

Appendix A

Basic Notions from Algebra, Analysis, and Geometry

In this appendix we summarize for the convenience of the reader some basic mathematical results that are assumed to be known in the main text. Of course, reading this appendix cannot substitute for a systematic study of the corresponding topics via standard textbooks.

A.1 Algebra

A.1.1 Matrices

Let A be an $n \times m$ matrix with complex elements $a_{jk} \in \mathbb{C}^1$:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix},$$

and let A^T denote its *transpose*:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & & & \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}.$$

The *product* of an $n \times m$ matrix A and an $m \times l$ matrix B is the $n \times l$ matrix $C = AB$ with the elements

$$c_{ij} = \sum_{k=1}^m a_{ik} b_{kj}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, l.$$

The following property holds:

$$(AB)^T = B^T A^T.$$

The *determinant* of a square $n \times n$ matrix A is the complex number defined by

$$\det A = \sum_{(i_1, i_2, \dots, i_n) \in S_n} (-1)^{\delta(i_1, i_2, \dots, i_n)} a_{i_1} a_{i_2} \cdots a_{i_n},$$

where S_n is the set of all permutations of n indices, and $\delta = 0$ when the multiindex (i_1, i_2, \dots, i_n) can be obtained from the multi-index $(1, 2, \dots, n)$ by an even number of one-step permutations; $\delta = 1$ otherwise. A square matrix A is called *nonsingular* if $\det A \neq 0$. For a nonsingular matrix A , there is the *inverse* matrix A^{-1} , such that $AA^{-1} = A^{-1}A = I$, where I is the identity $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$

The *rank* of an $n \times m$ matrix A is the order of the largest nonsingular square submatrix of A .

The *sum* of two $n \times m$ matrices A and B is the $n \times m$ matrix $C = A + B$ with the elements

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m.$$

The product of a complex number λ and an $n \times m$ matrix A is the $n \times m$ matrix $B = \lambda A$ with the elements

$$b_{ij} = \lambda a_{ij}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m.$$

Consider a function $x \mapsto f(x)$ defined by a convergent series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

(*analytic function*). Given a square matrix A , we can introduce a square matrix $f(A)$ by

$$f(A) = \sum_{k=0}^{\infty} f_k A^k,$$

where $A^0 = I$, $A^k = AA^{k-1}$, $k = 1, 2, \dots$. For example,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

A.1.2 Vector spaces and linear transformations

A complex $n \times 1$ matrix

$$v = (v_1, v_2, \dots, v_n)^T = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix}$$

is called a *vector*. The set of all such vectors is a *linear space* that can be identified with \mathbb{C}^n . In this space, addition between two elements and multiplication of an element by a complex number are defined component-wise.

A subset $X \subset \mathbb{C}^n$ is called the *linear subspace (hyperplane)* of \mathbb{C}^n if $x \in X$ and $y \in X$ imply $x + y \in X$ and $\lambda x \in X$ for any $\lambda \in \mathbb{C}^1$. A linear subspace Z is called the *sum* of two linear subspaces X and Y if any vector $z \in Z$ can be represented as $z = x + y$ for some vectors $x \in X$ and $y \in Y$. Symbolically: $Z = X + Y$. If such a representation is unique for each z , Z is called the *direct sum* of X and Y and is denoted by $Z = X \oplus Y$.

Vectors $\{a_1, a_2, \dots, a_k\}$ from \mathbb{C}^n are called *linearly independent* when

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_k a_k = 0,$$

if and only if $\alpha_j = 0$ for all $j = 1, 2, \dots, k$. The set

$$L = \text{span}\{a_1, a_2, \dots, a_k\} = \left\{ v \in \mathbb{C}^n : v = \sum_{i=1}^k \alpha_i a_i, \alpha_i \in \mathbb{C}^1 \right\}$$

is a linear subspace of \mathbb{C}^n . If $\{a_1, a_2, \dots, a_k\}$ are linearly independent, $\dim L = k$. A set of n linearly independent vectors is called a *basis*. The set of unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \end{pmatrix}$$

is the *standard basis* in \mathbb{C}^n . Any vector $v \in \mathbb{C}^n$ can be uniquely represented as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} = v_1 e_1 + v_2 e_2 + \dots + v_n e_n.$$

If $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ is another basis, any vector $v \in \mathbb{C}^n$ can also be represented as

$$v = u_1 \varepsilon_1 + u_2 \varepsilon_2 + \dots + u_n \varepsilon_n,$$

where $u_k \in \mathbb{C}^1$ are *components* of v in this basis. Denote $u = (u_1, u_2, \dots, u_n)^T$. Then

$$v = Cu,$$

where the $n \times n$ matrix C is nonsingular and has the elements c_{ij} that are the components of the basis vectors ε_j in the standard basis

$$\varepsilon_j = c_{1j} e_1 + c_{2j} e_2 + \dots + c_{nj} e_n, \quad j = 1, 2, \dots, n.$$

An $n \times n$ matrix A can be identified with a *linear transformation* of the space \mathbb{C}^n

$$v \mapsto Av.$$

In a basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ this transformation will have the form

$$u \mapsto Bu,$$

where the matrix B is given by

$$B = C^{-1}AC.$$

The matrices A and B are called *similar*. The determinants of similar matrices coincide.

