiv 0$. If ν is a positive measure with infinite support then ν is positive definite up to degree N for any N.

We say that two polynomials p and q are orthogonal with respect to a (signed) measure ν if

$$\langle p, q \rangle_\nu := \int p(t)q(t)\, d\nu(t) = 0.$$

For simplicity we shall (with a small abuse of notation) also write $\langle p(t), q(t) \rangle_\nu$ for $\langle p, q \rangle_\nu$.

In the following, we gather several fundamental results from the theory of orthogonal polynomials (although usually expressed in the context of a positive measure ν) [267, Theorems 3.2.1, 3.3.1 and 3.3.2].

Theorem A.5.2 *Let ν be a signed measure supported on $\mathbb{R}$ that is positive definite up to degree N. Then:*

(i) *there is a unique system $\{q_0, q_1, \ldots, q_{N+1}\}$ of monic polynomials that are pairwise orthogonal with respect to the measure ν and are such that $\deg(q_i) = i$, $i = 0, 1, \ldots, N+1$;*

(ii) *the system $\{q_0, q_1, \ldots, q_{N+1}\}$ satisfies a three-term recurrence relation:*

$$q_i(t) = (t + a_i)q_{i-1}(t) - b_i q_{i-2}(t), \qquad 1 \le i \le N+1, \tag{A.5.1}$$

where $q_{-1}(t) = 0$, $q_0(t) = 1$ and $b_1 = 0$;

(iii) *the following relation holds:*

$$\langle tq_{i-1}(t), q_i(t)\rangle_\nu = \langle q_{i-1}(t), tq_i(t)\rangle_\nu = \|q_i\|_\nu^2, \qquad 1 \le i \le N+1, \tag{A.5.2}$$

and the coefficients a_i *and* b_i *are given by*

$$a_i = \frac{\langle tq_{i-1}(t), q_{i-1}(t)\rangle_\nu}{\|q_{i-1}\|_\nu^2}, \qquad 1 \le i \le N+1, \tag{A.5.3}$$

and

$$b_i = \frac{\langle tq_{i-1}(t), q_{i-2}(t)\rangle_\nu}{\|q_{i-2}\|_\nu^2} = \frac{\|q_{i-1}\|_\nu^2}{\|q_{i-2}\|_\nu^2} > 0, \qquad 2 \le i \le N+1; \tag{A.5.4}$$

(iv) *for each* $0 \le i \le N+1$*, the polynomial* q_i *has* i *distinct simple real roots;*
(v) *for each* $1 \le i \le N+1$*, the roots of* q_i *and* q_{i-1} *are interlaced.*

Proof To establish the existence of such q_i's, apply the Gram–Schmidt orthogonalization procedure to the monomials $\{1, t, t^2, \ldots, t_{N+1}\}$:

$$q_i(t) = t^i - \sum_{j=0}^{i-1} \frac{\langle t^i, q_j(t)\rangle_\nu}{\|q_j\|_\nu^2} q_j(t), \qquad 0 \le i \le N+1, \tag{A.5.5}$$

observing that $\|q_j\|_\nu^2 > 0$ for $j \le N$ since ν is positive definite up to degree N. The uniqueness follows from the observation that q_i is the only monic polynomial of degree i orthogonal to all polynomials of smaller degree for $i = 0, \ldots, N+1$.

If $i > 2$ and $j < i-2$, then the degree of $tq_j(t)$ is less than $i-1$ and so $\langle tq_{i-1}(t), q_j(t)\rangle_\nu = \langle q_{i-1}(t), tq_j(t)\rangle_\nu = 0$. Therefore, $tq_{i-1}(t)$ is in the span of $q_i(t)$, $q_{i-1}(t)$, and $q_{i-2}(t)$ proving the three-term recurrence relation (A.5.1) (the cases $i = 1$ and 2 are trivial). Taking the ν-inner product of both sides of (A.5.1) with q_i, then q_{i-1} and finally q_{i-2} gives (A.5.2) and (A.5.3), and the first equality in (A.5.4), respectively. The second equality in (A.5.4) then follows using (A.5.4).

Finally, we show by induction that for $1 \le i \le N+1$, the polynomial q_i has i simple real roots and the zeros of q_i and q_{i-1} are interlaced. For $i = 1$, this statement is trivial. For a given $i \ge 2$, suppose that this statement holds with i replaced by j, where j is any of the numbers $1, 2, \ldots, i-1$. Then for each root r of q_{i-1} by (A.5.1) we have $q_i(r) = -b_i q_{i-2}(r)$. Since $b_i > 0$, the polynomials q_i and q_{i-2} have opposite signs at the roots of q_{i-1}. By the induction assumption, q_{i-1} has $i-1$ simple real zeros, q_{i-2} has $i-2$ simple real zeros, and between every two consecutive zeros of q_{i-1} there is a zero of q_{i-2}. Hence, the sign of q_{i-2} alternates at the zeros of q_{i-1} and so does the sign of q_i. Then between every two consecutive zeros of q_{i-1} there is a root of q_i. The polynomial q_{i-2} is strictly positive at the largest root r' of

q_{i-1} because all roots of q_{i-2} are less than r'. Then $q_i(r') < 0$ and, since the leading coefficient of q_i is positive, it must have a root on the interval (r', ∞). Since numbers i and $i-2$ have the same parity, q_i and q_{i-2} have the same sign if t is sufficiently large negative. Since q_i and q_{i-2} have opposite signs at the smallest root r'' of q_{i-1} and q_{i-2} preserves its sign on the interval $(-\infty, r'')$, q_i must have a root on the interval $(-\infty, r'')$. Thus, q_i has i simple real zeros, which are interlaced with the zeros of q_{i-1}. □

Let $\{P_i^{\alpha,\beta}\}_i$ for $\alpha, \beta > -1$ denote the monic Jacobi polynomials orthogonal with respect to the weight $d\nu^{\alpha,\beta}(t) := (1-t)^\alpha (1+t)^\beta \, dt$ on the interval $[-1, 1]$ as defined in Definition 2.6.5 (in Definition 2.6.5 the choice of normalization is not specified). The parameters a_i and b_i for $i \geq 1$ in the three-term recurrence (A.5.1) for monic Jacobi polynomials are (for example, see [78])

$$\begin{aligned} a_i &= \frac{\alpha^2 - \beta^2}{(2i+\alpha+\beta-2)(2i+\alpha+\beta)}, \\ b_i &= \frac{4(i-1)(i+\alpha-1)(i+\beta-1)(i+\alpha+\beta-1)}{(2i+\alpha+\beta-1)(2i+\alpha+\beta-2)^2(2i+\alpha+\beta-3)}. \end{aligned} \tag{A.5.6}$$

Using (A.5.4) and (A.5.6) we have

$$\lim_{i\to\infty} \frac{\|P_i^{\alpha,\beta}\|_{\nu^{\alpha,\beta}}}{\|P_{i-1}^{\alpha,\beta}\|_{\nu^{\alpha,\beta}}} = \lim_{i\to\infty} \sqrt{b_{i+1}} = \frac{1}{2},$$

which implies

$$\lim_{i\to\infty} \|P_i^{\alpha,\beta}\|_{\nu^{\alpha,\beta}}^{1/i} = \frac{1}{2}. \tag{A.5.7}$$

Theorem 4.6.2 then shows that the zeros of $P_i^{\alpha,\beta}(t)$ are asymptotically distributed according to the logarithmic equilibrium measure on the interval $[-1, 1]$ and so we obtain:

Proposition A.5.3 *Let $\alpha, \beta > -1$ and for $i \in \mathbb{N}$, let $\{x_{i,j}^{\alpha,\beta}\}_{j=1}^i$ denote the (simple) zeros of $P_i^{\alpha,\beta}(t)$. Then the normalized counting measures $\frac{1}{i}\sum_{j=1}^i \delta_{x_{i,j}^{\alpha,\beta}}$ converge weak* to $\frac{dx}{\pi\sqrt{1-x^2}}$, $x \in [-1, 1]$, as $i \to \infty$.*

As a consequence of the three-term recurrence relation (A.5.1) in Theorem A.5.2, we next obtain the well-known Christoffel–Darboux formulae.

Theorem A.5.4 (Christoffel–Darboux Formulae) *Let ν and $\{q_0, q_1, \ldots, q_{N+1}\}$ be as in Theorem A.5.2. For $k = 0, \ldots, N$,*

$$\sum_{j=0}^k \frac{q_j(x)q_j(y)}{\|q_j\|_\nu^2} = \frac{q_{k+1}(x)q_k(y) - q_{k+1}(y)q_k(x)}{(x-y)\|q_k\|_\nu^2}, \quad x \neq y, \tag{A.5.8}$$

$$\sum_{j=0}^{k} \frac{q_j(x)q_j(x)}{\|q_j\|_\nu^2} = \frac{q'_{k+1}(x)q_k(x) - q_{k+1}(x)q'_k(x)}{\|q_k\|_\nu^2}. \tag{A.5.9}$$

Proof Using (A.5.1) we obtain, for $1 \le j \le N+1$,

$$\begin{aligned} q_{j+1}(x)q_j(y) - q_{j+1}(y)q_j(x) &= ((x + a_{j+1})q_j(x) - b_{j+1}q_{j-1}(x))q_j(y) \\ &\quad - ((y + a_{j+1})q_j(y) - b_{j+1}q_{j-1}(y))q_j(x) \\ &= (x-y)q_j(x)q_j(y) + b_{j+1}\left(q_j(x)q_{j-1}(y) - q_j(y)q_{j-1}(x)\right), \end{aligned}$$

and so, using (A.5.4), we have

$$\begin{aligned} \frac{q_j(x)q_j(y)}{\|q_j\|_\nu^2} &= \frac{q_{j+1}(x)q_j(y) - q_{j+1}(y)q_j(x)}{\|q_j\|_\nu^2(x-y)} \\ &\quad - \frac{q_j(x)q_{j-1}(y) - q_j(y)q_{j-1}(x)}{\|q_{j-1}\|_\nu^2(x-y)}. \end{aligned} \tag{A.5.10}$$

Summing both sides of (A.5.10) over $1 \le j \le k$ and noting that

$$\frac{q_1(x)q_0(y) - q_1(y)q_0(x)}{\|q_0\|_\nu^2(x-y)} = \frac{1}{\|q_0\|_\nu^2} = \frac{q_0(x)q_0(y)}{\|q_0\|_\nu^2},$$

we obtain

$$\sum_{j=0}^{k} \frac{q_j(x)q_j(y)}{\|q_j\|_\nu^2} = \frac{q_{k+1}(x)q_k(y) - q_{k+1}(y)q_k(x)}{(x-y)\|q_k\|_\nu^2} - \frac{q_0(x)q_0(y)}{\|q_0\|_\nu^2}, \tag{A.5.11}$$

which proves (A.5.8). The relation (A.5.9) then follows from writing

$$\frac{q_{k+1}(x)q_k(y) - q_{k+1}(y)q_k(x)}{(x-y)} = \frac{(q_{k+1}(x) - q_{k+1}(y))q_k(y) + q_{k+1}(y)(q_k(y) - q_k(x))}{(x-y)}$$

on the right side of (A.5.8) and letting $y \to x$. □

The next statement deals with the zeros of polynomials of the form $q_n + \alpha q_{n-1}$ [267, Theorem 3.3.4].

Theorem A.5.5 *Let ν and $\{q_0, q_1, \ldots, q_{N+1}\}$ be as in Theorem A.5.2 and let $\alpha \in \mathbb{R}$ be arbitrary. Then for every $1 \le n \le N+1$, the polynomial $g := q_n + \alpha q_{n-1}$ has n simple real zeros that are interlaced with the zeros of q_{n-1}.*

Proof By Theorem A.5.2 (iii) and (iv), we have $t_1 < s_1 < t_2 < s_2 < \ldots < t_{n-1} < s_{n-1} < t_n$, where t_i are the zeros of q_n and s_i are the zeros of

q_{n-1}. Since $g(s_i) = q_n(s_i)$, $i = 1, \ldots, n-1$, and the sign of q_n alternates at $s_1, \ldots, s_{n-1}$, so does the sign of g. Then between each two consecutive zeros of q_{n-1} there is a zero of g. Since $g(s_{n-1}) = q_n(s_{n-1}) < 0$ and g is monic, there is a root of g in the interval (s_{n-1}, ∞). Observe that $g(s_1) = q_n(s_1)$ has the same sign as $(-1)^{n-1}$ while for every t sufficiently large negative, g has the same sign as $(-1)^n$. Then g has a root in the interval $(-\infty, s_1)$. Thus, g has n simple zeros that are interlaced with the ones of q_{n-1}. □

Throughout the rest of this section, we let μ be a positive Borel measure supported on $\mathbb{R}$ with infinite support such that all polynomials are integrable with respect to μ. If ν is replaced by μ in Theorem A.5.2, then the sequence of monic orthogonal polynomials q_i with respect to μ such that $\deg(q_i) = i$ for all i is infinite.

Throughout the rest of this section, we denote by $\{p_i\}_{i=0}^{\infty}$ the sequence of monic orthogonal polynomials with respect to μ such that $\deg(p_i) = i$, $i \geq 0$. The zeros of polynomials $p_n + \alpha p_{n-1}$ are the nodes of a quadrature formula with a high algebraic degree of precision as the following statement suggests [84, Lemma 3.6]. Formula (A.5.12) below can be considered as a minor variant of the Gauss–Jacobi quadrature, see [10, Theorem 5.3.2].

Lemma A.5.6 *Suppose that $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ are arbitrary and let $r_1 < r_2 < \ldots < r_n$ be the roots of the polynomial $p_n + \alpha p_{n-1}$. Then there are numbers $\lambda_1, \ldots, \lambda_n > 0$ such that every polynomial q with $\deg(q) \leq 2n-2$ satisfies*

$$\int q(t)\, d\mu(t) = \sum_{i=1}^{n} \lambda_i q(r_i). \tag{A.5.12}$$

Proof The system of equations

$$\sum_{i=1}^{n} \lambda_i r_i^k = \int t^k\, d\mu(t), \quad k = 0, 1, \ldots, n-1,$$

has a unique solution $(\lambda_1^*, \ldots, \lambda_n^*)$ because its determinant is the Vandermonde determinant, which is known to be nonzero. Then formula (A.5.12) holds for any polynomial q of degree at most $n-1$.

Now let q be any polynomial of degree at most $2n-2$. Then there exist a polynomial g of degree at most $n-2$ and a polynomial h of degree at most $n-1$ such that $q = (p_n + \alpha p_{n-1})g + h$. Since the polynomial $p_n + \alpha p_{n-1}$ is orthogonal to g, we have

$$\int q(t)\, d\mu(t) = \int h(t)\, d\mu(t).$$

Moreover, since the polynomial $p_n + \alpha p_{n-1}$ vanishes at every number r_i, we have $q(r_i) = h(r_i)$, $i = 1, \ldots, n$. Consequently,

$$\int q(t)\,d\mu(t) = \int h(t)\,d\mu(t) = \sum_{i=1}^{n} \lambda_i^* h(r_i) = \sum_{i=1}^{n} \lambda_i^* q(r_i).$$

To show the positivity of the weights $\lambda_1^*, \ldots, \lambda_n^*$, for every fixed $l = 1, \ldots, n$, we let $m_l(t) := \prod_{j:j\neq l}(t - r_j)^2$. Since m_l has degree at most $2n - 2$, we have

$$\int m_l(t)\,d\mu(t) = \sum_{i=1}^{n} \lambda_i^* m_l(r_i) = \lambda_l^* m_l(r_l).$$

Since μ has infinite support, we have $\int m_l(t)\,d\mu(t) > 0$. Since $m_l(r_l) > 0$, we have $\lambda_l^* > 0$, $l = 1, \ldots, n$. □

Recall that $r_1 < r_2 < \ldots < r_n$ denote the roots of the polynomial $p_n + \alpha p_{n-1}$. For $0 \leq j \leq n - 1$, define the measure $\mu_{n,j}$ by

$$d\mu_{n,j}(t) = \prod_{l=0}^{j-1}(r_{n-l} - t)\,d\mu(t).$$

Clearly, $\mu_{n,0} = \mu$.

Lemma A.5.7 *For $0 \leq j \leq n - 1$, the measure $\mu_{n,j}$ is positive definite up to degree $n - j - 1$.*

Proof Let q be any polynomial of degree at most $n - j - 1$. Then by Lemma A.5.6, there are positive numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\begin{aligned}\int (q(t))^2\,d\mu_{n,j}(t) &= \int (q(t))^2 \prod_{l=0}^{j-1}(r_{n-l} - t)\,d\mu(t) \\ &= \sum_{i=1}^{n} \lambda_i\,(q(r_i))^2 \prod_{l=0}^{j-1}(r_{n-l} - r_i) \qquad \text{(A.5.13)} \\ &= \sum_{i=1}^{n-j} \lambda_i\,(q(r_i))^2 \prod_{l=0}^{j-1}(r_{n-l} - r_i) \geq 0.\end{aligned}$$

If the expression in (A.5.13) equals zero, then q must vanish at $r_1, \ldots, r_{n-j}$. Since $\deg(q) \leq n - j - 1$, this is only possible if $q \equiv 0$. □

Throughout the rest of this section for $0 \leq j \leq n - 1$, we define $Q_j := \{q_{j,0}, q_{j,1}, \ldots, q_{j,n-j}\}$ to be the system of monic orthogonal polynomials for the measure $\mu_{n,j}$ such that $\deg(q_{j,i}) = i$, $i = 0, 1, \ldots, n - j$. In view of Lemma A.5.7, the system Q_j posesses the properties described in Theorem A.5.2.

