

# Appendix

## A Reminder of Some Functional Analysis and Operator Theory

This book is written in a functional-analytic spirit. Its main objects are operators on Banach spaces, and we use many, sometimes quite sophisticated, results and techniques from functional analysis and operator theory. As a rule, we refer to textbooks such as [Con85], [DS58], [Lan93], [RS72], [Rud73], [TL80], or [Yos65]. However, for the convenience of the reader we add this appendix, where we

- Introduce our notation,
- List some basic results, and
- Prove a few of them.

To start with, we introduce the following classical sequence and function spaces. Here,  $J$  is a real interval and  $\Omega$ , depending on the context, is a domain in  $\mathbb{R}^n$ , a locally compact metric space, or a measure space.

$$\ell^\infty := \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \sup_{n \in \mathbb{N}} \|x_n\| < \infty \right\}, \quad \|(x_n)_{n \in \mathbb{N}}\| := \sup_{n \in \mathbb{N}} \|x_n\|,$$

$$c := \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \lim_{n \rightarrow \infty} x_n \text{ exists} \right\} \subset \ell^\infty,$$

$$c_0 := \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \lim_{n \rightarrow \infty} x_n = 0 \right\} \subset c,$$

$$\ell^p := \left\{ (x_n)_{n \in \mathbb{N}} \subset \mathbb{C} : \sum_{n \in \mathbb{N}} |x_n|^p < \infty \right\}, \quad p \geq 1, \quad \|(x_n)_{n \in \mathbb{N}}\| := \left( \sum_{n \in \mathbb{N}} |x_n|^p \right)^{1/p},$$

$$C(\Omega) := \{f : \Omega \rightarrow \mathbb{K} \mid f \text{ is continuous}\},$$

$$\|f\|_\infty := \sup_{s \in \Omega} |f(s)| \quad (\text{if } \Omega \text{ is compact}),$$

$$C_0(\Omega) := \{f \in C(\Omega) : f \text{ vanishes at infinity}\}; \quad \text{cf. p. 20,}$$

$$C_b(\Omega) := \{f \in C(\Omega) : f \text{ is bounded}\},$$

$$C_c(\Omega) := \{f \in C(\Omega) : f \text{ has compact support}\}; \quad \text{cf. p. 21,}$$

$$C_{ub}(\Omega) := \{f \in C(\Omega) : f \text{ is bounded and uniformly continuous}\},$$

$$AC(J) := \{f : J \rightarrow \mathbb{K} \mid f \text{ is absolutely continuous}\},$$

$$C^k(J) := \{f \in C(J) : f \text{ is } k\text{-times continuously differentiable}\},$$

$$C^\infty(J) := \{f \in C(J) : f \text{ is infinitely many times differentiable}\},$$

$$L^p(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{K} \mid f \text{ is } p\text{-integrable on } \Omega\},$$

$$\|f\|_p := \left( \int_\Omega |f|^p(s) d\mu(s) \right)^{1/p},$$

$$L^\infty(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{K} \mid f \text{ is measurable and } \mu\text{-essentially bounded}\},$$

$$\|f\|_\infty := \text{ess sup } |f|; \quad \text{cf. p. 28,}$$

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) : \begin{array}{l} f \text{ is } k\text{-times distributionally differentiable} \\ \text{with } D^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k \end{array} \right\},$$

$$H^k(\Omega) := W^{k,2}(\Omega),$$

$$H_0^k(J) := \{f \in H^k(J) : f(s) = 0 \text{ for } s \in \partial J\},$$

$$\mathcal{S}(\mathbb{R}^n) := \text{Schwartz space of rapidly decreasing functions}; \quad \text{cf. p. 55.}$$

Clearly, we may combine the various sub- and superscripts for the spaces of continuous functions and obtain, e.g.,  $C_c^1(J) = C^1(J) \cap C_c(J)$ .

For an abstract complex Banach space  $X$  we denote its dual by  $X'$  and the *canonical bilinear form* by

$$\langle x, x' \rangle \quad \text{for } x \in X, \quad x' \in X'.$$

As usual, we also write  $x'(x)$  for  $\langle x, x' \rangle$  and denote by  $\sigma(X, X')$  the *weak topology* on  $X$  and by  $\sigma(X', X)$  the *weak\* topology* on  $X'$ . Then the following properties hold.

