Appendix A

Basic Notions from Algebra, Analysis, and Geometry

In this appendix we summarize for the convenience of the reader some basic mathematical results that are assumed to be known in the main text. Of course, reading this appendix cannot substitute for a systematic study of the corresponding topics via standard textbooks.

A.1 Algebra

A.1.1 Matrices

Let A be an $n \times m$ matrix with complex elements $a_{jk} \in \mathbb{C}$[1]:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix},$$

and let A^T denote its *transpose*:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & & & \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}.$$

The *product* of an $n \times m$ matrix A and an $m \times l$ matrix B is the $n \times l$ matrix $C = AB$ with the elements

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, l.$$

The following property holds:

$$(AB)^T = B^T A^T.$$

The *determinant* of a square $n \times n$ matrix A is the complex number defined by

$$\det A = \sum_{(i_1, i_2, \ldots, i_n) \in S_n} (-1)^{\delta(i_1, i_2, \ldots, i_n)} a_{i_1} a_{i_2} \cdots a_{i_n},$$

where S_n is the set of all permutations of n indices, and $\delta = 0$ when the multiindex $(i_1, i_2, \ldots, i_n)$ can be obtained from the multi-index $(1, 2, \ldots, n)$ by an even number of one-step permutations; $\delta = 1$ otherwise. A square matrix A is called *nonsingular* if $\det A \neq 0$. For a nonsingular matrix A, there is the *inverse* matrix A^{-1}, such that $AA^{-1} = A^{-1}A = I$, where I is the identity $n \times n$ matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

If A and B are two $n \times n$ matrices, then

$$\det(AB) = \det A \ \det B.$$

The *rank* of an $n \times m$ matrix A is the order of the largest nonsingular square submatrix of A.

The *sum* of two $n \times m$ matrices A and B is the $n \times m$ matrix $C = A + B$ with the elements

$$c_{ij} = a_{ij} + b_{ij}, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m.$$

The product of a complex number λ and an $n \times m$ matrix A is the $n \times m$ matrix $B = \lambda A$ with the elements

$$b_{ij} = \lambda a_{ij}, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m.$$

Consider a function $x \mapsto f(x)$ defined by a convergent series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k$$

(*analytic function*). Given a square matrix A, we can introduce a square matrix $f(A)$ by

$$f(A) = \sum_{k=0}^{\infty} f_k A^k,$$

where $A^0 = I,\ A^k = AA^{k-1},\ k = 1, 2, \ldots$. For example,

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

A.1.2 Vector spaces and linear transformations

A complex $n \times 1$ matrix

$$v = (v_1, v_2, \ldots, v_n)^T = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix}$$

is called a *vector*. The set of all such vectors is a *linear space* that can be identified with $\mathbb{C}^n$. In this space, addition between two elements and multiplication of an element by a complex number are defined component-wise.

A subset $X \subset \mathbb{C}^n$ is called the *linear subspace* (*hyperplane*) of $\mathbb{C}^n$ if $x \in X$ and $y \in X$ imply $x + y \in X$ and $\lambda x \in X$ for any $\lambda \in \mathbb{C}^1$. A linear subspace Z is called the *sum* of two linear subspaces X and Y if any vector $z \in Z$ can be represented as $z = x + y$ for some vectors $x \in X$ and $y \in Y$. Symbolically: $Z = X + Y$. If such a prepresentation is unique for each z, Z is called the *direct sum* of X and Y and is denoted by $Z = X \oplus Y$.

Vectors $\{a_1, a_2, \ldots, a_k\}$ from $\mathbb{C}^n$ are called *linearly independent* when

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_k a_k = 0,$$

if and only if $\alpha_j = 0$ for all $j = 1, 2, \ldots, k$. The set

$$L = \operatorname{span}\{a_1, a_2, \ldots, a_k\} = \left\{ v \in \mathbb{C}^n : v = \sum_{i=1}^{k} \alpha_i a_i,\ \ \alpha_i \in \mathbb{C}^1 \right\}$$

is a linear subspace of $\mathbb{C}^n$. If $\{a_1, a_2, \ldots, a_k\}$ are linearly independent, $\dim L = k$. A set of n linearly independent vectors is called a *basis*. The set of unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \cdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix}$$

is the *standard basis* in $\mathbb{C}^n$. Any vector $v \in \mathbb{C}^n$ can be uniquely represented as

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{pmatrix} = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n.$$

If $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ is another basis, any vector $v \in \mathbb{C}^n$ can also be represented as

$$v = u_1 \varepsilon_1 + u_2 \varepsilon_2 + \cdots + u_n \varepsilon_n,$$

where $u_k \in \mathbb{C}^1$ are *components* of v in this basis. Denote $u = (u_1, u_2, \ldots, u_n)^T$. Then

$$v = Cu,$$

where the $n \times n$ matrix C is nonsingular and has the elements c_{ij} that are the components of the basis vectors ε_j in the standard basis

$$\varepsilon_j = c_{1j} e_1 + c_{2j} e_2 + \cdots + c_{nj} e_n, \quad j = 1, 2, \ldots, n.$$

An $n \times n$ matrix A can be identified with a *linear transformation* of the space $\mathbb{C}^n$

$$v \mapsto Av.$$

In a basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ this transformation will have the form

$$u \mapsto Bu,$$

where the matrix B is given by

$$B = C^{-1} AC.$$

The matrices A and B are called *similar*. The determinants of similar matrices coincide.

A.1.3 Eigenvectors and eigenvalues

A nonzero complex vector

$$v = (v_1, v_2, \ldots, v_n)^T \in \mathbb{C}^n$$

is called an *eigenvector* of an $n \times n$ matrix A if

$$Av = \lambda v,$$

for some $\lambda \in \mathbb{C}^1$. The complex number λ is called an *eigenvalue* of A corresponding to the eigenvector v. The eigenvalues of A are roots of the *characteristic polynomial*

$$h(\lambda) = \det(A - \lambda I),$$

and every root is an eigenvalue. Thus, there are n eigenvalues if we count their multiplicities as the roots of $h(\lambda)$.

A.1.4 Invariant subspaces, generalized eigenvectors, and Jordan normal form

A linear subspace $X \subset \mathbb{C}^n$ is called an *invariant subspace* of the matrix A if $AX \subset X$, that is, if $w \in X$ implies $Aw \in X$.

If λ is a root of the characteristic polynomial, then there is an invariant subspace (*eigenspace*) of A that is spanned by the eigenvector $v \in \mathbb{C}^n$ associated with λ:

$$X = \{x \in \mathbb{C}^n : x = \omega v, \omega \in \mathbb{C}^1\}.$$

If λ is a multiple root of the characteristic polynomial of multiplicity m, then one can find $1 \le l \le m$ linearly independent eigenvectors $v_1, v_2, \dots, v_l$, corresponding to λ. For each eigenvector v_j, there is a *maximal chain* of complex vectors $\{w_1^{(j)}, w_2^{(j)}, \dots, w_{k_j}^{(j)}\}$, such that

$$\begin{aligned} Aw_1^{(j)} &= \lambda w_1^{(j)}, \\ Aw_2^{(j)} &= \lambda w_2^{(j)} + w_1^{(j)}, \\ &\cdots \\ Aw_{k_j}^{(j)} &= \lambda w_{k_j}^{(j)} + w_{k_j-1}^{(j)}. \end{aligned}$$

The chain can be composed of only one vector $w_1^{(j)}$, that is merely the eigenvector v_j. The vectors $w_k^{(j)}$ with $k \ge 2$ are called *generalized eigenvectors* of A corresponding to the eigenvalue λ. The subspace

$$X = \{x \in \mathbb{C}^n : x = \omega_1 w_1^{(j)} + \omega_2 w_2^{(j)} + \dots + \omega_k w_k^{(j)},\ \omega_j \in \mathbb{C}^1\}$$

is an invariant subspace of A.

Eigenvectors and generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. The vectors $\{w_1^{(j)}, w_2^{(j)}, \dots, w_{k_j}^{(j)}\}$ composing a chain corresponding to a multiple eigenvalue λ are also linearly independent.

Theorem A.1 (Jordan normal form) *The space $\mathbb{C}^n$ can be decomposed into linear invariant subspaces of the matrix A corresponding to its eigenvalues and spanned by the corresponding eigenvectors and generalized eigenvectors of A. In a basis given by all the eigenvectors and generalized eigenvectors, the matrix A has a block-diagonal form with square blocks*

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & \cdots & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda \end{pmatrix},$$

whose dimension is equal to the length of the corresponding chain. □

This form is called the *Jordan normal form* or *Jordan canonical form.* It also follows from this theorem that the product of all the eigenvalues of the matrix A is equal to its determinant:

$$\det A = \lambda_1 \lambda_2 \cdots \lambda_n.$$

If the matrix A is real, then it has linear invariant subspaces of $\mathbb{R}^n$ spanned by eigenvectors and generalized eigenvectors corresponding to its real eigenvalues and also by the real and imaginary parts of complex eigenvectors corresponding to its complex eigenvalues with, say, positive imaginary part. Such subspaces are called (*real*) *generalized eigenspaces* of A.

If λ is an eigenvalue of A, then $\mu = f(\lambda)$ is an eigenvalue of $B = f(A)$, where f is an analytic function.

A.1.5 Fredholm Alternative Theorem

Let A be a real $n \times m$ matrix and let $b \in \mathbb{R}^n$ be a real vector. The *null-space* of A is the linear subspace of $\mathbb{R}^m$ composed of all vectors $x \in \mathbb{R}^m$ for which $Ax = 0$. The *range* of A is the set of all $x \in \mathbb{R}^n$ for which $Ay = x$ for some $y \in \mathbb{R}^m$.

Theorem A.2 (Fredholm Alternative Theorem) *The equation $Ax = b$ has a solution if and only if $b^T v = 0$ for every vector $v \in \mathbb{R}^n$ satisfying $A^T v = 0$.* □

Notice that $b^T v = \sum_{j=1}^n b_j v_j$ is the standard *scalar product* in $\mathbb{R}^n$. The theorem means that the null-space of A^T is the orthogonal complement of the range of A and that together they span the whole $\mathbb{R}^n$. In other words, any vector $b \in \mathbb{R}^n$ can be uniquely decomposed as $b = b_r + b_0$, where b_r is in the range of A, b_0 is in the null-space of A^T, and b_r is orthogonal to b_0.

If A is a complex matrix and b is a complex vector, Theorem A.2 remains valid if we replace transposition by transposition composed with complex conjugation.

A.1.6 Groups

A set G is a *group* if a *product* "$\circ$": $G \times G \to G$ is defined which satisfies the following properties:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$ for all $f, g, h \in G$;
(ii) there is a *unit* element $e \in G$ such that $g \circ e = e \circ g = g$, for all $g \in G$;
(iii) for each $g \in G$, there is a unique element $g^{-1} \in G$ such that $g^{-1} \circ g = g \circ g^{-1} = e$.