A.1.3 Eigenvectors and eigenvalues

A nonzero complex vector

$$v = (v_1, v_2, \dots, v_n)^T \in \mathbb{C}^n$$

is called an *eigenvector* of an $n \times n$ matrix A if

$$Av = \lambda v,$$

for some $\lambda \in \mathbb{C}^1$. The complex number λ is called an *eigenvalue* of A corresponding to the eigenvector v . The eigenvalues of A are roots of the *characteristic polynomial*

$$h(\lambda) = \det(A - \lambda I),$$

and every root is an eigenvalue. Thus, there are n eigenvalues if we count their multiplicities as the roots of $h(\lambda)$.

A.1.4 Invariant subspaces, generalized eigenvectors, and Jordan normal form

A linear subspace $X \subset \mathbb{C}^n$ is called an *invariant subspace* of the matrix A if $AX \subset X$, that is, if $w \in X$ implies $Aw \in X$.

If λ is a root of the characteristic polynomial, then there is an invariant subspace (*eigenspace*) of A that is spanned by the eigenvector $v \in \mathbb{C}^n$ associated with λ :

$$X = \{x \in \mathbb{C}^n : x = \omega v, \omega \in \mathbb{C}^1\}.$$

If λ is a multiple root of the characteristic polynomial of multiplicity m , then one can find $1 \leq l \leq m$ linearly independent eigenvectors v_1, v_2, \dots, v_l , corresponding to λ . For each eigenvector v_j , there is a *maximal chain* of complex vectors $\{w_1^{(j)}, w_2^{(j)}, \dots, w_{k_j}^{(j)}\}$, such that

$$\begin{aligned} Aw_1^{(j)} &= \lambda w_1^{(j)}, \\ Aw_2^{(j)} &= \lambda w_2^{(j)} + w_1^{(j)}, \\ &\dots \\ Aw_{k_j}^{(j)} &= \lambda w_{k_j}^{(j)} + w_{k_j-1}^{(j)}. \end{aligned}$$

The chain can be composed of only one vector $w_1^{(j)}$, that is merely the eigenvector v_j . The vectors $w_k^{(j)}$ with $k \geq 2$ are called *generalized eigenvectors* of A corresponding to the eigenvalue λ . The subspace

$$X = \{x \in \mathbb{C}^n : x = \omega_1 w_1^{(j)} + \omega_2 w_2^{(j)} + \dots + \omega_k w_k^{(j)}, \omega_j \in \mathbb{C}^1\}$$

is an invariant subspace of A .

Eigenvectors and generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. The vectors $\{w_1^{(j)}, w_2^{(j)}, \dots, w_{k_j}^{(j)}\}$ composing a chain corresponding to a multiple eigenvalue λ are also linearly independent.

Theorem A.1 (Jordan normal form) *The space \mathbb{C}^n can be decomposed into linear invariant subspaces of the matrix A corresponding to its eigenvalues and spanned by the corresponding eigenvectors and generalized eigenvectors of A . In a basis given by all the eigenvectors and generalized eigenvectors, the matrix A has a block-diagonal form with square blocks*

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda \end{pmatrix},$$

whose dimension is equal to the length of the corresponding chain. \square

This form is called the *Jordan normal form* or *Jordan canonical form*. It also follows from this theorem that the product of all the eigenvalues of the matrix A is equal to its determinant:

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n.$$

If the matrix A is real, then it has linear invariant subspaces of \mathbb{R}^n spanned by eigenvectors and generalized eigenvectors corresponding to its real eigenvalues and also by the real and imaginary parts of complex eigenvectors corresponding to its complex eigenvalues with, say, positive imaginary part. Such subspaces are called (*real*) *generalized eigenspaces* of A .

If λ is an eigenvalue of A , then $\mu = f(\lambda)$ is an eigenvalue of $B = f(A)$, where f is an analytic function.

