A

Nomenclature of Unitals

There has been an assortment of conflicting names for the unitals constructed by Buekenhout [81]. In this book we have tried to develop a standard naming system for the different classes of unitals. We have included this appendix to emphasize the names we have used. Let $\mathcal{P}$ be a translation plane of order q^2 with kernel containing $\mathrm{GF}(q)$, and consider the Bruck-Bose representation of $\mathcal{P}$ in $\mathrm{PG}(4,q)$.

Throughout this book we use the names below.

- The **classical** or **Hermitian** unital corresponds to a nondegenerate Hermitian curve in $\mathrm{PG}(2,q^2)$.
- An **orthogonal-Buekenhout-Metz** unital in $\mathcal{P}$ corresponds to an orthogonal cone (elliptic cone) in $\mathrm{PG}(4,q)$; that is, an ovoidal cone with an elliptic quadric as base.
- A **Buekenhout-Tits** unital in $\mathcal{P}$ corresponds to an ovoidal cone in $\mathrm{PG}(4,q)$ with base a Tits ovoid, where $q \geq 8$ is an odd power of 2.
- An **ovoidal-Buekenhout-Metz** unital in $\mathcal{P}$ corresponds to an ovoidal cone in $\mathrm{PG}(4,q)$ with *any* ovoid as base. So the class of ovoidal-Buekenhout-Metz unitals includes the orthogonal-Buekenhout-Metz unitals and the Buekenhout-Tits unitals.
- A **nonsingular-Buekenhout** unital in $\mathcal{P}$ corresponds to a nonsingular parabolic quadric in $\mathrm{PG}(4,q)$.
- A **Buekenhout** unital in $\mathcal{P}$ corresponds in $\mathrm{PG}(4,q)$ to either a nonsingular parabolic quadric or to an ovoidal cone with *any* ovoid as base. That is, it is a unital arising from either one of Buekenhout's two constructions.

In the literature, several different names have been used. The term *Hermitian arc* refers to a unital embedded in a projective plane (for example, see [192]). Buekenhout used the names *parabolic unital* and *hyperbolic unital* to refer to unitals that were, respectively, tangent or secant to the line at infinity. These names have been used throughout the literature, and various authors use *parabolic Buekenhout unital* and *hyperbolic Buekenhout unital* to refer to an ovoidal-Buekenhout-Metz and a nonsingular-Buekenhout unital, respectively. Lefèvre-Percsy [163] first used the name *Buekenhout-Metz unital* to refer to a unital that corresponds to an ovoidal cone. Many subsequent researchers also used this term. Other authors (for example, see [108]) have used *Buekenhout-Metz* to refer to unitals which have an elliptic quadric as base. However, Metz's proof of the existence of nonclassical unitals is not specific to the case where the cone is an elliptic cone. The name *Buekenhout-Tits* was first used in [109] to specify those unitals whose base is a Tits ovoid. The term *orthogonal Buekenhout-Metz* has been used by authors who want a name for the class of unitals corresponding to orthogonal cones (for example, see [39], [89]). Finally, in [33] the unitals corresponding to a nonsingular quadric are called *Buekenhout unitals*.

B

Group Theoretic Characterizations of Unitals

In this appendix we give a brief summary of the main theorems in articles that give group theoretic characterizations of unitals, and also summarize related group theoretic results.

1971 Kantor [150]. Let $\mathcal{P}$ be a projective plane of order q^2, and let U be a unital obtained as the absolute points of some unitary polarity of $\mathcal{P}$. Let $G(U)$ be the group of collineations stabilizing U. If G is a subgroup of $G(U)$ that acts transitively on the flags (X, ℓ), where X is a point of U and ℓ is a secant line of U through X, then $\mathcal{P}$ is Desarguesian and $\mathrm{PSU}(3, q) \leq G$. Moreover, if there are at least three noncollinear points $X \in U$ such that $G(U)$ contains q (X, ℓ)-elations (where ℓ is the pole of X with respect to the polarity of U), then $\mathcal{P}$ is Desarguesian. In 1984 Camina and Gagen [83] noted that transitivity of G on secants of U implies flag-transitivity of G, thus improving the first result above.

1972 Hoffer [137]. If $\mathcal{P}$ is a projective plane of order q^2 and G is a collineation group of $\mathcal{P}$ that is isomorphic to $\mathrm{PSU}(3, q)$, then $\mathcal{P}$ is Desarguesian and G contains all elations of $\mathcal{P}$ that commute with a suitable unitary polarity.

1982 Biscarini [55]. Let G be a group of collineations acting on a unital U embedded in $\mathrm{PG}(2, q^2)$. If G is transitive on the secant lines of U and G is generated by involutions, then U is classical.

1984 Abatangelo [1]. A Buekenhout-Metz unital U of $\mathrm{PG}(2, q^2)$, q odd, is classical if and only if there is a cyclic linear collineation group of order $q^2 - 1$ that stabilizes U and fixes two distinct points of U. Applying this characterization, a new proof of Metz's result [172] is given.

1985 Kantor [151]. The classification of all finite simple groups is used to characterize designs with 2-transitive automorphism groups. An application to unitals shows that the only unitals with an automorphism group that is 2-transitive on points are the classical unitals and the Ree unitals.

1989 Biliotti and Korchmáros [51]. Let G be a collineation group acting on a unital U embedded in a projective plane $\mathcal{P}$ of order q^2, for some odd prime power $q \geq 5$. If G is transitive on the points of U and the socle of G has even order, then $\mathcal{P}$ is Desarguesian, U is classical, and $G \cong \mathrm{PSU}(3, q^2)$. A corollary of this result shows that if G acts primitively on the points of U, then $\mathcal{P}$ is Desarguesian, U is classical, and (for $q > 3$) $\mathrm{PSU}(3, q^2) \leq G \leq \mathrm{P\Gamma U}(3, q^2)$.

