Appendix

A Reminder of Some Functional Analysis and Operator Theory

This book is written in a functional-analytic spirit. Its main objects are operators on Banach spaces, and we use many, sometimes quite sophisticated, results and techniques from functional analysis and operator theory. As a rule, we refer to textbooks such as [Con85], [DS58], [Lan93], [RS72], [Rud73], [TL80], or [Yos65]. However, for the convenience of the reader we add this appendix, where we

- Introduce our notation,
- List some basic results, and
- Prove a few of them.

To start with, we introduce the following classical sequence and function spaces. Here, J is a real interval and Ω, depending on the context, is a domain in $\mathbb{R}^n$, a locally compact metric space, or a measure space.

$$\ell^\infty := \Big\{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\Big\}, \quad \|(x_n)_{n\in\mathbb{N}}\| := \sup_{n\in\mathbb{N}} \|x_n\|,$$

$$c := \Big\{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \lim_{n\to\infty} x_n \text{ exists}\Big\} \subset \ell^\infty,$$

$$c_0 := \Big\{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \lim_{n\to\infty} x_n = 0\Big\} \subset c,$$

$$\ell^p := \Big\{(x_n)_{n\in\mathbb{N}} \subset \mathbb{C} : \sum_{n\in\mathbb{N}} |x_n|^p < \infty\Big\},\ p \geq 1,\ \|(x_n)_{n\in\mathbb{N}}\| := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p},$$

$\mathrm{C}(\Omega) := \{f : \Omega \to \mathbb{K} \mid f \text{ is continuous}\},$

$\quad \|f\|_\infty := \sup_{s\in\Omega} |f(s)| \quad (\text{if } \Omega \text{ is compact}),$

$\mathrm{C}_0(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ vanishes at infinity}\};$ cf. p. 20,

$\mathrm{C}_\mathrm{b}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ is bounded}\},$

$\mathrm{C}_\mathrm{c}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ has compact support}\};$ cf. p. 21,

$\mathrm{C}_\mathrm{ub}(\Omega) := \{f \in \mathrm{C}(\Omega) : f \text{ is bounded and uniformly continuous}\},$

$\mathrm{AC}(J) := \{f : J \to \mathbb{K} \mid f \text{ is absolutely continuous}\},$

$\mathrm{C}^k(J) := \{f \in \mathrm{C}(J) : f \text{ is } k\text{-times continuously differentiable}\},$

$\mathrm{C}^\infty(J) := \{f \in \mathrm{C}(J) : f \text{ is infinitely many times differentiable}\},$

$\mathrm{L}^p(\Omega,\mu) := \big\{f : \Omega \to \mathbb{K} \mid f \text{ is } p\text{-integrable on } \Omega\big\},$

$\quad \|f\|_p := \Big(\int_\Omega |f|^p(s)\, d\mu(s)\Big)^{1/p},$

$\mathrm{L}^\infty(\Omega,\mu) := \big\{f : \Omega \to \mathbb{K} \mid f \text{ is measurable and } \mu\text{-essentially bounded}\big\},$

$\quad \|f\|_\infty := \operatorname{ess\,sup} |f|;$ cf. p. 28,

$\mathrm{W}^{k,p}(\Omega) := \left\{ f \in \mathrm{L}^p(\Omega) : \begin{array}{l} f \text{ is } k\text{-times distributionally differentiable} \\ \text{with } D^\alpha f \in \mathrm{L}^p(\Omega) \text{ for all } |\alpha| \le k \end{array} \right\},$

$\mathrm{H}^k(\Omega) := \mathrm{W}^{k,2}(\Omega),$

$\mathrm{H}^k_0(J) := \{f \in \mathrm{H}^k(J) : f(s) = 0 \text{ for } s \in \partial J\},$

$\mathscr{S}(\mathbb{R}^n) :=$ Schwartz space of rapidly decreasing functions; cf. p. 55.

Clearly, we may combine the various sub- and superscripts for the spaces of continuous functions and obtain, e.g., $\mathrm{C}^1_\mathrm{c}(J) = \mathrm{C}^1(J) \cap \mathrm{C}_c(J)$.

