Appendix A

The Application of the Expansion Formulas of the Second Essay to Some Physical Problems

A.1 The Shape of the Surface of a Spinning Beaker

Our first problem in this appendix is one whose solution can be obtained directly. We turn it into a perturbation problem only for the reader's benefit. Indeed, it may be a little too simple even for this.

Let a cylindrical beaker of radius R be filled with a liquid to a depth d and then be set into steady rotation about its axis of symmetry at an angular speed Ω. Our job is to determine the domain occupied by the liquid as a function of the angular speed at which the beaker turns, assuming that the liquid attains a state of rigid body rotation.

To do this, the position of the free surface of the liquid needs to be determined, but the cylindrical wall and the bottom of the beaker, not being displaced, do not come into the calculation.

Let the origin, O, lie on the axis of symmetry where it crosses the bottom of the beaker and let the axis Oz point upward. Figure A.1 indicates the important notation.

Cylindrical coordinates, r, θ and z will be used to identify the points of the liquid in its present configuration and its free surface will be denoted

$$z = Z(r,\ \Omega^2)$$

where the square of the angular speed will be used as the expansion variable in a perturbation calculation (viz., $\epsilon = \Omega^2$).

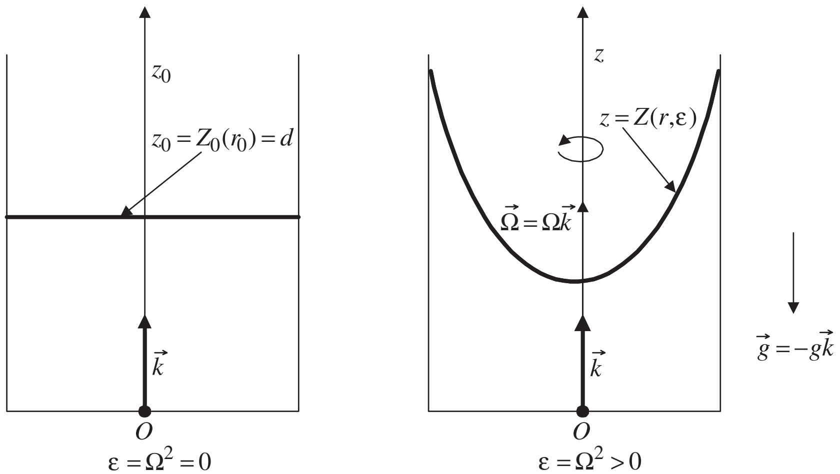

Figure A.1. *Some Notation*

To solve this problem, we must find the pressure throughout the liquid and the domain occupied by the liquid. The pressure must satisfy

$$\rho\vec{\Omega} \times \vec{\Omega} \times \vec{r} = -\nabla p + \rho\vec{g} \tag{A.1}$$

on the domain; at the free surface, it must satisfy

$$p = p_{ambient} \tag{A.2}$$

where $p_{ambient}$ is specified and held constant.

There is the additional demand that the volume of the liquid remain at its zero spin value for all $\Omega^2 > 0$, and this requires $Z(r,\ \Omega^2)$ to satisfy

$$\int_0^R 2\pi r Z(r,\ \Omega^2)dr = \pi R^2 d \tag{A.3}$$

Now, equation (A.1) can be written

$$\left.\begin{aligned} \frac{\partial p}{\partial r} &= \rho\Omega^2 r \\ \frac{\partial p}{\partial z} &= -\rho g \end{aligned}\right\} \quad 0 < r < R, \quad 0 < z < Z(r,\ \Omega^2)$$

and, before going on, observe that its solution is

$$p = \frac{1}{2}\rho\Omega^2 r^2 - \rho g z + C$$

where C remains to be determined. The requirement that the free surface be at constant pressure then determines its shape via

$$Z(r,\ \Omega^2) = \frac{1}{2}\frac{\Omega^2}{g}r^2 + D$$

where $D = \dfrac{C - p_{ambient}}{\rho g}$ and where, to satisfy the constant-volume requirement, D must be

$$d - \frac{1}{4}\frac{\Omega^2}{g}R^2$$

Now, return to the problem as it was originally stated and view it as a family of problems for increasing values of Ω^2. The simplest problem corresponds to $\Omega^2 = 0$ and this defines the reference problem. Its solution determines the reference domain. In terms of reference domain variables, this problem is

$$\left.\begin{aligned} \frac{\partial p_0}{\partial r_0} &= 0 \\ \frac{\partial p_0}{\partial z_0} &= -\rho g \end{aligned}\right\} \qquad 0 < r_0 < R, \quad 0 < z_0 < Z_0(r_0)$$

where at the free surface, $z_0 = Z_0(r_0)$, $0 < r_0 < R$, p_0 must satisfy

$$p_0 = p_{ambient}$$

and where, to maintain constant volume, Z_0 must satisfy

$$\int_0^R 2\pi r_0 Z_0(r_0)\ dr_0 = \pi R^2 d$$

The solution to this problem is

$$p_0 = -\rho g[z_0 - d] + p_{ambient}$$

and

$$Z_0(r_0) = d$$

whereupon the reference domain is the set of points (r_0, z_0) defined by $0 < r_0 < R$ and $0 < z_0 < d$.