1989 Biliotti and Korchmáros [52]. Let G be a collineation group acting on a unital U embedded in a projective plane $\mathcal{P}$ of even order q^2. If G is transitive on the points of U and the socle of G has even order, then G contains involutory elations, and hence by [150] $\mathcal{P}$ is Desarguesian, U is classical, and $\mathrm{PSU}(3, q^2) \leq G \leq \mathrm{P\Gamma U}(3, q^2)$. As a consequence, if G acts primitively on the points of U, then $\mathcal{P}$ is Desarguesian, U is classical, and (for $q > 2$) $\mathrm{PSU}(3, q^2) \leq G \leq \mathrm{P\Gamma U}(3, q^2)$.

1989 Doyen [107]. An investigation is conducted of 2-$(v, k, 1)$ designs with an automorphism group that satisfies certain transitivity assumptions. An application to unitals shows that the only line-transitive unitals are the classical unitals and the Ree unitals.

1991 Batten [40]. Let S be a blocking set in a finite projective plane $\mathcal{P}$ of order q^2, and let G be a collineation group of $\mathcal{P}$ that acts flag-transitively on S. Then $\mathcal{P} = \mathrm{PG}(2, q^2)$, S is either a Baer subplane of $\mathcal{P}$ or a classical unital of $\mathcal{P}$, and $\mathrm{PSU}(3, q) \lhd G$.

1991 Abatangelo and Larato [5]. Let U be the nonclassical unital in $\mathrm{PG}(2, q^2)$, q odd, constructed from a pencil of q suitable conics (see [21] or [134]). Then the projective group G stabilizing U has order $2q^3(q-1)$ and is the semidirect product of a normal elementary abelian subgroup of order q^3 and a cyclic group of order $2(q-1)$. Conversely, a unital whose projective stabilizer has the above structure is necessarily such a pencil of conics. These results were obtained independently in [22].

1992 Abatangelo [2]. Let U be a nonclassical orthogonal-Buekenhout-Metz unital with respect to some point P in $\mathrm{PG}(2, q^2)$, q even. Then the collineation group G stabilizing U fixes P, has a normal subgroup acting sharply transitively on the points of $U \backslash P$, and has point stabilizers of order $q-1$ for any point of $U \backslash P$. This property characterizes the orthogonal-Buekenhout-Metz unitals among nonclassical unitals in $\mathrm{PG}(2, q^2)$, q even. The first result was obtained independently in [108].

1996 Ebert and Wantz [112]. A unital U embedded in $\mathrm{PG}(2, q^2)$ is an orthogonal-Buekenhout-Metz unital if and only if U admits a linear collineation group that is the semidirect product of a subgroup of order q^3 by a subgroup of order $q-1$. This is the full linear collineation group stabilizing U except when U is a classical unital or the union of a partial pencil of conics (as constructed in [21] or [134]). As a corollary, one obtains the result that a unital U embedded in $\mathrm{PG}(2, q^2)$ is classical if and only if U admits a linear collineation group that is the semidirect product of a subgroup of order q^3 by a subgroup of order $q^2 - 1$.

1996 Abatangelo and Larato [6]. Let U be an ovoidal-Buekenhout-Metz unital embedded in $\mathrm{PG}(2, q^2)$, q even, and let G be the linear collineation group stabilizing U. Then U is an orthogonal-Buekenhout-Metz unital if and only if there is a point P of U such that the stabilizer of P in G has a subgroup that acts sharply transitively on $U \backslash P$. Furthermore, if U is a Buekenhout-Tits unital in $\mathrm{PG}(2, 2^r)$, where $r \geq 3$ is an odd integer, then the full linear collineation group leaving U invariant is an abelian group of order q^2 that fixes some point of U. The second result was independently obtained in [109].

1999 Abatangelo, Enea, Korchmáros, and Larato [3]. The collineation stabilizer is completely determined for the unital obtained as the set of absolute points for the natural unitary polarity of a commutative twisted field plane of odd order q^2.

1999 Biliotti [48]. A study is conducted of the structure of collineation groups which preserve a unital in a finite projective plane of order q^2, where $q \equiv 1 \pmod 4$. In particular, several results are obtained when the group is a 2-group.

2000 Cossidente, Ebert, and Korchmáros [91]. A unital in $\mathrm{PG}(2, q^2)$ is classical if and only if it is stabilized by a Singer subgroup of $\mathrm{PGL}(3, q^2)$ of order $q^2 - q + 1$.

2001 Cossidente, Ebert, and Korchmáros [92]. A unital in $\mathrm{PG}(2, q^2)$ is classical if and only if it is stabilized by a linear collineation group of order $6(q+1)^2$ which fixes neither a point nor a line in $\mathrm{PG}(2, q^2)$.

2001 Abatangelo, Korchmáros, and Larato [4]. Let U be a unital embedded in a translation plane $\mathcal{P}$ of odd order so that there is a collineation group G stabilizing U which fixes the point $P = U \cap \ell_\infty$ and acts transitively on the points of $U \backslash P$. If G contains an affine homology, then $\mathcal{P}$ is a semifield plane and G has a normal subgroup K that acts sharply transitively on $U \backslash P$. In addition, if K is abelian, then $\mathcal{P}$ is a commutative semifield plane.

2002 Giuzzi [123]. A new proof of the group theoretic characterization by Cossidente, Ebert, and Korchmáros [91] is given.

2002 Abatangelo and Larato [7]. Let $\mathcal{P}$ be a commutative Dickson semifield plane of odd order q^2, and suppose that U is a transitive parabolic unital (see [4]) obtained as the absolute points of a unitary polarity of $\mathcal{P}$. Then the sharply transitive normal subgroup K of the collineation stabilizer of U is non-abelian.

2006 Jha and Johnson [145]. Let U be a unital embedded in a translation plane $\mathcal{P}$ of order q^2 so that there is a collineation group G stabilizing U which fixes the point $P = U \cap \ell_\infty$ and acts transitively on the points of $U \backslash P$. Assume that q is a power of the prime p and G has order $q^3 u$, where u is a prime p-primitive divisor of $q^2 - 1$. Then $\mathcal{P}$ is Desarguesian. Other group theoretic results for such planes also are obtained.

2006 Johnson [146]. Let $\mathcal{P}$ be a semifield plane of order q^2 that is two-dimensional over its kernel. Then $\mathcal{P}$ admits a transitive parabolic unital; that is, a unital U with a collineation group G that fixes the point $P = U \cap \ell_\infty$ and acts transitively on the points of $U \setminus P$).