For an abstract complex Banach space X we denote its dual by X' and the *canonical bilinear form* by

$$\langle x, x'\rangle \quad \text{for } x \in X, \quad x' \in X'.$$

As usual, we also write $x'(x)$ for $\langle x, x'\rangle$ and denote by $\sigma(X, X')$ the *weak topology* on X and by $\sigma(X', X)$ the weak* topology on X'. Then the following properties hold.

A.1 Proposition.

(i) *For convex sets in X (in particular, for subspaces) the weak and norm closure coincide.*

(ii) *The closed, convex hull $\overline{\mathrm{co}}\, K$ of a weakly compact set K in X is weakly compact (Kreĭn's theorem).*

(iii) *The dual unit ball $U^0 := \{x' \in X' : \|x'\| \le 1\}$ is weak* compact (Banach–Alaoglu's theorem).*

The space of all bounded, linear operators on X is denoted[1] by $\mathcal{L}(X)$ and becomes a Banach space for the norm

$$\|T\| := \sup\{\|Tx\| : \|x\| \le 1\}, \quad T \in \mathcal{L}(X).$$

The operators $T \in \mathcal{L}(X)$ satisfying

$$\|Tx\| \le \|x\| \qquad \text{for all } x \in X$$

are called *contractions*, whereas *isometries* are defined by

$$\|Tx\| = \|x\| \qquad \text{for all } x \in X.$$

Besides the *uniform operator topology* on $\mathcal{L}(X)$, which is the one induced by the above operator norm, we frequently consider two more topologies on $\mathcal{L}(X)$.

We write $\mathcal{L}_s(X)$ if we endow $\mathcal{L}(X)$ with the *strong operator topology*, which is the topology of pointwise convergence on $(X, \|\cdot\|)$.

Finally, $\mathcal{L}_\sigma(X)$ denotes $\mathcal{L}(X)$ with the *weak operator topology*, which is the topology of pointwise convergence on $\big(X, \sigma(X, X')\big)$.

A net $(T_\alpha)_{\alpha \in \mathcal{A}} \subset \mathcal{L}(X)$ converges to $T \in \mathcal{L}(X)$ if and only if

(A.1) $\|T_\alpha - T\| \to 0$ (uniform operator topology),

(A.2) $\|T_\alpha x - Tx\| \to 0 \ \forall\, x \in X$ (strong operator topology),

(A.3) $|\langle T_\alpha x - Tx, x' \rangle| \to 0 \ \forall\, x \in X,\ x' \in X'$ (weak operator topology).

With these notions, the *principle of uniform boundedness* can be stated as follows.

A.2 Proposition. *For a subset $K \subset \mathcal{L}(X)$ the following properties are equivalent.*

(a) *K is bounded for the weak operator topology.*

(b) *K is bounded for the strong operator topology.*

(c) *K is uniformly bounded; i.e., $\|T\| \le c$ for all $T \in K$.*

Continuity with respect to the strong operator topology is shown frequently by using the following property (b) (see [Sch80, Sect. III.4.5]).

A.3 Proposition. *On bounded subsets of $\mathcal{L}(X)$, the following topologies coincide.*

(a) *The strong operator topology.*

(b) *The topology of pointwise convergence on a dense subset of X.*

(c) *The topology of uniform convergence on relatively compact subsets of X.*

[1] For the space of all bounded, linear operators between two normed spaces X and Y we use the notation $\mathcal{L}(X, Y)$.

The advantage of using the strong or weak operator topology instead of the norm topology on $\mathcal{L}(X)$ is that the former yield more continuity and more compactness. This becomes evident already from the definition of a strongly continuous semigroup in Section I.1.

As an example for the functional-analytic constructions made throughout the text, we consider the following setting.

Let $\mathcal{X}_{t_0} := \mathrm{C}\big([0,t_0], \mathcal{L}_s(X)\big)$ be the space of all functions on $[0,t_0]$ into $\mathcal{L}(X)$ that are continuous for the strong operator topology. For each $F \in \mathcal{X}_{t_0}$ and $x \in X$, the functions $t \mapsto F(t)x$ are continuous, hence bounded, on $[0,t_0]$. The uniform boundedness principle then implies

$$\|F\|_\infty := \sup_{s\in[0,t_0]} \|F(s)\| < \infty.$$

Clearly, this defines a norm making $\mathcal{X}_{t_0}$ a complete space.