All real nonsingular $n \times n$ matrices with the matrix product and the unit I form the *general linear group* denoted by $GL(n)$. All $n \times n$ matrices satisfying $A^T A = I$ compose its *orthogonal subgroup* $O(n)$.

A.2 Analysis

If $y = g(x),\ g : \mathbb{R}^n \to \mathbb{R}^m$, and $z = f(y),\ f : \mathbb{R}^m \to \mathbb{R}^k$, are two maps, then their *superposition* $h = f \circ g$ is a map $z = h(x),\ \mathbb{R}^n \to \mathbb{R}^k$, defined by the formula

$$h(x) = f(g(x)).$$

Let $f_y(y)$ denote the Jacobian matrix of f evaluated at a point $y \in \mathbb{R}^n$:

$$f_y(y) \quad = \quad \left(\frac{\partial f_i(y)}{\partial y_j}\right),$$

where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$. If we similarly define $h_x(x)$ and $g_x(x)$, then

$$h_x(x) = [f_y(y)]\big|_{y=g(x)}\, [g_x(x)]$$

(the *chain rule*).

A.2.1 Implicit and Inverse Function Theorems

Consider a map

$$(x, y) \mapsto F(x, y),$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,$$

is a smooth map defined in a neighborhood of $(x, y) = (0, 0)$ and such that $F(0, 0) = 0$. Let $F_x(0, 0)$ denote the matrix of first partial derivatives of F with respect to x evaluated at $(0, 0)$:

$$F_x(0,0) \quad = \quad \left(\frac{\partial F_i(x,y)}{\partial x_j}\right)\Bigg|_{(x,y)=(0,0)}.$$

Theorem A.3 (Implicit Function Theorem) *If the matrix $F_x(0, 0)$ is nonsingular, then there is a smooth locally defined function $y = f(x)$,*

$$f : \mathbb{R}^n \to \mathbb{R}^m,$$

such that

$$F(x, f(x)) = 0,$$

for all x in some neighborhood of the origin of $\mathbb{R}^n$. Moreover,

$$f_x(0) = -[F_x(0, 0)]^{-1} F_y(0, 0). \ \square$$

The degree of smoothness of the function f is the same as that of F.

Consider now a map

$$y = g(x),$$

where

$$g : \mathbb{R}^n \to \mathbb{R}^n$$

is a smooth function defined in a neighborhood of $x = 0$ and satisfying $g(0) = 0$. The following theorem is a consequence of the Implicit Function Theorem.

Theorem A.4 (Inverse Function Theorem) *If the matrix $g_x(0)$ is nonsingular, then there is a smooth locally defined function $x = f(y)$,*

$$f : \mathbb{R}^n \to \mathbb{R}^n,$$

such that

$$g(f(y)) = y$$

for all y in some neighborhood of the origin of $\mathbb{R}^n$. □

The function f is called the *inverse function* for g and is denoted by $f = g^{-1}$.

A.2.2 Taylor expansion

Let Ω be a region in $\mathbb{R}^n$ containing the origin $x = 0$. Denote by $C^k(\Omega, \mathbb{R}^m)$ the set of maps (vector-valued functions) $y = f(x)$, $f : \Omega \to \mathbb{R}^m$, having continuously differentiable components up to and including order $k \geq 0$. If $f \in C^k(\Omega, \mathbb{R}^m)$ with a sufficiently large k, the function f is called *smooth*. A C^∞ function has continuous partial derivatives of any order. Any function $f \in C^k(\Omega, \mathbb{R}^m)$ can be represented near $x = 0$ in the form (*Taylor expansion*)

$$f(x) = \sum_{|i|=0}^{k} \frac{1}{i_1! i_2! \cdots i_n!} \left. \frac{\partial^{|i|} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \right|_{x=0} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} + R(x),$$

where $|i| = i_1 + i_2 + \cdots + i_n$ and $R(x) = O(\|x\|^{k+1}) = o(\|x\|^k)$, namely,

$$\frac{\|R(x)\|}{\|x\|^k} \to 0$$

as $\|x\| \to 0$. Here $\|x\| = \sqrt{x^T x}$.

A C^∞-function f is called *analytic* near the origin if the corresponding *Taylor series*

$$\sum_{|i|=0}^{\infty} \frac{1}{i_1! i_2! \cdots i_n!} \left. \frac{\partial^{|i|} f(x)}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} \right|_{x=0} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

converges to $f(x)$ at any point x sufficiently close to $x = 0$.

A.2.3 Metric, normed, and other spaces

Let X be a set of elements for which addition and multiplication by (complex) numbers satisfying standard axioms are defined. The set X can consist of functions, for example, $X = C^k(\Omega, \mathbb{C}^m)$.

The set X is a *metric space* if a function $\rho : X \times X \to \mathbb{R}^1$ is defined such that:

(i) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(ii) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ if and only if $x = y$;
(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

The function ρ is called a *metric* (or *distance*). A sequence $\{x_k\}_{k=1}^{\infty}$ of elements $x_k \in X$ has a *limit* $x^0 \in X$ (*convergent*) if for any $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that

$$\rho(x_k, x_0) < \varepsilon,$$

for all $k \geq N(\varepsilon)$. Notation: $x^0 = \lim_{k\to\infty} x_k$. A function $f : X \to X$ is *continuous* at x^0 if

$$\lim_{k\to\infty} f(x_k) = f(x^0),$$

for all sequences such that $\lim_{k\to+\infty} x_k = x^0$. A function $g : X \to X$ is called *Hölder-continuous* at x^0 if there exist a constant L_0 and an *index* $0 < \beta \leq 1$, such that

$$\rho(g(x), g(x_0)) \leq L_0[\rho(x, x_0)]^{\beta}$$

for all x sufficiently close to x_0.

A set $S \subset X$ is *closed* if it contains the limits of all convergent sequences such that for any finite k, $x_k \in S$. A sequence $\{x_k\}_{k=1}^{\infty}$ of elements $x_k \in X$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there is an integer $N(\varepsilon)$ such that for every $n, m \geq N$,

$$\rho(x_n, x_m) < \varepsilon.$$

If the sequence $\{x_k\}_{k=1}^{\infty}$ has a limit, it is a Cauchy sequence. If any Cauchy sequence has a limit in X, the space X is called *complete*.

The set X is a *normed space* if a function $\|\cdot\| : X \to \mathbb{R}^1$ is defined such that:

(i) $\|x\| \geq 0$ and $\|x\| = 0$ implies $x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for any (complex) number α;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The function $\|\cdot\|$ is called a *norm*. Any normed space is a metric space with the metric $\rho(x, y) = \|x - y\|$. If X is complete in this metric, it is called a *Banach space*. The space of continuous functions $C^0(\Omega, \mathbb{C}^m)$ is a Banach in the norm

$$\|f\| = \max_{i=1,2,\ldots,m} \sup_{\xi\in\Omega} |f_i(\xi)|.$$

A set $S \subset X$ is *bounded* if $\|x\| < C$ with some $C > 0$ for all $x \in S$.

The set X is a *space with a scalar product* if for each pair of $(x, y) \in X$ a complex number $\langle x, y\rangle$ (called the *scalar product*) is defined so that the following properties hold:

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$;
(ii) $\langle x, \alpha y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in X$ and any complex number α;
(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$;
(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

Any space X with a scalar product is a normed space with $\|x\| = \sqrt{\langle x, x\rangle}$. If it is a Banach space in this norm, it is called a *Hilbert space*. The space $\mathbb{C}^n$ is a Hilbert space with the scalar product

$$\langle x, y\rangle = \bar{x}^T y = \sum_{k=1}^{n} \bar{x}_k y_k.$$

Thus, it is also a Banach space and a complete metric space. Note that

$$\langle x, Ay\rangle = \langle \bar{A}^T x, y\rangle,$$

for any $x, y \in \mathbb{C}^n$ and a complex matrix A. The space $C^0(\Omega, \mathbb{C}^m)$ is a space with the scalar product

$$\langle f, g\rangle = \int_\Omega \bar{f}^T(x) g(x)\, dx,$$

but is not a Hilbert space.

A.3 Geometry

A.3.1 Sets

To denote that x is an element of a *set* X, we write $x \in X$. A set A is a *subset* of X ($A \subset X$) if $x \in A$ implies $x \in X$. If A and B are two sets, then the set $A \cup B$ consists of all elements that belong to either A or B, while the set $A \cap B$ is composed of all elements that belong to both A and B. The set of all ordered pairs (a, b), such that $a \in A$ and $b \in B$, is called the *direct product* of two sets A and B and is denoted by $A \times B$.

The following notations are used for the standard sets:

$\mathbb{R}^1$: the set of all real numbers $-\infty < x < +\infty$; $\mathbb{R}^1_+$ denotes the set of all nonnegative real numbers $x \geq 0$;

$\mathbb{R}^n$: the direct product of n sets $\mathbb{R}^1$; an element $x \in \mathbb{R}^n$ is considered as a vector (one-column matrix) $x = (x_1, x_2, \ldots, x_n)^T$;

$\mathbb{C}^1$: the set of all complex numbers $z = x + iy$, where $x, y \in \mathbb{R}^1$, $i^2 = -1$. Any $z \in \mathbb{C}^1$ can be represented as $z = \rho e^{i\varphi} = \rho(\cos\varphi + i\sin\varphi)$, where $\rho = |z| = \sqrt{x^2 + y^2}$ and $\varphi = \arg z$; $\bar{z} = x - iy$;

$\mathbb{C}^n$: the direct product of n sets $\mathbb{C}^1$; an element $z \in \mathbb{C}^n$ is considered as a vector (one-column matrix) $z = (z_1, z_2, \ldots, z_n)^T$;

$\mathbb{Z}$: the set of all integer numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$; $\mathbb{Z}_+$ denotes the set of all nonnegative integers $k = 0, 1, 2, \ldots$;

$\mathbb{S}^1$: the unit circle: $\mathbb{S}^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$;

$\mathbb{T}^2$: the two-torus: $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.

A.3.2 Maps

Let X and Y be two sets. A (*single-valued*) *map* (or *function*)

$$f : X \to Y$$

is said to be defined from X to Y if, for any element $x \in X$, an element $y \in Y$ is specified. We write

$$y = f(x),$$

or

$$x \mapsto f(x).$$

A map $f : X \to X$ is called a *transformation* of X. A map $f : X \to Y$ can be defined only for elements of a subset $D \subset X$. In this case, D is called the *domain of definition* of f. The set $f^{-1}(Y_0)$ of all $x \in X$ such that $f(x) \in Y_0$ is called the *preimage* of $Y_0 \subset Y$.