References

1. L.M. Abatangelo. Una caratterizzazione gruppale delle curve Hermitiane. *Le Matematiche (Catania)*, **39** (1984) 101–110.
2. V. Abatangelo. On Buekenhout-Metz unitals in $\mathrm{PG}(2, q^2)$, q even. *Arch. Math.*, **59** (1992) 197–203.
3. V. Abatangelo, M.R. Enea, G. Korchmáros, and B. Larato. Ovals and unitals in commutative twisted field planes. Combinatorics (Assisi, 1996), *Discrete Math.*, **208/209** (1999) 3–8.
4. V. Abatangelo, G. Korchmáros, and B. Larato. Transitive parabolic unitals in translation planes of odd order. *Discrete Math.*, **231** (2001) 3–10.
5. V. Abatangelo and B. Larato. A group theoretic characterization of parabolic Buekenhout-Metz unitals. *Boll. U.M.I.*, **5**A (7) (1991) 195–206.
6. V. Abatangelo and B. Larato. A characterization of Buekenhout-Metz unitals in $\mathrm{PG}(2, q^2)$, q even. *Geom. Dedicata*, **59** (1996) 137–145.
7. V. Abatangelo and B. Larato. Polarity and transitive parabolic unitals in translation planes of odd order. *J. Geometry*, **74** (2002) 1–6.
8. V. Abatangelo, B. Larato, and L.A. Rosati. Unitals in planes derived from Hughes planes. *J. Combin. Inf. Syst. Sci.*, **15** (1990) 151–155.
9. A. Aguglia. Designs arising from Buekenhout-Metz unitals. *J. Combin. Des.*, **11** (2003) 79–88.
10. A. Aguglia and G.L. Ebert. A combinatorial characterization of classical unitals. *Arch. Math.*, **78** (2002) 166–172.
11. A.A. Albert. On non-associative division algebras. *Trans. Amer. Math. Soc.*, **72** (1952) 296–309.
12. J. André. Über nicht-Desarguessche Ebenen mit transitiver Translationgruppe. *Math. Z.*, **60** (1954) 156–186.
13. B.R. Andriamanalimanana. *Ovals, Unitals and Codes*. Ph.D. thesis, Lehigh University, 1979.
14. E.F. Assmus Jr. and J.D. Key. Arcs and ovals in the Hermitian and Ree unitals. *Eur. J. Combin.*, **10** (1989) 297–308.
15. E.F. Assmus Jr. and J.D. Key. Baer subplanes, ovals and unitals. *IMA Math. Appl. Coding Theory and Design Theory, Part I: Coding Theory*, **20** (1990) 1–8.
16. E.F. Assmus Jr. and J.D. Key. *Designs and Their Codes.* Cambridge University Press, 1992.

17. R. Baer. Polarities in finite projective planes. *Bull. Amer. Math. Soc.*, **52** (1946) 77–93.
18. R. Baer. *Linear Algebra and Projective Geometry.* Dover, 2005 (Academic Press, 1952).
19. B. Bagchi and S. Bagchi. Designs from pairs of finite fields. I. A cyclic unital $U(6)$ and other regular Steiner 2-designs. *J. Combin. Theory Ser. A*, **52** (1989) 51–61.
20. R.D. Baker. Polarities of elliptic semiplanes. *Proc. Ninth Southeast. Conf. Combin. Graph Theory Comp.*, (1978) 85–96.
21. R.D. Baker and G.L. Ebert. Intersections of unitals in the Desarguesian plane. *Congr. Numer.*, **70** (1990) 87–94.
22. R.D. Baker and G.L. Ebert. On Buekenhout-Metz unitals of odd order. *J. Combin. Theory Ser. A*, **60** (1992) 67–84.
23. R.D. Baker, G.L. Ebert, G. Korchmáros, and T. Szőnyi. Orthogonally divergent spreads of Hermitian curves. Finite Geometry and Combinatorics (Deinze, 1992), *London Math. Soc. Lecture Note Ser.*, **191** (1993) 17–30.
24. R.D. Baker and K.L. Wantz. Unitals in the code of the Hughes plane. *J. Combin. Des.*, **12** (2004) 35–38.
25. S. Ball. Polynomials in finite geometries. Surveys in Combinatorics (Canterbury, 1999), *London Math. Soc. Lecture Note Ser.*, **267** (1999) 17–35.
26. S. Ball, A. Blokhuis, and C.M. O'Keefe. On unitals with many Baer sublines. *Des. Codes Cryptogr.*, **17** (1999) 237–252.
27. S. Ball and M.R. Brown. The six semifield planes associated with a semifield flock. *Adv. Math.*, **189** (2004) 68–87.
28. A. Barlotti. Un'estensione del teorema di Segre-Kustaanheimo. *Boll. U.M.I.*, **10** (1955) 498–506.
29. A. Barlotti. Some topics in finite geometric structures. *Institute of Statistics Mimeo Series No. 439*, University of North Carolina, Chapel Hill, 1965.
30. A. Barlotti. Finite geometries and designs. Surveys in Combinatorics, *London Math. Soc. Lecture Note Ser.*, **124** (1987) 1–11.
31. A. Barlotti and G. Lunardon. Una classe di unitals nei Δ-piani. *Riv. Mat. Univ. Parma*, (4) **5** (1979) 781–785.
32. M. Barnabei and F. Bonetti. Two examples of finite Bolyai-Lobachevsky planes. *Rend. Mat.*, **12** (1979) 291–296.
33. S.G. Barwick. A characterization of the classical unital. *Geom. Dedicata*, **52** (1994) 175–180.
34. S.G. Barwick. A class of Buekenhout unitals in the Hall plane. *Bull. Belg. Math. Soc. Simon Stevin*, **3** (1996) 113–124.
35. S.G. Barwick. Unitals in the Hall plane. *J. Geometry*, **58** (1997) 26–42.
36. S.G. Barwick, L.R.A. Casse, and C.T. Quinn. The André/Bruck and Bose representation in $\mathrm{PG}(2h, q)$: unitals and Baer subplanes. *Bull. Belg. Math. Soc. Simon Stevin*, **7** (2000) 173–197.
37. S.G. Barwick and C.M. O'Keefe. Unitals and inversive planes. *J. Geometry*, **58** (1997) 43–52.
38. S.G. Barwick, C.M. O'Keefe, and L. Storme. Unitals which meet Baer subplanes in 1 modulo q points. *J. Geometry*, **68** (2000) 16–22.
39. S.G. Barwick and C.T. Quinn. Generalising a characterisation of Hermitian curves. *J. Geometry*, **70** (2001) 1–7.
40. L.M. Batten. Blocking sets with flag transitive collineation groups. *Arch. Math.*, **56** (1991) 412–416.