A.4 Proposition. *The space*

$$\mathcal{X}_{t_0} := \Big(\mathrm{C}\big([0,t_0], \mathcal{L}_s(X)\big), \|\cdot\|_\infty\Big)$$

is a Banach space.

PROOF. Let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{X}_{t_0}$. Then, by the definition of the norm in $\mathcal{X}_{t_0}$, $\big(F_n(\cdot)x\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{C}([0,t_0],X)$ for all $x \in X$. Because $\mathrm{C}([0,t_0],X)$ is complete, the limit $\lim_{n\to\infty} F_n(\cdot)x =: F(\cdot)x \in \mathrm{C}([0,t_0],X)$ exists, and we obtain $\lim_{n\to\infty} F_n = F$ in $\mathcal{X}_{t_0}$. □

Familiarity with linear operators, in particular unbounded operators, is essential for an understanding of our semigroups and their generators. The best introduction is still Kato's monograph [Kat80] (see also [DS58], [GGK90], [Gol66], [TL80], [Wei80]), but we briefly restate some of the basic definitions and properties.[2]

A.5 Definition. *A linear operator A with domain $D(A)$ in a Banach space X, i.e., $D(A) \subset X \to X$, is closed if it satisfies one of the following equivalent properties.*

(a) *If for the sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ the limits $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} Ax_n = y \in X$ exist, then $x \in D(A)$ and $Ax = y$.*

(b) *The graph $\mathcal{G}(A) := \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$.*

(c) *$X_1 := \big(D(A), \|\cdot\|_A\big)$ is a Banach space*[3] *for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

[2] Most of the following concepts also make sense for operators acting between different Banach spaces. However, for simplicity we state them for a single Banach space only and leave the straightforward generalization to the reader.

[3] This definition of X_1 also makes sense if A has an empty resolvent set. Because if $\rho(A) \neq \emptyset$, the graph norm and the norms $\|\cdot\|_{1,\lambda}$ from Exercise II.2.22.(1) are all equivalent, this definition of X_1 will not conflict with Definition II.2.14 for $n = 1$.

(d) *A is weakly closed; i.e., property* (a) *(or property* (b)*) holds for the $\sigma(X, X')$-topology on X.*

If $\lambda - A$ is injective for some $\lambda \in \mathbb{C}$, then the above properties are also equivalent to

(e) *$(\lambda - A)^{-1}$ is closed.*

Next we consider perturbations of closed operators. Whereas the additive perturbation of a closed operator A by a bounded operator $B \in \mathcal{L}(X)$ yields again a closed operator, the situation is slightly more complicated for multiplicative perturbations.

A.6 Proposition. *Let $\big(A, D(A)\big)$ be a closed operator and take $B \in \mathcal{L}(X)$. Then the following hold.*

(i) *AB with domain $D(AB) := \{x \in X : Bx \in D(A)\}$ is closed.*

(ii) *BA with domain $D(BA) := D(A)$ is closed if $B^{-1} \in \mathcal{L}(X)$.*

PROOF. (i) is easy to check and implies (ii) after the similarity transformation $BA = B(AB)B^{-1}$. □

It will be important to find closed extensions of not necessarily closed operators. Here are the relevant notions.

A.7 Definition. *An operator $\big(B, D(B)\big)$ is an extension of $\big(A, D(A)\big)$, in symbols $A \subset B$, if $D(A) \subset D(B)$ and $Bx = Ax$ for $x \in D(A)$. The smallest closed extension of A, if it exists, is called the closure of A and is denoted by $\overline{A}$. Operators having a closure are called closable.*

A.8 Proposition. *An operator $\big(A, D(A)\big)$ is closable if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ with $x_n \to 0$ and $Ax_n \to z$ one has $z = 0$. In that case, the graph of the closure is given by*

$$\mathcal{G}(\overline{A}) = \overline{\mathcal{G}(A)}.$$

A simple operator that is not closable is

$$Af := f'(0) \cdot \mathbb{1} \quad \text{with domain} \quad D(A) := \mathrm{C}^1[0, 1]$$

in the Banach space $X := \mathrm{C}[0, 1]$. This follows, e.g., from the following characterization of bounded linear forms and the fact that the kernel of a closed operator is always closed.[4]

A.9 Proposition. *Let X be a normed vector space and take a linear functional $x' : X \to \mathbb{C}$. Then x' is bounded if and only if its kernel $\ker x'$ is closed in X. Hence, x' is unbounded if and only if $\ker x'$ is dense in X.*

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[4] Here, for a linear map $\Phi : X \to Y$ between two vector spaces X and Y its kernel is defined by $\ker \Phi := \{x \in X : \Phi x = 0\}$.