To derive the equations holding on the reference domain and at its boundary, to orders zero, one, two, etc. in ϵ, introduce the mapping

$$r = r_0$$

and

$$z = z_0 + \epsilon z_1(r_0,\ z_0) + \frac{1}{2}\epsilon^2 z_2(r_0,\ z_0) + \ \cdots$$

which ties the present domain to the reference domain. Observe that at the boundary this is written

$$r = r_0$$

and

$$z = Z(r,\ \epsilon) = Z_0 + \epsilon Z_1(r_0) + \frac{1}{2}\epsilon^2 Z_2(r_0) + \ \cdots$$

Now, introduce the expansions of p, $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial z}$ along the mapping, namely

$$p(r,\ z,\ \epsilon) = p_0 + \epsilon\left[p_1 + z_1 \frac{\partial p_0}{\partial z_0}\right]$$

$$+\frac{1}{2}\epsilon^2\left[p_2 + 2z_1\frac{\partial p_1}{\partial z_0} + z_1^2\frac{\partial^2 p_0}{\partial z_0^2} + z_2\frac{\partial p_0}{\partial z_0}\right] + \ \cdots$$

$$\frac{\partial p}{\partial r}(r,\ z,\ \epsilon) = \frac{\partial p_0}{\partial r_0} + \epsilon\left[\frac{\partial p_1}{\partial r_0} + z_1\frac{\partial^2 p_0}{\partial r_0 \partial z_0}\right]$$

$$+\frac{1}{2}\epsilon^2\left[\frac{\partial p_2}{\partial r_0} + 2z_1\frac{\partial^2 p_1}{\partial r_0\partial z_0} + z_1^2\frac{\partial^3 p_0}{\partial r_0 \partial z_0^2} + z_2\frac{\partial^2 p_0}{\partial r_0\partial z_0}\right] + \cdots$$

and

$$\frac{\partial p}{\partial z}(r,\ z,\ \epsilon) = \frac{\partial p_0}{\partial z_0} + \epsilon\left[\frac{\partial p_1}{\partial z_0} + z_1\frac{\partial^2 p_0}{\partial z_0^2}\right]$$

$$+\frac{1}{2}\epsilon^2\left[\frac{\partial p_2}{\partial z_0} + 2z_1\frac{\partial^2 p_1}{\partial z_0^2} + z_1^2\frac{\partial^3 p_0}{\partial z_0^3} + z_2\frac{\partial^2 p_0}{\partial z_0^2}\right] + \ \cdots$$

and notice that all the variables on the right-hand side depend at most on r_0 and z_0. Then, substitute these expansions into equations (A.1) and (A.2) to get the problems satisfied by p_0 at zeroth order, p_1 at first order, etc.

At zeroth order, there obtains

$$\left.\begin{aligned} \frac{\partial p_0}{\partial r_0} &= 0 \\ \frac{\partial p_0}{\partial z_0} &= -\rho g \end{aligned}\right\} \quad 0 < r_0 < R, \quad 0 < z_0 < d \tag{A.4}$$

and

$$p_0 = p_{ambient}, \quad 0 < r_0 < R, \quad z_0 = d \tag{A.5}$$

At first order, there obtains

$$\left.\begin{aligned} \frac{\partial p_1}{\partial r_0} &= \rho r_0 \\ \frac{\partial p_1}{\partial z_0} &= 0 \end{aligned}\right\} \quad 0 < r_0 < R, \quad 0 < z_0 < d \tag{A.6}$$

and

$$p_1 + Z_1 \frac{\partial p_0}{\partial z_0} = 0, \quad 0 < r_0 < R, \quad z_0 = d \tag{A.7}$$

At second order, there obtains

$$\left.\begin{aligned} \frac{\partial p_2}{\partial r_0} &= 0 \qquad & 0 < r_0 < R \\ \frac{\partial p_2}{\partial z_0} &= 0 \qquad & 0 < z_0 < d \end{aligned}\right\} \tag{A.8}$$

and

$$p_2 + 2Z_1 \frac{\partial p_1}{\partial z_0} + Z_1^2 \frac{\partial^2 p_0}{\partial z_0^2} + Z_2 \frac{\partial p_0}{\partial z_0} = 0, \quad 0 < r_0 < R, \quad z_0 = d \tag{A.9}$$

etc.

Notice that on the reference domain, but not on its boundary, the zeroth-order equations eliminate the mapping in the first-order equations, the zeroth- and first-order equations eliminate the mapping at second order, etc.

Now, equations (A.4), (A.6), (A.8), $\cdots$ can be used to evaluate the derivatives appearing in equations (A.7), (A.9), $\cdots$ and doing this simplifies these equations to

$$\left.\begin{aligned} p_0 &= p_{ambient} \\ p_1 - Z_1 \rho g &= 0 \\ p_2 - Z_2 \rho g &= 0 \\ &\text{etc.} \end{aligned}\right\} \quad 0 < r_0 < R, \quad z_0 = d$$

where $Z_1, \ Z_2, \ \cdots$ depend on r_0.

Then, equations (A.4), (A.6), (A.8), $\cdots$ on the domain can be satisfied by writing

$$p_0 = -\rho g z_0 + C_0$$

$$p_1 = \frac{1}{2} \rho r_0^2 + C_1$$

$$p_2 = C_2$$

etc.

where the constants $C_0,\ C_1,\ C_2,\ \cdots$ along with the functions $Z_1,\ Z_2,\ \cdots$ remain to be determined by using equations (A.5), (A.7), (A.9), $\cdots$. These require

$$C_0 = p_{ambient} + \rho g d$$

$$Z_1 = \frac{1}{2g} r_0^2 + \frac{C_1}{\rho g}$$

$$Z_2 = \frac{C_2}{\rho g}$$

etc.

and it remains only to satisfy the requirement that the volume of the liquid remain fixed as its surface deflects.