Lemma A.5.8 *Let $n \in \mathbb{N}$. For every $1 \leq j \leq n - 1$, we have $q_{j,n-j}(t) = (t - r_1) \cdot \ldots \cdot (t - r_{n-j})$. For every $0 \leq j \leq n - 1$ and $0 \leq i \leq n - j - 1$, the* *polynomial* $q_{j+1,i}$ *is a linear combination of polynomials* $q_{j,0}, q_{j,1}, \ldots, q_{j,i}$ *with positive coefficients.*

Proof Let $1 \leq j \leq n-1$. For every polynomial $u(t)$ of degree at most $n-j-1$, we have

$$\int (t-r_1)\cdot\ldots\cdot(t-r_{n-j})u(t)d\mu_{n,j}(t) = (-1)^j \int (p_n(t)+\alpha p_{n-1}(t))u(t)d\mu(t) = 0.$$

Consequently, $q_{j,n-j}(t) = (t-r_1)\cdot\ldots\cdot(t-r_{n-j})$.

If $1 \leq j \leq n-1$, by Lemma A.5.7, the measure $\mu_{n,j}$ is positive definite of degree up to $n-j-1$. Then by Theorem A.5.2, for $1 \leq i \leq n-j$, the zeros of polynomials $q_{j,i}$ are interlaced. Then the largest zero of the polynomial $q_{j,i}$, $1 \leq i \leq n-j-1$, is less than r_{n-j}. Consequently, $q_{j,i}(r_{n-j}) > 0$ for such i. Clearly, we also have $q_{j,0}(r_{n-j}) = 1 > 0$.

In the case $j=0$ we have $q_{0,i} = p_i$, $0 \leq i \leq n$. Since by Theorem A.5.2 the zeros of p_i's are interlaced, the largest zero of $q_{0,i}$, $0 \leq j \leq n-1$, is less than or equal to the largest zero of p_{n-1}, which by Theorem A.5.5 is less than the largest zero r_n of the polynomial $p_n + \alpha p_{n-1}$. Consequently, $q_{0,i}(r_n) > 0$, $0 \leq i \leq n-1$.

Now let $0 \leq j \leq n-1$ and $0 \leq i \leq n-j-1$. From the argument above, $q_{j,i}(r_{n-j}) > 0$. Denote

$$v_i(t) := \frac{q_{j,i+1}(t) + \alpha_{j,i} q_{j,i}(t)}{t - r_{n-j}},$$

where $\alpha_{j,i}$ is chosen such that $q_{j,i+1}(r_{n-j}) + \alpha q_{j,i}(r_{n-j}) = 0$. Then v_i is a polynomial of degree i. We next show that $q_{j+1,i} = v_i$. Clearly, $q_{j+1,0}(t) = v_0(t) = 1$. For $1 \leq i \leq n-j-1$, let $u(t)$ be any polynomial of degree at most $i-1$. Then

$$\int v_i(t)u(t)\,d\mu_{n,j+1}(t) = -\int (q_{j,i+1}(t) + \alpha_{j,i} q_{j,i}(t))u(t)\,d\mu_{n,j}(t) = 0.$$

Since $\mu_{n,j+1}$ is positive definite up to degree $n-j-2$, by Theorem A.5.2, $\mu_{n,j+1}$ has a unique orthogonal polynomial of degree i. Thus, $q_{j+1,i} = v_i$, $0 \leq i \leq n-j-1$.

Now let $c_0, c_1, \ldots, c_i$ be defined by

$$q_{j+1,i}(t) = \sum_{l=0}^{i} c_l q_{j,l}(t), \quad 0 \leq i \leq n-j-1. \tag{A.5.14}$$

For every $0 \le k \le i$, we have

$$\int q_{j+1,i}(t)(q_{j,k}(t) - q_{j,k}(r_{n-j}))\, d\mu_{n,j}(t) = \int (q_{j,i+1}(t) + \alpha_{j,i} q_{j,i}(t)) \frac{q_{j,k}(t) - q_{j,k}(r_{n-j})}{t - r_{n-j}}\, d\mu_{n,j}(t) = 0.$$

Then

$$\int q_{j+1,i}(t) q_{j,k}(t)\, d\mu_{n,j}(t) = q_{j,k}(r_{n-j}) \int q_{j+1,i}(t)\, d\mu_{n,j}(t). \tag{A.5.15}$$

Substituting expansion (A.5.14) into (A.5.15) we will obtain

$$c_k \int (q_{j,k}(t))^2\, d\mu_{n,j}(t) = c_0 q_{j,k}(r_{n-j}) \int d\mu_{n,j}(t). \tag{A.5.16}$$

As we proved above, $q_{j,k}(r_{n-j}) > 0$. Since $\mu_{n,j}$ is positive definite up to degree $n - j - 1$, both integrals in (A.5.16) are strictly positive. We have $c_0 \neq 0$ since otherwise all c_k's would be zero making $q_{j+1,i} \equiv 0$. Then $c_0, c_1, \ldots, c_i$ are all nonzero and have the same sign. In formula (A.5.14), we have $c_i = 1$ because $q_{j,i}$ is the only polynomial on the right-hand side of (A.5.14) that contains the term t^i and $q_{j,i}$ and $q_{j+1,i}$ are both monic. Consequently, $c_0, c_1, \ldots, c_i > 0$. □

We are now ready to prove the main result of this section.

Theorem A.5.9 *Let $\{p_i\}_{i=0}^{\infty}$ be the sequence of monic orthogonal polynomials for μ such that $\deg(p_i) = i$, $i \ge 0$. Let $\alpha \in \mathbb{R}$ be any number and let $r_1 < r_2 < \cdots < r_n$ be the roots of the polynomial $p_n + \alpha p_{n-1}$. Then for every $1 \le k \le n-1$, the polynomial $\prod_{l=1}^{k}(t - r_l)$ is a linear combination of $p_0, p_1, \ldots, p_k$ with positive coefficients.*

Proof For a given $1 \le k \le n-1$, by Lemma A.5.8, we have $\prod_{l=1}^{k}(t - r_l) = q_{n-k,k}(t)$. First, applying Lemma A.5.8 with $j = n - k - 1$ we obtain that $q_{n-k,k}$ is a linear combination of polynomials $q_{n-k-1,i}$, $0 \le i \le k$, with positive coefficients. Then Lemma A.5.8 implies that each polynomial $q_{n-k-1,i}$, $0 \le i \le k$, is a linear combination of polynomials $q_{n-k-2,0}, \ldots, q_{n-k-2,i}$ with positive coefficients. Continuing this process inductively, we obtain that $q_{n-k,k}$ is a linear combination of polynomials $q_{0,l} = p_l$, $0 \le l \le k$, with positive coefficients, which is the assertion of the theorem. □

An important example of orthogonal polynomials is the sequence of Gegenbauer polynomials $\{P_n^{\lambda}\}_{n=0}^{\infty}$ defined in Section 5.1. The fact that Theorem A.5.9 holds for this sequence is used in the proof of Theorem 5.7.2, which is one of the main results presented in Chapter 5.

We next prove the Rodrigues formula for the Gegenbauer polynomials. Recall that $(a)_n$ denotes is the Pochhammer symbol of (6.5.5).

Theorem A.5.10 *For $\lambda > -1/2$ and $n \geq 0$, we have*

$$P_n^\lambda(t) = c_n (1-t^2)^{1/2-\lambda} \frac{d^n}{dt^n}(1-t^2)^{n+\lambda-1/2}, \quad t \in (-1,1),$$

where

$$c_n := \frac{1}{(-2)^n \, (\lambda+1/2)_n}.$$

Proof For $n \geq 0$, let

$$g_n(t) := c_n (1-t^2)^{1/2-\lambda} \frac{d^n}{dt^n}(1-t)^{n+\lambda-1/2}(1+t)^{n+\lambda-1/2}.$$

For $n = 0$, we have $g_0(t) = c_0 = 1$. For $n \geq 1$, we have

$$g_n(t) = c_n \sum_{k=0}^{n} C_n^k (-1)^k (n-k+\lambda+1/2)_k (1-t)^{n-k} (k+\lambda+1/2)_{n-k} (1+t)^k.$$

Then g_n is a polynomial of degree n with $g_n(1) = c_n(-1)^n(\lambda+1/2)_n \cdot 2^n = 1$. Let $v(t) := (1-t^2)^{n+\lambda-1/2}$ and $w(t) := (1-t^2)^{\lambda-1/2}$. Since $\lambda > -1/2$, we have $v^{(k)}(-1) = v^{(k)}(1) = 0,\ k = 0, 1, \ldots, n-1$. For any polynomial Q of degree up to $n-1$, integrating by parts n times we will have

$$\int_{-1}^{1} Q(t) g_n(t) w(t)\, dt = c_n \int_{-1}^{1} Q(t) v^{(n)}(t)\, dt = (-1)^n c_n \int_{-1}^{1} Q^{(n)}(t) v(t)\, dt = 0.$$

Thus, $\{g_n\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with the weight $w(t)$ and normalized by $g_n(1) = 1$. Consequently, $g_n(t) = P_n^\lambda(t)$, $t \in (-1,1)$. □

A.6 Auxiliary Facts Concerning Certain Special Functions

The generalized hypergeometric function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$, is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{z^n}{n!}, \qquad \text{(A.6.1)}$$

where we have utilized the Pochhammer symbol of (6.5.5). There are several sources that provide the properties of hypergeometric functions such as [110], which is the successor of [1, 10, 219]. We shall cite the following properties of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ that can be found in [1] or [110]. If $\operatorname{Re} c > \operatorname{Re} b > 0$, then

$$\int_0^1 \frac{u^{b-1}(1-u)^{c-b-1}}{(1-zu)^a}\, du = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \cdot {}_2F_1(a, b; c; z). \tag{A.6.2}$$

Formula (A.6.2) is often called the Euler's integral representation for ${}_2F_1$. If $\operatorname{Re}(c-a-b) < 0$, then

$$\lim_{z\to 1^-} \frac{{}_2F_1(a, b; c; z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{A.6.3}$$

If $\operatorname{Re}(c-a-b) > 0$, then

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.6.4}$$

For other argument unity relations, see [110, Section 15.4 (ii)]. The term-by-term differentiation of the series (A.6.1) yields the following derivative:

$$\frac{d}{dz}\,{}_2F_1(a, b; c; z) = \frac{ab}{c}\,{}_2F_1(a+1, b+1; c+1; z). \tag{A.6.5}$$

In Section 6.5, we used the constant

$$V_s := \frac{1}{\pi^s} \sum_{n=0}^{\infty} \frac{a_n(s)}{4^n(2n-s+1)}, \quad s \in \mathbb{C} \setminus (2\mathbb{N}-1),$$

where the coefficients $a_n(s)$, $s \in \mathbb{C}$, are defined by the expansion

$$h(s, z) := \left(\frac{\sin(\pi z)}{\pi z}\right)^{-s} = e^{-s\log(\operatorname{sinc}(\pi z))} = \sum_{n=0}^{\infty} a_n(s) z^{2n}, \quad |z| < 1, \tag{A.6.6}$$

where we choose the branch of the logarithm such that $\log 1 = 0$. Here, we present proofs of certain properties of the coefficients $a_n(s)$.

Proposition A.6.1 *We have $a_0(s) = 1$, $s \in \mathbb{C}$. For every $n \geq 1$, the coefficient $a_n(s)$ is a polynomial in the variable s of degree at most n.*

Proof The identity $a_0(s) = 1$ follows from (A.6.6) on letting $z \to 0$. Differentiating the Maclaurin expansion (A.6.6) with respect to z, we obtain

$$\frac{\partial h}{\partial z} = \sum_{n=1}^{\infty} 2n\, a_n(s) z^{2n-1} = -s \left(\frac{\sin \pi z}{\pi z}\right)^{-s-1} \left(\frac{\sin \pi z}{\pi z}\right)'$$
$$= -s \left(\sum_{n=0}^{\infty} a_n(s+1) z^{2n}\right) \left(\sum_{n=1}^{\infty} \frac{2(-1)^n n\pi^{2n}}{(2n+1)!} z^{2n-1}\right).$$

Using the term-by-term multiplication of the series and comparison of coefficients of odd powers of z, for every $m \geq 1$, we have

$$a_m(s) = -\frac{s}{2m} \sum_{k=1}^{m} \frac{2(-1)^k k\pi^{2k}}{(2k+1)!} a_{m-k}(s+1).$$

Taking into account the fact that $a_0(s) = a_0(s+1) = 1$, $s \in \mathbb{C}$, and using mathematical induction, we obtain that $a_m(s)$ is a polynomial of degree at most m. □

Proposition A.6.2 *For every $s \in \mathbb{C}$, the coefficients $a_n(s)$ in (A.6.6) satisfy the relation*

$$a_n'(s) = \sum_{m=0}^{n-1} a_m(s) \frac{\zeta(2n-2m)}{n-m}, \quad n \geq 1, \tag{A.6.7}$$

where ζ is the classical Riemann zeta function.

Proof On the one hand, differentiating expansion (A.6.6) with respect to s, we have

$$\frac{\partial h}{\partial s} = \sum_{m=0}^{\infty} a_m'(s) z^{2m}, \quad |z| < 1. \tag{A.6.8}$$

On the other hand, using the known Taylor expansion

$$\log\left(\operatorname{sinc}(\pi z)\right) = -\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} z^{2k},$$

from (A.6.6), we obtain

$$\frac{\partial h}{\partial s} = e^{-s \log(\operatorname{sinc}(\pi z))} \left(-\log\left(\operatorname{sinc}(\pi z)\right)\right)$$
$$= \left(\sum_{n=0}^{\infty} a_n(s) z^{2n}\right) \left(\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k} z^{2k}\right)$$
$$= \sum_{m=1}^{\infty} \left(\sum_{i=0}^{m-1} a_i(s) \cdot \frac{\zeta(2m-2i)}{m-i}\right) z^{2m}.$$

Taking into account (A.6.8), we obtain (A.6.7). □

The constant V_s is an analytic function of s in the open set $\mathbb{C} \setminus (2\mathbb{N}-1)$, see [69]. Recall that σ_1 denotes the normalized (probability) arc length measure on S^1. The following statement holds true.

Lemma A.6.3 *For every $s \in \mathbb{C}$ with $\operatorname{Re} s < 1$, we have $I_s[\sigma_1] = V_s$; i.e., V_s is analytic extension of the energy $I_s[\sigma_1]$ from the half-plane $\operatorname{Re} s < 1$ to the set $\mathbb{C} \setminus (2\mathbb{N}-1)$.*

Proof For every $s \in \mathbb{C}$ with $\operatorname{Re} s < 1$, the integral $I_s[\sigma_1]$ below converges. Using the fact that the function $\sin(t/2)$ is symmetric about the line $x = \pi$ and the fact that the series in (6.5.3) converges uniformly on $[0, 1/2]$, we have

$$\begin{aligned} I_s[\sigma_1] &= \int_{S^1}\int_{S^1} |x-y|^{-s}\, d\sigma_1(x)\, d\sigma_1(y) = \frac{1}{2\pi}\int_0^{2\pi} \left|1-e^{it}\right|^{-s} dt \\ &= \frac{1}{\pi}\int_0^{\pi} \left(2\sin\frac{t}{2}\right)^{-s} dt = \frac{1}{\pi}\int_0^{\pi} h\left(s, \frac{t}{2\pi}\right) t^{-s} dt \\ &= \frac{1}{\pi}\int_0^{\pi} \left(\sum_{n=0}^{\infty} \frac{a_n(s)}{(2\pi)^{2n}} t^{2n-s}\right) dt = \frac{1}{\pi^s}\sum_{n=0}^{\infty} \frac{a_n(s)}{4^n(2n-s+1)}. \end{aligned}$$ □

A.7 Elements of Spherical Geometry and Euler Characteristics

As in the main text, we will use the boldface font to denote points and vectors on the sphere. A closed hemisphere defined by a given vector $\mathbf{a} \in S^2$ is the set $H = \{\mathbf{x} \in S^2 : \mathbf{x} \cdot \mathbf{a} \geq 0\}$. A spherical digon on S^2 is a subset of S^2 of positive area obtained as the intersection of two closed hemispheres. The angle of the spherical digon is equal to π minus the angle between the vectors that define these two hemispheres. If D is a spherical digon with angle α, then the area $|D|$ of D is $\frac{\alpha}{2\pi} \cdot 4\pi = 2\alpha$.

A spherical triangle on S^2 is a subset of S^2 of positive area obtained by intersecting three closed hemispheres and which is not a spherical digon. Angles of a spherical triangle Π are the angles of the spherical digons formed by each pair of the hemispheres that define Π. We first establish a known formula for the area of a spherical triangle.