### A.1 Proposition.

- (i) For convex sets in  $X$  (in particular, for subspaces) the weak and norm closure coincide.
- (ii) The closed, convex hull  $\overline{\text{co}} K$  of a weakly compact set  $K$  in  $X$  is weakly compact (*Kreĭn's theorem*).
- (iii) The dual unit ball  $U^0 := \{x' \in X' : \|x'\| \leq 1\}$  is weak\* compact (*Banach-Alaoglu's theorem*).

The space of all bounded, linear operators on  $X$  is denoted<sup>1</sup> by  $\mathcal{L}(X)$  and becomes a Banach space for the norm

$$\|T\| := \sup\{\|Tx\| : \|x\| \leq 1\}, \quad T \in \mathcal{L}(X).$$

The operators  $T \in \mathcal{L}(X)$  satisfying

$$\|Tx\| \leq \|x\| \quad \text{for all } x \in X$$

are called *contractions*, whereas *isometries* are defined by

$$\|Tx\| = \|x\| \quad \text{for all } x \in X.$$

Besides the *uniform operator topology* on  $\mathcal{L}(X)$ , which is the one induced by the above operator norm, we frequently consider two more topologies on  $\mathcal{L}(X)$ .

We write  $\mathcal{L}_s(X)$  if we endow  $\mathcal{L}(X)$  with the *strong operator topology*, which is the topology of pointwise convergence on  $(X, \|\cdot\|)$ .

Finally,  $\mathcal{L}_\sigma(X)$  denotes  $\mathcal{L}(X)$  with the *weak operator topology*, which is the topology of pointwise convergence on  $(X, \sigma(X, X'))$ .

A net  $(T_\alpha)_{\alpha \in A} \subset \mathcal{L}(X)$  converges to  $T \in \mathcal{L}(X)$  if and only if

$$(A.1) \quad \|T_\alpha - T\| \rightarrow 0 \quad (\text{uniform operator topology}),$$

$$(A.2) \quad \|T_\alpha x - Tx\| \rightarrow 0 \quad \forall x \in X \quad (\text{strong operator topology}),$$

$$(A.3) \quad |\langle T_\alpha x - Tx, x' \rangle| \rightarrow 0 \quad \forall x \in X, x' \in X' \quad (\text{weak operator topology}).$$

With these notions, the *principle of uniform boundedness* can be stated as follows.

**A.2 Proposition.** *For a subset  $K \subset \mathcal{L}(X)$  the following properties are equivalent.*

- (a)  $K$  is bounded for the weak operator topology.
- (b)  $K$  is bounded for the strong operator topology.
- (c)  $K$  is uniformly bounded; i.e.,  $\|T\| \leq c$  for all  $T \in K$ .

Continuity with respect to the strong operator topology is shown frequently by using the following property (b) (see [Sch80, Sect. III.4.5]).

**A.3 Proposition.** *On bounded subsets of  $\mathcal{L}(X)$ , the following topologies coincide.*

- (a) *The strong operator topology.*
- (b) *The topology of pointwise convergence on a dense subset of  $X$ .*
- (c) *The topology of uniform convergence on relatively compact subsets of  $X$ .*

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<sup>1</sup> For the space of all bounded, linear operators between two normed spaces  $X$  and  $Y$  we use the notation  $\mathcal{L}(X, Y)$ .

The advantage of using the strong or weak operator topology instead of the norm topology on  $\mathcal{L}(X)$  is that the former yield more continuity and more compactness. This becomes evident already from the definition of a strongly continuous semigroup in Section I.1.

As an example for the functional-analytic constructions made throughout the text, we consider the following setting.

Let  $\mathcal{X}_{t_0} := C([0, t_0], \mathcal{L}_s(X))$  be the space of all functions on  $[0, t_0]$  into  $\mathcal{L}(X)$  that are continuous for the strong operator topology. For each  $F \in \mathcal{X}_{t_0}$  and  $x \in X$ , the functions  $t \mapsto F(t)x$  are continuous, hence bounded, on  $[0, t_0]$ . The uniform boundedness principle then implies

$$\|F\|_\infty := \sup_{s \in [0, t_0]} \|F(s)\| < \infty.$$

Clearly, this defines a norm making  $\mathcal{X}_{t_0}$  a complete space.