PROOF. If x' is bounded, then clearly $\ker(x')$ is closed. On the other hand, if $\ker x'$ is closed, then the quotient ${}^{X}/_{\ker x'}$ is a normed vector space of dimension 1. Moreover, we can decompose $x' = i\,\widehat{x'}$ by the canonical maps $i : {}^{X}/_{\ker x'} \to \mathbb{C}$ and $\widehat{x'} : X \to {}^{X}/_{\ker x'}$. Because $\|\widehat{x'}\| \le 1$, this proves that x' is bounded. The remaining assertions follow from the fact that for each linear form $x' \neq 0$ the codimension of $\ker x'$ in X is 1. □

A subspace D of $D(A)$ that is dense in $D(A)$ for the graph norm is called a *core* for A. If $\big(A, D(A)\big)$ is closed, one can recover A from its restriction to a core D; i.e., the closure of (A, D) becomes $\big(A, D(A)\big)$. See Exercise II.1.15.(2).

The closed graph theorem states that everywhere defined closed operators are already bounded. It can be phrased as follows.

A.10 Theorem. *For a closed operator $A : D(A) \subset X \to X$ the following properties are equivalent.*

(a) $\big(A, D(A)\big)$ *is a bounded operator; i.e., there exists $c \ge 0$ such that*

$$\|Ax\| \le c\,\|x\| \qquad \text{for all } x \in D(A).$$

(b) $D(A)$ *is a closed subspace of* X.

By the closed graph theorem, one obtains the following surprising result.

A.11 Corollary. *Let $A : D(A) \subset X \to X$ be closed and assume that a Banach space Y is continuously embedded in X such that the range* $\operatorname{rg} A := A\big(D(A)\big)$ *is contained in Y. Then A is bounded from $(D(A), \|\cdot\|_A)$ into Y.*

If an operator A has dense domain $D(A)$ in X, we can define its adjoint operator on the dual space X'.[5]

A.12 Definition. *For a densely defined operator $\big(A, D(A)\big)$ on X, we define the adjoint operator $\big(A', D(A')\big)$ on X' by*

$$D(A') := \big\{x' \in X' : \exists\, y' \in X' \text{ such that } \langle Ax, x'\rangle = \langle x, y'\rangle \ \forall\, x \in D(A)\big\},$$
$$A'x' := y' \text{ for } x' \in D(A').$$

A.13 Example. Take $A_p := {}^{d}/_{ds}$ on $X_p := \mathrm{L}^p(\mathbb{R})$, $1 \le p < \infty$, with domain $D(A_p) := \mathrm{W}^{1,p}(\mathbb{R}) := \{f \in X_p : f \text{ absolutely continuous}, f' \in X_p\}$. Then $A_p{}' = -A_q$ on X_q, where ${}^1/_p + {}^1/_q = 1$. For a proof and many more examples we refer to [Gol66, Sect. II.2 and Chap. VI] and [Kat80, Sect. III.5]. Compare also Exercise II.4.14.(11).

Although the adjoint operator is always closed, it may happen that $D(A') = \{0\}$ (e.g., take the nonclosable operator following Proposition A.8).

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[5] Similarly, one can define the Hilbert space adjoint A^* by replacing the canonical bilinear form $\langle\cdot,\cdot\rangle$ by the *inner product* $(\cdot\,|\,\cdot)$.

On reflexive Banach spaces there is a nice duality between densely defined and closable operators.

A.14 Proposition. *Let $\big(A, D(A)\big)$ be a densely defined operator on a reflexive Banach space X. Then the adjoint A' is densely defined if and only if A is closable. In that case, one has*

$$(A')' = \overline{A}.$$

We now prove a close relationship between inverses and adjoints.