Proposition A.7.1 *Let $\Pi \subset S^2$ be a spherical triangle with angles α_1, α_2, and α_3. Then the area of Π is given by $|\Pi| = \alpha_1 + \alpha_2 + \alpha_3 - \pi$.*

Proof Let H_1, H_2, and H_3 be hemispheres the intersection of the closures of which is Π and whose boundaries are such that the sets H_i and $-H_i$ form a partition of S^2, $i = 1, 2, 3$. Observe that $S^2 \setminus (H_1 \cup H_2 \cup H_3) = -(H_1 \cap H_2 \cap H_3)$. If $i = 1, 2$, or 3, let j and k denote the remaining two indices from the set $\{1, 2, 3\}$. Then $H_i \setminus (H_j \cup H_k) = -(H_j \cap H_k) \setminus H_i$, $i = 1, 2, 3$. Furthermore, the angle of the digon $D_i := \overline{H}_j \cap \overline{H}_k$ is one of the angles of Π. Consequently,

$$\begin{aligned}
4\pi = \left|S^2\right| &= |H_1 \cap H_2 \cap H_3| + \sum_{i=1}^{3} |(H_j \cap H_k) \setminus H_i| \\
&\quad + \sum_{i=1}^{3} |H_i \setminus (H_j \cup H_k)| + \left|S^2 \setminus (H_1 \cup H_2 \cup H_3)\right| \\
&= 2\,|H_1 \cap H_2 \cap H_3| + 2\sum_{i=1}^{3} |(H_j \cap H_k) \setminus H_i| \\
&= 2\,|\Pi| + 2\sum_{i=1}^{3} \left(|H_j \cap H_k| - |H_1 \cap H_2 \cap H_3|\right) \\
&= 2\sum_{i=1}^{3} |D_i| - 4\,|\Pi| = 4\sum_{i=1}^{3} \alpha_i - 4\,|\Pi|\,.
\end{aligned}$$

Then $\pi = \sum_{i=1}^{3} \alpha_i - |\Pi|$ and the required relation follows. □

Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be the vertices of a spherical triangle Π and let β_i be the angle of Π at the vertex $\mathbf{a}_i$, $i = 1, 2, 3$. Let i, j, k be any pairwise distinct indices from the set $\{1, 2, 3\}$. The following equality follows from known formulas of vector algebra or it can verified directly using the coordinates of the vectors:

$$\mathbf{a}_j \cdot \mathbf{a}_k = (\mathbf{a}_i \cdot \mathbf{a}_k)(\mathbf{a}_i \cdot \mathbf{a}_j) + (\mathbf{a}_i \times \mathbf{a}_j) \cdot (\mathbf{a}_i \times \mathbf{a}_k). \tag{A.7.1}$$

Let α_i be the angle between the vectors $\mathbf{a}_j$ and $\mathbf{a}_k$, where $j, k \neq i$ and observe that $\sin \beta_i$ is the the sine of the angle between vectors $\mathbf{a}_i \times \mathbf{a}_j$ and $\mathbf{a}_i \times \mathbf{a}_k$. Since $|\mathbf{a}_1| = |\mathbf{a}_2| = |\mathbf{a}_3| = 1$, relation (A.7.1) becomes

$$\cos \alpha_i = \cos \alpha_j \cos \alpha_k + \sin \alpha_k \sin \alpha_j \cos \beta_i. \tag{A.7.2}$$

Relation (A.7.2) is known as the First Cosine Theorem of Spherical Geometry. The spherical triangle Π' with vertices $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ that is *polar* with respect to the spherical triangle Π is obtained by choosing the vector $\mathbf{b}_i$ to be perpendicular to vectors $\mathbf{a}_j$ and $\mathbf{a}_k$ and so that $\mathbf{b}_i \cdot \mathbf{a}_i > 0$. Let γ_i be the angle at the vertex $\mathbf{b}_i$ of the spherical triangle Π' and let θ_k be the angle between vectors $\mathbf{b}_i$ and $\mathbf{b}_j$. It is not difficult to verify that $\theta_i = \pi - \beta_i$. Observe now that vector $\mathbf{a}_k$ is perpendicular to both vectors $\mathbf{b}_i$ and $\mathbf{b}_j$. Furthermore, $\mathbf{a}_k \cdot \mathbf{b}_k > 0$. Hence, the polar triangle for Π' is Π. Then we also have $\alpha_k = \pi - \gamma_k$. Applying relation (A.7.2) for the triangle Π', we will have

$$\cos\beta_i = -\cos\theta_i = -\cos\theta_j\cos\theta_k - \sin\theta_k\sin\theta_j\cos\gamma_i,$$

which yields the Second Cosine Theorem of Spherical Geometry:

$$\cos\beta_i = -\cos\beta_j\cos\beta_k + \sin\beta_k\sin\beta_j\cos\alpha_i. \tag{A.7.3}$$

The Second Cosine Theorem implies the following formulas for the area of a spherical triangle when one of its angles equals $\pi/2$ and two other angles being less than $\pi/2$.

Proposition A.7.2 *In the notation introduced above, let Π be a spherical triangle on S^2 with $0 < \beta_1, \beta_2 < \pi/2$ and $\beta_3 = \pi/2$. Then the area of Π is given by*

$$|\Pi| = \beta_1 - \arcsin(\cos\alpha_2\sin\beta_1) = \beta_1 - \arctan(\cos\alpha_3\tan\beta_1). \tag{A.7.4}$$

Proof Letting $i = 2$ in (A.7.3), we will obtain

$$\sin(\pi/2 - \beta_2) = \cos\beta_2 = \sin\beta_1\cos\alpha_2.$$

Then

$$|\Pi| = \beta_1 + \beta_2 + \beta_3 - \pi = \beta_1 + \beta_2 - \pi/2 = \beta_1 - \arcsin(\sin\beta_1\cos\alpha_2)$$

and the first equality in (A.7.4) follows. Letting $i = 3$ in (A.7.3) we will have $\cos\beta_1\cos\beta_2 = \sin\beta_1\sin\beta_2\cos\alpha_3$. Then

$$\tan(\pi/2 - \beta_2) = \cot\beta_2 = \tan\beta_1\cos\alpha_3$$

and

$$|\Pi| = \beta_1 + \beta_2 - \pi/2 = \beta_1 - \arctan(\tan\beta_1\cos\alpha_3),$$

which implies the second equality in (A.7.4). □

A subset P of S^2 with nonempty interior is called a *spherical N-gon* if it can be represented as an intersection of N closed hemispheres and the number N cannot be made smaller.

Proposition A.7.3 *Let P be a spherical N-gon with angles $\alpha_1, \ldots, \alpha_N$, $N \geq 2$. Then its area is given by $|P| = \alpha_1 + \ldots + \alpha_N - \pi(N-2)$.*

Proof We pick an interior point Q in P and join it with every vertex of P with a geodesic line. Summing the areas of the spherical triangles obtained and using Proposition A.7.1, we will have $|P| = \alpha_1 + \ldots + \alpha_N + 2\pi - \pi N$. □

We next state the well-known result due to Euler for convex polytopes.

Theorem A.7.4 *Let U be a convex polytope circumscribed about S^2 that has N faces, k edges, and v vertices. Then $N - k + v = 2$.*

Remark A.7.5 The Euler's formula asserted by Theorem A.7.4 is valid for any convex polyhedron in R^3 (not just the one circumscribed about a sphere). It also holds for any polyhedron in R^3, whose boundary is topologically equivalent to a sphere and whose faces are topologically equivalent to disks. We impose the additional assumption in Theorem A.7.4 since with it we can give a simple proof and we do not require a greater generality in the book.

Proof Let P_i be the radial projection of the ith face of U onto S^2 and let k_i be the number of edges of P_i, $i = 1, \ldots, N$. Taking into account Proposition A.7.3, we obtain

$$4\pi = \sum_{i=1}^{N} |P_i| = 2\pi v - \sum_{i=1}^{N} \pi(k_i - 2) = 2\pi v - 2\pi k + 2\pi N$$

and the required equality follows. □

A.8 Stereographic Projection

Let $\mathbf{y} \in S^d \subset \mathbb{R}^{d+1}$ be arbitrary fixed point and let $H_{\mathbf{y}} := \{\mathbf{y}\}^\perp = \{\mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{x} \cdot \mathbf{y} = 0\}$. The stereographic projection with center $\mathbf{y}$ is the mapping $K_{\mathbf{y}} : S^d \setminus \{\mathbf{y}\} \to H_{\mathbf{y}}$ defined by

$$K_{\mathbf{y}}(\mathbf{x}) = \mathbf{y} + \frac{2(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2}, \quad \mathbf{x} \in S^d \setminus \{\mathbf{y}\}. \tag{A.8.1}$$

We indeed have

$$K_{\mathbf{y}}(\mathbf{x}) \cdot \mathbf{y} = 1 + \frac{2(\mathbf{x} \cdot \mathbf{y} - 1)}{|\mathbf{x} - \mathbf{y}|^2} = 0,$$

i.e., $K_{\mathbf{y}}(\mathbf{x}) \in H_{\mathbf{y}}$. Moreover, definition (A.8.1) immediately implies that points $K_{\mathbf{y}}(\mathbf{x})$ and $\mathbf{x}$ lie on the same ray stemming from the point $\mathbf{y}$ and that

$$|K_{\mathbf{y}}(\mathbf{x}) - \mathbf{y}|\, |\mathbf{x} - \mathbf{y}| = 2, \quad x \in S^d \setminus \{\mathbf{y}\}. \tag{A.8.2}$$

Mapping $K_{\mathbf{y}}$ also has the following properties.

Proposition A.8.1 *For every $\mathbf{x}_1, \mathbf{x}_2 \in S^d \setminus \{\mathbf{y}\}$, there holds*

$$|K_{\mathbf{y}}(\mathbf{x}_1) - K_{\mathbf{y}}(\mathbf{x}_2)| = \frac{2\,|\mathbf{x}_1 - \mathbf{x}_2|}{|\mathbf{x}_1 - \mathbf{y}|\, |\mathbf{x}_2 - \mathbf{y}|}. \tag{A.8.3}$$

Furthermore, $K_{\mathbf{y}} : S^d \setminus \{\mathbf{y}\} \to H_{\mathbf{y}}$ *is a bijective mapping with the inverse* $K_{\mathbf{y}}^{-1}$ *defined by*

$$K_{\mathbf{y}}^{-1}(\mathbf{z}) = \mathbf{y} + \frac{2(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2}, \quad \mathbf{z} \in H_{\mathbf{y}}.$$

Proof For every $\mathbf{x}_1, \mathbf{x}_2 \in S^d \setminus \{\mathbf{y}\}$, we have

$$|K_{\mathbf{y}}(\mathbf{x}_1) - K_{\mathbf{y}}(\mathbf{x}_2)|^2 = 4\left|\frac{(\mathbf{x}_1-\mathbf{y})}{|\mathbf{x}_1-\mathbf{y}|^2} - \frac{(\mathbf{x}_2-\mathbf{y})}{|\mathbf{x}_2-\mathbf{y}|^2}\right|^2$$

$$= 4\left(\frac{1}{|\mathbf{x}_1-\mathbf{y}|^2} - \frac{2(\mathbf{x}_1-\mathbf{y})\cdot(\mathbf{x}_2-\mathbf{y})}{|\mathbf{x}_1-\mathbf{y}|^2\,|\mathbf{x}_2-\mathbf{y}|^2} + \frac{1}{|\mathbf{x}_2-\mathbf{y}|^2}\right)$$

$$= 4\frac{|(\mathbf{x}_2-\mathbf{y})-(\mathbf{x}_1-\mathbf{y})|^2}{|\mathbf{x}_1-\mathbf{y}|^2\,|\mathbf{x}_2-\mathbf{y}|^2} = \frac{4\,|\mathbf{x}-\mathbf{x}_2|^2}{|\mathbf{x}_1-\mathbf{y}|^2\,|\mathbf{x}_2-\mathbf{y}|^2},$$

which implies (A.8.3). Relation (A.8.3) implies that the mapping $K_{\mathbf{y}}$ is injective.

Now let $\mathbf{z} \in H_{\mathbf{y}}$ be arbitrary point. For $\mathbf{x}_0 := \mathbf{y} + \frac{2(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2}$, we have $\mathbf{x}_0 \neq \mathbf{y}$ and

$$|\mathbf{x}_0|^2 = 1 + \frac{4\mathbf{y}\cdot(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2} + \frac{4}{|\mathbf{z}-\mathbf{y}|^2} = 1;$$

i.e., $\mathbf{x}_0 \in S^d \setminus \{\mathbf{y}\}$. Furthermore,

$$K_{\mathbf{y}}(\mathbf{x}_0) = \mathbf{y} + \frac{2(\mathbf{x}_0-\mathbf{y})}{|\mathbf{x}_0-\mathbf{y}|^2} = \mathbf{y} + \frac{4(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2\,|\mathbf{x}_0-\mathbf{y}|^2} = \mathbf{y} + (\mathbf{z}-\mathbf{y}) = \mathbf{z},$$

which implies that $K_{\mathbf{y}}$ is surjective, and, hence, bijective. Moreover, $K_{\mathbf{y}}^{-1}(\mathbf{z}) = \mathbf{x}_0$. □

Equalities (A.8.3) and (A.8.2) imply the following inversion of the distance formula (A.8.3).

Corollary A.8.2 *For every* $\mathbf{x}_1, \mathbf{x}_2 \in S^d \setminus \{\mathbf{y}\}$, *there holds*

$$|\mathbf{x}_1 - \mathbf{x}_2| = \frac{2\,|K_{\mathbf{y}}(\mathbf{x}_1) - K_{\mathbf{y}}(\mathbf{x}_2)|}{|K_{\mathbf{y}}(\mathbf{x}_1) - \mathbf{y}|\,|K_{\mathbf{y}}(\mathbf{x}_2) - \mathbf{y}|}.$$

A.9 Homogeneous Polynomials and Spherical Harmonics

Let $\mathcal{P}_n$, $n \geq 0$, be the space of all degree n homogeneous polynomials in $d+1$ variables with real coefficients and H_n be the subspace of $\mathcal{P}_n$ consisting of polynomials harmonic on $\mathbb{R}^{d+1}$. The main goal of this section is to prove Theorem A.9.5.

Given polynomials $p(x) = \sum_{\alpha \in G_n} c_\alpha x^\alpha$ and $q(x) = \sum_{\alpha \in G_n} \widetilde{c}_\alpha x^\alpha$ in $\mathcal{P}_n$, we define

$$\langle p, q \rangle_n := \sum_{\alpha \in G_n} c_\alpha \widetilde{c}_\alpha \alpha!$$

where $\alpha! := \alpha_1! \cdot \ldots \cdot \alpha_{d+1}!$: and $G := \{(k_1, \ldots, k_{d+1}) \in \mathbb{Z}^{d+1} : k_i \geq 0, i = \overline{1, d+1}, \sum_{i=1}^{d+1} k_i \leq n\}$. It is not difficult to verify that $\langle p, q \rangle_n$ is a dot-product on $\mathcal{P}_n$. Observe that if p and q are monomials of degree n whose coefficients both equal 1, and $p \neq q$, then $\langle p, q \rangle_n = 0$. If $p(x) = q(x) = x^\alpha$, then $\langle p, q \rangle_n = \alpha!$.

We will use the following material from Linear Algebra.

Definition A.9.1 Let X and Y be inner product spaces and $L : X \to Y$ be a linear operator. An operator $L^* : Y \to X$ is called an *adjoint* for L if for every $u \in X$ and $v \in Y$, we have $\langle Lu, v \rangle_Y = \langle u, L^* v \rangle_X$.

Recall that $\operatorname{Ker}(L) = \{x \in X : L(x) = 0_Y\}$. The following statement is well known.

Theorem A.9.2 *Let X and Y be finite-dimensional inner product spaces and let $L : X \to Y$ be a linear operator. Then*

$$Y = L(X) \oplus \operatorname{Ker}(L^*),$$

where $\oplus$ is the direct orthogonal sum with respect to the dot-product in Y.

Define the mapping $T : \mathcal{P}_n \to \mathcal{P}_{n+2}$ by $T(p)(x) := |x|^2 p(x)$, $p \in \mathcal{P}_n$. Let $\Delta : \mathcal{P}_{n+2} \to \mathcal{P}_n$ be the Laplace operator. With the dot-product defined above the following statement holds true.

Lemma A.9.3 *For any $d \geq 1$ and $n \geq 0$, we have $T^* = \Delta$.*

Proof Let $p(x_1, \ldots, x_{d+1}) = x_1^{k_1} \cdot \ldots \cdot x_{d+1}^{k_{d+1}}$ and $q(x_1, \ldots, x_{d+1}) = x_1^{l_1} \cdot \ldots \cdot x_{d+1}^{l_{d+1}}$ be any monomials such that $k := (k_1, \ldots, k_{d+1}) \in G_n$ and $l := (l_1, \ldots, l_{d+1}) \in G_{n+2}$. Denote by $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}$ the standard basis vectors in $\mathbb{R}^{d+1}$. Then

$$T(p)(x) = \left(x_1^2 + \ldots + x_{d+1}^2\right) \cdot x_1^{k_1} \cdot \ldots \cdot x_{d+1}^{k_{d+1}} = \sum_{i=1}^{d+1} x^{k+2\mathbf{e}_i}.$$

Consequently, $\langle T(p), q\rangle_{n+2} = 0$ if and only if $k + 2\mathbf{e}_i \neq l$ for any i. If for some i, we have $k + 2\mathbf{e}_i = l$, then $\langle Tp, q\rangle_{n+2} = l! = l_1! \cdot \ldots \cdot l_{d+1}!$.

Observe that $\Delta q(x) = \sum_{j=1}^{d+1} l_j(l_j - 1)x^{l-2\mathbf{e}_j}$. If $k \neq l - 2\mathbf{e}_i$ for any i, then $\langle p, \Delta q\rangle_n = 0 = \langle Tp, q\rangle_{n+2}$. If $k = l - 2\mathbf{e}_i$ for some i, then $\langle p, \Delta q\rangle_n = k! = l_i(l_i - 1)(l - 2\mathbf{e}_i)! = l! = \langle Tp, q\rangle_{n+2}$. Thus, we have $\langle Tp, q\rangle_{n+2} = \langle p, \Delta q\rangle_n$ for any monomials $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_{n+2}$. By linearity, this equality can be extended to all pairs of polynomials $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_{n+2}$, which by Definition A.9.1 implies that $T^* = \Delta$. □

Observe that Ker $\Delta = H_{n+2}$. Then Theorem A.9.2 and Lemma A.9.3 immediately imply the following statement.