**A.4 Proposition.** *The space*

$$\mathcal{X}_{t_0} := \left( C([0, t_0], \mathcal{L}_s(X)), \|\cdot\|_\infty \right)$$

*is a Banach space.*

PROOF. Let  $(F_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{X}_{t_0}$ . Then, by the definition of the norm in  $\mathcal{X}_{t_0}$ ,  $(F_n(\cdot)x)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $C([0, t_0], X)$  for all  $x \in X$ . Because  $C([0, t_0], X)$  is complete, the limit  $\lim_{n \rightarrow \infty} F_n(\cdot)x =: F(\cdot)x \in C([0, t_0], X)$  exists, and we obtain  $\lim_{n \rightarrow \infty} F_n = F$  in  $\mathcal{X}_{t_0}$ .  $\square$

Familiarity with linear operators, in particular unbounded operators, is essential for an understanding of our semigroups and their generators. The best introduction is still Kato's monograph [Kat80] (see also [DS58], [GGK90], [Gol66], [TL80], [Wei80]), but we briefly restate some of the basic definitions and properties.<sup>2</sup>

**A.5 Definition.** *A linear operator  $A$  with domain  $D(A)$  in a Banach space  $X$ , i.e.,  $D(A) \subset X \rightarrow X$ , is closed if it satisfies one of the following equivalent properties.*

- (a) *If for the sequence  $(x_n)_{n \in \mathbb{N}} \subset D(A)$  the limits  $\lim_{n \rightarrow \infty} x_n = x \in X$  and  $\lim_{n \rightarrow \infty} Ax_n = y \in X$  exist, then  $x \in D(A)$  and  $Ax = y$ .*
- (b) *The graph  $\mathcal{G}(A) := \{(x, Ax) : x \in D(A)\}$  is closed in  $X \times X$ .*
- (c)  *$X_1 := (D(A), \|\cdot\|_A)$  is a Banach space<sup>3</sup> for the graph norm*

$$\|x\|_A := \|x\| + \|Ax\|, \quad x \in D(A).$$

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<sup>2</sup> Most of the following concepts also make sense for operators acting between different Banach spaces. However, for simplicity we state them for a single Banach space only and leave the straightforward generalization to the reader.

<sup>3</sup> This definition of  $X_1$  also makes sense if  $A$  has an empty resolvent set. Because if  $\rho(A) \neq \emptyset$ , the graph norm and the norms  $\|\cdot\|_{1,\lambda}$  from Exercise II.2.22.(1) are all equivalent, this definition of  $X_1$  will not conflict with Definition II.2.14 for  $n = 1$ .

- (d)  $A$  is weakly closed; i.e., property (a) (or property (b)) holds for the  $\sigma(X, X')$ -topology on  $X$ .

If  $\lambda - A$  is injective for some  $\lambda \in \mathbb{C}$ , then the above properties are also equivalent to

- (e)  $(\lambda - A)^{-1}$  is closed.

Next we consider perturbations of closed operators. Whereas the additive perturbation of a closed operator  $A$  by a bounded operator  $B \in \mathcal{L}(X)$  yields again a closed operator, the situation is slightly more complicated for multiplicative perturbations.

**A.6 Proposition.** Let  $(A, D(A))$  be a closed operator and take  $B \in \mathcal{L}(X)$ . Then the following hold.

- (i)  $AB$  with domain  $D(AB) := \{x \in X : Bx \in D(A)\}$  is closed.  
(ii)  $BA$  with domain  $D(BA) := D(A)$  is closed if  $B^{-1} \in \mathcal{L}(X)$ .

PROOF. (i) is easy to check and implies (ii) after the similarity transformation  $BA = B(AB)B^{-1}$ .  $\square$

It will be important to find closed extensions of not necessarily closed operators. Here are the relevant notions.