A.3.3 Manifolds

For our purposes, it is sufficient to consider the *manifold* $M \subset \mathbb{R}^n$ as a set of points in $\mathbb{R}^n$ that satisfy a system of m scalar equations:

$$F(x) = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ for some $m \leq n$. The manifold M is *smooth* (*differentiable*) if F is smooth and the rank of the Jacobian matrix F_x is equal to m at each point $x \in M$. At each point x of a smooth manifold M, an $(n-m)$-dimensional *tangent space* T_xM is defined. This space consists of all vectors $v \in \mathbb{R}^n$ that can be represented as $v = \dot{\gamma}(0)$, where $\gamma : \mathbb{R}^1 \to M$ is a smooth *curve* on the manifold satisfying $\gamma(0) = x$. Alternatively, T_xM can be characterized as the orthogonal complement to

$$\text{span}\{\nabla F_1, \nabla F_2, \ldots, \nabla F_m\},$$

where

$$\nabla F_k = \left(\frac{\partial F_k}{\partial x_1}, \frac{\partial F_k}{\partial x_2}, \ldots, \frac{\partial F_k}{\partial x_n} \right)^T, \quad k = 1, 2, \ldots, m,$$

are linear independent gradient vectors at point x. One can introduce $n-m$ coordinates near each point $x \in M$ by projecting to T_xM, so that a smooth manifold M is locally equivalent to $\mathbb{R}^{n-m}$.

A *region* $\Omega \in \mathbb{R}^n$ is a closed set of points in $\mathbb{R}^n$ bounded by a piecewise smooth $(n-1)$-dimensional manifold.

References

Afraimovich, V. & Shil'nikov, L. [1972], 'On singular trajectories of dynamical systems', *Uspekhi. Mat. Nauk* **5**, 189–190. In Russian.

Afraimovich, V. & Shil'nikov, L. [1974], 'On attainable transitions from Morse-Smale systems to systems with many periodic points', *Math. USSR-Izv.* **8**, 1235–1270.

Afraimovich, V. & Shil'nikov, L. [1982], 'Bifurcation of codimension 1, leading to the appearance of a countable set of tori', *Dokl. Akad. Nauk SSSR* **262**, 777–780. In Russian.

Allgower, E. & Georg, K. [1990], *Numerical Continuation Methods. An Introduction*, Springer-Verlag, New York.

Allgower, E. & Georg, K. [1993], Continuation and path following, *in* 'Acta numerica', Cambridge University Press, Cambridge, pp. 1–64.

Andronov, A. [1933], Mathematical problems of self-oscillation theory, *in* 'I All-Union Conference on Oscillations, November 1931', GTTI, Moscow-Leningrad, pp. 32–71. In Russian.

Andronov, A. & Leontovich, E. [1939], 'Some cases of the dependence of the limit cycles upon parameters', *Uchen. Zap. Gork. Univ.* **6**, 3–24. In Russian.

Andronov, A. & Pontryagin, L. [1937], 'Systèmes grossières', *Dokl. Akad. Nauk SSSR* **14**, 247–251. In Russian.

Andronov, A., Leontovich, E., Gordon, I. & Maier, A. [1973], *Theory of Bifurcations of Dynamical Systems on a Plane*, Israel Program for Scientific Translations, Jerusalem.

Anosov, D. [1967], 'Geodesic flows and closed Riemannian manifolds with negative curvature', *Proc. Steklov Inst. Math.* **90**, 1–212. In Russian.

Arnold, V. [1972], 'Lectures on bifurcations in versal families', *Russian Math. Surveys* **27**, 54–123.

Arnold, V. [1973], *Ordinary Differential Equations*, MIT Press, Cambridge, MA.

Arnold, V. [1977], 'Loss of stability of self-induced oscillations near resonance, and versal deformations of equivariant vector fields', *Functional Anal. Appl.* **11**, 85–92.

Arnold, V. [1983], *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York.

Arnold, V. [1984], *Catastrophe Theory*, Springer-Verlag, New York.

Arnol'd, V., Afraimovich, V., Il'yashenko, Y. & Shil'nikov, L. [1994], Bifurcation theory, *in* V. Arnold, ed., 'Dynamical Systems V. Encyclopaedia of Mathematical Sciences', Springer-Verlag, New York.

Arnold, V., Varchenko, A. & Guseyn-Zade, S. [1985], *Singularities of Differential Maps I*, Birkhäuser, Boston, MA.

Aronson, D., Chory, M., Hall, G. & McGehee, R. [1982], 'Bifurcations from an invariant circle for two-parameter families of maps on the plane: A computer assisted study', *Comm. Math. Phys.* **83**, 303–354.

Arrowsmith, D. & Place, C. [1990], *An Introduction to Dynamical Systems*, Cambridge University Press, Cambridge.

Arrowsmith, D., Cartwright, J., Lansbury, A. & Place, C. [1993], 'The Bogdanov map: Bifurcations, mode locking, and chaos in a dissipative system', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **3**, 803–842.

Auchmuty, J. & Nicolis, G. [1976], 'Bifurcation analysis of reaction–diffusion equations (III). Chemical oscillations', *Bull. Math. Biol.* **38**, 325–350.

Babenko, K. & Petrovich, V. [1983], On computer-assisted proofs, Preprint 83-133, Institute of Applied Mathematics of the USSR Academy of Sciences, Moscow.

Back, A., Guckenheimer, J., Myers, M., Wicklin, F. & Worfolk, P. [1992], 'DsTool: Computer assisted exploration of dynamical systems', *Notices Amer. Math. Soc.* **39**, 303–309.

Bai, F. & Champneys, A. [1996], 'Numerical detection and continuation of saddle-node homoclinic bifurcations of codimension one and two', *Dynam. Stability Systems* **11**, 325–346.

Bajaj, A. [1986], 'Resonant parametric perturbations of the Hopf bifurcation', *J. Math. Anal. Appl.* **115**, 214–224.

Balabaev, N. & Lunevskaya, L. [1978], Following a curve in an n-dimensional space. FORTRAN Software Series **1**, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Balakrishnan, A. [1976], *Applied Functional Analysis*, Springer-Verlag, Berlin.

Bautin, N. [1949], *Behavior of Dynamical Systems near the Boundaries of Stability Regions*, OGIZ GOSTEXIZDAT, Leningrad. In Russian.

Bautin, N. & Leontovich, E. [1976], *Methods and Rules for the Qualitative Study of Dynamical Systems on the Plane*, Nauka, Moscow. In Russian.

Bautin, N. & Shil'nikov, L. [1980], Supplement I: Behavior of dynamical systems near stability soundaries of equilibria and periodic motions, *in* 'The Limit Cycle Bifurcation and its Applications. Russian translation of the book by Marsden, J.E. and McCracken', Mir, Moscow. In Russian.

Bazykin, A. [1974], Volterra system and Michaelis-Menten equation, *in* 'Problems of Mathematical Genetics', NGU, Novosibirsk, pp. 103–143. In Russian.

Bazykin, A. [1985], *Mathematical Biophysics of Interacting Populations*, Nauka, Moscow. In Russian.

Bazykin, A. & Khibnik, A. [1981], 'On sharp exitation of self-oscillations in a Volterra-type model', *Biophysika* **26**, 851–853. In Russian.

Bazykin, A., Berezovskaya, F., Denisov, G. & Kuznetsov, Y. [1981], 'The influence of predator saturation effect and competition among predators on predator-prey system dynamics', *Ecol. Modelling* **14**, 39–57.

Bazykin, A., Kuznetsov, Y. & Khibnik, A. [1985], Bifurcation diagrams of planar dynamical systems, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Bazykin, A., Kuznetsov, Y. & Khibnik, A. [1989], *Portraits of Bifurcations: Bifurcation Diagrams of Dynamical Systems on the Plane*, Znanie, Moscow. In Russian.

Belitskii, G. [1973], 'Functional equations and conjugacy of local diffeomorphisms of a finite smoothness class', *Functional Anal. Appl.* **7**, 268–277.

Belitskii, G. [1979], *Normal Forms, Ivariants, and Local Mappings*, Naukova Dumka, Kiev. In Russian.

Belyakov, L. [1974], 'A case of the generation of a periodic motion with homoclinic curves', *Mat. Zametki* **15**, 336–341.

Belyakov, L. [1980], 'The bifurcation set in a system with a homoclinic saddle curve', *Mat. Zametki* **28**, 910–916.

Belyakov, L. [1984], 'Bifurcation of systems with homoclinic curve of a saddle-focus with saddle quantity zero', *Mat. Zametki* **36**, 838–843.

Berezovskaya, F. & Khibnik, A. [1979], On the problem of bifurcations of self-oscillations close to a 1:4 resonance (investigation of the model equation), Research Computing Centre, USSR Academy of Sciences, Pushchino, Moscow Region. In Russian (English translation: *Selecta Math. Soviet.* **13(3)**, 1994, 197-215).

Berezovskaya, F. & Khibnik, A. [1981], 'On the bifurcations of separatrices in the problem of stability loss of auto-oscillations near 1:4 resonance', *J. Appl. Math. Mech.* **44**, 938–942.

Berezovskaya, F. & Khibnik, A. [1985], Bifurcations of a dynamical second-order system with two zero eigenvalues and additional degeneracy, *in* 'Methods of Qualitative Theory of Differential Eqiations', GGU, Gorkii, pp. 128–138. In Russian.

Beyn, W.-J. [1990*a*], Global bifurcations and their numerical computation, *in* D. Roose, A. Spence & B. De Dier, eds, 'Continuation and Bifurcations: Numerical Techniques and Applications', Kluwer, Dordrecht, pp. 169–181.

Beyn, W.-J. [1990*b*], 'The numerical computation of connecting orbits in dynamical systems', *IMA J. Numer. Anal.* **9**, 379–405.

Beyn, W.-J. [1991], Numerical methods for dynamical systems, *in* W. Light, ed., 'Advances in Numerical Analysis, Vol. I, Nonlinear Partial Differential Equations and Dynamical Systems', Oxford University Press, Oxford, pp. 175–236.

Beyn, W.-J. [1994], 'Numerical analysis of homoclinic orbits emanating from a Takens-Bogdanov point', *IMA J. Numer. Anal.* **14**, 381–410.

Birkhoff, G. [1927], 'Dynamical systems', *Amer. Math. Soc. Colloq. Publ.*

Birkhoff, G. [1935], 'Nouvelles recherches sur les systèmes dynamiques', *Memoriae Pont. Acad. Sci. Novi. Lincaei, Ser. 3* **1**, 85–216.

Bogdanov, R. [1975], 'Versal deformations of a singular point on the plane in the case of zero eigenvalues', *Functional Anal. Appl.* **9**, 144–145.

