Conclusion: Other Groups, Other Horizons

Although the theory of zeta functions of Siegel modular forms is far from being complete, and many fundamental questions (such as analogues of the Shimura–Taniyama conjecture for abelian varieties) have not even been touched, one must keep in mind that the symplectic group is but one example from a big family of arithmetically significant groups. The most natural situation, when it is possible to approach both analytic properties and factorization into an Euler product of arithmetic zeta functions, is the case in which zeta functions can be associated with representations of suitable arithmetic discrete subgroups of Lie groups on related function spaces of automorphic forms. The typical example of this kind is provided by "zeta functions of bilinear forms." Let, for example,

$$\mathbf{q} = \mathbf{q}(X,Y) = \sum_{i,j=1,\ldots,m} q_{ij}x_iy_j = {}^tXQY \qquad \left(X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \right)$$

be a bilinear form of order m with integral nonsingular matrix $Q = (q_{ij})$. With the form $\mathbf{q}$ we associate the *automorph semigroup of* $\mathbf{q}$,

$$A(\mathbf{q}) = \left\{ D \in \mathbb{Z}_m^m \,\middle|\, \mathbf{q}(DX, DY) = \mu\mathbf{q}(X,Y) \quad \text{with } \mu = \mu(D) > 0 \right\},$$

and the *group of units of* $\mathbf{q}$,

$$E(\mathbf{q}) = \{D \in A(\mathbf{q}) | \, \mu(D) = 1\}.$$

The pair $(E(\mathbf{q}), A(\mathbf{q}))$ is quite often left-finite in the sense of Section 3.2, and we may define the Hecke–Shimura ring $\mathcal{D} = D(E(\mathbf{q}), A(\mathbf{q}))$ of the pair (over $\mathbb{Z}$), which is also called the *automorph class ring of* $\mathbf{q}$ and is denoted by $\mathcal{H}(\mathbf{q})$. Generally speaking, one has little to say about the ring $\mathcal{H}(\mathbf{q})$, and it should be replaced by the more complicated construction of a *matrix Hecke–Shimura ring*. Each bilinear form is the sum of uniquely defined *symmetric* and *skew-symmetric* forms, i.e., the forms $\mathbf{q}$ satisfying

$$\mathbf{q}(X,Y) = \mathbf{q}(Y,X) \quad \text{or} \quad \mathbf{q}(X,Y) = -\mathbf{q}(Y,X),$$

respectively, and corresponding groups and semigroups of automorphs are intersections of those for the components. In the case of either a pure symmetric or skew-symmetric form $\mathbf{q}$, each double coset of the semigroup $A(\mathbf{q})$ modulo the group $E(\mathbf{q})$ is a finite union of left cosets. In such a case, a theory can be developed that would resemble the Hecke–Shimura theory outlined above for the most interesting skew-symmetric case of the bilinear form with matrix $Q = J_n$ of order $2n$ defined in (0.1) (with the integral symplectic group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$ as the group of units and Siegel modular forms as representation spaces for symplectic Hecke–Shimura rings). In the symmetric case, when the form $\mathbf{q}$ is symmetric and the group of units $E(\mathbf{q})$ coincides with the proper integral orthogonal group of the quadratic form $\mathbf{q}(X,X)$, the theory of orthogonal matrix Hecke–Shimura rings is similar in many respects to the theory of symplectic rings, including the formal Euler factorization of generating Dirichlet series. In the orthogonal case, Hecke–Shimura rings operate on spaces of harmonic polynomials related to basic quadratic forms, and one can define zeta functions corresponding to these representations. It was revealed recently that in the case of positive definite quadratic forms in two and four variables, the relevant (orthogonal) zeta functions coincide with zeta functions corresponding to representations of symplectic Hecke–Shimura rings on spaces of theta series of genus 1 and 2 with harmonic coefficients, respectively. The coincidence is even more striking because Hecke–Shimura rings of quite different groups of units are involved: finite orthogonal groups and infinite symplectic groups. Other examples of relations between zeta functions of different arithmetic groups are provided by various lifting of automorphic forms and related zeta functions to similar groups of higher orders such as Saito–Kurokawa and Ikeda lifts, and various liftings of zeta functions of the general linear group to their symmetric degrees. There is no doubt that further progress in number theory will be closely connected with the investigation of relations among zeta functions of representations of Hecke–Shimura rings of various arithmetic discrete subgroups of Lie groups on automorphic functions.