41. C. Bernasconi and R. Vincenti. Spreads induced by varieties $\mathcal{V}_2^3$ of $\mathrm{PG}(4,q)$ and Baer subplanes. *Boll. Un. Mat. Ital. B*, **18** (1981) 821–830.
42. E. Bertini. Geometria proiettiva degli iperspasi. E. Spoerri, ed., Pisa, 1907.
43. A. Betten, D. Betten, and V.D. Tonchev. Unitals and codes. *Discrete Math.*, **267** (2003) 23–33.
44. A. Beutelspacher. Embedding the complement of a Baer subplane or a unital in a finite projective plane. *Mitt. Math. Sem. Giessen*, **163** (1984) 189–202.
45. A. Beutelspacher. Embedding linear spaces with two line degrees in finite projective planes. *J. Geometry*, **26** (1986) 43–61.
46. A. Beutelspacher and U. Rosenbaum. *Projective Geometry: From Foundations to Applications*. Cambridge University Press, 1998.
47. J. Bierbrauer. *Introduction to Coding Theory*. Chapman & Hall/CRC, 2005.
48. M. Biliotti. Collineation groups preserving a unital in a projective plane of order m^2 with $m \equiv 1(4)$. *J. London Math. Soc.*, **60** (1999) 589–606.
49. M. Biliotti, V. Jha, and N.L. Johnson. *Foundations of Translation Planes.* Marcel Dekker, 2001.
50. M. Biliotti, V. Jha, N.L. Johnson, and A. Montinaro. Two-transitive groups on a hyperbolic unital. Preprint.
51. M. Biliotti and G. Korchmáros. Collineation groups preserving a unital of a projective plane of odd order. *J. Algebra*, **122** (1989) 130–149.
52. M. Biliotti and G. Korchmáros. Collineation groups preserving a unital of a projective plane of even order. *Geom. Dedicata*, **31** (1989) 333–344.
53. M. Biliotti and G. Lunardon. Insiemi di derivazione e sottopiani di Baer. *Atti Accad. Lincei, Rend. Sci. Fis. Mat. Nat.*, Serie VIIIa, **LXIX** (1980) 135–141.
54. G. Birkhoff and J. von Neumann. The logic of quantum mechanics. *Ann. Math.*, **37** (1936) 823–843.
55. P. Biscarini. Hermitian arcs of $\mathrm{PG}(2,q^2)$ with a transitive collineation group on the set of $(q+1)$–secants. *Rend. Sem. Mat. Brescia*, **7** (1982) 111–124.
56. A. Blokhuis, A. Brouwer, and H. Wilbrink. Hermitian unitals are codewords. *Discrete Math.*, **97** (1991) 63–68.
57. A. Blokhuis and C.M. O'Keefe. On deriving unitals. 16th British Combinatorial Conference (London, 1997), *Discrete Math.*, **197/198** (1999) 137–141.
58. E. Boros and T. Szőnyi. On the sharpness of a theorem of B. Segre. *Combinatorica*, **6** (1986) 261–268.
59. R.C. Bose. On the construction of balanced incomplete block designs. *Ann. Eugenics*, **9** (1939) 353–399.
60. R.C. Bose. Mathematical theory of symmetrical factorial design. *Sankhyā*, **8** (1947) 107–166.
61. R.C. Bose. On a representation of the Baer subplanes of the Desarguesian plane $PG(2,q^2)$ in a projective five dimensional space $PG(5,q)$. *Proceedings of the International Colloquium on Combinatorial Theory*, Rome (1973) 381–391.
62. R.C. Bose, R.C. Freeman, and D.G. Glynn. On the intersection of two Baer subplanes in a finite projective plane. *Utilitas Math.*, **17** (1980) 65–77.
63. P.L.H. Brooke. On the Steiner system $S(2,4,28)$ and codes associated with the simple group of order 6048. *J. Algebra,* **97** (1985) 376–406.
64. A.E. Brouwer. Some unitals on 28 points and their embeddings in projective planes of order 9. Geometries and Groups, *Springer Lecture Notes*, **893** (1981) 183–188.
65. A.E. Brouwer. Some unitals on 28 points and their embeddings in projective planes of order 9. *Math. Centre Report ZW155181,* Amsterdam, 1981.