Bogdanov, R. [1976*a*], Bifurcations of a limit cycle for a family of vector fields on the plane, *in* 'Proceedings of Petrovskii Seminar, Vol. 2', Moscow State University, pp. 23–35. In Russian (English translation: *Selecta Math. Soviet.* **1(4)**, 1981, 373-388).

Bogdanov, R. [1976*b*], Versal deformations of a singular point on the plane in the case of zero eigenvalues, *in* 'Proceedings of Petrovskii Seminar, Vol. 2', Moscow State University, pp. 37–65. In Russian (English translation: *Selecta Math. Soviet.* **1(4)**, 1981, 389-421).

Borisyuk, R. [1981], Stationary solutions of a system of ordinary differential equations depending upon a parameter. FORTRAN Software Series **6**, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Broer, H. & Vegter, G. [1984], 'Subordinate Šil'nikov bifurcations near some singularities of vector fields having low codimension', *Ergodic Theory Dynamical Systems* **4**, 509–525.

Broer, H. W., Roussarie, R. & Simó, C. [1993], On the Bogdanov-Takens bifurcation for planar diffeomorphisms, *in* 'International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991)', World Scientific, River Edge, NJ, pp. 81–92.

Broyden, C. [1965], 'A class of method for solving nonlinear simultaneous equations', *Math. Comp.* **19**, 577–593.

Butenin, N., Neimark, J. & Fufaev, N. [1976], *Introduction to the Theory of Nonlinear Oscillations*, Nauka, Moscow. In Russian.

Carr, J. [1981], *Applications of Center Manifold Theory*, Springer-Verlag, New York.

Carr, J., Chow, S.-N. & Hale, J. [1985], 'Abelian integrals and bifurcation theory', *J. Differential Equations* **59**, 413–436.

Champneys, A. & Kuznetsov, Y. [1994], 'Numerical detection and continuation of codimension-two homoclinic bifurcations', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **4**, 795–822.

Champneys, A., Härterich, J. & Sandstede, B. [1996], 'A non-transverse homoclinic orbit to a saddle-node equilibrium', *Ergodic Theory Dynamical Systems* **16**, 431–450.

Champneys, A., Kuznetsov, Y. & Sandstede, B. [1995], HomCont: An AUTO86 driver for homoclinic bifurcation analysis. Version 2.0, Centrum voor Wiskunde en Informatica, Amsterdam, Report AM-R9516.

Champneys, A., Kuznetsov, Y. & Sandstede, B. [1996], 'A numerical toolbox for homoclinic bifurcation analysis', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **6**, 867–887.

Char, B., Geddes, K., Gonnet, G., Leong, B., Monagan, M. & Watt, S. [1991*a*], *Maple V Language Reference Manual*, Springer-Verlag, New York.

Char, B., Geddes, K., Gonnet, G., Leong, B., Monagan, M. & Watt, S. [1991*b*], *Maple V Library Reference Manual*, Springer-Verlag, New York.

Chenciner, A. [1981], 'Courbes invariantes non normalement hyperboliques au voisinage d'une bifurcation de Hopf dégénérée de difféomorphismes de $\mathbf{R}^2$', *C. R. Acad. Sci. Paris Sér. I Math.* **292**, 507–510.

Chenciner, A. [1982], 'Points homoclines au voisinage d'une bifurcation de Hopf dégénérée de difféomorphismes de $\mathbf{R}^2$', *C. R. Acad. Sci. Paris Sér. I Math.* **294**, 269–272.

Chenciner, A. [1983*a*], Bifurcations de difféomorphismes de $\mathbf{R}^2$ au voisinage d'un point fixe elliptique, *in* G. Iooss, R. Helleman & R. Stora, eds, 'Chaotic behavior of deterministic systems (Les Houches, 1981)', North-Holland, Amsterdam, pp. 273–348.

Chenciner, A. [1983*b*], 'Orbites périodiques et ensembles de Cantor invariants d'Aubry-Mather au voisinage d'une bifurcation de Hopf dégénérée de difféomorphismes de $\mathbf{R}^2$', *C. R. Acad. Sci. Paris Sér. I Math.* **297**, 465–467.

Chenciner, A. [1985*a*], 'Bifurcations de points fixes elliptiques. I. Courbes invariantes', *Inst. Hautes Études Sci. Publ. Math.* **61**, 67–127.

Chenciner, A. [1985*b*], 'Bifurcations de points fixes elliptiques. II. Orbites periodiques et ensembles de Cantor invariants', *Invent. Math.* **80**, 81–106.

Chenciner, A. [1988], 'Bifurcations de points fixes elliptiques. III. Orbites périodiques de "petites" périodes et élimination résonnante des couples de courbes invariantes', *Inst. Hautes Études Sci. Publ. Math.* **66**, 5–91.

Cheng, C.-Q. [1990], 'Hopf bifurcations in nonautonomous systems at points of resonance', *Sci. China Ser. A* **33**, 206–219.

Cheng, C.-Q. & Sun, Y.-S. [1992], 'Metamorphoses of phase portraits of vector fields in the case of symmetry of order 4', *J. Differential Equations* **95**, 130–139.

Chow, S.-N. & Hale, J. [1982], *Methods of Bifurcation Theory*, Springer-Verlag, New York.

Chow, S.-N. & Lin, X.-B. [1990], 'Bifurcation of a homoclinic orbit with a saddle-node equilibrium', *Differential Integral Equations* **3**, 435–466.

Chow, S.-N., Deng, B. & Fiedler, B. [1990], 'Homoclinic bifurcation at resonant eigenvalues', *J. Dynamics Differential Equations* **2**, 177–244.

Chow, S.-N., Drachman, B. & Wang, D. [1990], 'Computation of normal forms', *J. Comput. Appl. Math.* **29**, 129–143.

Chow, S.-N., Li, C. & Wang, D. [1989*a*], 'Erratum: "Uniqueness of periodic orbits of some vector fields with codimension two singularities"', *J. Differential Equations* **82**, 206.

Chow, S.-N., Li, C. & Wang, D. [1989*b*], 'Uniqueness of periodic orbits of some vector fields with codimension two singularities', *J. Differential Equations* **77**, 231–253.

Chow, S.-N., Li, C. & Wang, D. [1994], *Normal Forms and Bifurcations of Planar Vector Fields*, Cambridge University Press, Cambridge.

Chu, K., Govaerts, W. & Spence, A. [1994], 'Matrices with rank deficiency two in eigenvalue problems and dynamical systems', *SIAM J. Numer. Anal.* **31**, 524–539.

Coddington, E. & Levinson, N. [1955], *Theory of Ordinary Differential Equations*, McGraw-Hill, New York.

Collet, P. & Eckmann, J.-P. [1980], *Iterated Maps on the Interval as a Dynamical System*, Birkhauser, Boston, MA.

Cox, D., Little, J. & O'Shea, D. [1992], *Ideals, Varieties, and Algorithms*, Springer-Verlag, New York.

Cushman, R. & Sanders, J. [1985], 'A codimension two bifurcation with a third order Picard-Fuchs equation', *J. Differential Equations* **59**, 243–256.

de Boor, C. & Swartz, B. [1973], 'Collocation at Gaussian points', *SIAM J. Numer. Anal.* **10**, 582–606.

Deng, B. [1989], 'The Shil'nikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation', *J. Differential Equations* **79**, 189–231.

Deng, B. [1990], 'Homoclinic bifurcations with nonhyperbolic equilibria', *SIAM J. Math. Anal.* **21**, 693–720.

Deng, B. [1993*a*], 'Homoclinic twisting bifurcations and cusp horseshoe maps', *J. Dynamics Differential Equations* **5**, 417–467.

Deng, B. [1993*b*], 'On Shil'nikov's saddle-focus theorem', *J. Differential Equations* **102**, 305–329.

Deng, B. [1994], 'Constructing homoclinic orbits and chaotic attractors', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **4**, 823–841.

Deng, B. & Sakamoto, K. [1995], 'Šil'nikov-Hopf bifurcations', *J. Differential Equations* **119**, 1–23.

Denjoy, A. [1932], 'Sur les courbes définies par les équations différentielles à la surface du tore', *J. Math.* **17(IV)**, 333–375.

Dennis, J. & Schnabel, R. [1983], *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice-Hall, Englewood Cliffs, NJ.

Deuflhard, P., Fiedler, B. & Kunkel, P. [1987], 'Efficient numerical pathfollowing beyond critical points', *SIAM J. Numer. Anal.* **24**, 912–927.

Diekmann, O. & van Gils, S. [1984], 'Invariant manifolds for Volterra integral equations of convolution type', *J. Differential Equations* **54**, 139–180.

Diekmann, O., van Gils, S., Verduyn Lunel, S. & Walther, H.-O. [1995], *Delay Equations: Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York.

Diener, F. [1983], Quelques exemples de bifurcations et leurs canards, *in* 'Proceedings of the eleventh annual Iranian mathematics conference (Mashhad, 1980)', Univ. Mashhad, Mashhad, pp. 59–73.

Doedel, E. [1981], 'AUTO, a program for the automatic bifurcation analysis of autonomous systems', *Congr. Numer.* **30**, 265–384.

Doedel, E. & Friedman, M. [1989], 'Numerical computation of heteroclinic orbits', *J. Comput. Appl. Math.* **26**, 159–170.

Doedel, E. & Kernévez, J. [1986], AUTO: Software for continuation problems in ordinary differential equations with applications, California Institute of Technology, Applied Mathematics.

Doedel, E., Champneys, A., Fairgrieve, T., Kuznetsov, Y., Sandstede, B. & Wang, X.-J. [1986], AUTO97: Continuation and bifurcation software for ordinary differential equations (with HomCont), Computer Science, Concordia University, Montreal, Canada.

Doedel, E., Friedman, M. & Monteiro, A. [1994], 'On locating connecting orbits', *Appl. Math. Comput.* **65**, 231–239.

Doedel, E., Jepson, A. & Keller, H. [1984], Numerical methods for Hopf bifurcation and continuation of periodic solution paths, *in* R. Glowinski & J. Lions, eds, 'Computing Methods in Applied Science and Engineering VI', North-Holland, Amsterdam.

Doedel, E., Keller, H. & Kernévez, J. [1991*a*], 'Numerical analysis and control of bifurcation problems: (I) Bifurcation in finite dimensions', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **1**, 493–520.

Doedel, E., Keller, H. & Kernévez, J. [1991*b*], 'Numerical analysis and control of bifurcation problems: (II) Bifurcation in infinite dimensions', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **1**, 745–772.

Dulac, M. [1923], 'Sur les cycles limites', *Bull. Soc. Math. France* **51**, 45–188.

Dumortier, F. [1978], *Singularities of Vector Fields, Mathematical Monographs No. 32*, IMPA, Rio de Janeiro.