Theorem A.9.4 *For any $d \geq 1$ and $n \geq 0$, we have $\mathcal{P}_{n+2} = T(\mathcal{P}_n) \oplus H_{n+2}$.*

Theorem A.9.2 implies the following result used for characterizing spherical designs.

Theorem A.9.5 *Let $d \geq 1$, $n \geq 0$, and $q \in \mathcal{P}_n$. Then there exist spherical harmonics $Y_i \in \mathbb{H}^d_{n-2i}$, $i = 0, 1, \ldots, \lfloor n/2\rfloor$ such that*

$$q(x) = \sum_{i=0}^{\lfloor n/2\rfloor} Y_i(x), \quad x \in S^d.$$

Proof If $n = 0$ or $n = 1$, then $q \in H_n$ and the assertion of the theorem holds trivially. Assume that $n \geq 2$ and let $l := \lfloor n/2\rfloor$. By Theorem A.9.2, there are $q_1 \in \mathcal{P}_{n-2}$ and $p_0 \in H_n$ such that $q(y) = |y|^2 q_1(y) + p_0(y)$, $y \in \mathbb{R}^{d+1}$. Applying Theorem A.9.2 repeatedly (starting with polynomial q_1), one can find polynomials $q_l \in \mathcal{P}_{n-2l}$ and $p_i \in H_{n-2i}$, $i = 0, 1, \ldots, l-1$, such that

$$q(y) = |y|^{2l} q_l(y) + |y|^{2(l-1)} p_{l-1}(y) + \cdots + |y|^2 p_1(y) + p_0(y), \quad y \in \mathbb{R}^{d+1}.$$

Since q_ℓ has degree at most one, it is harmonic and, hence, $q_\ell|_{S^d} \in \mathbb{H}^d_{n-2l}$. Furthermore, $p_i|_{S^d} \in \mathbb{H}^d_{n-2i}$, $i = 0, 1, \ldots, l-1$. Since for every $x \in S^d$,

$$q(x) = q_l(x) + p_{l-1}(x) + \ldots + p_1(x) + p_0(x),$$

the assertion of the theorem follows. □

The previous theorem provides a method for computing the dimension of the linear space $\mathcal{Q}^{d+1}_n$ consisting of all polynomials in $d+1$ variables restricted to S^d and having degree up to and including n. First note that Theorem A.9.5 implies

$$\mathcal{Q}^{d+1}_n\big|_{S^d} = \mathbb{H}^{d+1}_0 \oplus \cdots \oplus \mathbb{H}^{d+1}_n.$$

Second, recalling from (5.1.2) and (5.3.6) that the dimension of $\mathbb{H}_k^d$ is

$$Z(d,k) = \binom{k+d-1}{d-1} + \binom{k+d-2}{d-1} = \binom{k+d}{d} - \binom{k+d-2}{d}, \quad \text{(A.9.1)}$$

we have

$$\dim \left(\mathcal{Q}_n^{d+1}\big|_{S^d} \right) = \sum_{k=0}^{n} Z(d,k) = \binom{n+d}{d} + \binom{n-1+d}{d} = Z(d+1,n). \quad \text{(A.9.2)}$$

Thus, for example, on S^2 we see from the formula (5.1.1) that the space of such polynomials has dimension $Z(3,n) = (n+1)^2$.

A.10 A Basic Fact on Smooth Manifolds

Recall that a set B in $\mathbb{R}^m$ is *bi-Lipschitz homeomorphic to a set* $D \subset \mathbb{R}^n$ *with a constant* $M \geq 1$, if there is a mapping $\varphi : B \to D$ such that $\varphi(B) = D$ and

$$M^{-1} |x-y| \leq |\varphi(x) - \varphi(y)| \leq M |x-y|, \quad x,y \in B.$$

We use Definition 9.5.2 as the definition of a d-dimensional C^1-manifold in $\mathbb{R}^p$. In Chapters 9 and 14, we use the following statement.

Lemma A.10.1 *Let* $U \subset \mathbb{R}^d$ *be a nonempty open set and* $f : U \to \mathbb{R}^p$, $p \geq d$, *be an injective* C^1*-mapping such that its inverse* $f^{-1} : f(U) \to U$ *is continuous and the Jacobian matrix*

$$J_x^f := \begin{bmatrix} \nabla f_1(x) \\ \cdots \\ \nabla f_p(x) \end{bmatrix} \quad \text{(A.10.1)}$$

of f *has rank* d *at any point* $x \in U$. *Then for every* $\epsilon > 0$ *and every point* $y_0 \in f(U)$, *there is a closed ball* $B \subset \mathbb{R}^p$ *centered at* y_0 *such that the set* $B \cap f(U)$ *is bi-Lipschitz homeomorphic to some compact set in* $\mathbb{R}^d$ *with a constant* $1 + \epsilon$.

Proof Let $x_0 \in U$ be the point such that $f(x_0) = y_0$. Choose any $\epsilon > 0$ and let $\delta = \delta(x_0, \epsilon) > 0$ be such that $B[x_0, \delta] \subset U$ and

$$|\nabla f_i(x) - \nabla f_i(x_0)| < \epsilon, \quad x \in B[x_0, \delta], \ i = 1, \ldots, p.$$

Let $x, y \in B[x_0, \delta]$ be two arbitrary points. Define the function $g_i(t) := f_i(x + t(y-x))$, $t \in [0,1]$. Then there exists $\xi_i \in (0,1)$ such that

$$f_i(y) - f_i(x) = g_i(1) - g_i(0) = g_i'(\xi_i) = \nabla f_i(z_i) \cdot (y - x)$$

$$= \nabla f_i(x_0) \cdot (y - x) + (\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x),$$

where $z_i = x + \xi_i(y - x)$, $i = 1, \ldots, p$. Since $z_i \in B[x_0, \delta]$, we have

$$|f_i(y) - f_i(x) - \nabla f_i(x_0) \cdot (y - x)|$$

$$= |(\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x)| \le \epsilon |y - x|, \quad i = 1, \ldots, p,$$

and hence (we treat x and y as vector-columns below),

$$\left|f(y) - f(x) - J_{x_0}^f (y - x)\right| \le \tau |y - x|, \quad x, y \in B[x_0, \delta], \tag{A.10.2}$$

where $\tau := \epsilon\sqrt{p}$ (differentiability of f at x_0 immediately gives only an upper bound for the left-hand side of (A.10.2) in terms of $|x - x_0|$ and $|y - x_0|$ and not in terms of $|x - y|$, which is why we had to do the additional work). Since the matrix $J_{x_0}^f$ has rank d, for every standard basis vector e_i from $\mathbb{R}^d$, there is a vector $v_i \in R^p$ such that $(J_{x_0}^f)^T v_i = e_i$, $i = 1, \ldots, d$, where $(J_{x_0}^f)^T$ denotes the transpose of the matrix $J_{x_0}^f$. Then the $d \times p$ matrix $Z := [v_1, \ldots, v_d]^T$ satisfies $Z J_{x_0}^f = I_d$, where I_d is the $d \times d$ identity matrix. Taking into account (A.10.2) we have

$$\left|f(y) - f(x) - J_{x_0}^f (y - x)\right| \le \tau \left|Z J_{x_0}^f (y - x)\right|$$

$$\le \tau \|Z\| \left|J_{x_0}^f (y - x)\right|, \quad x, y \in B[x_0, \delta],$$

where $\|Z\| := \max\{|Zu| : u \in \mathbb{R}^p,\ |u| = 1\}$. Consequently,

$$(1 - \tau\|Z\|) \left|J_{x_0}^f (y - x)\right| \le |f(y) - f(x)|$$

$$\le (1 + \tau\|Z\|) \left|J_{x_0}^f (y - x)\right|, \quad x, y \in B[x_0, \delta].$$

Let $u_1, \ldots, u_d$ be an orthonormal basis in the subspace H of R^p spanned by the columns of the matrix $J_{x_0}^f$ and let $D := [u_1, \ldots, u_d]$ be the $p \times d$ matrix with columns $u_1, \ldots, u_d$. Since the columns of $J_{x_0}^f$ also form a basis in H, there exists an invertible $d \times d$ matrix Q such that $D = J_{x_0}^f Q$.

Let $V \subset \mathbb{R}^d$ be the open set such that $\Phi(V) = B(x_0, \delta)$, where $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is the linear mapping given by $\Phi(v) = Qv$. Since the columns of the matrix D are orthonormal, for every $u, v \in \overline{V}$, we will have

$$|f \circ \Phi(u) - f \circ \Phi(v)| = |f(Qu) - f(Qv)|$$

$$\le (1 + \tau\|Z\|) \left|J_{x_0}^f Q(u - v)\right| = (1 + \tau\|Z\|) |D(u - v)|$$

$$= (1 + \tau\|Z\|)\,|u - v|\,.$$

Similarly,

$$|f \circ \Phi(u) - f \circ \Phi(v)| \geq (1 - \tau\|Z\|)\,|u - v|\,, \quad u, v \in \overline{V},$$

which implies that for $0 < \epsilon < (\sqrt{p}\|Z\|)^{-1}$, the restriction of the mapping $\psi := f \circ \Phi$ to the set $\overline{V}$ is a bi-Lipschitz mapping onto the set $f(\Phi(\overline{V})) = f(B[x_0, \delta])$ with constant $M_\epsilon := \max\{1 + \tau\|Z\|, (1 - \tau\|Z\|)^{-1}\}$.

Since f is a homeomorphism of U onto $f(U)$, the set $f(B(x_0, \delta))$ is open relative to $f(U)$. Then there is a closed ball B in $\mathbb{R}^p$ centered at $y_0 = f(x_0)$ such that $B \cap f(U) \subset f(B(x_0, \delta))$. Then the set $B \cap f(U) = B \cap f(B[x_0, \delta])$ is bi-Lipschitz homeomorphic (with constant M_ϵ) to the set

$$V_1 := \psi^{-1}(B \cap f(U)) = \psi^{-1}(B \cap f(B[x_0, \delta])),$$

which is compact in $\mathbb{R}^d$. Since $M_\epsilon \to 1$ as $\epsilon \to 0^+$, the assertion of the lemma follows. □

A.11 Energy and Potential of the Normalized Surface Area Measure on S^d

In this section, we calculate the energy

$$I_{K_f}[\sigma_d] = \int\limits_{S^d}\int\limits_{S^d} f\left(|\mathbf{x} - \mathbf{y}|^2\right)\, d\sigma_d(\mathbf{x})\, d\sigma_d(\mathbf{y}), \tag{A.11.1}$$

of the normalized surface area (probability) measure σ_d on S^d with respect to the kernel $K_f(\mathbf{x}, \mathbf{y}) = f(|\mathbf{x} - \mathbf{y}|^2)$ and the potential of σ_d with respect to this kernel.

Proposition A.11.1 *Let $f : (0, \infty) \to \mathbb{R}$ be a continuous function such that the integral (A.11.1) is finite. Then for every $\mathbf{y} \in \mathbb{R}^{d+1}$ with $|\mathbf{y}| = R > 0$, there holds*

$$U^{\sigma_d}_{K_f}(\mathbf{y}) = 2^{d-1}\gamma_d \int_0^1 f\left((R+1)^2 - 4Ru\right) u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\, du,$$

where $\gamma_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)}$.

Proof Using the Funk–Hecke formula (see formula (5.1.9)) we will obtain

$$U_{K_f}^{\sigma_d}(\mathbf{y}) = \int\limits_{S^d} f\left(|\mathbf{x}-\mathbf{y}|^2\right)\, d\sigma_d(\mathbf{x}) = \int\limits_{S^d} f\left(R^2+1-2R\mathbf{x}\cdot\frac{\mathbf{y}}{R}\right)\, d\sigma_d(\mathbf{x})$$
$$= \gamma_d \int_{-1}^{1} f\left(R^2+1-2Rt\right)\left(1-t^2\right)^{\frac{d}{2}-1}\, dt.$$

Making the substitution $2u = 1+t$ in the last integral, we have

$$U_{K_f}^{\sigma_d}(\mathbf{y}) = 2^{d-1}\gamma_d \int_0^1 f\left((R+1)^2-4Ru\right) u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\, du. \qquad \square$$

Proposition A.11.1 implies the following statement.

Proposition A.11.2 *Let $f:(0,4]\to\mathbb{R}$ be a continuous function such that the integral $I_{K_f}[\sigma_d]$ is finite. Then*

$$I_{K_f}[\sigma_d] = \frac{2^{d-1}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2}\right)} \int\limits_0^1 f(4u)u^{\frac{d-2}{2}}(1-u)^{\frac{d-2}{2}}\, du.$$

Proof Fix any $\mathbf{y}\in S^2$ $(R=1)$. Taking into account Proposition A.11.1 and making the substitution $u=1-t$, we will obtain

$$I_{K_f}[\sigma_d] = U_{K_f}^{\sigma_d}(\mathbf{y}) = 2^{d-1}\gamma_d \int_0^1 f\left(4-4t\right) t^{\frac{d}{2}-1}(1-t)^{\frac{d}{2}-1}\, dt$$
$$= 2^{d-1}\gamma_d \int_0^1 f(4u)u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\, du,$$

which implies the required identity. $\square$

When K_f is the Riesz s-kernel, $0<s<d$, the potential $U_{K_f}^{\sigma_d}$ has the following representation.

Proposition A.11.3 *Let $d\in\mathbb{N}$. Then for $0<s<d$ and every $\mathbf{y}\in\mathbb{R}^{d+1}$ with $R=|\mathbf{y}|>0$, we have*

$$U_s^{\sigma_d}(\mathbf{y}) = \int\limits_{S^d} \frac{d\sigma_d(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^s} = (R+1)^{-s}\,{}_2F_1\left(\frac{s}{2},\frac{d}{2};d;\frac{4R}{(R+1)^2}\right). \qquad \text{(A.11.2)}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined in (A.6.1).

Proof From Proposition A.11.1 taking into account (A.6.2), we obtain

$$U_s^{\sigma_d}(\mathbf{y}) = \frac{2^{d-1}\gamma_d}{(R+1)^s}\int_0^1 \left(1-\frac{4Ru}{(R+1)^2}\right)^{-\frac{s}{2}} u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\,du$$
$$= \frac{2^{d-1}\gamma_d\left(\Gamma\left(\frac{d}{2}\right)\right)^2}{\Gamma\left(d\right)(R+1)^s}\cdot {}_2F_1\left(\frac{s}{2},\frac{d}{2};d;\frac{4R}{(R+1)^2}\right).$$

By the duplication formula for the Gamma-function (see Section 1.9), we have

$$\frac{\gamma_d\left(\Gamma\left(\frac{d}{2}\right)\right)^2}{\Gamma\left(d\right)} = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma(d)} = 2^{1-d},$$

and (A.11.2) follows. □

For the Riesz and logarithmic potentials, energy integral (A.11.1) has the following value.

Corollary A.11.4 *Let $d\in\mathbb{N}$. Then for $0<s<d$, we have*

$$I_s[\sigma_d] = \frac{2^{d-s-1}}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-s}{2}\right)}{\Gamma\left(d-\frac{s}{2}\right)} = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(d-s)}{\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(d-\frac{s}{2}\right)}, \tag{A.11.3}$$

and for $-2<s<0$, there holds

$$I_s[\sigma_d] = -\frac{2^{d-s-1}}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-s}{2}\right)}{\Gamma\left(d-\frac{s}{2}\right)} = -\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(d-s)}{\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(d-\frac{s}{2}\right)}. \tag{A.11.4}$$

Furthermore,

$$I_{\log}[\sigma_d] = -\log 2 + \frac{1}{2}\left(\psi(d)-\psi\left(\frac{d}{2}\right)\right), \tag{A.11.5}$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function. In particular,

$$I_s[\sigma_2] = \frac{2^{1-s}}{2-s},\quad 0<s<2,$$

and

$$I_{\log}[\sigma_2] = 1/2-\log 2.$$

Proof When $0<s<d$, applying Proposition A.11.2 with $f(t)=t^{-s/2}$, we will have

$$I_s[\sigma_d] = \frac{2^{d-s-1}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}B\left(\frac{d-s}{2},\frac{d}{2}\right) = \frac{2^{d-s-1}}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d-s}{2}\right)}{\Gamma\left(d-\frac{s}{2}\right)},$$

and the first equality in (A.11.3) follows. In particular, when $d = 2$ we have

$$I_s[\sigma_2] = \frac{2^{1-s}\Gamma\left(\frac{3}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\sqrt{\pi}\Gamma\left(2-\frac{s}{2}\right)} = \frac{2^{1-s}}{2-s}, \quad 0 < s < 2.$$

Using the duplication formula (see Section 1.9), we have $\Gamma\left(\frac{d-s}{2}\right)\Gamma\left(\frac{d-s+1}{2}\right) = 2^{1-d+s}\sqrt{\pi}\Gamma(d-s)$, which yields the second equality in (A.11.3). To show (A.11.4) we apply Proposition A.11.2 with $f(t) = -t^{-s/2}$ and apply the same argument. Finally, to show (A.11.5), we apply Proposition A.11.2 with $f(t) = \frac{1}{2}\log\frac{1}{t}$ and obtain

$$\begin{aligned} I_{\log}[\sigma_d] &= \frac{2^{d-2}\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} \int_0^1 \log\frac{1}{4u} \cdot u^{\frac{d}{2}-1}(1-u)^{\frac{d}{2}-1}\,du \\ &= \frac{2\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} \cdot \frac{d}{ds}\int_0^1 (4u)^{\frac{d-s}{2}-1}(1-u)^{\frac{d}{2}-1}\,du\,\Bigg|_{s=0} \\ &= \frac{2\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)} \cdot \frac{d}{ds}B\left(\frac{d-s}{2},\frac{d}{2}\right)\Bigg|_{s=0} \\ &= \frac{2^{d-1}}{\sqrt{\pi}}\Gamma\left(\frac{d+1}{2}\right) \cdot \frac{d}{ds}\frac{\Gamma\left(\frac{d-s}{2}\right)}{2^s\Gamma\left(d-\frac{s}{2}\right)}\Bigg|_{s=0} \\ &= \frac{2^{d-1}\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma(d)}\left(-\log 2 + \frac{\Gamma'(d)}{2\Gamma(d)} - \frac{\Gamma'\left(\frac{d}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)}\right). \end{aligned} \tag{A.11.6}$$

The duplication formula (see Section 1.9) implies that $\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d}{2}\right) = 2^{1-d}\sqrt{\pi}\Gamma(d)$. Consequently, $I_{\log}[\sigma_d] = -\log 2 + (\psi(d) - \psi(d/2))/2$. Finally, when $d = 2$, from the first equality in (A.11.6) we have

$$I_{\log}[\sigma_2] = \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}}\int_0^1 \log\frac{1}{4u}\,du = \frac{1}{2} - \log 2. \qquad \square$$

A.12 The Limiting Position of Five Minimal s-Energy Points on S^2

In this section of the Appendix, we present the result by Bondarenko, Hardin, and Saff [37] on the limit of s-energy minimizing 5-point configurations on the sphere S^2 as s gets large.