66. A.E. Brouwer. A unital in the Hughes plane of order nine. *Discrete Math.*, **77** (1989) 55–56.
67. A.E. Brouwer, W.H. Haemers, and H.A. Wilbrink. Some 2-ranks. *Discrete Math.*, **106/107** (1992) 83–92.
68. M.R. Brown. Ovoids of $\mathrm{PG}(3,q)$, q even, with a conic section. *J. London Math. Soc.*, **62** (2000) 569–582.
69. R.H. Bruck. Difference sets in a finite group. *Trans. Amer. Math. Soc.*, **78** (1955) 464–481.
70. R.H. Bruck. Quadratic extensions of cyclic planes. *Proc. Symp. Appl. Math.*, **10** (1960) 15–44.
71. R.H. Bruck. Construction problems of finite projective planes. *Conference on Combinatorial Mathematics and its Applications*, University of North Carolina Press, (1969) 426–514.
72. R.H. Bruck. Circle geometry in higher dimensions. *A Survey of Combinatorial Theory.* North Holland, (1973) 69–77.
73. R.H. Bruck and R.C. Bose. The construction of translation planes from projective spaces. *J. Algebra*, **1** (1964) 85–102.
74. R.H. Bruck and R.C. Bose. Linear representations of projective planes in projective spaces. *J. Algebra*, **4** (1966) 117–172.
75. A. Bruen. Baer subplanes and blocking sets. *Bull. Amer. Math. Soc.*, **76** (1970) 342–344.
76. A. Bruen. Blocking sets in finite projective planes. *SIAM J. Appl. Math.*, **21** (1971) 380–392.
77. A.A. Bruen and J.C. Fisher. An observation on certain point-line configurations in classical planes. *Discrete Math.*, **106/107** (1992) 93–96.
78. A.A. Bruen and J.W.P. Hirschfeld. Intersections in projective spaces I: Combinatorics. *Math. Z.*, **193** (1986) 215–225.
79. A.A. Bruen and R. Silverman. Arcs and blocking sets II. *Eur. J. Combin.*, **8** (1987) 351–356.
80. A.A. Bruen and J.A. Thas. Blocking sets. *Geom. Dedicata*, **6** (1977) 193–203.
81. F. Buekenhout. Existence of unitals in finite translation planes of order q^2 with a kernel of order q. *Geom. Dedicata*, **5** (1976) 189–194.
82. F. Buekenhout. Characterizations of semi quadrics. *Atti Conv. Lincei*, **17** (1976) 393–421.
83. A.R. Camina and T.M. Gagen. Block transitive automorphism groups and designs. *J. Algebra*, **86** (1984) 549–554.
84. J. Cannon and C. Playoust. *An Introduction to MAGMA*. University of Sydney Press, 1993.
85. L.R.A. Casse. *A Guide to Projective Geometry.* Oxford University Press, 2006.
86. L.R.A. Casse and G. Lunardon. On Buekenhout-Metz unitals. Finite Geometry and Combinatorics (Deinze, 1997), *Bull. Belg. Math. Soc. Simon Stevin*, **5** (1998) 237–240.
87. L.R.A. Casse, C.M. O'Keefe, and T. Penttila. Characterisations of Buekenhout-Metz unitals. *Geom. Dedicata*, **59** (1996) 29–42.
88. L.R.A. Casse and C.T. Quinn. The Bruck-Bose map and Buekenhout-Metz unitals. R. C. Bose Memorial Conference (Fort Collins, CO, 1995), *J. Stat. Plann. Inference*, **72** (1998) 89–98.
89. L.R.A. Casse and C.T. Quinn. Ruled cubic surfaces in $\mathrm{PG}(4,q)$, Baer subplanes of $\mathrm{PG}(2,q^2)$ and Hermitian curves. *Discrete Math.*, **248** (2002) 17–25.

90. H.S.M. Coxeter. *Projective Geometry, Second Edition.* Springer-Verlag, 1987.
91. A. Cossidente, G.L. Ebert, and G. Korchmáros. A group-theoretic characterization of classical unitals. *Arch. Math.*, **74** (2000) 1–5.
92. A. Cossidente, G.L. Ebert, and G. Korchmáros. Unitals in finite Desarguesian planes. *J. Algebraic Combin.*, **14** (2001) 119–125.
93. M. deFinis. On k-sets of type (m, n) in projective planes of square order. Finite Geometries and Designs, *London Math. Soc. Lecture Note Ser.*, **49** (1981) 98–103.
94. M.J. de Resmini. There exists at least three non isomorphic $S(2, 4, 28)$'s. *J. Geometry*, **16** (1981) 148–151.
95. M.J. de Resmini and N. Hamilton. Hyperovals and unitals in Figueroa planes. *Eur. J. Combin.*, **19** (1998) 215–220.
96. P. Dembowski. *Finite Geometries.* Springer-Verlag, 1968 (reprinted 1997).
97. L.J. Dickey. Embedding the complement of a unital in a projective plane. *Atti Conv. Geom. Appl.*, Perugia, (1971) 199–203.
98. L.E. Dickson. *Linear Groups with an Exposition of the Galois Field Theory.* Teubner, 1901 (Dover, 1958).
99. L.E. Dickson. On finite algebras. *Nachr. kgl. Ges. Wiss. Göttingen*, (1905) 358–393.
100. G. Donati and N. Durante. Some subsets of the Hermitian curve. *Eur. J. Combin.*, **24** (2003) 211–218.
101. G. Donati and N. Durante. On the intersection of Hermitian curves and of Hermitian surfaces. *Discrete Math.*, in press.
102. J. Dover. Some design-theoretic properties of Buekenhout unitals. *J. Combin. Des.*, **4** (1996) 449–456.
103. J.M. Dover. A family of non-Buekenhout unitals in the Hall planes. Mostly Finite Geometries (Iowa City, IA, 1996), *Lecture Notes Pure Appl. Math.*, **190** (1997) 197–205.
104. J.M. Dover. Spreads and resolutions of Ree unitals. *Ars Combin.*, **54** (2000) 301–309.
105. J.M. Dover. Subregular spreads of Hermitian unitals. *Des. Codes Cryptogr.*, **39** (2006) 5–15.
106. J.M. Dover. A search for spreads of Hermitian unitals. Private communication.
107. J. Doyen. Designs and automorphism groups. Surveys in Combinatorics, *London Math. Soc. Lecture Note Ser.*, **141** (1989) 74–83.
108. G.L. Ebert. On Buekenhout-Metz Unitals of even order. *Eur. J. Combin.*, **13** (1992) 109–117.
109. G.L. Ebert. Buekenhout-Tits unitals. *J. Algebraic Combin.*, **6** (1997) 133–140.
110. G.L. Ebert. Hermitian arcs. *Rend. Circ. Mat. Palermo*, **51** (1998) 87–105.
111. G.L. Ebert. Buekenhout unitals. Combinatorics (Assisi, 1996), *Discrete Math.*, **208/209** (1999) 247–260.
112. G.L. Ebert and K. Wantz. A group-theoretic characterization of Buekenhout-Metz unitals. *J. Combin. Des.*, **4** (1996) 143–152.
113. G. Faina. Sylow p-subgroups of $PSU(3, q^2)$. *Atti Sem. Mat. Fis. Univ. Modena*, **32** (1983) 255–259.
114. G. Faina. A characterization of the tangent lines to a Hermitian curve. *Rend. Mat.*, **3** (1983) 553–557.
115. G. Faina. Intrinsic characterizations of Hermitian curves. *Simon Stevin*, **65** (1991) 165–178.
116. G. Faina and G. Korchmáros. A graphic characterization of Hermitian curves. *Ann. Discrete Math.*, **18** (1983) 335–342.