Dumortier, F. & Rousseau, R. [1990], 'Cubic Liénard equations with linear damping', *Nonlinearity* **3**, 1015–1039.

Dumortier, F., Roussarie, R. & Sotomayor, J. [1987], 'Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension 3', *Ergodic Theory Dynamical Systems* **7**, 375–413.

Dumortier, F., Roussarie, R., Sotomayor, J. & Żołądek, H. [1991], *Bifurcations of Planar Vector Fields. Nilpotent Singularities and Abelian Integrals*, Vol. 1480 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

Evans, J., Fenichel, N. & Feroe, J. [1982], 'Double impulse solutions in nerve axon equations', *SIAM J. Appl. Math.* **42**, 219–234.

Fairgrieve, T. & Jepson, A. [1991], 'O.K. Floquet multipliers', *SIAM J. Numer. Anal.* **28**, 1446–1462.

Feichtinger, G. [1992], 'Hopf bifurcation in an advertising diffusion model', *J. Econom. Behavior Organization* **17**, 401–411.

Feigenbaum, M. [1978], 'Quantitative universality for a class of nonlinear transformations', *J. Statist. Phys.* **19**, 25–52.

Fenichel, N. [1971], 'Persistence and smoothness of invariant manifolds for flows', *Indiana Univ. Math. J.* **21**, 193–226.

Feroe, J. [1981], 'Travelling waves of infinitely many pulses in nerve equations', *Math. Biosci.* **55**, 189–204.

Feudel, U. & Jansen, W. [1992], 'CANDYS/QA – A software system for qualitative analysis of nonlinear dynamical systems', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **4**, 773–794.

Fiedler, B. [1988], *Global Bifurcations of Periodic Solutions with Symmetry*, Vol. 1309 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

FitzHugh, R. [1961], 'Impulses and physiological states in theoretical models of nerve membrane', *Biophys. J.* **1**, 445–446.

Friedman, M. & Doedel, E. [1991], 'Numerical computation and continuation of invariant manifolds connecting fixed points', *SIAM J. Numer. Anal.* **28**, 789–808.

Friedman, M. & Doedel, E. [1993], 'Computational methods for global analysis of homoclinic and heteroclinic orbits: A case study', *J. Dynamics Differential Equations* **5**, 37–57.

Frouzakis, C., Adomaitis, R. & Kevrekidis, I. [1991], 'Resonance phenomena in an adaptively-controlled system', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **1**, 83–106.

Gambaudo, J. [1985], 'Perutbation of a Hopf bifurcation by an external time-periodic forcing', *J. Differential Equations* **57**, 172–199.

Gamero, E., Freire, E. & Rodríguez-Luis, A. J. [1993], Hopf-zero bifurcation: normal form calculation and application to an electronic oscillator, *in* 'International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991)', World Scientific, River Edge, NJ, pp. 517–524.

Gaspard, P. [1983], 'Generation of a countable set of homoclinic flows through bifurcation', *Phys. Lett. A* **97**, 1–4.

Gaspard, P. [1993], 'Local birth of homoclinic chaos', *Physica D* **62**, 94–122.

Gaspard, P., Kapral, R. & Nicolis, G. [1984], 'Bifurcation phenomena near homoclinic systems: a two- parameter analysis', *J. Statist. Phys.* **35**, 687–727.

Gatermann, K. & Hohmann, A. [1991], 'Symbolic exploitation of symmetry in numerical pathfollowing', *Impact Comput. Sci. Engrg.* pp. 330–365.

Gavrilov, N. [1978], On some bifurcations of an equilibrium with one zero and a pair of pure imaginary roots, *in* 'Methods of Qualitative Theory of Differential Equations', GGU, Gorkii. In Russian.

Gavrilov, N. [1980], On bifurcations of an equilibrium with two pairs of pure imaginary roots, *in* 'Methods of Qualitative Theory of Differential Equations', GGU, Gorkii, pp. 17–30. In Russian.

Gavrilov, N. & Shilnikov, A. [1996], A "Blue Sky Catastrophe" model, Abstract of Int. Conference "Contemporary Problems in the Theory of Dynamical Systems", Nizhnhy Novgorod, Russia, July 1–6, 1996.

Gavrilov, N. & Shilnikov, L. [1972], 'On three-dimensional systems close to systems with a structurally unstable homoclinic curve: I', *Math. USSR-Sb.* **17**, 467–485.

Gavrilov, N. & Shilnikov, L. [1973], 'On three-dimensional systems close to systems with a structurally unstable homoclinic curve: II', *Math. USSR-Sb.* **19**, 139–156.

Ghezzi, L. & Kuznetsov, Y. [1994], 'Strong resonances and chaos in a stock market model', *Internat. J. Systems Sci.* **11**, 1941–1955.

Glendinning, P. [1988], Global bifurcations in flows, *in* T. Bedford & J. Swift, eds, 'New Directions in Dynamical Systems', Vol. 127 of *London Math. Soc. Lecture Note Ser.*, Cambridge University Press, Cambridge, pp. 120–149.

Glendinning, P. & Sparrow, C. [1984], 'Local and global behaviour near homoclinic orbits', *J. Stat. Phys.* **35**, 645–696.

Golden, M. & Ydstie, B. [1988], 'Bifurcation in model reference adaptive control systems', *Systems Control Lett.* **11**, 413–430.

Golubitsky, M. & Schaeffer, D. [1985], *Singularities and Groups in Bifurcation Theory I*, Springer-Verlag, New York.

Golubitsky, M., Stewart, I. & Schaeffer, D. [1988], *Singularities and Groups in Bifurcation Theory II*, Springer-Verlag, New York.

Govaerts, W. & Pryce, J. [1993], 'Mixed block elimination for linear systems with wider borders', *IMA J. Numer. Anal.* **13**, 161–180.

Govaerts, W., Guckenheimer, J. & Khibnik, A. [1997], 'Defining functions for multiple Hopf bifurcations', *SIAM J. Numer. Anal.* **34**, 1269–1288.

Griewank, A. & Reddien, G. [1984], 'Characterization and computation of generalized turning points', *SIAM J. Numer. Anal.* **21**, 176–184.

Grobman, D. [1959], 'Homeomorphisms of systems of differential equations', *Dokl. Akad. Nauk SSSR* **128**, 880–881. In Russian.

Guckenheimer, J. [1981], On a codimension two bifurcation, *in* D. Rand & L. Young, eds, 'Dynamical Systems and Turbulence', Vol. 898 of *Lecture Notes in Mathematics*, Springer-Verlag, New York.

Guckenheimer, J. & Holmes, P. [1983], *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer-Verlag, New York.

Guckenheimer, J. & Myers, M. [1996], 'Computing Hopf bifurcations. II. Three examples from neurophysiology', *SIAM J. Sci. Comput.* **17**, 1275–1301.

Guckenheimer, J. & Worfolk, P. [1993], Dynamical systems: some computational problems, *in* D. Schlomiuk, ed., 'Bifurcations and Periodic Orbits of Vector Fields', Vol. 408 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, Kluwer, Dordrecht, pp. 241–278.

Guckenheimer, J., Myers, M. & Sturmfels, B. [1997], 'Computing Hopf bifurcations. I', *SIAM J. Numer. Anal.* **34**, 1–21.

Hadamard, J. [1901], 'Sur l'itération et let solutions asymptotiques des équations diffiérentialles', *Proc. Soc. Math. France* **29**, 224–228.

Hale, J. [1971], *Functional Differential Equations*, Springer-Verlag, New York.

Hale, J. [1977], *Theory of Functional Differential Equations*, Springer-Verlag, New York.

Hale, J. & Koçak, H. [1991], *Dynamics and Bifurcations*, Springer-Verlag, New York.

Hale, J. & Verduyn Lunel, S. [1993], *Introduction to Functional Differential Equations*, Springer-Verlag, New York.

Hartman, P. [1963], 'On the local linearization of differential equations', *Proc. Amer. Math. Soc.* pp. 568–573.

Hartman, P. [1964], *Ordinary Differential Equations*, Wiley, New York.

Hassard, B. [1980], Computation of invariant manifolds, *in* P. Holmes, ed., 'New Approaches to Nonlinear Problems in Dynamics', SIAM, Philadelphia, PA, pp. 27–42.

Hassard, B., Kazarinoff, N. & Wan, Y.-H. [1981], *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, London.

Hastings, S. [1976], 'On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations', *Quart. J. Math. (Oxford)* **27**, 123–134.

Hearn, A. [1993], *REDUCE User's Manual, Version 3.5*, The RAND Corporation, Santa Monica.

Henry, D. [1981], *Geometric Theory of Semilinear Parabolic Equations*, Vol. 840 of *Lecture Notes in Mathematics*, Springer-Verlag, New York.

Hille, E. & Phillips, R. [1957], *Functional Analysis and Semigroups*, American Mathematical Society, Providence, RI.

Hirsch, M. & Smale, S. [1974], *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York.

Hirsch, M., Pugh, C. & Shub, M. [1977], *Invariant Manifolds*, Vol. 583 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin.

Hirschberg, P. & Knobloch, E. [1993], 'Šil'nikov–Hopf bifurcation', *Physica D* **62**, 202–216.

Holling, C. [1965], 'The functional response of predators to prey density and its role in mimicry and population regulation', *Mem. Entomol. Soc. Canada* **45**, 5–60.

Holmes, P. & Rand, D. [1978], 'Bifurcations of the forced van der Pol oscillator', *Quart. Appl. Math.* **35**, 495–509.

Holmes, P. & Whitley, D. [1984], 'Bifurcations of one- and two-dimensional maps', *Philos. Trans. Roy. Soc. London, Ser. A* **311**, 43–102.

Holodniok, M. & Kubiček, M. [1984*a*], 'DERPER - An algorithm for the continuation of periodic solutions in ordinary differential equations', *J. Comput. Phys.* **55**, 254–267.

Holodniok, M. & Kubiček, M. [1984*b*], 'New algorithms for the evaluation of complex bifurcation points in ordinary differential equations. A comparative numerical study', *Appl. Math. Comput.* **15**, 261–274.

Homburg, A. [1993], Some global aspects of homoclinic bifurcations of vector fields, PhD thesis, Department of Mathematics, University of Groningen.

Homburg, A., Kokubu, H. & Krupa, M. [1994], 'The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit', *Ergodic Theory Dynamical Systems* **14**, 667–693.

Hopf, E. [1942], 'Abzweigung einer periodischen Lösung von einer stationaren Lösung eines Differetialsystems', *Ber. Math.-Phys. Kl. Sachs, Acad. Wiss. Leipzig* **94**, 1–22.