Theorem A.12.1 *Let Q' be a cluster point of a family of* 5*-point s-energy minimizing configurations on S^2 as $s \to \infty$. Then Q' is isometric to the configuration*

$$Q = \mathrm{SBP}(\infty) := \{\mathrm{e}_1, -\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3, -\mathrm{e}_3\}, \tag{A.12.1}$$

where $\mathrm{e}_1 = (1, 0, 0)$, $\mathrm{e}_2 = (0, 1, 0)$, *and* $\mathrm{e}_3 = (0, 0, 1)$.

We start the proof with an upper estimate for the minimal 5-point s-energy on S^2.

Lemma A.12.2

$$\limsup_{s\to\infty} 2^{s/2}\mathcal{E}_s(S^2, 5) \le 8.$$

Proof For arbitrary $0 < t < 1$, we define the following 5-point configuration on S^2:

$$Q_t := \{(\pm\sqrt{1-t^2}, -t, 0), (0, -t, \pm\sqrt{1-t^2}), \mathrm{e}_2\}, \tag{A.12.2}$$

which, for a suitable choice of t (depending on s), is a conjectured minimal energy configuration on S^2 for every s large enough. The s-energy of this configuration is given by

$$E_s(Q_t) := 4 \cdot 2^{-s}(1-t^2)^{-s/2} + 8 \cdot 2^{-s/2}(1-t^2)^{-s/2} + 8 \cdot 2^{-s/2}(1+t)^{-s/2}.$$

Letting now $t = s^{-2/3}$, we obtain that

$$\lim_{s\to\infty}(1-t^2)^{-s/2} = 1 \quad \text{and} \quad \lim_{s\to\infty}(1+t)^{-s/2} = 0,$$

and so

$$\limsup_{s\to\infty} 2^{s/2}\mathcal{E}_s(S^2, 5) \le \lim_{s\to\infty} 2^{s/2}E_s(Q_t)$$

$$= \lim_{s\to\infty}(4 \cdot 2^{-s/2}(1-t^2)^{-s/2} + 8(1-t^2)^{-s/2} + 8(1+t)^{-s/2}) = 8. \qquad \square$$

We further need the following statement.

Lemma A.12.3 *Let A, B, and M be fixed positive constants. Then*

$$f(x) := M(1-Ax)^{-s} + (1+Bx)^{-s} \ge M + \min\{1, AM/B\}$$

for every $x \in [0, 1/A)$ and $s > 0$.

Proof It is not difficult to see that f attains its minimum on $[0, 1/A)$ at the point $x_0 = 0$ if $B \le AM$ and at the point

$$x_1 = \frac{(B/(AM))^{1/(s+1)} - 1}{B + A(B/(AM))^{1/(s+1)}}$$

if $B > AM$. In the first case we have

$$f(x) \geq f(0) = M + 1, \quad x \in [0, 1/A), \quad s > 0.$$

In the second case, since

$$x_1 \leq \frac{1}{B}\left[(B/(AM))^{1/(s+1)} - 1\right],$$

we have

$$f(x) \geq f(x_1) \geq M + (1 + Bx_1)^{-s} \geq M + (B/(AM))^{-s/(s+1)} > M + AM/B$$

for all $x \in [0, 1/A)$ and $s > 0$. Combining the results in both cases, we obtain the assertion of the lemma. □

Proof of Theorem A.12.1. As we mentioned in Proposition 3.1.2, any cluster point of a family of s-energy minimizing configurations as $s \to \infty$ is a best-packing configuration. Thus, by Theorem 3.3.2, it is sufficient to show that no 5-point configuration consisting of two opposite poles and an acute triangle on the equator (which we call an *acute configuration*) could be such a cluster point. We will prove this by contradiction. For s large, consider a minimal s-energy configuration that is 'close' to a fixed acute configuration. We may assume that this minimal s-energy configuration $\omega_5(s)$ consists of three points

$$A_1 = A_{1s} = (a_{11s}, a_{12s}, h), \qquad A_2 = A_{2s} = (a_{21s}, a_{22s}, h),$$

$$A_3 = A_{3s} = (a_{31s}, a_{32s}, h),$$

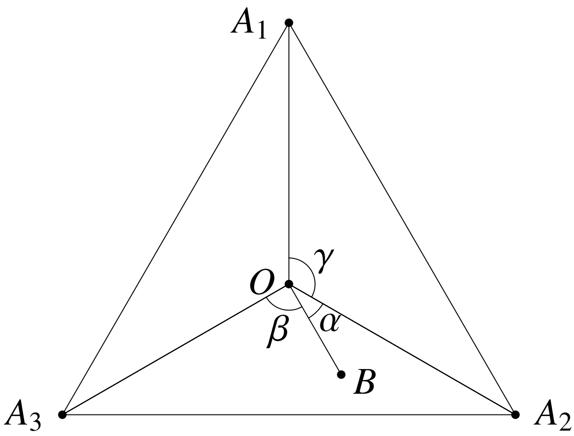
where $h = h_s = o(1)$ as $s \to \infty$, that are close to the vertices of a fixed acute triangle on the equator, and two points $A_4 = A_{4s}$ and $A_5 = A_{5s}$ that are close to $(0, 0, 1)$ and $(0, 0, -1)$, respectively. Denote by

$$E_1 := E_{1s} = \sum_{i=1}^{3} |A_4 - A_i|^{-s}, \quad \text{and} \quad E_2 := E_{2s} = \sum_{i=1}^{3} |A_5 - A_i|^{-s}.$$

Clearly, the total s-energy $E_s(\omega_5(s)) > 2E_1 + 2E_2$.

Let us first estimate E_1 from below. Denote by O the point $(0, 0, h)$, by B the projection of A_4 to the plane $A_1A_2A_3$, and by x the length $|O - B|$. Without loss of generality, we may assume that B lies in the triangle OA_2A_3. Here we use the facts that $x = x_s = o(1)$ as $s \to \infty$, and that $A_1A_2A_3$

Fig. A.1 Projection B of A_4 on horizontal $A_1A_2A_3$ plane



is "close" to a fixed acute triangle implying that O lies inside the triangle $A_1A_2A_3$. Denote by $\alpha = \alpha_s$, $\beta = \beta_s$, and $\gamma = \gamma_s$ the angles A_2OB, A_3OB, A_2OA_1, respectively (see Figure A.1).

Since

$$E_1 = \sum_{i=1}^{3} |A_4 - A_i|^{-s} = \sum_{i=1}^{3} \left(|B - A_4|^2 + |B - A_i|^2\right)^{-s/2},$$

we have, by the law of cosines and the fact that $|B - A_4| = \sqrt{1 - x^2} - h$,

$$\begin{aligned} E_1 &= (2 - 2h\sqrt{1 - x^2} - 2x\sqrt{1 - h^2}\cos\alpha)^{-s/2} \\ &\quad + (2 - 2h\sqrt{1 - x^2} - 2x\sqrt{1 - h^2}\cos\beta)^{-s/2} \\ &\quad + (2 - 2h\sqrt{1 - x^2} - 2x\sqrt{1 - h^2}\cos(\alpha + \gamma))^{-s/2}. \end{aligned}$$

The crucial observation is the fact that $\alpha + \beta < \tau < \pi$, for some τ that does not depend on s. Now monotonicity and convexity of the function $t^{-s/2}$, $t > 0$, immediately imply

$$\begin{aligned} E_1 &\geq 2\Big(2 - 2h\sqrt{1 - x^2} - x\sqrt{1 - h^2}(\cos\alpha + \cos\beta)\Big)^{-s/2} \\ &\qquad + (2 - 2h\sqrt{1 - x^2} + 2x)^{-s/2} \\ &\geq 2\Big(2 - 2h\sqrt{1 - x^2} - x\sqrt{1 - h^2}(1 + \cos\tau)\Big)^{-s/2} \\ &\qquad + (2 - 2h\sqrt{1 - x^2} + 2x)^{-s/2}. \end{aligned} \tag{A.12.3}$$

From the facts that $x = o(1)$, and $h = o(1)$ as $s \to \infty$ and the inequality $1 - x \leq \sqrt{1 - x^2} \leq 1$, we get that

$$E_1 \geq 2\,(2 - 2h - \theta_1 x)^{-s/2} + (2 - 2h + 3x)^{-s/2},$$

for some absolute constant $\theta_1 > 0$. Then, by Lemma A.12.3,

$$E_1 \geq (2 + \theta_2)(2 - 2h)^{-s/2},$$

for some absolute constant $\theta_2 > 0$. Similarly we obtain

$$E_2 \geq (2 + \theta_2)(2 + 2h)^{-s/2},$$

and so again applying the convexity of $t^{-s/2}$ we finally deduce that, for s sufficiently large,

$$E_s(\omega_5(s)) > 2(E_1 + E_2) \geq (8 + 4\theta_2)\, 2^{-s/2}. \tag{A.12.4}$$

On the other hand, from Lemma A.12.2, we know that $\mathcal{E}_s(S^2, 5) \leq (8 + o(1))2^{-s/2}$. Therefore, by (A.12.4), an acute configuration cannot be a cluster point of minimal s-energy configurations as $s \to \infty$. □

The following statement is a consequence of Theorem A.12.1.

Proposition A.12.4 *For $N = 5$ and every $s > 0$ sufficiently large, the triangular bipyramid P defined by (2.5.1) is not an s-energy minimizing configuration on S^2.*

Proof Assume to the contrary that P is an s-energy minimizing configuration for every s from some positive sequence $\mathcal{S}$ that increases to ∞. Then P is the cluster point of a family of s-energy minimizing configurations with $s \in \mathcal{S}$. By Theorem A.12.1, P must be isometric to configuration in (A.12.1), which is a contradiction. □

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. National Bureau of Standards Applied Mathematics Series, vol. 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. (1964)
2. A. Abrams, J. Cantarella, J.H.G. Fu, M. Ghomi, R. Howard, Circles minimize most knot energies. Topology **42**(2), 381–394 (2003)
3. J. Aczel, Solution to problem 35. Mat. Lapok **3**, 94–95 (1952)
4. R. Alexander, On the sum of distances between n points on a sphere. Acta Math. Acad. Sci. Hung. **23**, 443–448 (1972)
5. G. Ambrus, Analytic and probabilistic problems in discrete geometry. Thesis (Ph.D.)–University College London (2009)
6. G. Ambrus, K.M. Ball, T. Erdélyi, Chebyshev constants for the unit circle. Bull. Lond. Math. Soc. **45**(2), 236–248 (2013)
7. N.N. Andreev, An extremal property of the icosahedron. East J. Approx. **2**(4), 459–462 (1996)
8. N.N. Andreev, Location of points on a sphere with minimal energy. Tr. Mat. Inst. Steklova **219**(Teor. Priblizh. Garmon. Anal.), 27–31 (1997)
9. N.N. Andreev, A minimal design of order 11 on the three-dimensional sphere. Mat. Zametki **67**(4), 489–497 (2000)
10. G.E. Andrews, R. Askey, R. Roy, *Special Functions*. Encyclopedia of Mathematics and Its Applications, vol. 71 (Cambridge University Press, Cambridge, 1999)
11. S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*. Graduate Texts in Mathematics, vol. 137, 2nd edn. (Springer, New York, 2001)
12. K. Ball, A lower bound for the optimal density of lattice packings. Int. Math. Res. Not. **10**, 217–221 (1992)
13. R. Bauer, Distribution of points on a sphere with application to star catalogs. J. Guid. Control. Dyn. **23**(1), 130–137 (2000)
14. A. Bausch, M. Bowick, A. Cacciuto, A. Dinsmore, M. Hsu, D. Nelson, M. Nikolaides, A. Travesset, D. Weitz, Grain boundary scars and spherical crystallography. Science **299**(5613), 1716–1718 (2003)
15. J. Beck, Some upper bounds in the theory of irregularities of distribution. Acta Arith. **43**(2), 115–130 (1984)

S. V. Borodachov et al., *Discrete Energy on Rectifiable Sets*,
Springer Monographs in Mathematics,
https://doi.org/10.1007/978-0-387-84808-2

16. J. Beck, Sums of distances between points on a sphere—an application of the theory of irregularities of distribution to discrete geometry. Mathematika **31**(1), 33–41 (1984)
17. J. Beck, W.W.L. Chen, *Irregularities of Distribution*. Cambridge Tracts in Mathematics, vol. 89 (Cambridge University Press, Cambridge, 2008). Reprint of the 1987 original [MR0903025]
18. C. Beltrán, Harmonic properties of the logarithmic potential and the computability of elliptic Fekete points. Constr. Approx. **37**(1), 135–165 (2013)
19. C. Beltrán, N. Corral, J.G. Criado del Rey, Discrete and continuous green energy on compact manifolds. J. Approx. Theory **237**, 160–185 (2019)
20. C. Beltrán, U. Etayo, A generalization of the spherical ensemble to even-dimensional spheres
21. C. Beltrán, U. Etayo, The projective ensemble and distribution of points in odd-dimensional spheres. Constr. Approx. **48**(1), 163–182 (2018)
22. C. Beltrán, U. Etayo, J. Marzo, J. Ortega-Cerdà, A sequence of polynomials with optimal condition number (2019)
23. C. Beltrán, J. Marzo, J. Ortega-Cerdà, Energy and discrepancy of rotationally invariant determinantal point processes in high dimensional spheres. J. Complex. **37**, 76–109 (2016)
24. J.L. Bentley, D.F. Stanat, E.H. Williams Jr., The complexity of finding fixed-radius near neighbors. Inf. Process. Lett. **6**(6), 209–212 (1977)
25. D. Bergersen, B. Boal, P. Palffy-Muhoray, Equilibrium configurations of particles on the sphere: the case of logarithmic interactions. J. Phys. A: Math. Gen. **27**, 2579–2586 (1994)
26. R. Berman, S. Boucksom, D. Witt Nyström, Fekete points and convergence towards equilibrium measures on complex manifolds. Acta Math. **207**(1), 1–27 (2011)
27. L. Bétermin, M. Petrache, Dimension reduction techniques for the minimization of theta functions on lattices. J. Math. Phys. **58**(7), 071902, 40 (2017)
28. L. Bétermin, E. Sandier, Renormalized energy and asymptotic expansion of optimal logarithmic energy on the sphere. Constr. Approx. **47**(1), 39–74 (2018)
29. D. Bilyk, F. Dai, R. Matzke, The Stolarsky principle and energy optimization on the sphere. Constr. Approx. **48**(1), 31–60 (2018)
30. G. Björck, Distributions of positive mass, which maximize a certain generalized energy integral. Arkiv für Matematik **3**, 255–269 (1956). https://doi.org/10.1007/BF02589412
31. X. Blanc, M. Lewin, The crystallization conjecture: a review. EMS Surv. Math. Sci. **2**(2), 225–306 (2015)
32. H.-P. Blatt, E.B. Saff, M. Simkani, Jentzsch-Szego type theorems for the zeros of best approximants. J. Lond. Math. Soc. (2) **38**(2), 307–316 (1988)
33. H.F. Blichfeldt, The minimum value of quadratic forms, and the closest packing of spheres. Math. Ann. **101**(1), 605–608 (1929)
34. S. Bochner, Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse. Math. Ann. **108**(1), 378–410 (1933)
35. A. Bondarenko, D. Radchenko, M. Viazovska, Optimal asymptotic bounds for spherical designs. Ann. Math. (2) **178**(2), 443–452 (2013)
36. A. Bondarenko, D. Radchenko, M. Viazovska, Well-separated spherical designs. Constr. Approx. **41**(1), 93–112 (2015)
37. A.V. Bondarenko, D.P. Hardin, E.B. Saff, Mesh ratios for best-packing and limits of minimal energy configurations. Acta Math. Hung. **142**(1), 118–131 (2014)
38. A.V. Bondarenko, M.S. Viazovska, Spherical designs via Brouwer fixed point theorem. SIAM J. Discret. Math. **24**(1), 207–217 (2010)
39. K. Böröczky, The problem of Tammes for $n = 11$. Stud. Sci. Math. Hung. **18**(2–4), 165–171 (1983)
40. K. Böröczky Jr., *Finite Packing and Covering*. Cambridge Tracts in Mathematics, vol. 154 (Cambridge University Press, Cambridge, 2004)