A.15 Proposition. *Let $\big(A, D(A)\big)$ be a densely defined closed operator on X. Then the inverse $A^{-1} \in \mathcal{L}(X)$ exists if and only if the inverse $(A')^{-1} \in \mathcal{L}(X')$ exists. In that case, one has*

$$(A')^{-1} = (A^{-1})'.$$

Proof. Assume $A^{-1} \in \mathcal{L}(X)$. Because $(A^{-1})' \in \mathcal{L}(X')$, one has

$$\big\langle x, (A^{-1})'A'x' \big\rangle = \big\langle A^{-1}x, A'x' \big\rangle = \big\langle AA^{-1}x, x' \big\rangle = \langle x, x' \rangle$$

for all $x \in X$, $x' \in D(A')$; i.e., A' has a left inverse. Similarly,

$$\big\langle Ax, (A^{-1})'x' \big\rangle = \big\langle A^{-1}Ax, x' \big\rangle = \langle x, x' \rangle$$

holds for all $x \in D(A)$, $x' \in X'$; i.e., $(A^{-1})'x' \in D(A')$ and $A'(A^{-1})'x' = x'$.

On the other hand, assume $(A')^{-1} \in \mathcal{L}(X')$. Then

$$\big\langle Ax, (A')^{-1}x' \big\rangle = \big\langle x, A'(A')^{-1}x' \big\rangle = \langle x, x' \rangle$$

for all $x \in D(A)$ and $x' \in X'$. For every $x \in D(A)$, choose $x' \in X'$ such that $\|x'\| = 1$ and $|\langle x, x' \rangle| = \|x\|$ and obtain

$$\|x\| = \big|\big\langle Ax, (A')^{-1}x' \big\rangle\big| \le \|Ax\| \cdot \big\|(A')^{-1}\big\|.$$

This shows that A is injective and its inverse satisfies

$$\big\|A^{-1}\big\| \le \big\|(A')^{-1}\big\|,$$

hence is bounded. By Theorem A.10, $D(A^{-1}) = \operatorname{rg} A$ must be closed. A simple Hahn–Banach argument shows that $\operatorname{rg} A = X$, hence $A^{-1} \in \mathcal{L}(X)$. □

A.16 Corollary. *For a densely defined closed operator $\big(A, D(A)\big)$ the spectra of A and of A' coincide; i.e.,*

$$\sigma(A) = \sigma(A')$$

and $R(\lambda, A)' = R(\lambda, A')$ for all $\lambda \in \rho(A)$.

Now we turn again to the unbounded situation and define iterates of unbounded operators.

A.17 Definition. *The nth power A^n of an operator $A : D(A) \subset X \to X$ is defined successively as*

$$A^n x := A(A^{n-1}x),$$
$$D(A^n) := \{x \in D(A) : A^{n-1}x \in D(A)\}.$$

In general, it may happen that $D(A^2) = \{0\}$ even if A is densely defined and closed. However, if $A^{-1} \in \mathcal{L}(X)$ exists (or if $\rho(A) \neq \emptyset$), the infinite intersection

$$D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n)$$

is still dense. This is proved in Proposition II.1.8 for semigroup generators and in [Len94] or [AEMK94, Prop. 6.2] for the general case.

Next, we give some results concerning the continuity and differentiability of products of operator-valued functions.

A.18 Lemma. *Let J be some real interval and P, $Q : I \to \mathcal{L}(X)$ be two strongly continuous operator-valued functions defined on J. Then the product $(PQ)(\cdot) : J \to \mathcal{L}(X)$, defined by $(PQ)(t) := P(t)Q(t)$, is strongly continuous as well.*

Proof. We fix $x \in X$ and $t \in J$ and take a sequence $(t_n)_{n\in\mathbb{N}} \subset J$ with $\lim_{n\to\infty} t_n = t$. Then, by the uniform boundedness principle, the set $\{P(t_n) : n \in \mathbb{N}\} \subset \mathcal{L}(X)$ is bounded, and therefore

$$\begin{aligned}\|P(t_n)Q(t_n)x - P(t)Q(t)x\| \le \|P(t_n)\| \cdot \|Q(t_n)x - Q(t)x\| \\ + \big\|\big(P(t_n) - P(t)\big)Q(t)x\big\|,\end{aligned}$$

where the right-hand side converges to zero as $n \to \infty$. □

A.19 Lemma. *Let J be some real interval and P, $Q : J \to \mathcal{L}(X)$ be two strongly continuous operator-valued functions defined on J. Moreover, assume that $P(\cdot)x : J \to X$ and $Q(\cdot)x : J \to X$ are differentiable for all $x \in D$ for some subspace D of X, which is invariant under Q. Then $(PQ)(\cdot)x : J \to X$, defined by $(PQ)(t)x := P(t)Q(t)x$, is differentiable for every $x \in D$ and*

$$\tfrac{d}{dt}\Big(P(\cdot)Q(\cdot)x\Big)(t_0) = \tfrac{d}{dt}\Big(P(\cdot)Q(t_0)x\Big)(t_0) + P(t_0)\Big(\tfrac{d}{dt}Q(\cdot)x\Big)(t_0).$$