Horozov, E. [1979], Versal deformations of equivariant vector fields for the cases of symmetry of order 2 and 3, *in* 'Proceedings of Petrovskii Seminar, Vol. 5', Moscow State University, pp. 163–192. In Russian.

Iooss, G. [1979], *Bifurcations of Maps and Applications*, North-Holland, Amsterdam.

Iooss, G. & Adelmeyer, M. [1992], *Topics in Bifurcation Theory and Applications*, World Scientific, Singapore.

Iooss, G. & Joseph, D. [1980], *Elementary Stability and Bifurcation Theory*, Springer-Verlag, New York.

Iooss, G., Arneodo, A., Coullet, P. & Tresser, C. [1981], Simple computation of bifurcating invariant circles for mappings, *in* D. Rand & L.-S. Young, eds, 'Dynamical Systems and Turbulence', Vol. 898 of *Lecture Notes in Mathematics*, Springer-Verlag, New York.

Irwin, M. [1980], *Smooth Dynamical Systems*, Academic Press, New York.

Kaas-Petersen, C. [1989], PATH - User's Guide, University of Leeds, England.

Kantorovich, L. & Akilov, G. [1964], *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford.

Keener, J. [1981], 'Infinite period bifurcation and global bifurcation branches', *SIAM J. Appl. Math.* **41**, 127–144.

Keller, H. [1977], Numerical solution of bifurcation and nonlinear eigenvalue problems, *in* P. Rabinowitz, ed., 'Applications of Bifurcation Theory', Academic Press, New York, pp. 359–384.

Kelley, A. [1967], 'The stable, center stable, center, center unstable and unstable manifolds', *J. Differential Equations* **3**, 546–570.

Khibnik, A. [1990], LINLBF: A program for continuation and bifurcation analysis of equilibria up to codimension three, *in* D. Roose, B. d. Dier & A. Spence, eds, 'Continuation and Bifurcations: Numerical Techniques and Applications', Kluwer, Dordrecht, pp. 283–296.

Khibnik, A., Kuznetsov, Y., Levitin, V. & Nikolaev, E. [1993], 'Continuation techniques and interactive software for bifurcation analysis of ODEs and iterated maps', *Physica D* **62**, 360–371.

Khibnik, I. [1979], Periodic solutions of a system of differential equations, FORTRAN Software Series **5**, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Kisaka, M., Kokubu, H. & Oka, H. [1993*a*], 'Bifurcation to n-homoclinic orbits and n-periodic orbits in vector fields', *J. Dynamics Differential Equations* **5**, 305–358.

Kisaka, M., Kokubu, H. & Oka, H. [1993*b*], Supplement to homoclinic doubling bifurcation in vector fields, *in* R. Bamon, J. Labarca, J. Lewowicz & J. Palis, eds, 'Dynamical Systems', Longman, London, pp. 92–116.

Kolmogorov, A. [1957], La théorie générale des systeèmes dynamique et méchanique classique, *in* 'Proc. 1954 Int. Congr. Math. Amsterdam', North-Holland, Amsterdam, pp. 315–333.

Krauskopf, B. [1994*a*], 'Bifurcation sequences at 1:4 resonance: an inventory', *Nonlinearity* **7**, 1073–1091.

Krauskopf, B. [1994*b*], 'The bifurcation set for the 1:4 resonance problem', *Experimental Mathematics* **3**, 107–128.

Krauskopf, B. [1997], 'Bifurcations at ∞ in a model for 1:4 resonance', *Ergodic Theory Dynamical Systems* **17**, 899–931.

Kubiček, M. [1976], 'Algorothm 502. Dependence of solutions of nonlinear systems on a parameter', *ACM Trans. Math. Software* **2**, 98–107.

Kuznetsov, Y. [1983], One-dimensional invariant manifolds in ordinary differential equations depending upon parameters. FORTRAN Software Series **8**, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Kuznetsov, Y. [1984], Andronov-Hopf bifurcation in a four-dimensional system with circular symmetry, Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Kuznetsov, Y. [1985], Auto-waves in a reaction-diffusion system with circular symmetry (center manifold approach), Research Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Kuznetsov, Y. [1990], Computation of invariant manifold bifurcations, *in* D. Roose, A. Spence & B. De Dier, eds, 'Continuation and Bifurcations: Numerical Techniques and Applications', Kluwer, Dordrecht, pp. 183–195.

Kuznetsov, Y. [1991], 'Numerical analysis of the orientability of homoclinic trajectories', *Internat. Ser. Numer. Math.* **97**, 237–241.

Kuznetsov, Y. [1997], Explicit normal form corfficients for all codim 2 bifurcations of equilibria in ODEs, Centrum voor Wiskunde en Informatica, Amsterdam. Report MAS-R9730.

Kuznetsov, Y. & Panfilov, A. [1981], Stochastic waves in the FitzHugh-Nagumo system, Reseach Computing Centre, USSR Academy of Sciences, Pushchino. In Russian.

Kuznetsov, Y. & Piccardi, C. [1994*a*], 'Bifurcation analysis of periodic SEIR and SIR epidemic models', *J. Math. Biol.* **32**, 109–121.

Kuznetsov, Y. & Piccardi, C. [1994*b*], 'Bifurcations and chaos in a periodically forced prototype adaptive control system', *Kybernetika* **30**, 121–128.

Kuznetsov, Y. & Rinaldi, S. [1991], 'Numerical analysis of the flip bifurcation of maps', *Appl. Math. Comput.* **43**, 231–236.

Kuznetsov, Y., Muratori, S. & Rinaldi, S. [1992], 'Bifurcations and chaos in a periodic predator-prey model', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **2**, 117–128.

Kuznetsov, Y., Muratori, S. & Rinaldi, S. [1995], 'Homoclinic bifurcations in slow-fast second order systems', *Nonlinear Analysis, Theory, Methods & Applications* **25**, 747–762.

Lanford, O. [1980], 'A computer-assisted proof of the Feigenbaum conjectures', *Bull. Amer. Math. Soc.* **6**, 427–434.

Langford, W. [1979], 'Periodic and steady mode interactions lead to tori', *SIAM J. Appl. Math.* **37**, 22–48.

Lefever, R. & Prigogine, I. [1968], 'Symmetry-breaking instabilities in dissipative systems (II)', *J. Chem. Phys.* **48**, 1695–1700.

Lin, X.-B. [1990], 'Using Melnikov's method to solve Silnikov's problems', *Proc. Roy. Soc. Edinburgh Sect. A* **116**, 295–325.

Lorenz, E. [1963], 'Deterministic non-periodic flow', *J. Atmos. Sci.* **20**, 130–141.

Lorenz, E. [1984], 'Irregularity: A fundamental property of the atmosphere', *Tellus A* **36**, 98–110.

Lukyanov, V. [1982], 'Bifurcations of dynamical systems with a saddle-point separatrix loop', *Differential Equations* **18**, 1049–1059.

Lyapunov, A. [1892], *General Problem of Stability of Motion*, Mathematics Society of Kharkov, Kharkov.

Marsden, J. & McCracken, M. [1976], *Hopf Bifurcation and its Applications*, Springer-Verlag, New York.

May, R. [1974], 'Biological populations with nonoverlapping generations: Stable points, stable cycles, and chaos', *Science* **17**, 645–647.

Maynard Smith, J. [1968], *Mathematical Ideas in Biology*, Cambridge University Press, London.

Medvedev, V. [1980], 'On a new type of bifurcations on manifolds', *Mat. Sbornik* **113**, 487–492. In Russian.

Melnikov, V. [1962], Qualitative description of resonance phenomena in nonlinear systems, P-1013, OIJaF, Dubna. In Russian.

Melnikov, V. [1963], 'On the stability of the center for time periodic perturbations', *Trans. Moscow Math. Soc.* **12**, 1–57.

Mishchenko, E. & Rozov, N. [1980], *Differential Equations with Small Parameters and Relaxation Oscillations*, Plenum, New York.

Miura, R. [1982], 'Accurate computation of the stable solitary wave for the FitzHugh-Nagumo equations', *J. Math. Biol.* **13**, 247–269.

Moon, F. & Rand, R. [1985], Parametric stiffness control of flexible structures, *in* 'Proceedings of the Workshop on Identification and Control of Flexible Space Structures, Vol. II', Jet Propulsion Laboratory Publication 85-29, Pasadena, pp. 329–342.

Moore, G. & Spence, A. [1980], 'The calculation of turning points of nonlinear equations', *SIAM J. Numer. Anal.* **17**, 567–576.

Moore, G., Garret, T. & Spence, A. [1990], The numerical detection of Hopf bifurcation points, *in* D. Roose, B. d. Dier & A. Spence, eds, 'Continuation and Bifurcations: Numerical Techniques and Applications', Kluwer, Dordrecht, pp. 227–246.

Moser, J. [1968], 'Lectures on Hamiltonian systems', *Mem. Amer. Math. Soc.* **81**, 1–60.

Moser, J. [1973], *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, NJ.

Nagumo, J., Arimoto, S. & Yoshizawa, S. [1962], 'An active pulse transmission line simulating nerve axon', *Proc. IRE* **50**, 2061–2070.

Namachchivaya, S. & Ariaratnam, S. [1987], 'Periodically perturbed Hopf bifurcation', *SIAM J. Appl. Math.* **47**, 15–39.

Nayfeh, A. & Balachandran, B. [1995], *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods*, Wiley, New York.

Neimark, J. [1959], 'On some cases of periodic motions depending on parameters', *Dokl. Akad. Nauk SSSR* **129**, 736–739. In Russian.

Neimark, J. [1967], 'Motions close to doubly-asymptotic motion', *Soviet Math. Dokl.* **8**, 228–231.

Neimark, J. [1972], *Method of Point Transformations in the Theory of Nonlinear Oscillations*, Nauka, Moscow. In Russian.

Neimark, J. & Shil'nikov, L. [1965], 'A case of generation of periodic motions', *Soviet Math. Dokl.* **6**, 305–309.

Neishtadt, A. [1978], 'Bifurcations of the phase pattern of an equation system arising in the problem of stability loss of self-oscillations close to 1 : 4 resonance', *J. Appl. Math. Mech.* **42**, 896–907.

Nemytskii, V. & Stepanov, V. [1949], *Qualitative Theory of Differential Equations*, GITTL, Moscow-Leningrad. In Russian.

Newhouse, S., Palis, J. & Takens, F. [1983], 'Bifurcations and stability of families of diffeomorphisms', *Inst. Hautes Études Sci. Publ. Math.* **57**, 5–71.

Nikolaev, E. [1992], On bifurcations of closed orbits in the presence of involutory symmetry, Institute of Mathematical Problems of Biology, Russian Academy of Sciences, Pushchino, Moscow Region.

Nikolaev, E. [1994], Periodic motions in systems with a finite symmetry group, Institute of Mathematical Problems of Biology, Russian Academy of Sciences, Pushchino, Moscow Region.