Notes

Introduction

For the Riemann zeta function, see, e.g., [Ti86]. On zeta functions of algebraic varieties one can read [Shi71, Chapter 7]. For the Birch–Swinnerton-Dyer conjectures see [BSD63/65]. For modular forms in one variable and corresponding Dirichlet series see [Ogg69]. The Hecke theory of the Euler product factorization of Dirichlet series of modular forms is stated in [He37]. The original Atkin–Lehner theory was set forth in [AtL70]. For the history of the Shimura–Taniyama conjecture see [Shi89]. For the proof of Fermat's last theorem and related questions see [Wil95].

Chapter 1

The general references on this chapter, where one can find all omitted details and much more, are [An87, Chapters 1 and 2] and [AnZ90/95]. The later book also treats modular forms of half-integer weight. For a classical introduction to Siegel modular forms and Dirichlet series see [Kl90]. More details on the theory of Siegel modular forms and functions can be found in [Fr83], [Ma55], and [Si39]. For the treatment of automorphic forms on Lie groups see [SC58]. For modular forms in one variable see the books [Ogg69] and [Ma64]; for elementary presentations see [Ser70] and [Ga62]; relations to algebraic geometry are considered in [Fr83] and [Shi71]. Relations of modular forms to integral quadratic forms are considered in [Ki86] (numbers of representations of quadratic forms by quadratic forms) and in [An87] (multiplicative properties of the representations).

Section 1.2. For details on the reduction theory of positive definite quadratic forms see, e.g., [Ca78, Chapter 12]. For the construction of the fundamental domain of the modular group see also [Si73].

Section 1.3. The notation (1.29) is a variation of the notation introduced by H. Petersson. The Koecher effect was discovered in [Ko54/55]. For the rather complicated proof of the estimate (1.52) see, e.g., [An87, Theorem 2.3.4]. The scalar product of modular forms in one variable was introduced by H. Petersson in [Pe39/41]; the consideration of the general case is based on a similar idea and was initiated by H. Maass in [Ma51].

Chapter 2

Section 2.1. Radial Dirichlet series (2.10) are not the only kind of Dirichlet series constructed with the help of Fourier coefficients of common eigenfunctions for regular Hecke operators that have an Euler product factorization and good analytic properties. Another kind of such Dirichlet series is given by series of the form

$$\sum_{M \in \mathrm{SL}_n(\mathbb{Z}) \backslash \{M \in \mathbb{Z}_n^n \mid \det M > 0\}} \frac{\psi(\det M) f(MA\,{}^tM)}{(\det M)^s},$$

where $n \geq 1$, f are Fourier coefficients of a common eigenfunction for regular Hecke operators on the space $\mathfrak{M}_k(\Gamma_0^n(q), \boldsymbol{\chi})$ with integral or half-integral k, ψ is a Dirichlet character, and A is a fixed even positive definite matrix of order n. For $n = 1$ these series were considered in [Ra39], [Se40], and [Shi75]; Euler product factorizations for an arbitrary n and integral k were discussed in [An87, §4.3.3], whereas both integral and half-integral k were considered in [AnZ90/95, §3.3]; analytic properties of corresponding Euler products (the *standard zeta functions of eigenforms F*) of level $q = 1$ with $\psi = 1$ were considered in [AnK78] with certain restrictions, and in [Bo85] without restrictions.

Section 2.3. Our exposition of the theory of radial Dirichlet series corresponding to the ray $mA_1 = 2m \cdot 1_2 (m = 1, 2, \ldots)$ of the matrix of a quadratic form $x_1^2 + x_2^2$ was outlined first in [An71, §3]; the general case of radial series corresponding to rays of matrices of arbitrary positive definite integral primitive binary quadratic forms was considered in detail in [An74, §§3.3–3.7].

On Chapter 3

Section 3.1. The multiplicative properties of the Fourier coefficients of the modular form $\Delta'(z) \in \mathfrak{N}_{12}(\Gamma)$ cited in Exercise 3.1(3) were observed (!) and conjectured by Ramanujan and a little later proved by Mordell in [Mo17]. Mordell's proof actually contains the idea of Hecke operators for the particular case of the space $\mathfrak{N}_{12}(\Gamma)$, an idea that had to wait twenty more years to be reborn in the case of more general spaces of modular forms by Hecke in [He37].