A.1.5 Fredholm Alternative Theorem

Let A be a real $n \times m$ matrix and let $b \in \mathbb{R}^n$ be a real vector. The *null-space* of A is the linear subspace of \mathbb{R}^m composed of all vectors $x \in \mathbb{R}^m$ for which $Ax = 0$. The *range* of A is the set of all $x \in \mathbb{R}^n$ for which $Ay = x$ for some $y \in \mathbb{R}^m$.

Theorem A.2 (Fredholm Alternative Theorem) *The equation $Ax = b$ has a solution if and only if $b^T v = 0$ for every vector $v \in \mathbb{R}^n$ satisfying $A^T v = 0$. \square*

Notice that $b^T v = \sum_{j=1}^n b_j v_j$ is the standard *scalar product* in \mathbb{R}^n . The theorem means that the null-space of A^T is the orthogonal complement of the range of A and that together they span the whole \mathbb{R}^n . In other words, any vector $b \in \mathbb{R}^n$ can be uniquely decomposed as $b = b_r + b_0$, where b_r is in the range of A , b_0 is in the null-space of A^T , and b_r is orthogonal to b_0 .

If A is a complex matrix and b is a complex vector, Theorem A.2 remains valid if we replace transposition by transposition composed with complex conjugation.

A.1.6 Groups

A set G is a *group* if a *product* “ \circ ”: $G \times G \rightarrow G$ is defined which satisfies the following properties:

- (i) $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in G$;
- (ii) there is a *unit* element $e \in G$ such that $g \circ e = e \circ g = g$, for all $g \in G$;
- (iii) for each $g \in G$, there is a unique element $g^{-1} \in G$ such that $g^{-1} \circ g = g \circ g^{-1} = e$.

All real nonsingular $n \times n$ matrices with the matrix product and the unit I form the *general linear group* denoted by $GL(n)$. All $n \times n$ matrices satisfying $A^T A = I$ compose its *orthogonal subgroup* $O(n)$.

A.2 Analysis

If $y = g(x)$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$, and $z = f(y)$, $f : \mathbb{R}^m \rightarrow \mathbb{R}^k$, are two maps, then their *superposition* $h = f \circ g$ is a map $z = h(x)$, $\mathbb{R}^n \rightarrow \mathbb{R}^k$, defined by the formula

$$h(x) = f(g(x)).$$

Let $f_y(y)$ denote the Jacobian matrix of f evaluated at a point $y \in \mathbb{R}^m$:

$$f_y(y) = \left(\frac{\partial f_i(y)}{\partial y_j} \right),$$

where $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. If we similarly define $h_x(x)$ and $g_x(x)$, then

$$h_x(x) = [f_y(y)]|_{y=g(x)} [g_x(x)]$$

(the *chain rule*).

A.2.1 Implicit and Inverse Function Theorems

Consider a map

$$(x, y) \mapsto F(x, y),$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n,$$

is a smooth map defined in a neighborhood of $(x, y) = (0, 0)$ and such that $F(0, 0) = 0$. Let $F_x(0, 0)$ denote the matrix of first partial derivatives of F with respect to x evaluated at $(0, 0)$:

$$F_x(0, 0) = \left(\frac{\partial F_i(x, y)}{\partial x_j} \right) \Big|_{(x, y) = (0, 0)}.$$

Theorem A.3 (Implicit Function Theorem) *If the matrix $F_x(0, 0)$ is nonsingular, then there is a smooth locally defined function $y = f(x)$,*

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^m,$$

such that

$$F(x, f(x)) = 0,$$

for all x in some neighborhood of the origin of \mathbb{R}^n . Moreover,

$$f_x(0) = -[F_x(0, 0)]^{-1} F_y(0, 0). \quad \square$$

The degree of smoothness of the function f is the same as that of F .

Consider now a map

$$y = g(x),$$

where

$$g : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

is a smooth function defined in a neighborhood of $x = 0$ and satisfying $g(0) = 0$. The following theorem is a consequence of the Implicit Function Theorem.

Theorem A.4 (Inverse Function Theorem) *If the matrix $g_x(0)$ is non-singular, then there is a smooth locally defined function $x = f(y)$,*

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

such that

$$g(f(y)) = y$$

for all y in some neighborhood of the origin of \mathbb{R}^n . \square

The function f is called the *inverse function* for g and is denoted by $f = g^{-1}$.