Nikolaev, E. [1995], 'Bifurcations of limit cycles of differential equations admitting an involutive symmetry', *Sbornik: Mathematics* **186**, 611–627.

Nitecki, Z. [1971], *Differentiable Dynamics*, MIT Press, Cambridge, MA.

Nozdrachova, V. [1982], 'Bifurcation of a noncourse seperatrix loop', *Differential Equations* **18**, 1098–1104.

Ovsyannikov, I. & Shil'nikov, L. [1987], 'On systems with a saddle-focus homoclinic curve', *Math. USSR-Sb.* **58**, 557–574.

Palis, J. & Pugh, C. [1975], Fifty problems in dynamical systems, *in* 'Dynamical systems –Warwick 1974', Vol. 468 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, pp. 345–353.

Palmer, K. [1984], 'Exponential dichotomies and transversal homoclinic points', *J. Differential Equations* **55**, 225–256.

Pavlou, S. & Kevrekidis, I. [1992], 'Microbial predation in a periodically operated chemostat: A global study of the interaction between natural and externally imposed frequencies', *Math. Biosci.* **108**, 1–55.

Peckham, B. & Kevrekidis, I. [1991], 'Period doubling with higher-order degeneracies', *SIAM J. Math. Anal.* **22**, 1552–1574.

Peixoto, M. [1962], 'Structural stability on two dimensional manifolds', *Topology* **1**, 101–120.

Perron, O. [1930], 'Die stabilitätsfrage bei differentialgleichungen', *Math. Z.* **32**, 703–728.

Pliss, V. [1964], 'A reduction principle in stability theory of motions', *Izv. Akad. Nauk SSSR, Ser. Mat.* **28**, 1297–1324.

Poincaré, H. [1879], *Sur les propriétés des fonctions définies par les équations aux différences partielles. Thèse*, Gauthier-Villars, Paris.

Poincaré, H. [1892,1893,1899], *Les Méthodes Nouvelles de la Méchanique Céleste*, Gauthier-Villars, Paris.

Pontryagin, L. [1934], 'On dynamical systems close to Hamiltonian systems', *J. Exptl. Theoret. Phys.* **4**, 234–238. In Russian.

Pontryagin, L. [1962], *Ordinary Differential Equations*, Pergamon Press, London.

Poston, T. & Stewart, I. [1978], *Catastrophe Theory and its Applications*, Pitman, San Francisco.

Rheinboldt, W. [1982], 'Computation of critical boundaries on equilibrium manifolds', *SIAM J. Numer. Anal.* **19**, 653–669.

Rheinboldt, W. [1986], *Numerical Analysis of Parametrized Nonlinear Equations*, Wiley, New York.

Rheinboldt, W. & Burkardt, J. [1983], 'Algorithm 596: A program for a locally-parametrized continuation process', *ACM Trans. Math. Software* **9**, 236–241.

Richtmyer, R. [1978], *Principles of Advanced Mathematical Physics I*, Springer-Verlag, New York.

Richtmyer, R. [1981], *Principles of Advanced Mathematical Physics II*, Springer-Verlag, New York.

Ricker, W. [1954], 'Stock and recruitment', *J. Fish. Res. Board Canada* **11**, 559–663.

Rodríguez-Luis, A., Freire, E. & Ponce, E. [1990], A method for homoclinic and heteroclinic continuation in two and three dimensions, *in* D. Roose, A. Spence & B. De Dier, eds, 'Continuation and Bifurcations: Numerical Techniques and Applications', Kluwer, Dordrecht, pp. 197–210.

Roose, D. & Hlavaček, V. [1985], 'A direct method for the computation of Hopf bifurcation points', *SIAM J. Appl. Math.* **45**, 897–894.

Roschin, N. [1973], 'On dynamics of an optic quantum generator with controllable resonator', *Izv. Vuzov, Radiofizika.* In Russian.

Roschin, N. [1978], 'Dangerous stability boundaries in the Lorenz model', *Prikl. Mat. Mekh.* **42**, 950–952. In Russian.

Rössler, O. [1979], Continuous chaos—four prototype equations, *in* O. Gurel & O. Rossler, eds, 'Bifurcation theory and applications in scientific disciplines (Papers, Conf., New York, 1977)', Vol. 316 of *Ann. New York Acad. Sci.*, New York Acad. Sci., New York, pp. 376–392.

Ruelle, D. [1973], 'Bifurcation in the presence of a symmetry group', *Arch. Rational Mech. Anal.* **51**, 136–152.

Ruelle, D. & Takens, F. [1971], 'On the nature of turbulence', *Comm. Math. Phys.* **20**, 167–192.

Sacker, R. [1964], On invariant surfaces and bifurcation of periodic solutions of ordinary differential equations, IMM-NUY 333, New York State University.

Sacker, R. [1965], 'A new approach to the perturbation theory of invariant surfaces', *Comm. Pure Appl. Math.* **18**, 717–732.

Salam, F. & Bai, S. [1988], 'Complicated dynamics of a prototype continuous-time adaptive control system', *IEEE Trans. Circuits and Systems* **35**, 842–849.

Sanders, J. [1982], 'Melnikov's method and averaging', *Celestial Mech.* **28**, 171–181.

Sanders, J. [1994], Versal normal form computation and representation theory, *in* E. Tournier, ed., 'Computer Algebra and Differential Equations', Cambrige University Press, Cambrige, pp. 185–210.

Sandstede, B. [1993], Verzweigungstheorie homokliner Verdopplungen, PhD thesis, Institut für Angewandte Analysis und Stochastik, Berlin, Germany.

Sandstede, B. [1995], Center manifolds for homoclinic solutions, Institut für Angewandte Analysis und Stochastik, Berlin, Germany. Preprint No. 186.

Sandstede, B. [1997*a*], 'Constructing dynamical systems having homoclinic bifurcation points of codimension two', *J. Dynamics Differential Equations* **9**, 269–288.

Sandstede, B. [1997*b*], 'Convergence estimates for the numerical approximation of homoclinic solutions', *IMA J. Numer. Anal.* **17**, 437–462.

Sandstede, B., Scheel, A. & Wulff, C. [1997], 'Dynamics of spiral waves on unbounded domains using center-manifold reductions', *J. Differential Equations* **141**, 122–149.

Schecter, S. [1993], 'Numerical computation of saddle-node homoclinic bifurcation points', *SIAM J. Numer. Anal.* **30**, 1155–1178.

Schuko, S. [1973], 'Computer implementation of an algorithm for distinguishing center from focus', *Trudy Gorkii Inst. Inzh. Vodn. Transp.* In Russian.

Serebryakova, N. [1959], 'On the behavior of dynamical systems with one degree of freedom near that point of the stability boundary, where soft bifurcation turns into sharp', *Izv. Akad. Nauk SSSR.* In Russian.

Seydel, R. [1988], *From Equilibrium to Chaos. Practical Bifurcation and Stability Analysis*, Elsevier, New York.

Seydel, R. [1991], 'Tutorial on continuation', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **1**, 3–11.

Shapiro, A. [1974], Mathematical models of competition, *in* 'Control and Information, Vol. 10', DVNC AN SSSR, Vladivostok, pp. 5–75. In Russian.

Shil'nikov, A., Nicolis, G. & Nicolis, C. [1995], 'Bifurcation and predictability analysis of a low-order atmospheric circulation model', *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **5**, 1701–1711.

Shil'nikov, L. [1963], 'Some cases of generation of periodic motions from singular trajectories', *Mat. Sbornik* **61(103)**, 443–466. In Russian.

Shil'nikov, L. [1965], 'A case of existence of a countable number of periodic motions', *Soviet Math. Dokl.* **6**, 163–166.

Shil'nikov, L. [1966], 'On the generation of a periodic motion from a trajectory which leaves and re-enters a saddle-saddle state of equilibrium', *Soviet Math. Dokl.* **7**, 1155–1158.

Shil'nikov, L. [1967*a*], 'The existence of a denumerable set of periodic motions in four-dimensional space in an extended neighborhood of a saddle-focus', *Soviet Math. Dokl.* **8**, 54–58.

Shil'nikov, L. [1967*b*], 'On a Poincaré–Birkhoff problem', *Math. USSR-Sb.* **3**, 353–371.

Shil'nikov, L. [1968], 'On the generation of periodic motion from trajectories doubly asymptotic to an equilibrium state of saddle type', *Math. USSR-Sb.* **6**, 427–437.

Shil'nikov, L. [1969], 'On a new type of bifurcation of multidimensional dynamical systems', *Sov. Math. Dokl.* **10**, 1368–1371.

Shil'nikov, L. [1970], 'A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type', *Math. USSR-Sb.* **10**, 91–102.

Shoshitaishvili, A. [1975], Bifurcations of topological type of a vector field near a singular point, *in* 'Proceedings of Petrovskii Seminar, Vol. 1', Moscow State University, pp. 279–309. In Russian.

Smale, S. [1961], 'On gradient dynamical systems', *Ann. of Math.* **74**, 199–206.

Smale, S. [1963], Diffeomorphisms with many periodic points, *in* S. Carins, ed., 'Differential and Combinatorial Topology', Princeton University Press, Princeton, NJ, pp. 63–80.

Smale, S. [1966], 'Structurally stable systems are not dense', *Amer. J. Math.* **88**, 491–496.

Smale, S. [1967], 'Differentiable dynamical systems', *Bull. Amer. Math. Soc.* **73**, 747–817.

Stéphanos, C. [1900], 'Sur une extension du calcul des substitutions linéares', *J. Math. Pures Appl.* **6**, 73–128.

Sternberg, S. [1957], 'Local contractions and a theorem of Poincaré', *Amer. J. Math.* **79**, 809–824.

Szmolyan, P. [1991], 'Transversal heteroclinic and homoclinic orbits in singular perturbed problems', *J. Differential Equations* **92**, 252–281.

Takens, F. [1973], 'Unfoldings of certain singularities of vector fields: generalized Hopf bifurcations', *J. Differential Equations.* **14**, 476–493.

Takens, F. [1974*a*], 'Forced oscillations and bifurcations', *Comm. Math. Inst., Rijkuniversiteit Utrecht* **2**, 1–111.

Takens, F. [1974*b*], 'Singularities of vector fields', *Inst. Hautes Études Sci. Publ. Math.* **43**, 47–100.

Taylor, M. & Kevrekidis, I. [1990], Interactive AUTO: A graphical interface for AUTO86, Department of Chemical Engineering, Princeton University.

Taylor, M. & Kevrekidis, I. [1991], 'Some common dynamic features of coupled reacting systems', *Physica D* **51**, 274–292.

Thom, R. [1972], *Stabilité Structurelle et Morphogénèse*, Benjamin, New York.