Proof. Let $t_0 \in J$ and $(h_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} h_n = 0$ and $t_0 + h_n \in J$ for all $n \in \mathbb{N}$. Then, for $x \in D$, we have

$$\begin{aligned}&\frac{P(t_0+h_n)Q(t_0+h_n)x - P(t_0)Q(t_0)x}{h_n} \\ &\quad = P(t_0+h_n)\,\frac{Q(t_0+h_n)x - Q(t_0)x}{h_n} + \frac{P(t_0+h_n) - P(t_0)}{h_n}\,Q(t_0)x \\ &\quad =: L_1(n,x) + L_2(n,x).\end{aligned}$$

Clearly, the sequence $\big(L_2(n,x)\big)_{n\in\mathbb{N}}$ converges for all $x \in D$ and its limit is $\lim_{n\to\infty} L_2(n,x) = P'(t_0)Q(t_0)x$. In order to show that $\big(L_1(n,x)\big)_{n\in\mathbb{N}}$ converges for $x \in D$, note that

$$\left\{\frac{Q(t_0+h_n)x - Q(t_0)x}{h_n} : n \in \mathbb{N}\right\}$$

is relatively compact in X and that $\{P(t_0+h_n) : n \in \mathbb{N}\}$ is bounded. Because by Proposition A.3 the topologies of pointwise convergence and of uniform convergence on relatively compact sets coincide, we conclude that $\big(L_1(n,x)\big)_{n\in\mathbb{N}}$ converges for $x \in D$ and

$$\lim_{n\to\infty} L_1(n,x) = P(t_0)Q'(t_0)x.$$

This completes the proof. $\square$

In the context of operators on spaces of vector-valued functions it is convenient to use the following *tensor product* notation.

Assume that X, Y are Banach spaces, $\mathrm{F}(J,Y)$ is a Banach space of Y-valued functions defined on an interval $J \subseteq \mathbb{R}$, $T \in \mathcal{L}(X,Y)$ is a bounded linear operator, and $f : J \to \mathbb{C}$ is a complex-valued function. If the map $f \otimes y : J \ni s \mapsto f(s)\cdot y \in Y$ belongs to $\mathrm{F}(J,Y)$ for all $y \in Y$, then we define the linear operator $f \otimes T : X \to \mathrm{F}(J,Y)$ by

$$\big((f\otimes T)x\big)(s) := (f \otimes Tx)(s) = f(s)\cdot Tx$$

for all $x \in X$, $s \in J$.

Independently, for a Banach space X and elements $x \in X$, $x' \in X'$, we frequently use the tensor product notation $x \otimes x'$ for the rank-one operator on X defined by

$$(x \otimes x')\,v := x'(v)\cdot x, \qquad v \in X.$$

We conclude this appendix with the following vector-valued version of the Riemann–Lebesgue lemma.

A.20 Theorem. *If* $f \in \mathrm{L}^1(\mathbb{R},X)$*, then* $\widehat{f} \in \mathrm{C}_0(\mathbb{R},X)$*; i.e., we have* $\lim_{s\to\pm\infty} \widehat{f}(s) = 0$.

For the proof it suffices to consider step functions, for which, as in the scalar case, the assertion follows by integration by parts.

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Smith/Kahanpää/ Kekäläinen/Traves: An Invitation to Algebraic Geometry
Smorynski: Self-Reference and Modal Logic
Stillwell: Geometry of Surfaces
Stroock: An Introduction to the Theory of Large Deviations
Sunder: An Invitation to von Neumann Algebras
Tondeur: Foliations on Riemannian Manifolds
Toth: Finite Möbius Groups, Minimal Immersions of Spheres, and Moduli
Van Brunt: The Calculus of Variations
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