41. K. Böröczky Jr., G. Wintsche, Covering the sphere by equal spherical balls, *Discrete and Computational Geometry*. Algorithms and Combinatorics, vol. 25 (Springer, Berlin, 2003), pp. 235–251
42. S. Borodachov, Polarization problem on a high-dimensional sphere for a simplex. Submitted
43. S. Borodachov, On minimization of the energy of varying range interactions on one- and multidimensional conductors. Ph.D. thesis. Vanderbilt University, Nashville, TN (2006)
44. S. Borodachov, Asymptotics for the minimum Riesz energy and best-packing on sets of finite packing premeasure. Publ. Mat. **56**(1), 225–254 (2012)
45. S.V. Borodachov, Lower order terms of the discrete minimal Riesz energy on smooth closed curves. Can. J. Math. **64**(1), 24–43 (2012)
46. S.V. Borodachov, N. Bosuwan, Asymptotics of discrete Riesz d-polarization on subsets of d-dimensional manifolds. Potential Anal. **41**(1), 35–49 (2014)
47. S.V. Borodachov, D.P. Hardin, A. Reznikov, E.B. Saff, Optimal discrete measures for Riesz potentials. Trans. Am. Math. Soc. **370**(10), 6973–6993 (2018)
48. S.V. Borodachov, D.P. Hardin, E.B. Saff, Asymptotically d-energy minimizing sequences of configurations on d-dimensional sets (submitted)
49. S.V. Borodachov, D.P. Hardin, E.B. Saff, Asymptotics of best-packing on rectifiable sets. Proc. Am. Math. Soc. **135**(8), 2369–2380 (electronic) (2007)
50. S.V. Borodachov, D.P. Hardin, E.B. Saff, Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets. Trans. Am. Math. Soc. **360**(3), 1559–1580 (electronic) (2008)
51. S.V. Borodachov, D.P. Hardin, E.B. Saff, Low complexity methods for discretizing manifolds via Riesz energy minimization. Found. Comput. Math. **14**(6), 1173–1208 (2014)
52. N. Bosuwan, Two problems in asymptotic analysis pade-orthogonal approximation and Riesz polarization constants and configurations. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Vanderbilt University
53. S. Boumova, P. Boyvalenkov, D. Danev, Necessary conditions for existence of some designs in polynomial metric spaces. Eur. J. Comb. **20**(3), 213–225 (1999)
54. J. Bourgain, J. Lindenstrauss, Distribution of points on spheres and approximation by zonotopes. Isr. J. Math. **64**(1), 25–31 (1988)
55. J. Bourgain, J. Lindenstrauss, V. Milman, Approximation of zonoids by zonotopes. Acta Math. **162**(1–2), 73–141 (1989)
56. M.J. Bowick, D.R. Nelson, A. Travesset, Interacting topological defects on frozen topographies. Phys. Rev. B **62**(13), 8738 (2003)
57. P. Boyvalenkov, D. Danev, I. Landgev, On maximal spherical codes. II. J. Comb. Des. **7**(5), 316–326 (1999)
58. P.G. Boyvalenkov, D.P. Danev, S.P. Bumova, Upper bounds on the minimum distance of spherical codes. IEEE Trans. Inf. Theory **42**(5), 1576–1581 (1996)
59. P.G. Boyvalenkov, P.D. Dragnev, D.P. Hardin, E.B. Saff, M.M. Stoyanova, Universal lower bounds for potential energy of spherical codes. Constr. Approx. **44**(3), 385–415 (2016)
60. P.G. Boyvalenkov, P.D. Dragnev, D.P. Hardin, E.B. Saff, M.M. Stoyanova, Energy bounds for codes in polynomial metric spaces. Anal. Math. Phys. (2019)
61. P.G. Boyvalenkov, P.D. Dragnev, D.P. Hardin, E.B. Saff, M.M. Stoyanova, Next levels universal bounds for spherical codes: the Levenshtein framework lifted. Manuscript (2019)
62. J.S. Brauchart, About the second term of the asymptotics for optimal Riesz energy on the sphere in the potential-theoretical case. Integral Transform. Spec. Funct. **17**(5), 321–328 (2006)
63. J.S. Brauchart, Optimal logarithmic energy points on the unit sphere. Math. Comput. **77**(263), 1599–1613 (2008)

64. J.S. Brauchart, J. Dick, A characterization of Sobolev spaces on the sphere and an extension of Stolarsky's invariance principle to arbitrary smoothness. Constr. Approx. **38**(3), 397–445 (2013)
65. J.S. Brauchart, P. Dragnev, E.B. Saff, Riesz external field problems on the hypersphere and optimal point separation. Potential Anal. **41**(3), 647–678 (2014)
66. J.S. Brauchart, P.J. Grabner, W.B. Kusner, Hyperuniform point sets on the sphere: deterministic constructions (2018)
67. J.S. Brauchart, P.J. Grabner, W.B. Kusner, J. Ziefle, Hyperuniform point sets on the sphere: probabilistic aspects (2018)
68. J.S. Brauchart, D.P. Hardin, E.B. Saff, The support of the limit distribution of optimal Riesz energy points on sets of revolution in $\mathbb{R}^3$. J. Math. Phys. **48**(12), 122901, 24 (2007)
69. J.S. Brauchart, D.P. Hardin, E.B. Saff, The Riesz energy of the Nth roots of unity: an asymptotic expansion for large N. Bull. Lond. Math. Soc. **41**(4), 621–633 (2009)
70. J.S. Brauchart, D.P. Hardin, E.B. Saff, Discrete energy asymptotics on a Riemannian circle. Unif. Distrib. Theory **7**(2), 77–108 (2012)
71. J.S. Brauchart, D.P. Hardin, E.B. Saff, The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere, *Recent Advances in Orthogonal Polynomials, Special Functions, and Their Applications*. Contemporary Mathematics, vol. 578 (American Mathematical Society, Providence, 2012), pp. 31–61
72. J.S. Brauchart, A.B. Reznikov, E.B. Saff, I.H. Sloan, Y.G. Wang, R.S. Womersley, Random point sets on the sphere—hole radii, covering, and separation. Exp. Math. **27**(1), 62–81 (2018)
73. A.M. Bruckner, J.B. Bruckner, B.S. Thomson, *Real Analysis* (Prentice-Hall International, Upper Saddle River, 1997)
74. E. Capiński, M. Kopp, *Measure, Integral and Probability*. Springer Undergraduate Mathematics Series (Springer, Berlin, 1999)
75. L. Carleson, *Selected Problems on Exceptional Sets*. Van Nostrand Mathematical Studies, vol. 13 (D. Van Nostrand Co., Inc., Princeton, 1967)
76. J.W.S. Cassels, On a problem of Rankin about the Epstein zeta-function. Proc. Glasg. Math. Assoc. **4**, 73–80 (1959)
77. U. Cegrell, S. Kolodziej, N. Levenberg, Two problems on potential theory for unbounded sets. Math. Scand. **83**(2), 265–276 (1998)
78. T.S. Chihara, *An Introduction to Orthogonal Polynomials*. Mathematics and Its Applications, vol. 13 (Gordon and Breach Science Publishers, New York, 1978)
79. C. Choirat, R. Seri, Numerical properties of generalized discrepancies on spheres of arbitrary dimension. J. Complex. **29**(2), 216–235 (2013)
80. G. Choquet, Diamètre transfini et comparaison de diverses capacités. Technical report, Faculté des Sciences de Paris (1958)
81. H. Cohn, J.H. Conway, N.D. Elkies, A. Kumar, The D_4 root system is not universally optimal. Exp. Math. **16**(3), 313–320 (2007)
82. H. Cohn, M. de Courcy-Ireland, The Gaussian core model in high dimensions. Duke Math. J. **167**(13), 2417–2455 (2018)
83. H. Cohn, N. Elkies, New upper bounds on sphere packings. I. *Ann. of Math. (2)* **157**(2), 689–714 (2003)
84. H. Cohn, A. Kumar, Universally optimal distribution of points on spheres. J. Am. Math. Soc. **20**(1), 99–148 (2007)
85. H. Cohn, A. Kumar, Optimality and uniqueness of the Leech lattice among lattices. Ann. Math. (2) **170**(3), 1003–1050 (2009)
86. H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, M. Viazovska, The sphere packing problem in dimension 24. Ann. Math. (2) **185**(3), 1017–1033 (2017)
87. H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, M. Viazovska, Universal optimality of the E_8 and Leech lattices and interpolation formulas, arXiv:1902.05438.pdf

88. H. Cohn, J. Woo, Three-point bounds for energy minimization. J. Am. Math. Soc. **25**(4), 929–958 (2012)
89. H. Cohn, Y. Zhao, Sphere packing bounds via spherical codes. Duke Math. J. **163**(10), 1965–2002 (2014)
90. S.D. Conte, C.W. de Boor, *Elementary Numerical Analysis: An Algorithmic Approach*, 3rd edn. (McGraw-Hill Higher Education, New York, 1980)
91. J.B. Conway, *A Course in Abstract Analysis*. Graduate Studies in Mathematics, vol. 141 (American Mathematical Society, Providence, 2012)
92. J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 290, 3rd edn. (Springer, New York, 1999). With additional contributions by E. Bannai, R.E. Borcherds, J. Leech, S.P. Norton, A.M. Odlyzko, R.A. Parker, L. Queen and B.B. Venkov
93. R. Coulangeon, Spherical designs and zeta functions of lattices. Int. Math. Res. Not. IMRN **2006**, Art. ID 49620, 16 (2006)
94. R. Coulangeon, A. Schürmann, Energy minimization, periodic sets and spherical designs. Int. Math. Res. Not. IMRN **2012**(4), 829–848 (2012)
95. B.E.J. Dahlberg, On the distribution of Fekete points. Duke Math. J. **45**(3), 537–542 (1978)
96. F. Dai, Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls* Springer Monographs in Mathematics (Springer, New York, 2013)
97. L. Dalla, D.G. Larman, P. Mani-Levitska, C. Zong, The blocking numbers of convex bodies. Discret. Comput. Geom. **24**(2–3), 267–277 (2000). The Branko Grünbaum birthday issue
98. S.B. Damelin, V. Maymeskul, On point energies, separation radius and mesh norm for s-extremal configurations on compact sets in $\mathbb{R}^n$. J. Complex. **21**(6), 845–863 (2005)
99. L. Danzer, Finite point-sets on S^2 with minimum distance as large as possible. Discret. Math. **60**, 3–66 (1986)
100. H. Davenport, Problems of packing and covering. Univ. e Politec. Torino Rend. Sem. Mat. **24**, 41–48 (1964/1965)
101. H. Davenport, G. Hajós, Solution to problem 35. Mat. Lapok **2**, 68 (1951)
102. H. Davenport, C.A. Rogers, Hlawka's theorem in the geometry of numbers. Duke Math. J. **14**, 367–375 (1947)
103. G. David, S. Semmes, Singular integrals and rectifiable sets in $\boldsymbol{R}^n$: beyond Lipschitz graphs. Astérisque **193**, 152 (1991)
104. D. de Laat, Moment methods in energy minimization: new bounds for Riesz minimal energy problems: beyond Lipschitz graphs (2016)
105. P. Delsarte, Bounds for unrestricted codes, by linear programming. Philips Res. Rep. **27**, 272–289 (1972)
106. P. Delsarte, J.M. Goethals, J.J. Seidel, Spherical codes and designs. Geom. Dedicata **6**(3), 363–388 (1977)
107. M. DeVos, A. Georgakopoulos, B. Mohar, R. Šámal, An eberhard-like theorem for pentagons and heptagons. Discret. Comput. Geom. **44**(4), 931–945 (2010)
108. P.H. Diananda, Notes on two lemmas concerning the Epstein zeta-function. Proc. Glasg. Math. Assoc. **6**(202–204), 1964 (1964)
109. J. Dick, F. Pillichshammer, *Digital Nets and Sequences: Discrepancy Theory and Quasi-Monte Carlo Integration* (Cambridge University Press, Cambridge, 2010)
110. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16. F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller and B.V. Saunders (eds.)
111. W.F. Donoghue Jr., *Monotone Matrix Functions and Analytic Continuation*. Die Grundlehren der mathematischen Wissenschaften, vol. 207 (Springer, New York, 1974)

112. P.D. Dragnev, On the separation of logarithmic points on the sphere, *Approximation Theory, X (St. Louis, MO, 2001)*. Innovations in Applied Mathematics (Vanderbilt University Press, Nashville, 2002), pp. 137–144
113. P.D. Dragnev, D.A. Legg, D.W. Townsend, Discrete logarithmic energy on the sphere. Pac. J. Math. **207**(2), 345–358 (2002)
114. P.D. Dragnev, E.B. Saff, Riesz spherical potentials with external fields and minimal energy points separation. Potential Anal. **26**(2), 139–162 (2007)
115. M. Drmota, R.F. Tichy, *Sequences, Discrepancies and Applications*. Lecture Notes in Mathematics, vol. 1651 (Springer, Berlin, 1997)
116. A. Dubickas, On the maximal product of distances between points on a sphere. Liet. Mat. Rink. **36**(3), 303–312 (1996)
117. V. Eberhard, *Zur morphologie der polyeder* (BG Teubner, Berlin, 1891)
118. G. Edgar, *Measure, Topology, and Fractal Geometry*. Undergraduate Texts in Mathematics, 2nd edn. (Springer, New York, 2008)
119. M. Eisen, *Introduction to Mathematical Probability Theory* (Prentice-Hall Inc., Englewood Cliffs, 1969)
120. V. Ennola, A lemma about the Epstein zeta-function. Proc. Glasg. Math. Assoc. **6**(198–201), 1964 (1964)
121. P. Epstein, Zur Theorie allgemeiner Zetafunctionen. Math. Ann. **56**(4), 615–644 (1903)
122. A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi, *Higher Transcendental Functions*, vol. II (Robert E. Krieger Publishing Co., Inc., Melbourne, 1981). Based on notes left by Harry Bateman, Reprint of the 1953 original
123. T. Erdélyi, E.B. Saff, Riesz polarization inequalities in higher dimensions. J. Approx. Theory **171**, 128–147 (2013)
124. B. Farkas, B. Nagy, Transfinite diameter, chebyshev constant and energy on locally compact spaces. Potential Anal. **28**, 241–260 (2008)
125. B. Farkas, B. Nagy, S.G. Révész, A minimax problem for sums of translates on the torus. Trans. Lond. Math. Soc. **5**(1), 1–46 (2018)
126. H. Federer, *Geometric Measure Theory*. Die Grundlehren der mathematischen Wissenschaften, vol. 153 (Springer, New York, 1969)
127. U. Feige, G. Schechtman, On the optimality of the random hyperplane rounding technique for MAX CUT. Random Struct. Algorithms **20**(3), 403–440 (2002). Probabilistic methods in combinatorial optimization
128. G. Fejes Tóth, Kreisüberdeckungen der Sphäre. Stud. Sci. Math. Hung. **4**, 225–247 (1969)
129. L. Fejes Tóth, Über einen geometrischen Satz. Math. Z. **46**, 83–85 (1940)
130. L. Fejes Tóth, Uber die abschätzung des kürzesten abstandes zweier punkte eines auf einer kugelfläche liegenden punktsystems. Jb. Dtsch. Math. Ver. **53**, 66–68 (1943)
131. L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*. Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, vol. LXV (Springer, Berlin, 1953)
132. L. Fejes Tóth, On the sum of distances determined by a pointset. Acta Math. Acad. Sci. Hung. **7**, 397–401 (1956)
133. L. Fejes Tóth, Über eine Punktverteilung auf der Kugel. Acta Math. Acad. Sci. Hung. **10**, 13–19 (unbound insert) (1959)
134. L. Fejes Tóth, *Regular Figures*. Springer Undergraduate Mathematics Series (Pergamon, New York, 1964)
135. M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math. Z. **17**(1), 228–249 (1923)
136. M.H. Freedman, Z.-X. He, Z. Wang, Möbius energy of knots and unknots. Ann. Math. (2) **139**(1), 1–50 (1994)
137. S.I. Galiev, Multiple packings and coverings of a sphere. Diskret. Mat. **8**(3), 148–160 (1996)