Substituting the expansion of the surface shape, namely

$$Z(r,\ \epsilon) = Z_0(r_0) + \epsilon Z_1(r_0) + \frac{1}{2}\epsilon^2 Z_2(r_0) + \ \cdots$$

into

$$\pi R^2 d = \int_0^R 2\pi r Z(r,\ \epsilon) dr$$

and using $Z_0 = d$ produces the conditions

$$\int_0^R r_0 Z_1(r_0) dr_0 = 0$$

$$\int_0^R r_0 Z_2(r_0) dr_0 = 0$$

etc.

which can be used to determine the constants $C_1,\ C_2,\ \cdots$. The result is

$$C_1 = -\frac{1}{4}\rho R^2$$

$$C_2 = 0$$

etc.

whence

$$p_0 = -\rho g[z_0 - d] + p_{ambient}, \quad Z_0 = d$$

$$p_1 = \frac{1}{2}\rho r_0^2 - \frac{1}{4}\rho R^2, \quad Z_1 = \frac{1}{2g} r_0^2 - \frac{1}{4g} R^2$$

$$p_2 = 0, \quad Z_2 = 0$$

etc.

The reader can go on and discover that p_3 and Z_3 must satisfy

$$\left.\begin{aligned} \frac{\partial p_3}{\partial r_0} &= 0 \\ \frac{\partial p_3}{\partial z_0} &= 0 \end{aligned}\right\} \quad 0 < r_0 < R, \quad 0 < z_0 < d$$

$$p_3 - Z_3 \rho g = 0, \quad 0 < r_0 < R, \quad z_0 = d$$

and

$$\int_0^R r_0 Z_3 dr_0 = 0$$

whence $p_3 = 0 = Z_3$. This result obtains (viz., $p_i = 0 = Z_i$), for all $i = 2,\ 3, \cdots$.

The present domain can now be determined. The shape of its free surface is given by

$$Z(r,\ \Omega^2) = Z_0(r_0) + \epsilon Z_1(r_0) + \frac{1}{2}\epsilon^2 Z_2(r_0) + \cdots$$

$$= Z_0(r_0) + \Omega^2 Z_1(r_0) = d + \Omega^2 \left[\frac{r_0^2}{2g} - \frac{R^2}{4g}\right]$$

and this turns out to be the correct formula for Z, as obtained earlier.

It remains to determine the pressure throughout this domain [viz., throughout $0 < r < R$, $0 < z < Z(r,\ \Omega^2)$]. The problem we face in doing this is indicated in Figure A.2. It is this: While some of the points of the present domain are also points of the reference domain, others are not. The first set of points is shaded in the figure, and at these points, p can be estimated by carrying z straight back to z_0, as explained in the essay. By doing this, p is obtained as

$$p(r,\ z,\ \epsilon) = p_0(r_0,\ z) + \epsilon p_1(r_0,\ z) + \cdots$$

where $r = r_0$, whence at these points the value of p is found to be

$$p(r,\ z,\ \Omega^2) = p_{ambient} - \rho g[z - d] + \Omega^2 \left[\frac{1}{2}\rho r^2 - \frac{1}{4}\rho R^2\right]$$

Again, this turns out to be correct, as $p_3,\ p_4, \cdots$ all vanish. It also turns out to be correct if it is used at the unshaded points, but this is only because it is correct on the shaded points. Indeed, if the correct result can

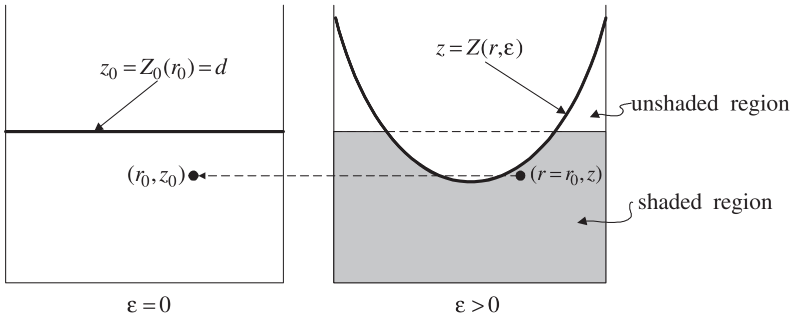

Figure A.2. *Points That Can Be Carried Straight Back to the Reference Domain and Points That Cannot*

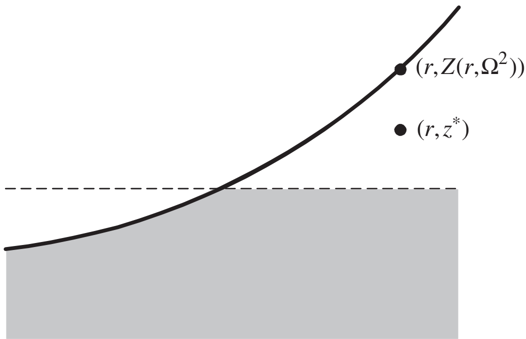

Figure A.3. *A Point Near the Boundary That Cannot Be Carried Straight Back to the Reference Domain*

be obtained on a part of a domain, it extends to the entire domain by analytic continuation. The weakness of this example is that it is so simple that its solution, rather than an estimate of its solution, is obtained after just a few terms in the expansion.

Nonetheless, the values of p at points of the unshaded region can be estimated, as they would have been if this lucky accident had not occurred, by observing that p and all its derivatives can be evaluated at points of the boundary of the present domain and by observing that these points must be neighbors of the points in the unshaded region. This is illustrated in Figure A.3.