**A.7 Definition.** An operator  $(B, D(B))$  is an extension of  $(A, D(A))$ , in symbols  $A \subset B$ , if  $D(A) \subset D(B)$  and  $Bx = Ax$  for  $x \in D(A)$ . The smallest closed extension of  $A$ , if it exists, is called the closure of  $A$  and is denoted by  $\bar{A}$ . Operators having a closure are called closable.

**A.8 Proposition.** An operator  $(A, D(A))$  is closable if and only if for every sequence  $(x_n)_{n \in \mathbb{N}} \subset D(A)$  with  $x_n \rightarrow 0$  and  $Ax_n \rightarrow z$  one has  $z = 0$ . In that case, the graph of the closure is given by

$$\mathcal{G}(\bar{A}) = \overline{\mathcal{G}(A)}.$$

A simple operator that is not closable is

$$Af := f'(0) \cdot \mathbb{1} \quad \text{with domain} \quad D(A) := C^1[0, 1]$$

in the Banach space  $X := C[0, 1]$ . This follows, e.g., from the following characterization of bounded linear forms and the fact that the kernel of a closed operator is always closed.<sup>4</sup>

**A.9 Proposition.** Let  $X$  be a normed vector space and take a linear functional  $x' : X \rightarrow \mathbb{C}$ . Then  $x'$  is bounded if and only if its kernel  $\ker x'$  is closed in  $X$ . Hence,  $x'$  is unbounded if and only if  $\ker x'$  is dense in  $X$ .

<sup>4</sup> Here, for a linear map  $\Phi : X \rightarrow Y$  between two vector spaces  $X$  and  $Y$  its kernel is defined by  $\ker \Phi := \{x \in X : \Phi x = 0\}$ .

PROOF. If  $x'$  is bounded, then clearly  $\ker(x')$  is closed. On the other hand, if  $\ker x'$  is closed, then the quotient  $X/\ker x'$  is a normed vector space of dimension 1. Moreover, we can decompose  $x' = i\widehat{x'}$  by the canonical maps  $i : X/\ker x' \rightarrow \mathbb{C}$  and  $\widehat{x'} : X \rightarrow X/\ker x'$ . Because  $\|\widehat{x'}\| \leq 1$ , this proves that  $x'$  is bounded. The remaining assertions follow from the fact that for each linear form  $x' \neq 0$  the codimension of  $\ker x'$  in  $X$  is 1.  $\square$

A subspace  $D$  of  $D(A)$  that is dense in  $D(A)$  for the graph norm is called a *core* for  $A$ . If  $(A, D(A))$  is closed, one can recover  $A$  from its restriction to a core  $D$ ; i.e., the closure of  $(A, D)$  becomes  $(A, D(A))$ . See Exercise II.1.15.(2).

The closed graph theorem states that everywhere defined closed operators are already bounded. It can be phrased as follows.

**A.10 Theorem.** *For a closed operator  $A : D(A) \subset X \rightarrow X$  the following properties are equivalent.*

- (a)  $(A, D(A))$  is a bounded operator; i.e., there exists  $c \geq 0$  such that

$$\|Ax\| \leq c\|x\| \quad \text{for all } x \in D(A).$$

- (b)  $D(A)$  is a closed subspace of  $X$ .

By the closed graph theorem, one obtains the following surprising result.

**A.11 Corollary.** *Let  $A : D(A) \subset X \rightarrow X$  be closed and assume that a Banach space  $Y$  is continuously embedded in  $X$  such that the range  $\text{rg } A := A(D(A))$  is contained in  $Y$ . Then  $A$  is bounded from  $(D(A), \|\cdot\|_A)$  into  $Y$ .*

If an operator  $A$  has dense domain  $D(A)$  in  $X$ , we can define its adjoint operator on the dual space  $X'$ .<sup>5</sup>

**A.12 Definition.** *For a densely defined operator  $(A, D(A))$  on  $X$ , we define the adjoint operator  $(A', D(A'))$  on  $X'$  by*

$$D(A') := \{x' \in X' : \exists y' \in X' \text{ such that } \langle Ax, x' \rangle = \langle x, y' \rangle \ \forall x \in D(A)\},$$

$$A'x' := y' \text{ for } x' \in D(A').$$

**A.13 Example.** Take  $A_p := d/ds$  on  $X_p := L^p(\mathbb{R})$ ,  $1 \leq p < \infty$ , with domain  $D(A_p) := W^{1,p}(\mathbb{R}) := \{f \in X_p : f \text{ absolutely continuous, } f' \in X_p\}$ . Then  $A_p' = -A_q$  on  $X_q$ , where  $1/p + 1/q = 1$ . For a proof and many more examples we refer to [Gol66, Sect. II.2 and Chap. VI] and [Kat80, Sect. III.5]. Compare also Exercise II.4.14.(11).