A.2.2 Taylor expansion

Let Ω be a region in \mathbb{R}^n containing the origin $x = 0$. Denote by $C^k(\Omega, \mathbb{R}^m)$ the set of maps (vector-valued functions) $y = f(x)$, $f : \Omega \rightarrow \mathbb{R}^m$, having continuously differentiable components up to and including order $k \geq 0$. If $f \in C^k(\Omega, \mathbb{R}^m)$ with a sufficiently large k , the function f is called *smooth*. A C^∞ function has continuous partial derivatives of any order. Any function $f \in C^k(\Omega, \mathbb{R}^m)$ can be represented near $x = 0$ in the form (*Taylor expansion*)

$$f(x) = \sum_{|i|=0}^k \frac{1}{i_1! i_2! \cdots i_n!} \frac{\partial^{|i|} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \Big|_{x=0} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + R(x),$$

where $|i| = i_1 + i_2 + \cdots + i_n$ and $R(x) = O(\|x\|^{k+1}) = o(\|x\|^k)$, namely,

$$\frac{\|R(x)\|}{\|x\|^k} \rightarrow 0$$

as $\|x\| \rightarrow 0$. Here $\|x\| = \sqrt{x^T x}$.

A C^∞ -function f is called *analytic* near the origin if the corresponding *Taylor series*

$$\sum_{|i|=0}^{\infty} \frac{1}{i_1! i_2! \cdots i_n!} \frac{\partial^{|i|} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \Big|_{x=0} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

converges to $f(x)$ at any point x sufficiently close to $x = 0$.

A.2.3 Metric, normed, and other spaces

Let X be a set of elements for which addition and multiplication by (complex) numbers satisfying standard axioms are defined. The set X can consist of functions, for example, $X = C^k(\Omega, \mathbb{C}^m)$.

The set X is a *metric space* if a function $\rho : X \times X \rightarrow \mathbb{R}^1$ is defined such that:

- (i) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
- (ii) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$;
- (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

The function ρ is called a *metric* (or *distance*). A sequence $\{x_k\}_{k=1}^{\infty}$ of elements $x_k \in X$ has a *limit* $x^0 \in X$ (*convergent*) if for any $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that

$$\rho(x_k, x_0) < \varepsilon,$$

for all $k \geq N(\varepsilon)$. Notation: $x^0 = \lim_{k \rightarrow \infty} x_k$. A function $f : X \rightarrow X$ is *continuous* at x^0 if

$$\lim_{k \rightarrow \infty} f(x_k) = f(x^0),$$

for all sequences such that $\lim_{k \rightarrow +\infty} x_k = x^0$. A function $g : X \rightarrow X$ is called *Hölder-continuous* at x^0 if there exist a constant L_0 and an *index* $0 < \beta \leq 1$, such that

$$\rho(g(x), g(x_0)) \leq L_0[\rho(x, x_0)]^\beta$$

for all x sufficiently close to x_0 .

A set $S \subset X$ is *closed* if it contains the limits of all convergent sequences such that for any finite k , $x_k \in S$. A sequence $\{x_k\}_{k=1}^{\infty}$ of elements $x_k \in X$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that for every $n, m \geq N$,

$$\rho(x_n, x_m) < \varepsilon.$$

If the sequence $\{x_k\}_{k=1}^{\infty}$ has a limit, it is a Cauchy sequence. If any Cauchy sequence has a limit in X , the space X is called *complete*.

The set X is a *normed space* if a function $\|\cdot\| : X \rightarrow \mathbb{R}^1$ is defined such that:

- (i) $\|x\| \geq 0$ and $\|x\| = 0$ implies $x = 0$;
- (ii) $\|\alpha x\| = |\alpha| \|x\|$ for any (complex) number α ;
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The function $\|\cdot\|$ is called a *norm*. Any normed space is a metric space with the metric $\rho(x, y) = \|x - y\|$. If X is complete in this metric, it is called a *Banach space*. The space of continuous functions $C^0(\Omega, \mathbb{C}^m)$ is a Banach in the norm

$$\|f\| = \max_{i=1,2,\dots,m} \sup_{\xi \in \Omega} |f_i(\xi)|.$$

A set $S \subset X$ is *bounded* if $\|x\| < C$ with some $C > 0$ for all $x \in S$.

The set X is a *space with a scalar product* if for each pair of $(x, y) \in X$ a complex number $\langle x, y \rangle$ (called the *scalar product*) is defined so that the following properties hold:

- (i) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in X$;
- (ii) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle$ for all $x, y \in X$ and any complex number α ;
- (iii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in X$;
- (iv) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Any space X with a scalar product is a normed space with $\|x\| = \sqrt{\langle x, x \rangle}$. If it is a Banach space in this norm, it is called a *Hilbert space*. The space \mathbb{C}^n is a Hilbert space with the scalar product

$$\langle x, y \rangle = \bar{x}^T y = \sum_{k=1}^n \bar{x}_k y_k.$$

Thus, it is also a Banach space and a complete metric space. Note that

$$\langle x, Ay \rangle = \langle \bar{A}^T x, y \rangle,$$

for any $x, y \in \mathbb{C}^n$ and a complex matrix A . The space $C^0(\Omega, \mathbb{C}^m)$ is a space with the scalar product

$$\langle f, g \rangle = \int_{\Omega} \bar{f}^T(x) g(x) dx,$$

but is not a Hilbert space.