138. G. Gigante, P. Leopardi, Diameter bounded equal measure partitions of Ahlfors regular metric measure spaces. Discret. Comput. Geom. **57**(2), 419–430 (2017)
139. D.V. Gorbachev, An extremal problem for entire functions of exponential spherical type, which is connected with the Levenshteĭn bound for the density of a packing of $\mathbb{R}^n$ by balls. Izv. Tul. Gos. Univ. Ser. Mat. Mekh. Inform. **6**(1, Matematika), 71–78 (2000)
140. K. Gorski, E. Hivon, A. Banday, B. Wandelt, F. Hansen, M. Reinecke, M. Bartelman, Healpix – a framework for for high resolution discretization, and fast analysis of the data distributed on the sphere **622**, 759–771 (2005)
141. M. Götz, On the Riesz energy of measures. J. Approx. Theory **122**(1), 62–78 (2003)
142. M. Götz, E.B. Saff, Note on d-extremal configurations for the sphere in $\mathbb{R}^{d+1}$, *Recent Progress in Multivariate Approximation (Witten-Bommerholz, 2000)*. International Series of Numerical Mathematics, vol. 137 (Birkhäuser, Basel, 2001), pp. 159–162
143. P.J. Grabner, Erdös-Turán type discrepancy bounds. Monatsh. Math. **111**(2), 127–135 (1991)
144. S. Graf, H. Luschgy, *Foundations of Quantization for Probability Distributions*. Lecture Notes in Mathematics, vol. 1730 (Springer, Berlin, 2000)
145. B. Grünbaum, *Convex Polytopes* (Springer, Berlin, 1967)
146. B. Grünbaum, T.S. Motzkin, The number of hexagons and the simplicity of geodesics on certain polyhedra. Can. J. Math. **15**, 744–751 (1963)
147. W. Habicht, B.L. van der Waerden, Lagerung von Punkten auf der Kugel. Math. Ann. **123**, 223–234 (1951)
148. W. Habicht, B.L. van der Waerden, Punkte auf der kugel. Math. Ann. **125**, 213–222 (1952)
149. T.C. Hales, Cannonballs and honeycombs. Not. Am. Math. Soc. **47**(4), 440–449 (2000)
150. T.C. Hales, A proof of the Kepler conjecture. Ann. Math. (2) **162**(3), 1065–1185 (2005)
151. D. Hardin, E.B. Petrache, M. Saff, Unconstrained polarization (Chebyshev) problems: basic properties and Riesz kernel asymptotics (2018)
152. D.P. Hardin, A.P. Kendall, E.B. Saff, Polarization optimality of equally spaced points on the circle for discrete potentials. Discret. Comput. Geom. **50**(1), 236–243 (2013)
153. D.P. Hardin, T.J. Michaels, E.B. Saff, A comparison of popular point configurations on s^2. Dolomites Res. Notes Approx. **9**, 16–49 (2016)
154. D.P. Hardin, T.J. Michaels, E.B. Saff, Asymptotic linear programming lower bounds for the energy of minimizing Riesz and Gauss configurations. Mathematika **65**(1), 157–180 (2019)
155. D.P. Hardin, E.B. Saff, Discretizing manifolds via minimum energy points. Not. Am. Math. Soc. **51**(10), 1186–1194 (2004)
156. D.P. Hardin, E.B. Saff, Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds. Adv. Math. **193**(1), 174–204 (2005)
157. D.P. Hardin, E.B. Saff, B. Simanek, Periodic discrete energy for long-range potentials. J. Math. Phys. **55**(12), 123509, 27 (2014)
158. D.P. Hardin, E.B. Saff, B.Z. Simanek, Y. Su, Next order energy asymptotics for Riesz potentials on flat tori. Int. Math. Res. Not. IMRN **12**, 3529–3556 (2017)
159. D.P. Hardin, E.B. Saff, H. Stahl, Support of the logarithmic equilibrium measure on sets of revolution in $\mathbb{R}^3$. J. Math. Phys. **48**(2), 022901, 14 (2007)
160. D.P. Hardin, E.B. Saff, O.V. Vlasiuk, Generating point configurations via hypersingular Riesz energy with an external field. SIAM J. Math. Anal. **49**(1), 646–673 (2017)
161. D.P. Hardin, E.B. Saff, J.T. Whitehouse, Quasi-uniformity of minimal weighted energy points on compact metric spaces. J. Complex. **28**(2), 177–191 (2012)
162. L. Hárs, The Tammes problem for $n = 10$. Stud. Sci. Math. Hung. **21**(3–4), 439–451 (1986)

163. E. Hlawka, Zur Geometrie der Zahlen. Math. Z. **49**, 285–312 (1943)
164. E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Chelsea Publishing Company, New York, 1955)
165. A. Holhoş, D. Roşca, An octahedral equal area partition of the sphere and near optimal configurations of points. Comput. Math. Appl. **67**(5), 1092–1107 (2014)
166. X. Hou, J. Shao, Spherical distribution of 5 points with maximal distance sum. Discret. Comput. Geom. **46**(1), 156–174 (2011)
167. J.B. Hough, M. Krishnapur, Y. Peres, B. Virág, *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. University Lecture Series, vol. 51 (American Mathematical Society, Providence, 2009)
168. W.-Y. Hsiang, A simple proof of a theorem of Thue on the maximal density of circle packings in E^2. Enseign. Math. (2) **38**(1–2), 125–131 (1992)
169. W.T.M. Irvine, M.J. Bowick, P.M. Chaikin, Fractionalization of interstitials in curved colloidal crystals. Nat. Mater. **11**, 948–951 (2012)
170. M. Jiang, On the sum of distances along a circle. Discret. Math. **308**(10), 2038–2045 (2008)
171. G.A. Kabatjanskiĭ, V.I. Levenšteĭn, Bounds for packings on the sphere and in space. Probl. Peredači Inf. **14**(1), 3–25 (1978)
172. J. Kepler, *Vom sechseckigen Schnee. Ostwalds Klassiker der Exakten Wissenschaften [Ostwald's Classics of the Exact Sciences]*, vol. 273 (Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1987). Strena seu de Nive sexangula, Translated from the Latin and with an introduction and notes by Dorothea Goetz
173. R. Kershner, The number of circles covering a set. Am. J. Math. **61**, 665–671 (1939)
174. A.N. Kolmogorov, S.V. Fomīn, *Introductory Real Analysis* (Dover Publications Inc., New York, 1975) Translated from the second Russian edition and edited by Richard A Silverman, Corrected reprinting
175. A.N. Kolmogorov, V.M. Tihomirov, ε-entropy and ε-capacity of sets in function spaces. Uspehi Mat. Nauk **14**(2(86)), 3–86 (1959)
176. A.V. Kolushov, V.A. Yudin, On the Korkin-Zolotarev construction. Diskret. Mat. **6**(1), 155–157 (1994)
177. A.V. Kolushov, V.A. Yudin, Extremal dispositions of points on the sphere. Anal. Math. **23**(1), 25–34 (1997)
178. J. Korevaar, J.L.H. Meyers, Spherical Faraday cage for the case of equal point charges and Chebyshev-type quadrature on the sphere. Integral Transform. Spec. Funct. **1**(2), 105–117 (1993)
179. A.B.J. Kuijlaars, E.B. Saff, Asymptotics for minimal discrete energy on the sphere. Trans. Am. Math. Soc. **350**(2), 523–538 (1998)
180. A.B.J. Kuijlaars, E.B. Saff, X. Sun, On separation of minimal Riesz energy points on spheres in Euclidean spaces. J. Comput. Appl. Math. **199**(1), 172–180 (2007)
181. L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences* Pure and Applied Mathematics (Wiley-Interscience, New York, 1974)
182. N.S. Landkof, *Foundations of Modern Potential Theory* (Springer, New York, 1972)
183. P. Leopardi, A partition of the unit sphere into regions of equal area and small diameter. Electron. Trans. Numer. Anal. **25**, 309–327 (electronic) (2006)
184. P. Leopardi, Diameter bounds for equal area partitions of the unit sphere. Electron. Trans. Numer. Anal. **35**, 1–16 (2009)
185. P. Leopardi, Discrepancy, separation and Riesz energy of finite point sets on the unit sphere. Adv. Comput. Math. **39**(1), 27–43 (2013)
186. V.I. Levenshtein, Bounds for packings of metric spaces and some of their applications. Probl. Kibernet. **40**, 43–110 (1983)
187. V.I. Levenshtein, Designs as maximum codes in polynomial metric spaces. Acta Appl. Math. **25**, 1–82 (1992)
188. V.I. Levenshtein, The maximal density of filling an n-dimensional Euclidean space with equal balls. Mat. Zametki **18**(2), 301–311 (1975)

189. V.I. Levenshtein, Boundaries for packings in n-dimensional Euclidean space. Dokl. Akad. Nauk SSSR **245**(6), 1299–1303 (1979)
190. V.I. Levenshtein, Universal bounds for codes and designs, in *Handbook of Coding Theory*, eds. by V.S Pless, W.C. Huffman (Elsevier, 1998), pp. 499–648
191. P. Lipowsky, M.J. Bowick, J.H. Meinke, D.R. Nelson, A.R. Bausch, Direct visualization of dislocation dynamics in grain-boundary scars. Nat. Mater. **4**, 407–411 (2005)
192. A. Martínez-Finkelshtein, V. Maymeskul, E.A. Rakhmanov, E.B. Saff, Asymptotics for minimal discrete Riesz energy on curves in $\mathbb{R}^d$. Can. J. Math. **56**(3), 529–552 (2004)
193. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*. Cambridge Studies in Advanced Mathematics, vol. 44 (Cambridge University Press, Cambridge, 1995)
194. T.W. Melnyk, O. Knop, W.R. Smith, Extremal arrangements of points and unit charges on a sphere: equilibrium configurations revisited. Can. J. Chem. **55**(10), 1745–1761 (1977)
195. T. Michaels, Equidistributed icosahedral configurations on the sphere. Comput. Math. Appl. **74**(4), 605–612 (2017)
196. H. Minkowski, Diskontinuitätsbereich für arithmetische Äquivalenz. J. Reine Angew. Math. **129**, 220–274 (1905)
197. H. Minkowski, Gesammelte abhandlungen. **1**, 265, 270 and 277 (1911)
198. H.L. Montgomery, Minimal theta functions. Glasg. Math. J. **30**(1), 75–85 (1988)
199. F. Morgan, *Geometric Measure Theory* (Elsevier/Academic, Amsterdam, 2009). A beginner's guide
200. C. Müller, *Spherical Harmonics*. Lecture Notes in Mathematics, vol. 17 (Springer, Berlin, 1966)
201. O.R. Musin, The kissing problem in three dimensions. Discret. Comput. Geom. **35**(3), 375–384 (2006)
202. O.R. Musin, The kissing number in four dimensions. Ann. Math. (2) **168**(1), 1–32 (2008)
203. O.R. Musin, A.S. Tarasov, The strong thirteen spheres problem. Discret. Comput. Geom. **48**(1), 128–141 (2012)
204. O.R. Musin, A.S. Tarasov, The Tammes problem for $N = 14$. Exp. Math. **24**(4), 460–468 (2015)
205. F.J. Narcowich, X. Sun, J.D. Ward, Z. Wu, LeVeque type inequalities and discrepancy estimates for minimal energy configurations on spheres. J. Approx. Theory **162**(6), 1256–1278 (2010)
206. H. Niederreiter, Random number generation and quasi-Monte Carlo methods Soc. Ind. Appl. Math. (1992)
207. N. Nikolov, R. Rafailov, On the sum of powered distances to certain sets of points on the circle. Pac. J. Math. **253**(1), 157–168 (2011)
208. N. Nikolov, R. Rafailov, On extremums of sums of powered distances to a finite set of points. Geom. Dedicata **167**, 69–89 (2013)
209. S. Nonnenmacher, A. Voros, Chaotic eigenfunctions in phase space. J. Stat. Phys. **92**(3–4), 431–518 (1998)
210. A.M. Odlyzko, N.J.A. Sloane, New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. J. Comb. Theory Ser. A **26**(2), 210–214 (1979)
211. J. O'Hara, Energy of a knot. Topology **30**(2), 241–247 (1991)
212. J. O'Hara, Family of energy functionals of knots. Topol. Appl. **48**(2), 147–161 (1992)
213. J. O'Hara, Energy functionals of knots. II. Topol. Appl. **56**(1), 45–61 (1994)
214. M. Ohtsuka, On various definitions of capacity and related notions. Nagoya Math. J. **30**, 121–127 (1967)
215. O.N. Osychenko, G.E. Astrakharchik, J. Boronat, Ewald method for polytropic potentials in arbitrary dimensionality. Mol. Phys. **110**, 227–247 (2012)
216. M. Petrache, S. Serfaty, Next order asymptotics and renormalized energy for Riesz interactions. J. Inst. Math. Jussieu **16**(3), 501–569 (2017)

217. M.A. Pinsky, *Introduction to Fourier Analysis and Wavelets*. Graduate Studies in Mathematics, vol. 102 (American Mathematical Society, Providence, 2009). Reprint of the 2002 original
218. G. Polya, G. Szegő, über den transfiniten durchmesser (kapazittskonstante) von ebenen und rumlichen punktmengen. J. Reine Angew. Math. **165**, 4–49 (1931)
219. A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev, *Integrals and Series*, vol. 3 (Gordon and Breach Science Publishers, New York, 1990). More special functions, Translated from the Russian by G.G Gould
220. E.A. Rakhmanov, E.B. Saff, Y.M. Zhou, Minimal discrete energy on the sphere. Math. Res. Lett. **1**(6), 647–662 (1994)
221. E.A. Rakhmanov, E.B. Saff, Y.M. Zhou, Electrons on the sphere, *Computational Methods and Function Theory 1994 (Penang)*. Approximations and Decompositions, vol. 5 (World Scientific Publishing, River Edge, 1995), pp. 293–309
222. R.A. Rankin, A minimum problem for the Epstein zeta-function. Proc. Glasg. Math. Assoc. **1**, 149–158 (1953)
223. R.A. Rankin, The closest packing of spherical caps in n dimensions. Proc. Glasg. Math. Assoc. **2**, 139–144 (1955)
224. M. Reimer, Quadrature rules for the surface integral of the unit sphere based on extremal fundamental systems. Math. Nachr. **169**, 235–241 (1994)
225. M. Reimer, Spherical polynomial approximations: a survey, *Advances in Multivariate Approximation (Witten-Bommerholz, 1998)*. Mathematical Research, vol. 107 (Wiley-VCH, Berlin, 1999), pp. 231–252
226. A. Reznikov, E.B. Saff, The covering radius of randomly distributed points on a manifold. Int. Math. Res. Not. IMRN **19**, 6065–6094 (2016)
227. A. Reznikov, E.B. Saff, O. Vlasiuk, A minimal principle for potentials with application to Chebyshev constants. Potential Anal (2017)
228. A. Reznikov, E.B. Saff, A. Volberg, Covering and separation of Chebyshev points for non-integrable Riesz potentials (2017), arXiv:1703.00106
229. A. Reznikov, O. Vlasiuk, Riesz energy on self-similar sets (2018)
230. R.M. Robinson, Arrangements of 24 points on a sphere. Math. Ann. **144**, 17–48 (1961)
231. C.A. Rogers, Existence theorems in the geometry of numbers. Ann. Math. **2**(48), 994–1002 (1947)
232. C.A. Rogers, The packing of equal spheres. Proc. Lond. Math. Soc. **3**(8), 609–620 (1958)
233. C.A. Rogers, *Packing and Covering*. Cambridge Tracts in Mathematics and Mathematical Physics, vol. 54 (Cambridge University Press, New York, 1964)
234. W. Rudin, *Principles of Mathematical Analysis*. International Series in Pure and Applied Mathematics, 3rd edn. (McGraw-Hill Book Co., New York, 1976)
235. W. Rudin, *Real and Complex Analysis*, 3rd edn. (McGraw-Hill Book Co., New York, 1987)
236. S.S. Ryškov, On the question of the final ζ-optimality of lattices that yield the densest packing of n-dimensional balls. Sibirsk. Mat. Ž. **14**, 1065–1075, 1158 (1973)
237. E.B. Saff, J.C. Snader, The error for quadrature methods: a complex variables approach. Am. Math. Mon. **94**(2), 175–180 (1987)
238. E.B. Saff, V. Totik, *Logarithmic Potentials with External Fields*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 316 (Springer, Berlin, 1997). Appendix B by Thomas Bloom
239. E. Sandier, S. Serfaty, From the Ginzburg-Landau model to vortex lattice problems. Commun. Math. Phys. **313**(3), 635–743 (2012)
240. P. Sarnak, A. Strömbergsson, Minima of Epstein's zeta function and heights of flat tori. Invent. Math. **165**(1), 115–151 (2006)
241. E. Schmutz, Rational points on the unit sphere. Cent. Eur. J. Math. **6**(3), 482–487 (2008)