To determine the value of p at the point $(r,\ z^*)$, substitute into

$$p(r,\ z^*,\ \epsilon) = p(r, Z, \epsilon) + \frac{\partial p}{\partial z}(r,\ Z,\ \epsilon)[z^* - Z]$$

$$+\frac{1}{2}\frac{\partial^2 p}{\partial z^2}(r,\ Z,\ \epsilon)[z^* - Z]^2 + \ \cdots$$

p and its derivatives determined according to

$$p(r,\ Z,\ \epsilon) = p_0(r_0, Z_0) + \epsilon\left[p_1(r_0,\ Z_0) + Z_1(r_0)\frac{\partial p_0}{\partial z_0}(r_0,\ Z_0)\right] + \cdots$$

$$\frac{\partial p}{\partial z}(r,\ Z,\ \epsilon) = \frac{\partial p_0}{\partial z_0}(r_0, Z_0) + \epsilon\left[\frac{\partial p_1}{\partial z_0}(r_0,\ Z_0) + Z_1(r_0)\frac{\partial^2 p_0}{\partial z_0^2}(r_0,\ Z_0)\right] + \cdots$$

etc.

Doing this and using equations (A.4), (A.5), (A.6) and (A.7) to write

$$p_0 = p_{ambient}$$

$$\frac{\partial p_0}{\partial z_0} = -\rho g$$

$$p_1 + Z_1\frac{\partial p_0}{\partial z_0} = 0$$

and

$$\frac{\partial p_1}{\partial z_0} + Z_1\frac{\partial^2 p_0}{\partial z_0^2} = 0$$

at points $(r_0,\ Z_0)$ along the boundary of the reference domain, produces

$$p(r,\ z^*,\ \Omega^2) = p_{ambient} - \rho g[z^* - Z]$$

where $r = r_0$ and where $Z(r,\ \Omega^2) = d + \Omega^2\left[\frac{r_0^2}{2g} - \frac{R^2}{4g}\right]$. Again, this is the correct result.

It might be worthwhile to explain a little more about this last calculation. It requires the use of several series. Going back to the notation in the second essay, the problem is to calculate u at a point $(x,\ y^*)$ lying near a boundary point $(x,\ Y(x,\ \epsilon))$. To do this, the series

$$u(x,\ y^*,\ \epsilon) = u(x,\ Y,\ \epsilon) + \frac{\partial u}{\partial y}(x,\ Y,\ \epsilon)[y^* - Y]$$

$$+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}(x,\ Y,\ \epsilon)[y^* - Y]^2 + \ \cdots \tag{A.10}$$

must be used, into which the several series, namely

$$Y = Y_0 + \epsilon Y_1 + \frac{1}{2}\epsilon^2 Y_2 + \ \cdots \tag{A.11}$$

$$u(x,\ Y,\ \epsilon) = u_0 + \epsilon\left[u_1 + Y_1\frac{\partial u_0}{\partial y_0}\right] + \cdots \tag{A.12}$$

$$\frac{\partial u}{\partial y}(x,\ Y,\ \epsilon) = \frac{\partial u_0}{\partial y_0} + \epsilon\left[\frac{\partial u_1}{\partial y_0} + Y_1\frac{\partial^2 u_0}{\partial y_0^2}\right] + \cdots \tag{A.13}$$

etc.

must be substituted, where all the variables on the right-hand sides of these series are evaluated at x_0 and $y_0 = Y_0(x_0)$, and where, on the left-hand side, $x = x_0$.

Then, to obtain $u(x,\ y^*,\ \epsilon)$ to a certain order in ϵ, all of the terms in the first series are required, each to the assigned order in ϵ. To illustrate this, let u be required to first order in ϵ; then, substituting equations (A.11), (A.12), (A.13), $\cdots$ into equation (A.10), there obtains

$$u(x,\ y^*,\ \epsilon) = \left[u_0 + \frac{\partial u_0}{\partial y_0}[y^* - Y_0] + \frac{1}{2}\frac{\partial^2 u_0}{\partial y_0^2}[y^* - Y_0]^2 + \cdots\right]$$

$$+\epsilon\left[u_1 + \frac{\partial u_1}{\partial y_0}[y^* - Y_0] + \frac{1}{2}\frac{\partial^2 u_1}{\partial y_0^2}[y^* - Y_0]^2 + \cdots\right]$$

Hence, to get $u(x,\ y^*,\ \epsilon)$ to order ϵ requires only that the sums of these two series be estimated with sufficient accuracy. Presumably, y^* lies close enough to Y_0 that this does not present a new difficulty. Indeed, if $y^* - Y_0$ is of order ϵ, then $u(x,\ y^*,\ \epsilon)$ can be estimated to order ϵ by

$$u_0 + \frac{\partial u_0}{\partial y_0}[y^* - Y] + \epsilon u_1$$

A.2 The Position of a Melting Front

Turning now to a second example, which is physically and mathematically more interesting, we try to determine the configuration of a heat conducting body where the shape of the body depends upon the rate at which it is losing heat. To define the problem, let ice at its melting point occupy the half-space $x > 0$. Then, introduce heat across the wall bounding the ice at $x = 0$. As a result of doing this, the ice originally lying between $x = 0$ and $x = d$ is melted and the temperature of the water formed is raised above its freezing point. At time $t = 0$, the heating is stopped and the wall at $x = 0$ is thereafter insulated. The system is turned over to us at $t = 0$. At that time, it is made up of a layer of water occupying the region $0 < x < d$, exhibiting a positive distribution of temperature, and the unmelted ice, at zero temperature, occupying the region $x > d$. As time begins to run, the water begins to cool, and in so doing, it begins to melt the ice. Our job is to determine the position of the water–ice interface, as well as the temperature of the water, as they depend on time.

The temperature of the ice remains always at zero and so it is only the water that is of interest. To take the simplest possible case, the water is bounded by an insulated plane wall on the left and by the water–ice coexistence plane on the right. The heat conduction is one dimensional.