Although the adjoint operator is always closed, it may happen that  $D(A') = \{0\}$  (e.g., take the nonclosable operator following Proposition A.8).

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<sup>5</sup> Similarly, one can define the Hilbert space adjoint  $A^*$  by replacing the canonical bilinear form  $\langle \cdot, \cdot \rangle$  by the *inner product*  $(\cdot | \cdot)$ .

On reflexive Banach spaces there is a nice duality between densely defined and closable operators.

**A.14 Proposition.** *Let  $(A, D(A))$  be a densely defined operator on a reflexive Banach space  $X$ . Then the adjoint  $A'$  is densely defined if and only if  $A$  is closable. In that case, one has*

$$(A')' = \bar{A}.$$

We now prove a close relationship between inverses and adjoints.

**A.15 Proposition.** *Let  $(A, D(A))$  be a densely defined closed operator on  $X$ . Then the inverse  $A^{-1} \in \mathcal{L}(X)$  exists if and only if the inverse  $(A')^{-1} \in \mathcal{L}(X')$  exists. In that case, one has*

$$(A')^{-1} = (A^{-1})'.$$

PROOF. Assume  $A^{-1} \in \mathcal{L}(X)$ . Because  $(A^{-1})' \in \mathcal{L}(X')$ , one has

$$\langle x, (A^{-1})' A' x' \rangle = \langle A^{-1} x, A' x' \rangle = \langle A A^{-1} x, x' \rangle = \langle x, x' \rangle$$

for all  $x \in X$ ,  $x' \in D(A')$ ; i.e.,  $A'$  has a left inverse. Similarly,

$$\langle Ax, (A^{-1})' x' \rangle = \langle A^{-1} Ax, x' \rangle = \langle x, x' \rangle$$

holds for all  $x \in D(A)$ ,  $x' \in X'$ ; i.e.,  $(A^{-1})' x' \in D(A)$  and  $A'(A^{-1})' x' = x'$ .

On the other hand, assume  $(A')^{-1} \in \mathcal{L}(X')$ . Then

$$\langle Ax, (A')^{-1} x' \rangle = \langle x, A'(A')^{-1} x' \rangle = \langle x, x' \rangle$$

for all  $x \in D(A)$  and  $x' \in X'$ . For every  $x \in D(A)$ , choose  $x' \in X'$  such that  $\|x'\| = 1$  and  $|\langle x, x' \rangle| = \|x\|$  and obtain

$$\|x\| = |\langle Ax, (A')^{-1} x' \rangle| \leq \|Ax\| \cdot \|(A')^{-1}\|.$$

This shows that  $A$  is injective and its inverse satisfies

$$\|A^{-1}\| \leq \|(A')^{-1}\|,$$

hence is bounded. By Theorem A.10,  $D(A^{-1}) = \text{rg } A$  must be closed. A simple Hahn–Banach argument shows that  $\text{rg } A = X$ , hence  $A^{-1} \in \mathcal{L}(X)$ .  $\square$

**A.16 Corollary.** *For a densely defined closed operator  $(A, D(A))$  the spectra of  $A$  and of  $A'$  coincide; i.e.,*

$$\sigma(A) = \sigma(A')$$

and  $R(\lambda, A)' = R(\lambda, A')$  for all  $\lambda \in \rho(A)$ .

Now we turn again to the unbounded situation and define iterates of unbounded operators.