A.3 Geometry

A.3.1 Sets

To denote that x is an element of a *set* X , we write $x \in X$. A set A is a *subset* of X ($A \subset X$) if $x \in A$ implies $x \in X$. If A and B are two sets, then the set $A \cup B$ consists of all elements that belong to either A or B , while the set $A \cap B$ is composed of all elements that belong to both A and B . The set of all ordered pairs (a, b) , such that $a \in A$ and $b \in B$, is called the *direct product* of two sets A and B and is denoted by $A \times B$.

The following notations are used for the standard sets:

\mathbb{R}^1 : the set of all real numbers $-\infty < x < +\infty$; \mathbb{R}_+^1 denotes the set of all nonnegative real numbers $x \geq 0$;

\mathbb{R}^n : the direct product of n sets \mathbb{R}^1 ; an element $x \in \mathbb{R}^n$ is considered as a vector (one-column matrix) $x = (x_1, x_2, \dots, x_n)^T$;

\mathbb{C}^1 : the set of all complex numbers $z = x + iy$, where $x, y \in \mathbb{R}^1$, $i^2 = -1$. Any $z \in \mathbb{C}^1$ can be represented as $z = \rho e^{i\varphi} = \rho(\cos \varphi + i \sin \varphi)$, where $\rho = |z| = \sqrt{x^2 + y^2}$ and $\varphi = \arg z$; $\bar{z} = x - iy$;

\mathbb{C}^n : the direct product of n sets \mathbb{C}^1 ; an element $z \in \mathbb{C}^n$ is considered as a vector (one-column matrix) $z = (z_1, z_2, \dots, z_n)^T$;

\mathbb{Z} : the set of all integer numbers $\{\dots, -2, -1, 0, 1, 2, \dots\}$; \mathbb{Z}_+ denotes the set of all nonnegative integers $k = 0, 1, 2, \dots$;

\mathbb{S}^1 : the unit circle: $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$;

\mathbb{T}^2 : the two-torus: $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

A.3.2 Maps

Let X and Y be two sets. A (*single-valued*) *map* (or *function*)

$$f : X \rightarrow Y$$

is said to be defined from X to Y if, for any element $x \in X$, an element $y \in Y$ is specified. We write

$$y = f(x),$$

or

$$x \mapsto f(x).$$

A map $f : X \rightarrow X$ is called a *transformation* of X . A map $f : X \rightarrow Y$ can be defined only for elements of a subset $D \subset X$. In this case, D is called the *domain of definition* of f . The set $f^{-1}(Y_0)$ of all $x \in X$ such that $f(x) \in Y_0$ is called the *preimage* of $Y_0 \subset Y$.

A.3.3 Manifolds

For our purposes, it is sufficient to consider the *manifold* $M \subset \mathbb{R}^n$ as a set of points in \mathbb{R}^n that satisfy a system of m scalar equations:

$$F(x) = 0,$$

where $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for some $m \leq n$. The manifold M is *smooth* (*differentiable*) if F is smooth and the rank of the Jacobian matrix F_x is equal to m at each point $x \in M$. At each point x of a smooth manifold M , an $(n - m)$ -dimensional *tangent space* $T_x M$ is defined. This space consists of all vectors $v \in \mathbb{R}^n$ that can be represented as $v = \dot{\gamma}(0)$, where $\gamma : \mathbb{R}^1 \rightarrow M$ is a smooth *curve* on the manifold satisfying $\gamma(0) = x$. Alternatively, $T_x M$ can be characterized as the orthogonal complement to

$$\text{span}\{\nabla F_1, \nabla F_2, \dots, \nabla F_m\},$$

where

$$\nabla F_k = \left(\frac{\partial F_k}{\partial x_1}, \frac{\partial F_k}{\partial x_2}, \dots, \frac{\partial F_k}{\partial x_n} \right)^T, \quad k = 1, 2, \dots, m,$$

are linear independent gradient vectors at point x . One can introduce $n - m$ coordinates near each point $x \in M$ by projecting to $T_x M$, so that a smooth manifold M is locally equivalent to \mathbb{R}^{n-m} .

A *region* $\Omega \in \mathbb{R}^n$ is a closed set of points in \mathbb{R}^n bounded by a piecewise smooth $(n - 1)$ -dimensional manifold.

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