At any time, t, let u denote the temperature, as it depends on position, at the points presently occupied by water and let X denote the position of the water–ice interface. The speed at which this interface moves is then given by $\frac{dX}{dt}$.

Let $f(x)$, $0 < x < d$, indicate the way in which the temperature is initially distributed throughout the water. Then, express the initial condition of the water by

$$u(t = 0) = \epsilon f(x), \quad 0 < x < d$$

where ϵ is a measure of the amount of heat, over that just required to melt the ice, that the water receives while the initial state is being created. This is the amount of sensible heat on the water side that is available to melt the ice once the system is allowed to seek its equilibrium configuration.

By introducing ϵ in this way, a family of problems is created and the family member corresponding to $\epsilon = 0$ is especially simple. Each member of the family is defined by a specific value of ϵ. Its domain the current domain, current in terms of ϵ, not t, is not the spatial domain occupied by the water at some time t but the entire history of these spatial domains corresponding to a specified value of ϵ. The current domain is then defined by three boundaries:

(i) the insulated wall at $x = 0$ for all $t \geq 0$
(ii) the initial spatial domain of the water at $t = 0$ for all x, $0 < x < d$
(iii) the water–ice coexistence plane at $x = X(t, \epsilon)$ for all $t \geq 0$, where the function $X(t, \epsilon)$ determining this part of the boundary of the current domain is part of the solution of the problem

This is illustrated in Figure A.4.

Then, u and X must be determined. On the current domain and on its boundary, $u(t, x, \epsilon)$ and $X(t, \epsilon)$ must satisfy[1]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < X(t, \epsilon)$$

[1] It may be observed that as long as the right-hand phase is maintained at the phase change temperature, the fact that the density of the two phases differ is of no account. The flow produced by this takes place in the right-hand phase, but it is at uniform temperature.

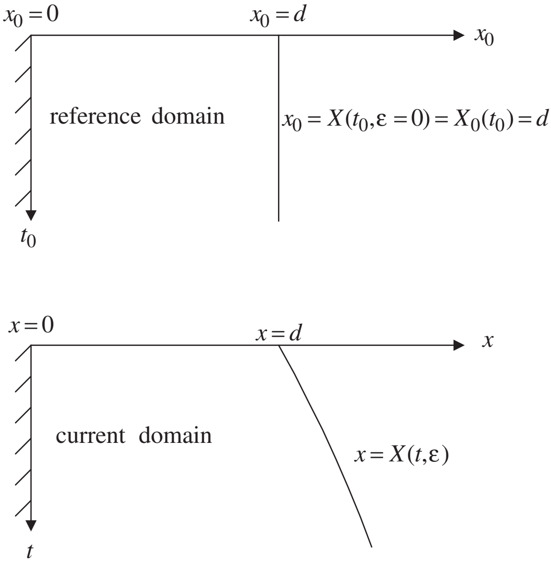

Figure A.4. *The Current and the Reference Domains*

$$\frac{\partial u}{\partial x}=0,\quad t>0,\quad x=0$$

$$\left.\begin{aligned} u &= 0 \\ \frac{\partial u}{\partial x} &= -\frac{\partial X}{\partial t} \end{aligned}\right\} \qquad t>0,\quad x=X(t,\ \epsilon)$$

and

$$u=\epsilon f,\quad t=0,\quad 0<x<d$$

where $X(t=0,\ \epsilon)=d$.

This is the nonlinear problem and its solution at $\epsilon=0$ defines the reference domain. In terms of reference domain variables, $u_0,\ X_0,\ t_0$ and x_0, this solution is

$$u_0=0$$

and

$$X_0=d$$

The reference domain is then the simple domain $0\le x_0\le d$, $t_0\ge 0$.

To introduce the mapping that ties the reference domain to the current domain, write

$$t=t_0$$

and

$$x = g(t_0,\ x_0,\ \epsilon) = x_0 + \epsilon x_1(t_0,\ x_0) + \frac{1}{2}\epsilon^2 x_2(t_0,\ x_0) + \ \cdots$$

and observe that all parts of the boundary of the reference domain must be carried by this mapping into the corresponding parts of the boundary of the present domain. This introduces certain requirements that the mapping must meet. At the insulated wall, the points $(t_0,\ x_0 = 0)$ must be carried into the points $(t,\ x = 0)$ whence

$$x_1(t_0,\ x_0 = 0) = 0$$

$$x_2(t_0,\ x_0 = 0) = 0$$

etc.

Then, requiring the points $(t_0 = 0,\ x_0)$ of the initial spatial domain of the reference configuration to be carried into the corresponding points $(t = 0,\ x = x_0)$ of the initial spatial domain of the present configuration leads to

$$x_1(t_0 = 0,\ x_0) = 0$$

$$x_2(t_0 = 0,\ x_0) = 0$$

etc.

Turn now to the water–ice interface. In the reference configuration, it is denoted $x_0 = X_0(t_0) = d$, while in the present configuration, it is denoted $x = X(t,\ \epsilon)$. The expansion of $X(t,\ \epsilon)$ along the mapping is written

$$X(t,\ \epsilon) = X_0(t_0) + \epsilon X_1(t_0) + \frac{1}{2}\epsilon^2 X_2(t_0) + \ \cdots$$

and $X(t = 0, \epsilon) = d = X_0(t_0 = 0)$ requires

$$X_1(t_0 = 0) = 0$$

$$X_2(t_0 = 0) = 0$$

etc.

where

$$X_1(t_0) = x_1(t_0,\ x_0 = X_0(t_0) = d)$$

$$X_2(t_0) = x_2(t_0,\ x_0 = X_0(t_0) = d)$$

etc.