**A.17 Definition.** The  $n$ th power  $A^n$  of an operator  $A : D(A) \subset X \rightarrow X$  is defined successively as

$$A^n x := A(A^{n-1}x),$$

$$D(A^n) := \{x \in D(A) : A^{n-1}x \in D(A)\}.$$

In general, it may happen that  $D(A^2) = \{0\}$  even if  $A$  is densely defined and closed. However, if  $A^{-1} \in \mathcal{L}(X)$  exists (or if  $\rho(A) \neq \emptyset$ ), the infinite intersection

$$D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n)$$

is still dense. This is proved in Proposition II.1.8 for semigroup generators and in [Len94] or [AEMK94, Prop. 6.2] for the general case.

Next, we give some results concerning the continuity and differentiability of products of operator-valued functions.

**A.18 Lemma.** Let  $J$  be some real interval and  $P, Q : I \rightarrow \mathcal{L}(X)$  be two strongly continuous operator-valued functions defined on  $J$ . Then the product  $(PQ)(\cdot) : J \rightarrow \mathcal{L}(X)$ , defined by  $(PQ)(t) := P(t)Q(t)$ , is strongly continuous as well.

PROOF. We fix  $x \in X$  and  $t \in J$  and take a sequence  $(t_n)_{n \in \mathbb{N}} \subset J$  with  $\lim_{n \rightarrow \infty} t_n = t$ . Then, by the uniform boundedness principle, the set  $\{P(t_n) : n \in \mathbb{N}\} \subset \mathcal{L}(X)$  is bounded, and therefore

$$\begin{aligned} \|P(t_n)Q(t_n)x - P(t)Q(t)x\| &\leq \|P(t_n)\| \cdot \|Q(t_n)x - Q(t)x\| \\ &\quad + \|(P(t_n) - P(t))Q(t)x\|, \end{aligned}$$

where the right-hand side converges to zero as  $n \rightarrow \infty$ . □

**A.19 Lemma.** Let  $J$  be some real interval and  $P, Q : J \rightarrow \mathcal{L}(X)$  be two strongly continuous operator-valued functions defined on  $J$ . Moreover, assume that  $P(\cdot)x : J \rightarrow X$  and  $Q(\cdot)x : J \rightarrow X$  are differentiable for all  $x \in D$  for some subspace  $D$  of  $X$ , which is invariant under  $Q$ . Then  $(PQ)(\cdot)x : J \rightarrow X$ , defined by  $(PQ)(t)x := P(t)Q(t)x$ , is differentiable for every  $x \in D$  and

$$\frac{d}{dt} \left( P(\cdot)Q(\cdot)x \right) (t_0) = \frac{d}{dt} \left( P(\cdot)Q(t_0)x \right) (t_0) + P(t_0) \left( \frac{d}{dt} Q(\cdot)x \right) (t_0).$$

PROOF. Let  $t_0 \in J$  and  $(h_n)_{n \in \mathbb{N}} \subset \mathbb{R}$  be a sequence such that  $\lim_{n \rightarrow \infty} h_n = 0$  and  $t_0 + h_n \in J$  for all  $n \in \mathbb{N}$ . Then, for  $x \in D$ , we have

$$\begin{aligned} &\frac{P(t_0 + h_n)Q(t_0 + h_n)x - P(t_0)Q(t_0)x}{h_n} \\ &= P(t_0 + h_n) \frac{Q(t_0 + h_n)x - Q(t_0)x}{h_n} + \frac{P(t_0 + h_n) - P(t_0)}{h_n} Q(t_0)x \\ &=: L_1(n, x) + L_2(n, x). \end{aligned}$$

Clearly, the sequence  $(L_2(n, x))_{n \in \mathbb{N}}$  converges for all  $x \in D$  and its limit is  $\lim_{n \rightarrow \infty} L_2(n, x) = P'(t_0)Q(t_0)x$ . In order to show that  $(L_1(n, x))_{n \in \mathbb{N}}$  converges for  $x \in D$ , note that

$$\left\{ \frac{Q(t_0 + h_n)x - Q(t_0)x}{h_n} : n \in \mathbb{N} \right\}$$

is relatively compact in  $X$  and that  $\{P(t_0 + h_n) : n \in \mathbb{N}\}$  is bounded. Because by Proposition A.3 the topologies of pointwise convergence and of uniform convergence on relatively compact sets coincide, we conclude that  $(L_1(n, x))_{n \in \mathbb{N}}$  converges for  $x \in D$  and

$$\lim_{n \rightarrow \infty} L_1(n, x) = P(t_0)Q'(t_0)x.$$

This completes the proof.  $\square$

In the context of operators on spaces of vector-valued functions it is convenient to use the following *tensor product* notation.