117. R. Figueroa. A family of not (V,ℓ)-transitive projective planes of order q^3, $q \not\equiv 1 \pmod 3$ and $q > 2$. *Math. Z.*, **181** (1982) 471–479.
118. J.C. Fisher, J.W.P. Hirschfeld, and J.A. Thas. Complete arcs in planes of square order. *Ann. Discrete Math.*, **30** (1986) 243–250.
119. J.W. Freeman. Reguli and pseudo–reguli in $PG(3,s^2)$. *Geom. Dedicata,* **9** (1980) 267–280.
120. M.J. Ganley. Polarities in translation planes. *Geom. Dedicata,* **1** (1972) 28–40.
121. M.J. Ganley. A class of unitary block designs. *Math. Z.*, **128** (1972) 34–42.
122. L. Giuzzi. Collineation groups of the intersection of two classical unitals. *J. Combin. Des.*, **9** (2001) 445–459.
123. L. Giuzzi. A characterisation of classical unitals. *J. Geometry*, **74** (2002) 86–89.
124. V.D. Goppa. Algebraic-geometric codes. *Math. USSR-Izv.*, **21** (1983) 75–91.
125. T. Grundhöfer. A synthetic construction of Figueroa planes. *J. Geometry*, **26** (1986) 191–201.
126. K. Grüning. Das Kleinste Ree-Unital. *Arch. Math.*, **46** (1986) 473–480.
127. K. Grüning. A class of unitals of order q which can be embedded in two different translation planes of order q^2. *J. Geometry,* **29** (1987) 61–77.
128. M. Hall. *Theory of Groups, Second Edition.* Chelsea Publishing Co., 1976.
129. R. Hill. *A First Course in Coding Theory*. Oxford University Press, 1986.
130. J.W.P. Hirschfeld. *Finite Projective Spaces of Three Dimensions.* Oxford University Press, 1985.
131. J.W.P. Hirschfeld. *Projective Geometry over Finite Fields, Second Edition.* Oxford University Press, 1998.
132. J.W.P. Hirschfeld, L. Storme, J.A. Thas, and J.F. Voloch. A characterization of Hermitian curves. *J. Geometry*, **41** (1991) 72–78.
133. J.W.P. Hirschfeld and T. Szőnyi. Sets in a finite plane with few intersection numbers and a distinguished point. *Discrete Math.*, **97** (1991) 229–242.
134. J.W.P. Hirschfeld and J.A. Thas. *General Galois Geometries.* Oxford University Press, 1991.
135. G. Hiss. On the incidence matrix of the Ree unital. *Des. Codes Cryptogr.*, **10** (1997) 57–62.
136. G. Hiss. Hermitian function fields, classical unitals, and representations of 3-dimensional unitary groups. *Indag. Math. (N.S.)*, **15** (2004) 223–243.
137. A.R. Hoffer. On unitary collineation groups. *J. Algebra*, **22** (1972) 211–218.
138. G. Hölz. Constructions of designs which contain a unital. *Arch. Math.*, **37** (1981) 179–183.
139. D.R. Hughes. A class of non-Desarguesian projective planes. *Canad. J. Math.*, **9** (1957) 378–388.
140. D.R. Hughes and F.C. Piper. *Projective Planes.* Springer-Verlag, 1973.
141. D.R. Hughes and F.C. Piper. *Design Theory.* Cambridge University Press, 1985.
142. W.A. Jackson, C.M. O'Keefe, and T. Penttila. On the unitals of Mathon, Bagchi and Bagchi. *Univ. West. Aust. Research Report 46*, University of Western Australia, 1990.
143. W.A. Jackson, F.C. Piper, and P.R. Wild. Non-embedding of non prime power unitals with point–regular groups. *Discrete Math.*, **98** (1991) 23–28.
144. D.B. Jaffe and V.D. Tonchev. Computing linear codes and unitals. *Des. Codes Cryptogr.*, **14** (1998) 39–52.
145. V. Jha and N.L. Johnson. Translation planes of order q^2 admitting collineation groups of order q^3u preserving a parabolic unital. *Note di Mat.*, **26** (2006) 119–137.