Section 3.2. Our definition of an abstract ring of double cosets by means of multiplication of right-invariant linear combinations of left cosets is equivalent to the definition given by Shimura in [Shi63] with the help of the direct multiplication of double cosets, but in some respects it turns out to be more convenient.

Section 3.3. The rings of double cosets of the general linear group are discussed here and in [An87, §3.2] in the spirit of the fundamental paper [Ta63] of Tamagawa. For the explicit formulas for spherical functions on GL_n see [An70].

Section 3.4. The symplectic case is described as in [An87, §3.3]. The structure of Hecke–Shimura rings for Γ^n was indicated first in [Sa63] and [Shi63]. I cannot judge who was the first. The passage to congruence subgroups goes back to Hecke's ideas in [He37]. Our presentation uses common sense and an analogy with the case of the general linear group rather than any historical reminiscences. For the theory of singular Hecke–Shimura rings see [An99]. The theory of spherical mappings in the form of a general theory of zonal spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields is due to Satake; see [Sa63]. We use an elementary approach based on explicit formulas. The series (3.107) and (3.109) were summed up in [Shi63]; similar series for arbitrary n were computed in [An69] and [An70].

Section 3.5. The embedding of symplectic Hecke–Shimura rings into rings of triangular-symplectic double cosets allows one to split their elements into elementary components naturally related to the general linear group. This gives impetus to the theory of factorization of standard symplectic polynomials in triangular extensions, presented in detail in [An87, §§3.4–3.5]. Here we use only the elementary aspects of the theory.

Chapter 4

General references are [An87, §§4.1–4.2] or [AnZ90/95].

Section 4.1. Hecke operators for the group $\Gamma^1 = \mathrm{SL}_2(\mathbb{Z})$ and some congruence subgroups were introduced in [He37]. Hecke operators on Siegel modular forms were first introduced by Sugawara in [Su37] and [Su38] and after the war were studied by Maass in [Ma51]. The existence of a basis of common eigenfunctions for Hecke operators acting on cusp forms in one variable was proved by Petersson in [Pe39/41]. For Siegel modular forms, Petersson's ideas were developed by Maass in [Ma51]. For diagonalization of singular Hecke operators see [AtL70] in the case of genus $n = 1$, and [An99], [An03] in the general case.

Section 4.3. Zharkovskaya commutation relations for the group Γ^n and the unit character are due to Natasha Zharkovskaya [Zha74], who generalized the Maass relation for Γ^2 obtained in [Ma51]. Our consideration follows the same idea.

Chapter 5

Section 5.1. Relations of Fourier coefficients of eigenforms with eigenvalues can be found in [He37].

Section 5.2. For more details on the relations of binary quadratic forms and modules in quadratic number fields see, e.g., [An87, Appendix 3], or the excellent book on algebraic numbers [BSh63, Chapter 2].

Section 5.3. It is proved in [An74, Theorem 3.1.1] without any restrictions that the zeta function $\zeta(s,F)$ of an arbitrary eigenfunction for all Hecke operators $F \in \mathfrak{M}_k^2$ of integral weight $k \geq 0$ can be continued to the whole s-plane as a meromorphic function; the function

$$\Psi(s,F) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\zeta(s,F),$$

where Γ is the gamma function, is meromorphic on the s-plane with the only possible poles at $s = 0, k-2, k, 2k-2$ and satisfies the functional equation

$$\Psi(2k-2-s,F) = (-1)^k\Psi(s,F).$$

This does not contradict the functional equation (5.67), because the assumptions of Theorem 5.35 imply that k there must be even.

The relation between the eigenvalues of Hecke operators and Fourier coefficients of eigenfunctions for Siegel modular forms of genus n discovered by Zharkovskaya in [Zha75] can possibly provide the first step to approach the still open problem of analytic properties of similar zeta functions of cusp form for genera $n > 2$.

Conclusion

On relations of zeta functions of orthogonal and symplectic groups see [An06]. For a general outlook on problems of Euler products attached to automorphic forms see Langlands' lectures [La67] and [La69].

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