Assume that  $X, Y$  are Banach spaces,  $F(J, Y)$  is a Banach space of  $Y$ -valued functions defined on an interval  $J \subseteq \mathbb{R}$ ,  $T \in \mathcal{L}(X, Y)$  is a bounded linear operator, and  $f : J \rightarrow \mathbb{C}$  is a complex-valued function. If the map  $f \otimes y : J \ni s \mapsto f(s) \cdot y \in Y$  belongs to  $F(J, Y)$  for all  $y \in Y$ , then we define the linear operator  $f \otimes T : X \rightarrow F(J, Y)$  by

$$((f \otimes T)x)(s) := (f \otimes Tx)(s) = f(s) \cdot Tx$$

for all  $x \in X, s \in J$ .

Independently, for a Banach space  $X$  and elements  $x \in X, x' \in X'$ , we frequently use the tensor product notation  $x \otimes x'$  for the rank-one operator on  $X$  defined by

$$(x \otimes x')v := x'(v) \cdot x, \quad v \in X.$$

We conclude this appendix with the following vector-valued version of the Riemann–Lebesgue lemma.

**A.20 Theorem.** *If  $f \in L^1(\mathbb{R}, X)$ , then  $\widehat{f} \in C_0(\mathbb{R}, X)$ ; i.e., we have  $\lim_{s \rightarrow \pm\infty} \widehat{f}(s) = 0$ .*

For the proof it suffices to consider step functions, for which, as in the scalar case, the assertion follows by integration by parts.

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# List of Symbols and Abbreviations

(ACP)	Abstract Cauchy Problem .....	18, 77, 110, 112, 113, 185
(CSMT)	Circular Spectral Mapping Theorem .....	193
(DE)	Differential Equation .....	15
(FE)	Functional Equation .....	2, 11
(IE)	Integral Equation/Variation of Parameter Formula .	119
(IE*)	Integral Equation/Variation of Parameter Formula .	120
(SBeGB)	Spectral Bound equal Growth Bound Condition .....	177, 183, 185
(SMT)	Spectral Mapping Theorem .....	176, 182, 183
(WSMT)	Weak Spectral Mapping Theorem .....	176, 183, 184
$\mathbb{1}$	constant one function .....	21
$\ \cdot\ _A$	graph norm for $A$ .....	39, 225
$\ \cdot\ _{\text{ess}}$	essential norm .....	166
$\ \cdot\ _n$	Sobolev norm of order $n$ .....	57
$\langle \cdot, \cdot \rangle$	canonical bilinear form .....	223
$(\cdot   \cdot)$	inner product .....	227
$f \otimes y$	element of a space of vector-valued functions .....	230
$f \otimes T$	operator on a space of vector-valued functions .....	230
$x \otimes x'$	rank-one operator .....	230
$0 \leq f$	positive element in a Banach lattice .....	205
$0 < f$	positive nonzero element in a Banach lattice .....	205
$0 \ll f$	strictly positive element in a Banach lattice .....	211
$A'$	(Banach space) adjoint of $A$ .....	227
$A^*$	(Hilbert space) adjoint of $A$ .....	227
$\overline{A}$	closure of $A$ .....	226