Tournier, E., ed. [1994], *Computer Algebra and Differential Equations*, London Mathematical Society Lecture Note Series, Vol. 193, Cambrige University Press, Cambrige.

Tresser, C. [1984], 'About some theorems by L.P.Sil'nikov', *Ann. Inst. H. Poincaré* **40**, 441–461.

Turaev, D. & Shil'nikov, L. [1995], 'Blue Sky Catastrophes', *Dokl. Math.* **51**, 404–407.

Vaĭnberg, M. M. & Trenogin, V. A. [1974], *Theory of branching of solutions of non-linear equations*, Noordhoff International Publishing, Leyden. Israel Program for Scientific Translations.

van Gils, S. [1982], On a formula for the direction of Hopf bifurcation, Centre for Mathematics and Computer Science, Report TW/225.

van Gils, S. [1985], 'A note on "Abelian integrals and bifurcation theory"', *J. Differential Equations* **59**, 437–441.

van Gils, S. & Mallet-Paret, J. [1986], 'Hopf bifurcation and symmetry: Travelling and standing waves on the circle', *Proc. Roy. Soc. Edinburgh Sect. A* **104**, 279–307.

van Strien, S. [1979], 'Center manifolds are not C^∞', *Math. Z.* **166**, 143–145.

van Strien, S. [1991], Interval dynamics, *in* E. van Groesen & E. de Jager, eds, 'Structures in Dynamics. Studies in Math. Phys., Vol. 2', North-Holland, Amsterdam, pp. 111–160.

Vance, W. & Ross, J. [1991], 'Bifurcation structures of periodically forced oscillators', *Chaos* **1**, 445–453.

Vanderbauwhede, A. [1989], 'Centre manifolds, normal forms and elementary bifurcations', *Dynamics Reported* **2**, 89–169.

Volterra, V. [1931], *Lecònse sur la Théorie Mathematique de la Lutte pur la Via*, Gauthier-Villars, Paris.

Wan, Y.-H. [1978*a*], 'Bifurcation into invariant tori at points of resonance', *Arch. Rational Mech. Anal.* **68**, 343–357.

Wan, Y.-H. [1978*b*], 'Computations of the stability condition for the Hopf bifurcation of diffeomorphisms on $\mathbf{R}^2$', *SIAM J. Appl. Math.* **34**, 167–175.

Wang, D. [1993], A recursive formula and its application to computations of normal forms and focal values, *in* S.-T. Liao, T.-R. Ding & Y.-Q. Ye, eds, 'Nankai Series in Pure. Appl. Math. and Theor. Phys., Vol. 4, Dynamical Systems, Sep. 90 – June 91', World Scientific, Singapore, pp. 238–247.

Werner, B. [1996], 'Computation of Hopf bifurcations with bordered matrices', *SIAM J. Numer. Anal.* **33**, 435–455.

Whitley, D. [1983], 'Discrete dynamical systems in dimensions one and two', *Bull. London Math. Soc.* **15**, 177–217.

Wiggins, S. [1988], *Global Bifurcations and Chaos*, Springer-Verlag, New York.

Wiggins, S. [1990], *Introduction to Applied Non-linear Dynamical Systems and Chaos*, Springer-Verlag, New York.

Wolfram, S. [1991], *Mathematica. A System for Doing Mathematics by Computer*, Addison-Wesley, Redwood City, CA.

Yanagida, E. [1987], 'Branching of double pulse solutions from single pulse solutions in nerve axon equations', *J. Differential Equations* **66**, 243–262.

Zegeling, A. [1993], 'Equivariant unfoldings in the case of symmetry of order 4', *Serdica* **19**, 71–79.

Żołądek, H. [1984], 'On the versality of a family of symmetric vector-fields in the plane', *Math. USSR-Sb.* **48**, 463–498.

Żołądek, H. [1987], 'Bifurcations of certain family of planar vector fields tangent to axes', *J. Differential Equations* **67**, 1–55.

Index

Applied Mathematical Sciences

(continued from page ii)

61. *Sattinger/Weaver:* Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics.
62. *LaSalle:* The Stability and Control of Discrete Processes.
63. *Grasman:* Asymptotic Methods of Relaxation Oscillations and Applications.
64. *Hsu:* Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems.
65. *Rand/Armbruster:* Perturbation Methods, Bifurcation Theory and Computer Algebra.
66. *Hlavácek/Haslinger/Necasl/Lovísek:* Solution of Variational Inequalities in Mechanics.
67. *Cercignani:* The Boltzmann Equation and Its Applications.
68. *Temam:* Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed.
69. *Golubitsky/Stewart/Schaeffer:* Singularities and Groups in Bifurcation Theory, Vol. II.
70. *Constantin/Foias/Nicolaenko/Temam:* Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations.
71. *Catlin:* Estimation, Control, and the Discrete Kalman Filter.
72. *Lochak/Meunier:* Multiphase Averaging for Classical Systems.
73. *Wiggins:* Global Bifurcations and Chaos.
74. *Mawhin/Willem:* Critical Point Theory and Hamiltonian Systems.
75. *Abraham/Marsden/Ratiu:* Manifolds, Tensor Analysis, and Applications, 2nd ed.
76. *Lagerstrom:* Matched Asymptotic Expansions: Ideas and Techniques.
77. *Aldous:* Probability Approximations via the Poisson Clumping Heuristic.
78. *Dacorogna:* Direct Methods in the Calculus of Variations.
79. *Hernández-Lerma:* Adaptive Markov Processes.
80. *Lawden:* Elliptic Functions and Applications.
81. *Bluman/Kumei:* Symmetries and Differential Equations.
82. *Kress:* Linear Integral Equations, 2nd ed.
83. *Bebernes/Eberly:* Mathematical Problems from Combustion Theory.
84. *Joseph:* Fluid Dynamics of Viscoelastic Fluids.
85. *Yang:* Wave Packets and Their Bifurcations in Geophysical Fluid Dynamics.
86. *Dendrinos/Sonis:* Chaos and Socio-Spatial Dynamics.
87. *Weder:* Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media.
88. *Bogaevski/Povzner:* Algebraic Methods in Nonlinear Perturbation Theory.
89. *O'Malley:* Singular Perturbation Methods for Ordinary Differential Equations.
90. *Meyer/Hall:* Introduction to Hamiltonian Dynamical Systems and the N-body Problem.
91. *Straughan:* The Energy Method, Stability, and Nonlinear Convection.
92. *Naber:* The Geometry of Minkowski Spacetime.
93. *Colton/Kress:* Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed.
94. *Hoppensteadt:* Analysis and Simulation of Chaotic Systems, 2nd ed.
95. *Hackbusch:* Iterative Solution of Large Sparse Systems of Equations.
96. *Marchioro/Pulvirenti:* Mathematical Theory of Incompressible Nonviscous Fluids.
97. *Lasota/Mackey:* Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, 2nd ed.
98. *de Boor/Höllig/Riemenschneider:* Box Splines.
99. *Hale/Lunel:* Introduction to Functional Differential Equations.
100. *Sirovich (ed):* Trends and Perspectives in Applied Mathematics.
101. *Nusse/Yorke:* Dynamics: Numerical Explorations, 2nd ed.
102. *Chossat/Iooss:* The Couette-Taylor Problem.
103. *Chorin:* Vorticity and Turbulence.
104. *Farkas:* Periodic Motions.
105. *Wiggins:* Normally Hyperbolic Invariant Manifolds in Dynamical Systems.
106. *Cercignani/Illner/Pulvirenti:* The Mathematical Theory of Dilute Gases.
107. *Antman:* Nonlinear Problems of Elasticity.
108. *Zeidler:* Applied Functional Analysis: Applications to Mathematical Physics.
109. *Zeidler:* Applied Functional Analysis: Main Principles and Their Applications.
110. *Diekmann/van Gils/Verduyn Lunel/Walther:* Delay Equations: Functional-, Complex-, and Nonlinear Analysis.
111. *Visintin:* Differential Models of Hysteresis.
112. *Kuznetsov:* Elements of Applied Bifurcation Theory, 2nd ed.
113. *Hislop/Sigal:* Introduction to Spectral Theory: With Applications to Schrödinger Operators.
114. *Kevorkian/Cole:* Multiple Scale and Singular Perturbation Methods.
115. *Taylor:* Partial Differential Equations I, Basic Theory.
116. *Taylor:* Partial Differential Equations II, Qualitative Studies of Linear Equations.
117. *Taylor:* Partial Differential Equations III, Nonlinear Equations.

(continued on next page)

Applied Mathematical Sciences

(continued from previous page)

118. *Godlewski/Raviart:* Numerical Approximation of Hyperbolic Systems of Conservation Laws.
119. *Wu:* Theory and Applications of Partial Functional Differential Equations.
120. *Kirsch:* An Introduction to the Mathematical Theory of Inverse Problems.
121. *Brokate/Sprekels:* Hysteresis and Phase Transitions.
122. *Gliklikh:* Global Analysis in Mathematical Physics: Geometric and Stochastic Methods.
123. *Le/Schmitt:* Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems.
124. *Polak:* Optimization: Algorithms and Consistent Approximations.
125. *Arnold/Khesin:* Topological Methods in Hydrodynamics.
126. *Hoppensteadt/Izhikevich:* Weakly Connected Neural Networks.
127. *Isakov:* Inverse Problems for Partial Differential Equations.
128. *Li/Wiggins:* Invariant Manifolds and Fibrations for Perturbed Nonlinear Schrödinger Equations.
129. *Müller:* Analysis of Spherical Symmetries in Euclidean Spaces.
130. *Feintuch:* Robust Control Theory in Hilbert Space.
131. *Ericksen:* Introduction to the Thermodynamics of Solids, Revised ed.
132. *Ihlenburg:* Finite Element Analysis of Acoustic Scattering.
133. *Vorovich:* Nonlinear Theory of Shallow Shells.
134. *Vein/Dale:* Determinants and Their Applications in Mathematical Physics.
135. *Drew/Passman:* Theory of Multicomponent Fluids.
136. *Cioranescu/Saint Jean Paulin:* Homogenization of Reticulated Structures.
137. *Gurtin:* Configurational Forces as Basic Concepts of Continuum Physics.
138. *Haller:* Chaos Near Resonance.
139. *Sulem/Sulem:* The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse.
140. *Cherkaev:* Variational Methods for Structural Optimization.
141. *Naber:* Topology, Geometry, and Gauge Fields: Interactions.
142. *Schmid/Henningson:* Stability and Transition in Shear Flows.
143. *Sell/You:* Dynamics of Evolutionary Equations.
144. *Nédélec:* Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems.
145. *Newton:* The *N*-Vortex Problem: Analytical Techniques.
146. *Allaire*: Shape Optimization by the Homogenization Method.