146. N.L. Johnson. Transitive parabolic unitals in semifield planes. *J. Geometry*, **85** (2006) 61–71.
147. N.L. Johnson, V. Jha, and M. Biliotti. *Handbook of Finite Translation Planes.* Chapman and Hall, 2007.
148. N.L. Johnson and G. Lunardon. On the Bose-Barlotti-planes. *Geom. Dedicata*, **49** (1994) 173–182.
149. N.L. Johnson and R. Pomareda. Collineation groups of translation planes admitting hyperbolic Buekenhout or parabolic Buekenhout-Metz unitals. *J. Combin. Theory Ser. A*, **114** (2007) 658–680.
150. W.M. Kantor. On unitary polarities of finite projective planes. *Canad. J. Math.*, **23** (1971) 1060–1077.
151. W.M. Kantor. Homogeneous designs and geometric lattices. *J. Combin. Theory Ser. A*, **38** (1985) 66–74.
152. B.C. Kestenband. Unital intersections in finite projective planes. *Geom. Dedicata*, **11** (1981) 107–117.
153. B.C. Kestenband. Degenerate unital intersections in finite projective planes. *Geom. Dedicata*, **13** (1982) 101–106.
154. B.C. Kestenband. A family of unitals in the Hughes plane. *Canad. J. Math.*, **42** (1990) 1067–1083.
155. B.C. Kestenband. New families of balanced incomplete block designs with block size $q+1$. *J. Stat. Plann. Inference*, **30** (1992) 83–91.
156. J.D. Key. Some applications of Magma in designs and codes: oval designs, Hermitian unitals and generalized Reed-Muller codes. *J. Symbolic Comput.*, **31** (2001) 37–53.
157. J.D. Key and N.K.A. Rostom. Unitary designs with regular sets of points. *Discrete Math.*, **69** (1988) 235–239.
158. P.B. Kleidman. The finite flag transitive linear spaces with an exceptional automorphism group. Finite Geometries and Combinatorial Designs, *Contemp. Math.*, **111** (1987) 117–136.
159. G. Korchmáros. Inherited arcs in finite affine planes. *J. Combin. Theory Ser. A*, **42** (1986) 140–143.
160. B. Larato. Una caraterizzazione degli unitals parabolici di Buekenhout-Metz. *Le Matematiche (Catania)*, **38** (1983) 95–98.
161. B. Larato. Strutture d'incidenza associate a curve hermitiane. *Boll. U.M.I.*, **3**A (1984) 87–95.
162. C. Lefèvre. Tallini sets in projective spaces. *Atti Accad. Naz. Lincei Rend.*, **59** (1975) 392–400.
163. C. Lefèvre-Percsy. Characterization of Buekenhout-Metz unitals. *Arch. Math.*, **36** (1981) 565–568.
164. C. Lefèvre-Percsy. Characterization of Hermitian curves. *Arch. Math.*, **39** (1982) 476–480.
165. K.H. Leung and Q. Xiang. On the dimensions of the binary codes of a class of unitals. *Discrete Math.*, to appear.
166. H. Lüneburg. Some remarks concerning the Ree groups of type (G_2). *J. Algebra*, **3** (1966) 256–259.
167. R. Maddock. *The Bose Representation of* $\mathrm{PG}(2,q^2)$ *and Some Associated Varieties*. Ph.D. thesis, University of Adelaide, 2000.
168. G. McGuire, V.D. Tonchev, and H.N. Ward. Characterizing the Hermitian and Ree unitals on 28 points. *Des. Codes Cryptogr.*, **13** (1998) 57–61.

169. R. Mathon. Constructions for cyclic Steiner 2-designs. *Ann. Discrete Math.*, **34** (1987) 353–362.
170. R. Mathon and T. van Trung. Unitals and unitary polarities in symmetric designs. *Des. Codes Cryptogr.*, **10** (1997) 237–250.
171. K. Metsch. A note on Buekenhout-Metz unitals. Geometry, Combinatorial Designs and Related Structures (Spetses, 1996), *London Math. Soc. Lecture Note Ser.*, **245** (1997) 177–180.
172. R. Metz. On a class of unitals. *Geom. Dedicata*, **8** (1979) 125–126.
173. B. Mortimer. The modular permutation representations of the known doubly transitive groups. *Proc. London Math. Soc.*, **41** (1980) 1–20.
174. M.E. O'Nan. Automorphisms of unitary block designs. *J. Algebra*, **20** (1972) 495–511.
175. W.F. Orr. *The Miquelian Inversive Plane $IP(q)$ and the Associated Projective Planes.* Ph.D. thesis, University of Wisconsin - Madison, 1973.
176. G. Panella. Caratterizzazione delle quadriche di uno spazio (tri-dimensionale) lineare sopra un corpo finito. *Boll. Un. Mat. Ital.*, **10** (1955) 507–513.
177. S.E. Payne and J.A. Thas. *Finite Generalized Quadrangles.* Pitman, 1984.
178. T. Penttila and G.F. Royle. Sets of type (m, n) in the affine and projective planes of order nine. *Des. Codes Cryptogr.*, **6** (1995) 229–245.
179. F.C. Piper. Polarities in the Hughes plane. *Bull. London Math. Soc.*, **2** (1970) 209–213.
180. F.C. Piper. Unitary block designs. Graph Theory and Combinatorics, *Res. Notes Math.*, **34** (1979) 98–105.
181. O. Polverino. Linear representation of Buekenhout-Metz unitals. *Discrete Math.*, **267** (2003) 247–252.
182. C.T. Quinn. *Baer Structures, Unitals and Associated Finite Geometries.* Ph.D. thesis, University of Adelaide, 1996.
183. C.T. Quinn and L.R.A. Casse. Concerning a characterisation of Buekenhout-Metz unitals. *J. Geometry*, **52** (1995) 159–167.
184. B. Qvist. Some remarks concerning curves of the second degree in a finite plane. *Ann. Acad. Sci. Fenn. Ser. A*, **134** (1952).
185. A. Rahilly. Embeddings of generalized unitals and semi-regular polarities. *Ars Combin.*, **23** (1987) 273–302.
186. M. Richardson. On finite projective games. *Proc. Amer. Math. Soc.*, **7** (1956) 453–465.
187. G. Rinaldi. Hyperbolic unitals in the Hall planes. *J. Geometry*, **54** (1995) 148–154.
188. G. Rinaldi. Constructions of unitals in the Hall planes. *Geom. Dedicata,* **56** (1995) 249–255.
189. B. Rokowska. A new construction of the block systems $B(4, 1, 25)$ and $B(4, 1, 28)$. *Coll. Math.*, **38** (1977) 165–167.
190. L.A. Rosati. I gruppi di collineazioni dei piani di Hughes. *Boll. U.M.I.*, **13** (1958) 505–513.
191. L.A. Rosati. Designi unitari nei piani di Hughes. *Geom. Dedicata*, **27** (1988) 295–299.
192. M. Tallini Scafati. $\{k; n\}$-archi di un piano grafico finito, con particolare riguardo a quelli con due caratteri. *Atti Acad. Naz. Lincei Rend.*, **40** (1966) Nota I, 812–818, Nota II, 1020–1025.
193. M. Tallini Scafati. Caratterizzazione grafica delle forme hermitiane di un $S_{r,q}$. *Rend. Mat. Appl.*, **26** (1967) 273–303.