To produce the equations satisfied by $u_0,\ u_1,\ u_2$, etc. on the reference domain, substitute into $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0$ the expansions

$$\frac{\partial^2 u}{\partial x^2}(t,\ x,\ \epsilon) = \frac{\partial^2 u_0}{\partial x_0^2} + \epsilon\left[\frac{\partial^2 u_1}{\partial x_0^2} + x_1 \frac{\partial^3 u_0}{\partial x_0^3}\right] + \ \cdots$$

and

$$\frac{\partial u}{\partial t}(t,\ x,\ \epsilon) = \frac{\partial u_0}{\partial t_0} + \epsilon\left[\frac{\partial u_1}{\partial t_0} + x_1 \frac{\partial^2 u_0}{\partial t_0 \partial x_0}\right] + \ \cdots$$

and set the coefficient of each power of ϵ to zero. The result is

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial t_0} - \frac{\partial^2 u_0}{\partial x_0^2} = 0 \\ &\frac{\partial u_1}{\partial t_0} - \frac{\partial^2 u_1}{\partial x_0^2} = 0 \\ &\text{etc.} \end{aligned}\right\} \quad t_0 > 0, \quad 0 < x_0 < d$$

where the first equation has been used to eliminate the mapping in the second, etc.

Likewise, the equations at the boundary of the reference domain obtain via substitution of the expansions of u and its derivatives. Along the insulated wall, substitution of

$$\frac{\partial u}{\partial x}(t,\ x,\ \epsilon) = \frac{\partial u_0}{\partial x_0} + \epsilon\left[\frac{\partial u_1}{\partial x_0} + x_1 \frac{\partial^2 u_0}{\partial x_0^2}\right] + \ \cdots$$

into

$$\frac{\partial u}{\partial x}(t,\ x = 0,\ \epsilon) = 0$$

and use of $x_1(t_0,\ x_0 = 0) = 0,\ x_2(t_0,\ x_0 = 0) = 0$, etc. leads to

$$\left.\begin{aligned} &\frac{\partial u_0}{\partial x_0} = 0 \\ &\frac{\partial u_1}{\partial x_0} = 0 \\ &\text{etc.} \end{aligned}\right\} \quad t_0 > 0, \quad x_0 = 0$$

while on the initial spatial domain, $t = 0,\ 0 < x < d$, substitution of

$$u(t,\ x,\ \epsilon) = u_0 + \epsilon\left[u_1 + x_1 \frac{\partial u_0}{\partial x_0}\right] + \ \cdots$$

into

$$u(t = 0,\ x,\ \epsilon) = \epsilon f(x)$$

and use of $x_1(t_0 = 0,\ x_0) = 0,\ x_2(t_0 = 0,\ x_0) = 0$, etc. produces

$$\left.\begin{aligned} u_0 &= 0 \\ u_1 &= f \\ &\text{etc.} \end{aligned}\right\} \qquad t_0 = 0, \quad 0 < x_0 < d$$

At the water–ice interface, two equations must be satisfied in the present configuration, one to make certain that the heat conduction problem in the water is properly posed, the other to determine the speed at which ice is turned into water. Substitution of the expansion of $u(t,\ x,\ \epsilon)$ into

$$u(t,\ X,\ \epsilon) = 0$$

and use of $x_1(t_0,\ x_0 = X_0) = X_1,\ x_2(t_0,\ x_0 = X_0) = X_2$, etc. requires

$$\left.\begin{aligned} u_0 &= 0 \\ u_1 &= -X_1\frac{\partial u_0}{\partial x_0} \\ &\text{etc.} \end{aligned}\right\} \qquad t_0 > 0, \quad x_0 = d$$

The substitution of the expansions of $\dfrac{\partial u}{\partial x}(t,\ x,\ \epsilon)$ and $X(t,\ \epsilon)$ into

$$\frac{\partial u}{\partial x}(t,\ X,\ \epsilon) = -\frac{\partial X}{\partial t}(t,\ \epsilon)$$

and, again, the use of $x_1(t_0,\ x_0 = X_0) = X_1, x_2(t_0,\ x_0 = X_0) = X_2$, etc. requires

$$\left.\begin{aligned} \frac{\partial u_0}{\partial x_0} &= -\frac{dX_0}{dt_0} \\ \frac{\partial u_1}{\partial x_0} + X_1\frac{\partial^2 u_0}{\partial x_0^2} &= -\frac{dX_1}{dt} \\ &\text{etc.} \end{aligned}\right\} \qquad t_0 > 0, \quad x_0 = d$$

The zeroth-order problem on the reference domain is then

$$\frac{\partial u_0}{\partial t_0} = \frac{\partial^2 u_0}{\partial x_0^2}, \quad t_0 > 0, \quad 0 < x_0 < X_0 = d$$

$$u_0(t_0 = 0) = 0$$

$$\frac{\partial u_0}{\partial x_0}(x_0 = 0) = 0$$

$$u_0(x_0 = d) = 0$$

and

$$\frac{\partial u_0}{\partial x_0}(x_0 = d) = -\frac{dX_0}{dt_0}$$

where

$$X_0(t_0 = 0) = d$$

Its solution is

$$u_0 = 0 \quad \text{and} \quad X_0 = d$$

Again, the zeroth-order problem is the original problem. This is always the case. The solution to the zeroth-order problem is always a solution to the original nonlinear problem.