242. I.J. Schoenberg, Positive definite functions on spheres. Duke Math. J. **9**, 96–108 (1942)
243. K. Schütte, Überdeckungen der Kugel mit höchstens acht Kreisen. Math. Ann. **129**, 181–186 (1955)
244. K. Schütte, B.L. van der Waerden, Auf welcher Kugel haben 5, 6, 7, 8 oder 9 Punkte mit Mindestabstand Eins Platz? Math. Ann. **123**, 96–124 (1951)
245. K. Schütte, B.L. van der Waerden, Das Problem der dreizehn Kugeln. Math. Ann. **125**, 325–334 (1953)
246. R.E. Schwartz, The phase transition in five point energy minimization (2016), arXiv:1610.03303v3
247. B. Segre, K. Mahler, On the densest packing of circles. Am. Math. Mon. **51**, 261–270 (1944)
248. S. Serfaty, Emergence of the Abrikosov lattice in several models with two dimensional Coulomb interaction, in *European Congress of Mathematics* (European Mathematical Society, Zürich, 2013), pp. 119–135
249. M. Shub, S. Smale, Complexity of Bézout's theorem. I. Geometric aspects. J. Am. Math. Soc. **6**(2), 459–501 (1993)
250. M. Shub, S. Smale, Complexity of Bezout's theorem. II. Volumes and probabilities, *Computational Algebraic Geometry (Nice, 1992)*. Progress in Mathematics, vol. 109 (Birkhäuser, Boston, 1993), pp. 267–285
251. M. Shub, S. Smale, Complexity of Bezout's theorem. III. Condition number and packing. J. Complex. **9**(1), 4–14 (1993). Festschrift for Joseph F. Traub, Part I
252. V.M. Sidel′nikov, The densest packing of balls on the surface of the n-dimensional Euclidean sphere, and the number of vectors of a binary code with prescribed code distance. Dokl. Akad. Nauk SSSR **213**, 1029–1032 (1973)
253. C.L. Siegel, A mean value theorem in geometry of numbers. Ann. Math. **2**(46), 340–347 (1945)
254. B. Simanek, Asymptotically optimal configurations for Chebyshev constants with an integrable kernel. N. Y. J. Math. **22**, 667–675 (2016)
255. M.M. Skriganov, Point distributions in compact metric spaces. Mathematika **63**(3), 1152–1171 (2017)
256. I.H. Sloan, R.S. Womersley, Extremal systems of points and numerical integration on the sphere. Adv. Comput. Math. **21**(1–2), 107–125 (2004)
257. S. Smale, Mathematical problems for the next century. Math. Intell. **20**(2), 7–15 (1998)
258. S. Smale, Mathematical problems for the next century, *Mathematics: Frontiers and Perspectives* (American Mathematical Society, Providence, 2000), pp. 271–294
259. M. Spivak, *Calculus on Manifolds. A Modern Approach to Classical Theorems of Advanced Calculus* (W. A. Benjamin, Inc., New York, 1965)
260. S. Steinerberger, Exponential sums and Riesz energies. J. Number Theory **182**, 37–56 (2018)
261. J. Stewart, Positive definite functions and generalizations, an historical survey. Rocky Mt. J. Math. **6**(3), 409–434 (1976)
262. K.B. Stolarsky, Sums of distances between points on a sphere. II. Proc. Am. Math. Soc. **41**, 575–582 (1973)
263. K.B. Stolarsky, Spherical distributions of N points with maximal distance sums are well spaced. Proc. Am. Math. Soc. **48**, 203–206 (1975)
264. K.B. Stolarsky, The sum of the distances to certain pointsets on the unit circle. Pac. J. Math. **59**(1), 241–251 (1975)
265. Y. Su, Discrete minimal energy on flat tori and four-point maximal polarization on S^2. Ph.D. dissertation, Vanderbilt University (2015)
266. R. Swinbank, R.J. Purser, Fibonacci grids: a novel approach to global modeling. Q. J. R. Meteorol. Soc. **132**(619), 1769–1793 (2006)

267. G. Szegő, *Orthogonal Polynomials*, 4th edn. (American Mathematical Society, Providence, 1975). American Mathematical Society, Colloquium Publications, vol. XXIII
268. T. Szele, Solution to problem 35. Mat. Lapok **3**, 95 (1952)
269. P. Tammes, On the origin of number and arrangement of the places of exit on pollen-grains. Diss. Groningen (1930)
270. F. team, The Flyspeck project. Announcement of completion (2014), https://code.google.com/p/flyspeck/wiki/AnnouncingCompletion
271. A. Terras, *Harmonic Analysis on Symmetric Spaces—Higher Rank Spaces, Positive Definite Matrix Space and Generalizations*, 2nd edn. (Springer, New York, 2016)
272. T.M. Thompson, *From Error-Correcting Codes Through Sphere Packings to Simple Groups*. Carus Mathematical Monographs, vol. 21 (Mathematical Association of America, Washington, 1983)
273. J.J. Thomson, Cathode rays. Philos. Mag. **44**, 293–316 (1897)
274. J.J. Thomson, On the structure of the atom: an investigation of the stability and periods of oscillation of a number of corpuscles arranged at equal intervals around the circumference of a circle; with application of the results to the theory of atomic structure. Philos. Mag. **7**(6), 237–265 (1904)
275. A. Thue, Om nogle geometrisk taltheoretiske theoremer. Forhdl. Skand. Naturforsk **14**, 352–353 (1892)
276. A. Thue, über die dichteste zuzammenstellung von kongruenten kreisen in der ebene. Norske Vid. Selsk. Skr. **1**, 1–9 (1910)
277. S. Torquato, Hyperuniform states of matter. Phys. Rep. **745**, 1–95 (2018)
278. R.J. Trudeau, *Introduction to Graph Theory* (Dover Publications, Inc., New York, 1993). Corrected reprint of the 1976 original
279. A. Tumanov, Minimal biquadratic energy of five particles on a 2-sphere. Indiana Univ. Math. J. **62**(6), 1717–1731 (2013)
280. J.D. Vaaler, Some extremal functions in Fourier analysis. Bull. Am. Math. Soc. (N.S.) **12**(2), 183–216 (1985)
281. S. Vance, Improved sphere packing lower bounds from Hurwitz lattices. Adv. Math. **227**(5), 2144–2156 (2011)
282. A. Venkatesh, A note on sphere packings in high dimension. Int. Math. Res. Not. IMRN **7**, 1628–1642 (2013)
283. M.S. Viazovska, The sphere packing problem in dimension 8. Ann. Math. (2) **185**(3), 991–1015 (2017)
284. G. Wagner, On the product of distances to a point set on a sphere. J. Austral. Math. Soc. Ser. A **47**(3), 466–482 (1989)
285. G. Wagner, On means of distances on the surface of a sphere (lower bounds). Pac. J. Math. **144**(2), 389–398 (1990)
286. G. Wagner, On means of distances on the surface of a sphere. II. Upper bounds. Pac. J. Math. **154**(2), 381–396 (1992)
287. L.L. Whyte, Unique arrangements of points on a sphere. Am. Math. Mon. **59**, 606–611 (1952)
288. D.V. Widder, *The Laplace Transform*. Princeton Mathematical Series, vol. 6 (Princeton University Press, Princeton, 1941)
289. L. Wimmer, Covering the sphere with equal circles. Discret. Comput. Geom. **57**(4), 763–781 (2017)
290. T.H. Wolff, *Lectures on Harmonic Analysis*. University Lecture Series, vol. 29 (American Mathematical Society, Providence, 2003). With a foreword by Charles Fefferman and preface by Izabella Łaba, Edited by Łaba and Carol Shubin
291. V.A. Yudin, Minimum potential energy of a point system of charges. Diskret. Mat. **4**(2), 115–121 (1992)
292. Y. Zhou, Arrangements of points on the sphere. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–University of South Florida

List of Symbols

#	cardinality of a multiset (counting the multiplicity), 50 .
$\xrightarrow{*}$	weak* convergence of measures, 30.
$\mu * \nu$ $(\mu * f)$	convolution of measures and/or functions, 43.
$\lfloor \cdot \rfloor$	floor function symbol, 527.
$\\| \cdot \\|_A$	the sup-norm of functions defined on A, 33.
$\langle \cdot, \cdot \rangle_K$	mutual K-energy o f two measures, 134.
∂A	boundary of a set A, 380.
$\partial_\infty A$	outer boundary of a set A, 170.
$\partial_B A$	boundary of a set A relative to a set B, 34.
$\overline{A}$	the closure of a set A, 34.
A°	the interior of a set A (also see $\mathrm{int}(A)$), 35.
$A(d, \theta)$	θ-complexity of S^d, 250.
$A(\epsilon)$	ϵ-neighborhood of a set A, 19.
$\mathcal{A}_d(h)$	nonnegative definite functions lower bounding h, 220.
$\mathcal{B}^d$	unit ball in $\mathbb{R}^d$, 7.
$B[x, r]$	closed ball of radius r centered at x, 13.
$B(x, r)$	open ball of radius r centered at x, 13.
$\mathrm{B}(p, q)$	beta function, 41.
$\mathrm{cap}_K(A)$	K-capacity of a set A, 129.
$\mathrm{cap}_s(A)$	s-capacity of A, 145.
$\mathrm{cap}_{\log}(A)$	logarithmic capacity of A, 146.
$C(\mathbf{y}, \rho)$	spherical cap with center $\mathbf{y}$ of Euclidean radius ρ, 273.
$C(\mathbf{a}, \varphi)$	spherical cap with center $\mathbf{a}$ and angular radius φ, 93.
$C_{s,p}$	Riesz s-energy (normalized) limit of the unit cube in $\mathbb{R}^p$, 371.
$\mathrm{diam}\, A$	diameter of a set A, 17.

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$\dim A$	Hausdorff dimension of a set A, 19.
$\overline{\dim}_M A$, $\underline{\dim}_M A$	upper and lower Minkowski dimension, 20.
D_4	the checkerboard lattice, 251.
$D(d,m)$	Delsarte–Goethals–Seidel lower bound, 218.
$D(r)$	Intersection of sets $D_i(r)$, 273.
$D_i(r)$	sphere S^d without the spherical cap centered at x_i of radius $rN^{-1/d}$, 273.
E_8	Korkin–Zolotarev lattice, 252.
$\mathcal{E}_K(A,N)$	minimal N-point K-energy of A, 50.
$E_K(\omega_N)$	K-energy of ω_N, 50.
$\mathcal{E}_s(A,N)$	minimal N-point Riesz s-energy of A, 52.
$E_s(\omega_N)$	Riesz s-energy of ω_N, 52.
$\mathcal{E}_{\log}(A,N)$	minimal N-point logarithmic energy of A, 53.
$E_{\log}(\omega_N)$	logarithmic energy of ω_N, 53.
$E^f(\omega_N)$	energy of $\omega_N \subset S^d$ for zonal kernel $\widetilde{K}_f(x,y) := f(\mathbf{x}\cdot\mathbf{y})$, 205.
$\mathcal{E}^f(S^d,N)$	minimal N-point E^f-energy of S^d, 205.
$\overline{E}_s^w(\omega_N;r)$	the r-truncated (w,s)-energy of ω_N, 497..
$\overline{\mathcal{E}}_s^w(A,N;r)$	the minimal N-point r-truncated (w,s)-energy of A, 497.
$F_{f,\Lambda}$	Λ-periodic potential generated by f, 442.
$\overline{g}_{s,d}(A), \underline{g}_{s,d}(A)$, $g_{s,d}(A)$	limits of the normalized minimal s-energy, 380.
$\overline{g}^w_{s,d}(A), \underline{g}^w_{s,d}(A)$, $g^w_{s,d}(A)$	limits of the normalized minimal (w,s)-energy, 481.
$G_t(x,y)$	Gaussian kernel, 54.
$\mathbb{H}_n^{d+1}$	the space of all spherical harmonics on S^d of degree n, 194.
$\mathcal{H}_d$	d-dimensional Hausdorff measure, $d\in\mathbb{N}$, 18.
$\mathcal{H}_d^A$	d-dimensional Hausdorff measure, $d\in\mathbb{N}$, 378.
$\overline{h}_{s,d}(A), \underline{h}_{s,d}(A)$, $h_{s,d}(A)$	limits of the normalized maximal s-polarization, $s>d$, 571.
$\mathcal{H}_\alpha$	α-dimensional Hausdorff measure, $\alpha\notin\mathbb{N}$, 18.
$\mathrm{int}(A)$	set of interior points of A (also see A°), 598.
$\mathcal{H}_\alpha^\infty$	set function $\mathcal{H}_\alpha^\infty$, 18.
$I_K[\mu]$	continuous K-energy of μ, 129.
$I_{\log}[\mu]$	continuous logarithmic energy of μ, 145.
$I_s[\mu]$	continuous s-energy of μ, 145.
$I^w(v,\delta)$	infimum of the ratio of the weight v to the weight w near the diagonal, 507.
$J(\rho)$,	integral of the Riesz d-potential over S^d with a cap of radius ρ removed, 274.
$\mathcal{K}_d$	a family of rectifiable sets, see Definition 9.4.2, 420.
$K_f(x,y)$	kernel of form $f(\|x-y\|^2)$, 269.
$K_s(x,y)$	Riesz s-kernel, 51.

$K_{\log}(x,y)$	logarithmic kernel, 53.
$\log N$	natural logarithm of N.
$L(d,s)$	Levenshtein bound, 225.
$\mathcal{L}_p$	Lebesgue measure in $\mathbb{R}^p$, 15.
$L^q(A)$	space of qth power Lebesgue integrable functions over A, 27.
$\mathcal{M}(A)$	set of all Borel probability measures supported on A, 32, 128.
$\mathcal{M}_{\text{sign}}(\mathbb{R}^p)$	set of all finite signed measures on $\mathbb{R}^p$, 42.
$\mathcal{M}_\alpha(A)$	Minkowski content of a set A, 19.
$\overline{\mathcal{M}}_\alpha(A)$, $\underline{\mathcal{M}}_\alpha(A)$	upper and lower Minkowski content of a set A, 19.
$\mathcal{P}_K(A,N)$	Nth K-Chebyshev (polarization) constant on A, 540.
$\mathcal{P}_{\log}(A,N)$	Nth logarithmic Chebyshev (polarization) constant on A, 66.
$P_n^{(d)}$	Gegenbauer polynomial, 197.
$P_n^{a,b}$	adjacent Gegenbauer polynomial, 207.
$\mathcal{P}_s(A,N)$	N-th Riesz s-Chebyshev (polarization) constant on A, 544.
$P_{K,A}(\omega_N)$	infimum over A of the K-potential of ω_N, 540.
$P_n = P_n^{(0,0)}$	Legendre polynomial, 66.
$P_n^{(\alpha,\beta)}$	Jacobi polynomial, 66.
$P_{\log,A}(\omega_N)$	infimum over A of the logarithmic potential of ω_N, 544.
$P_{s,A}(\omega_N)$	infimum over A of the Riesz s-potential of ω_N, 544.
$P_{\mathbb{H}_n^d} f$	orthogonal projection of f onto $\mathbb{H}_n^d$, 196.
$Q_n(x,y)$	kernel for $P_{\mathbb{H}_n^d}$, 196.
S^d	unit sphere in $\mathbb{R}^{d+1}$, 59.
$S^w(v,\delta)$	supremum of the ratio of the weight v to the weight w near the diagonal, 507.
$S(x,r)$	sphere centered at point x of radius r, 167.
$\operatorname{supp}\mu$	support of a measure μ, 15.
$T_K(A)$	continuous polarization (Chebyshev) constant of A, 563.
$T_K(A,\mu)$	infimum on A of K-potential for μ, 563.
T_n	Chebyshev polynomial, 66.
$U_K^\mu(x)$	K-potential of μ, 128.
$U_{\log}^\mu(x)$	logarithmic potential of μ, 145.
$U_s^\mu(x)$	s-potential of μ, 145.
$w_\lambda(t)$	the weight for Gegenbauer polynomials, 197.
$W_K(A)$	Wiener constant of a set A relative to a kernel K, 129.

$W_{\log}(A)$	Wiener constant of A for the logarithmic kernel, 145.
$W_s(A)$	Wiener constant of A for the Riesz s-kernel, 145.
$W_n^{a,b}(\cdot,\cdot)$	reproducing kernel for adjacent polynomials, 208.
Y_{nk}	elements of the orthonormal basic in $\mathbb{H}_n^d$, 196.
$Z(d,n)$	the dimension of $\mathbb{H}_n^d$, 195.
β_p	volume of the unit ball in $\mathbb{R}^p$, 42.
γ	Euler–Mascheroni constant, 42.
γ_d	ratio of $\varOmega_{d-1}$ to $\varOmega_d$, 197.
$\gamma(\omega_N, A)$	mesh-separation ratio of ω_N relative to A, 84.
$\varGamma_p$	minimal covering density in $\mathbb{R}^p$, 103.
$\varGamma(s)$	gamma function, 40.
δ_x	Dirac measure centered at a point x, 132.
$\delta(\omega_N)$	minimal pairwise Euclidean distance between distinct points in ω_N, 78.
$\delta_a(\omega_N)$	angular separation of $\omega_N \subset S^d$, 222.
$\delta^\rho(\omega_N)$	minimal pairwise distance between points in ω_N in the metric ρ, 78.
$\delta_N(A)$	minimal N-point best-packing distance on a set A, 78.
$\delta_N^\rho(A)$	minimal N-point best-packing distance on a metric space (A,ρ), 78.
$\varDelta f$	Laplace operator of f, 168.
$\varDelta(\mathcal{B})$	density of a packing $\mathcal{B}$, 101.
$\varDelta_p$	largest sphere packing density in $\mathbb{R}^p$, 102.
$\zeta(s)$	Riemann zeta function, 40.
$\zeta(s,q)$	Hurwitz zeta function, 40.
$\zeta_\varLambda(s)$	Epstein zeta function of a lattice $\varLambda$, 40.
$\zeta_\varLambda(s,x)$	Epstein–Hurwitz zeta function of a lattice $\varLambda$, 447.
$\kappa(p)$	kissing number in $\mathbb{R}^p$, 250.
$\eta_N(A)$	minimal N-point covering radius of A, 82.
$\eta_N^D(A)$	minimal N-point covering radius of A relative to D, 82.
$\eta(\omega_N, A)$	covering radius of ω_N relative to a set A, 82.
$\varLambda^*$ $(\varLambda_2^*)$	equi-triangular lattice, 100.
$\widehat{\varLambda}$	dual lattice of a lattice $\varLambda$, 39.
$\hat{\mu}$	Fourier transform of a measure μ, 42.
$\mu_{K,A}$	equilibrium measure for a set A relative to a kernel K, 131.
$\mu_{s,A}$	s-equilibrium measure for A, 145.
$\mu_{\log,A}$	equilibrium measure for A for the logarithmic kernel, 145.
$f * \mu$	convolution of a function f and a measure μ, 172.
$\nu(\omega_N)$	normalized counting measure of ω_N, 132.

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