194. B. Segre. Curve razionali normali e k-archi negli spazi finiti. *Ann. Mat. Pura Appl.*, **39** (1955) 357–379.
195. B. Segre. Ovals in a finite projective plane. *Canad. J. Math.*, **7** (1955) 414–416.
196. B. Segre. On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two. *Acta Arith.*, **5** (1959) 315–322.
197. B. Segre. Teoria de Galois, fibrazioni proiettive e geometrie non-desarguesiane. *Ann. Mat. Pura Appl.*, **64** (1964) 1–76.
198. B. Segre. Forme e geometrie hermitiane con particolare riguardo al caso finito. *Ann. Mat.*, **70** (1965) 1–201.
199. B. Segre. Introduction to Galois geometries. *Atti Accad. Naz. Lincei Mem.*, **8** (1967) 133–236. (edited by J.W.P. Hirschfeld.)
200. M. Seib. Unitäre Polaritäten endlicher projectiver Ebenen. *Arch. Math.*, **21** (1970) 103–112.
201. J.G. Semple and L. Roth. *Introduction to Algebraic Geometry.* Oxford University Press, 1949.
202. M.A. Shokrollahi. Codes on Hermitian curves. *Lecture Notes Comput. Sci.*, **307** (1988) 168–176.
203. S.D. Stoichev and V. Tonchev. Unital designs in planes of order 16. Coding, Cryptography and Computer Security (Lethbridge, 1998), *Discrete Appl. Math.*, **102** (2000) 151–158.
204. M. Sved. On configurations of Baer subplanes of the projective plane over a finite field of square order. *Lecture Notes Math.*, **952** (1982) 423–443.
205. G. Tallini. Sulle k-calotte uno spazio lineare finito. *Ann. Mat. Pura Appl.*, **42** (1956) 119–164.
206. D.E. Taylor. Unitary block designs. *J. Combin. Theory Ser. A*, **16** (1974) 51–56.
207. J.A. Thas. Semipartial geometries and spreads of classical polar spaces. *J. Combin. Theory Ser. A*, **35** (1983) 58–66.
208. J.A. Thas. Solution of a classical problem on finite inversive planes. *Finite Geometries, Buildings and Related Topics* (Pingree Park, CO, 1988), 145–159.
209. J.A. Thas. A combinatorial characterization of Hermitian curves. *J. Algebraic Combin.*, **1** (1992) 97–102.
210. J.A. Thas, P.J. Cameron, and A. Blokhuis. On a generalization of a theorem of B. Segre. *Geom. Dedicata*, **43** (1992) 299–305.
211. H.J. Tiersma. Remarks on codes from Hermitian curves. *IEEE Trans. Inf. Theory*, **33** (1987) 605–609.
212. J. Tits. Ovoides et groupes du Suzuki. *Arch. Math.*, **13** (1962) 187–198.
213. V.D. Tonchev. Unitals in the Hölz design on 28 points. *Geom. Dedicata*, **38** (1991) 357–363.
214. J.H. Van Lint and T.A. Springer. Generalized Reed-Solomon codes from algebraic geometry. *IEEE Trans. Inf. Theory*, **33** (1987) 305–309.
215. O. Veblen and J.H.M. Wedderburn. Non Desarguesian and non Pascalian geometries. *Trans. Amer. Math. Soc.*, **8** (1907) 379–380.
216. R. Vincenti. Fibrazioni di un $S_{3,q}$ indotte da fibrazioni di un S_{3,q^2} e rappresentazione di sottopiani di Baer di un piano proiettivo. *Atti Mem. Accad. Sci. Lett. Arti Modena*, Serie VI, **XIX** (1977) 1–8.
217. R. Vincenti. Alcuni tipi di varieta $\mathcal{V}_2^3$ di $S_{4,q}$ e sottopiani di Baer. *Suppl. BUMI*, **2** (1980) 31–44.
218. R. Vincenti. Varieties representing Baer subplanes of a translation plane of the class V. *Atti Sem. Mat. Fis. Univ. Modena*, **XXIV** (1980) 48–59.

219. R. Vincenti. A survey on varieties of $PG(4, q)$ and Baer subplanes of translation planes. *Ann. Discrete Math.*, **18** (1983) 775–780.
220. J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior, Second Edition*. Princeton University Press, 1947.
221. K.L. Wantz. A new class of unitals in the Hughes plane. *Geom. Dedicata*, **70** (1998) 125–138.
222. K.L. Wantz. Unitals in the regular nearfield planes. *J. Geometry*, to appear.
223. H. Wilbrink. A characterization of the classical unitals. Finite Geometries, *Lecture Notes Pure Appl. Math.* (Marcel Dekker), **82** (1983) 445-454.

Notation Index

$a, b, c, \ldots$	elements in $\mathrm{GF}(q)$
$\alpha, \beta, \gamma, \ldots$	elements in $\mathrm{GF}(q^2)$
σ	the conjugate map $x \to x^q$ for $x \in \mathrm{GF}(q^2)$
ζ	a primitive element in $\mathrm{GF}(q^2)$ with primitive polynomial $x^2 - t_1 x - t_0$
$\boldsymbol{x}'$	transpose of $\boldsymbol{x}$
X^*	dual space of X
$\mathrm{PG}(2, q^2)$	classical/field/Desarguesian plane of order q^2
$\mathrm{Hall}(q^2)$	Hall plane of order q^2
$\mathrm{Hgh}(q^2)$	Hughes plane of order q^2
$\mathrm{Fig}(q^6)$	Figueroa plane of order q^6
$\mathcal{H} = \mathcal{H}(2, q^2)$	classical unital in $\mathrm{PG}(2, q^2)$
$\mathcal{A}$	an affine plane
$\mathcal{D}(\mathcal{A})$	the derived affine plane
$\mathcal{P}$	a projective plane
$\mathcal{D}(\mathcal{P})$	the derived projective plane
$\mathcal{S}$	a spread in $\mathrm{PG}(3, q)$
$\mathcal{P}(\mathcal{S})$	the projective plane constructed from the spread $\mathcal{S}$ via the Bruck-Bose representation
U	a unital in $\mathrm{PG}(2, q^2)$
$\mathcal{U}$	the corresponding point set in $\mathrm{PG}(4, q)$
$\ell, m, \ldots$	lines in a projective space
$\pi, \ldots$	planes in a projective space
$\Sigma, \Pi, \ldots$	3-spaces in a projective space

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