The first-order problem begins to yield some information. It is

$$\frac{\partial u_1}{\partial t_0} = \frac{\partial^2 u_1}{\partial x_0^2}, \qquad t_0 > 0,\ 0 < x_0 < d$$

$$u_1(t_0 = 0) = f$$

$$\frac{\partial u_1}{\partial x_0}(x_0 = 0) = 0$$

$$u_1(x_0 = d) = -X_1 \frac{\partial u_0}{\partial x_0}(x_0 = d)$$

and

$$\frac{\partial u_1}{\partial x_0}(x_0 = d) = -X_1 \frac{\partial^2 u_0}{\partial x_0^2}(x_0 = d) - \frac{dX_1}{dt_0}$$

where

$$X_1(t_0 = 0) = 0$$

This problem must be solved on the reference domain (viz., $0 < x_0 < d$, $t_0 > 0$), and u_1 and X_1 must be determined together. They depend on u_0 and X_0. Then, at the next order, u_2 and X_2 must be determined together and they depend on u_1 and X_1, as well as on u_0 and X_0. This is the way the calculation proceeds and it is the main purpose of this example to make just this point. The position of the water–ice interface then becomes known as the terms in the series

$$X(t,\ \epsilon) = X_0(t_0) + \epsilon X_1(t_0) + \frac{1}{2}\epsilon^2 X_2(t_0) + \ \cdots$$

become known.

It might be worth observing that, in this particular case, u_0 is so simple that the first four equations at first order can be used to determine u_1 and then the last two can be used to determine X_1. Because u_0 and all its derivatives vanish, this uncoupling obtains at all orders. The readers can satisfy themselves that u_2 satisfies a simple heat conduction problem on the reference domain and that, once it is determined, X_2 can be obtained by an integration. The only difference between u_1 and u_2 is that u_1 is driven by a source at $t_0 = 0$, whereas u_2 is driven by a source at $x_0 = d$. Like u_2, all higher-order u's are driven by sources at $x_0 = d$ that depend on what is discovered at lower orders.

It remains to explain how it is that a point of the current domain can be evaluated in terms of $u_0,\ u_1,\ u_2,\ \cdots$ known only at points of the reference domain. But, as this ought to present little difficulty, we go on and rework this problem from another point of view.

A.3 Carrying the Melting Front Forward in Time

When this problem was first stated, it did not come equipped with a ready-made small parameter. It was simply a problem defined on a domain where the domain was at first not known. We, ourselves, introduced the parameter ϵ; it lies wholly outside the original statement of the problem. By doing this, we traded a problem on an unknown domain for many problems on a known and simple domain. In many ways, this is a fair trade, but it raises the question: Just what is ϵ? There is no one answer to this question. Sometimes, the problem itself determines what ϵ must be. Other times, we invent ϵ.

Even in a definite problem, many ways of introducing the expansion parameter will ordinarily present themselves. The definition ought to depend on what we are trying to discover. To illustrate this, we rework the foregoing problem, taking ϵ to be a difference in time.

To do this, let the solution be available to us at some reference time, denoted t_0. Then, our job is simply to advance it to a later time t. Looking at it in this way, the reference domain becomes the spatial domain of the water at the time t_0, described as $0 \le x_0 \le X_0 = X(t_0)$, where X_0 is assigned. The current domain becomes the spatial domain of the water at the later time t of interest, described as $0 \le x \le X(t)$, where $X(t)$ must be determined in the course of turning $u(x_0,\ t_0)$ on the reference domain into $u(x,\ t)$ on the present domain. The expansion parameter ϵ is now $t - t_0$ and the corresponding family of problems is now just what the problem itself was as originally stated. All this is illustrated in Figure A.5.

Our goal, now, is not, as it was before, to determine the reference domain and then to determine the equations satisfied by a sequence of functions

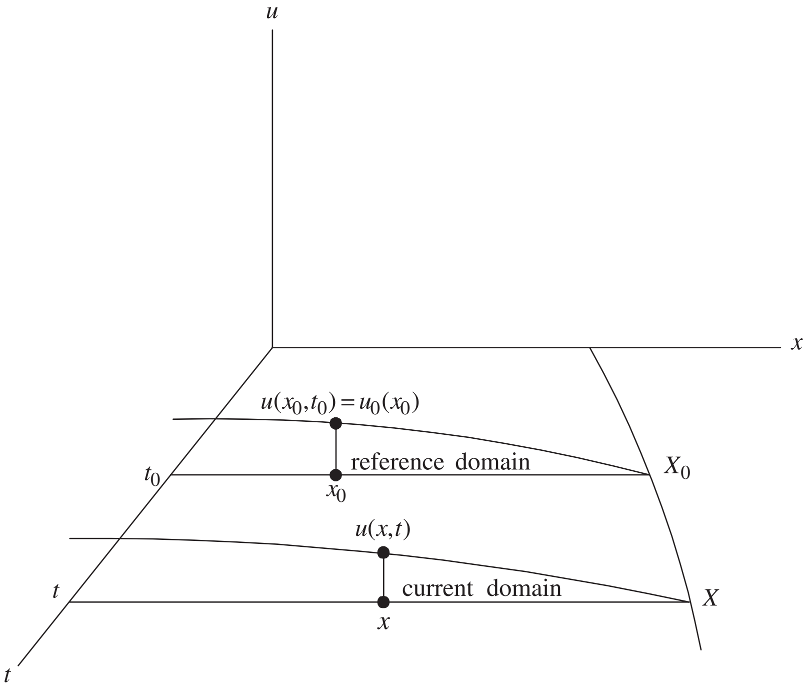

Figure A.5. *The Current and the Reference Domain If the Melting Front Is Being Carried Forward in Time*

$u_0,\ u_1,\ u_2$, etc. on the reference domain. Instead, it is to use u_0, which is now assigned, to determine $u_1,\ u_2$, etc., as well as, $X_1,\ X_2$, etc., and thereby to advance the solution in time from the reference domain to the present domain.

To do this, we introduce a mapping of the reference domain into the present domain and then introduce the expansions of u and its derivatives along the mapping.