$A^n$	$n$ th power of $A$ .....	229
$A_n$	part/extension of $A$ in $X_n$ .....	58, 59
$A \subset B$	$A$ is a restriction of $B$ .....	226
$A _Y$	part of $A$ in $Y$ .....	47
$AC(J)$	space of absolutely continuous functions .....	223
$A\sigma(A)$	approximate point spectrum of $A$ .....	160
$c$	space of convergent sequences .....	222
$c_0,$	space of null sequences .....	222
$C^\infty(J)$	space of infinitely many times differentiable functions	223
$C^k(J)$	space of $k$ -times continuously differentiable functions	223
$C(\Omega)$	space of continuous functions .....	223
$C_0(\Omega)$	space of continuous functions vanishing at infinity ...	20, 223
$C_b(\Omega)$	space of bounded continuous functions .....	223
$C_c(\Omega)$	space of continuous functions having compact support	21, 223
$C_{ub}(\Omega)$	space of bounded, uniformly continuous functions ..	223
$\overline{\text{co}}(K)$	closed convex hull of $K$ .....	223
$D(A)$	domain of $A$ .....	36
$D(A^\infty)$	intersection of the domains of all powers of $A$ ....	40, 229
$D(A^n)$	domain of $A^n$ .....	40, 229
ess sup	essential supremum .....	28
$\text{fix}(T(t))_{t \geq 0}$	fixed space of $(T(t))_{t \geq 0}$ .....	194
$\mathcal{G}(A)$	graph of $A$ .....	225
$H^k(\Omega)$	classical Sobolev space of order $(k, 2)$ .....	223
$H_0^k(J)$	classical Sobolev space of order $(k, 2)$ .....	223
$\mathcal{J}(x)$	duality set for $x \in X$ .....	81
$\ker(\Phi)$	kernel of $\Phi$ .....	226
$\mathcal{K}(X)$	space of all compact linear operators on $X$ .....	166
$\ell^\infty,$	space of bounded sequences .....	222
$\ell^p$	space of $p$ -summable sequences .....	222
lin $M$	linear span of the set $M$ .....	41
$L^\infty(\Omega, \mu)$	space of measurable, essentially bounded functions .	223
$L^p(\Omega, \mu)$	space of $p$ -integrable functions .....	223
$\mathcal{L}(X), \mathcal{L}(X, Y)$	space of bounded linear operators .....	224, 227
$M_q$	multiplication operator associated with $q$ .....	20
$\omega_0(\mathcal{T})$	growth bound of the semigroup $\mathcal{T}$ .....	5, 188
$P\sigma(A)$	point spectrum of $A$ .....	159
$q_{\text{ess}}(\Omega)$	essential range of the function $q$ .....	27
$r(A)$	spectral radius of $A$ .....	158
$r_{\text{ess}}(T)$	essential spectral radius of $T$ .....	166
$R(\lambda, A)$	resolvent of $A$ in $\lambda$ .....	157
$R\sigma(A)$	residual spectrum of $A$ .....	161
$\text{rg}(A)$	range of $A$ .....	227
$\rho(A)$	resolvent set of $A$ .....	157

$\rho_F(T)$	Fredholm domain of $T$ .....	166
$\mathbb{R}_+$	nonnegative real numbers .....	1
$\mathcal{S}(\mathbb{R}^N)$	Schwartz space of rapidly decreasing functions .....	223
$s(A)$	spectral bound of $A$ .....	44, 168
$\Sigma_\delta$	sector in $\mathbb{C}$ of angle $\delta$ .....	90
$\sigma(A)$	spectrum of $A$ .....	157
$\sigma_{\text{ess}}(T)$	essential spectrum of $T$ .....	166
$\sigma(X, X')$	weak topology .....	223
$\sigma(X', X)$	weak* topology .....	223
$\text{supp } f$	support of $f$ .....	21
$(T(t) _Y)_{t \geq 0}$	quotient semigroup of $(T(t))_{t \geq 0}$ in $X/Y$ .....	48
$(T(t) _Y)_{t \geq 0}$	subspace semigroup of $(T(t))_{t \geq 0}$ in $Y$ .....	47
$(T_n(t))_{t \geq 0}$	restricted/extrapolated semigroup of $(T(t))_{t \geq 0}$ in $X_n$ .....	57, 59
$(T_l(t))_{t \geq 0}$	left translation semigroup .....	30
$(T_r(t))_{t \geq 0}$	right translation semigroup .....	30
$(T(z))_{z \in \Sigma_\delta \cup \{0\}}$	analytic semigroup of angle $\delta$ .....	95
$W^{k,p}(\Omega, \mu)$	classical Sobolev space of order $(k, p)$ .....	223
$X_n$	abstract Sobolev space of order $n$ .....	57, 59, 225
$Y \hookrightarrow X$	$Y$ continuously embedded in $X$ .....	47

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