The mapping can be written

$$x = f(x_0,\ \epsilon) = x_0 + \epsilon x_1(x_0) + \frac{1}{2}\epsilon^2 x_2(x_0) + \ \cdots$$

where at the water–ice interface

$$x = X(\epsilon) = X_0 + \epsilon X_1 + \frac{1}{2}\epsilon^2 X_2 + \ \cdots$$

whence

$$x_1(x_0 = X_0) = X_1$$

$$x_2(x_0 = X_0) = X_2$$

etc.

while at the insulated wall

$$x = 0$$

whence

$$x_1(x_0 = 0) = 0$$

$$x_2(x_0 = 0) = 0$$

etc.

Again, denote by $u_0(x_0)$ the assigned values of $u(x_0,\ t_0)$ on the reference domain $0 \leq x_0 \leq X_0$. The goal, then, is to determine the values of $u(x,\ t)$, denoted by $u(x,\ \epsilon)$, on the present domain $0 \leq x \leq X(\epsilon)$, where $X(\epsilon)$ must also be determined. To do this, expand u and its derivatives along the mapping.

The expansion of u is

$$u(x,\ \epsilon) = u(\epsilon = 0) + \epsilon \frac{du}{d\epsilon}(\epsilon = 0) + \frac{1}{2}\epsilon^2 \frac{d^2u}{d\epsilon^2}(\epsilon = 0) + \ \cdots$$

where $d/d\epsilon$, which is also d/dt, denotes a total derivative holding x_0 fixed. On carrying out the calculation, much as before, this reduces to

$$u(x,\ \epsilon) = u_0 + \epsilon \left[u_1 + x_1 \frac{du_0}{dx_0} \right]$$

$$+ \frac{1}{2}\epsilon^2 \left[u_2 + 2x_1 \frac{du_1}{dx_0} + x_1^2 \frac{d^2u_0}{dx_0^2} + x_2 \frac{du_0}{dx_0} \right] + \ \cdots$$

where all variables on the right-hand side depend only on x_0, namely

$$u_0(x_0) = u(x_0,\ \epsilon = 0)$$

$$u_1(x_0) = \frac{\partial u}{\partial t}(x_0,\ \epsilon = 0)$$

$$u_2(x_0) = \frac{\partial^2 u}{\partial t^2}(x_0,\ \epsilon = 0)$$

etc.

The expansions of $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ likewise take their familiar forms

$$\frac{\partial u}{\partial x}(x,\ \epsilon) = \frac{du_0}{dx_0} + \epsilon \left[\frac{du_1}{dx_0} + x_1 \frac{d^2u_0}{dx_0^2} \right] + \ \cdots$$

and

$$\frac{\partial^2 u}{\partial x^2}(x,\ \epsilon) = \frac{d^2u_0}{dx_0^2} + \epsilon \left[\frac{d^2u_1}{dx_0^2} + x_1 \frac{d^3u_0}{dx_0^3} \right] + \ \cdots$$

but the expansion of $\dfrac{\partial u}{\partial t}(x,\ \epsilon)$, which is also $\dfrac{\partial u}{\partial \epsilon}(x,\ \epsilon)$, might seem to be new and so we work it out. The main idea is simply to calculate $\partial/\partial\epsilon$ of both sides of the expansion of $u(x,\ \epsilon)$, where x must be held fixed. This presents the usual problem on the right-hand side: x_0 cannot remain fixed if x is fixed, and in carrying out the differentiation, x_0 must be viewed as a function of ϵ at fixed x. This function, x_0 versus ϵ at fixed x, is specified by the mapping, and differentiating the mapping leads to a formula for $\partial x_0/\partial\epsilon$. Calling this $\partial x_0/\partial t$, the formula is

$$0 = \frac{\partial x_0}{\partial t} + x_1 + \epsilon\frac{\partial x_1}{\partial t} + \epsilon x_2 + \frac{1}{2}\epsilon^2\frac{\partial x_2}{\partial t} + \cdots$$

Then, differentiating the expansion of u with respect to t produces

$$\frac{\partial u}{\partial t}(x,\ \epsilon) = \frac{du_0}{dx_0}\frac{\partial x_0}{\partial t} + \left[u_1 + x_1\frac{du_0}{dx_0}\right]$$

$$+\epsilon\left[\frac{du_1}{dx_0}\frac{\partial x_0}{\partial t} + x_1\frac{d^2u_0}{dx_0^2}\frac{\partial x_0}{\partial t} + \frac{\partial x_1}{\partial t}\frac{du_0}{dx_0}\right]$$

$$+\epsilon\left[u_2 + 2x_1\frac{du_1}{dx_0} + x_1^2\frac{d^2u_0}{dx_0^2} + x_2\frac{du_0}{dx_0}\right] + \text{terms of order } \epsilon^2$$

whence, eliminating $\dfrac{\partial x_0}{\partial t}$ via

$$\frac{\partial x_0}{\partial t} = -x_1 - \epsilon\frac{\partial x_1}{\partial t} - \epsilon x_2 - \frac{1}{2}\epsilon^2\frac{\partial x_2}{\partial t}$$

there obtains

$$\frac{\partial u}{\partial t}(x,\ \epsilon) = u_1 + \epsilon\left[u_2 + x_1\frac{du_1}{dx_0}\right] + \cdots$$

The plan, now, is to substitute our expansions into the equations satisfied by u on the present domain and on its boundary to discover how $u_1,\ u_2,\ \cdots$ can be obtained from u_0.

Taking the domain equation, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, first, we get

$$u_1 = \frac{d^2u_0}{dx_0^2}$$

$$u_2 = \frac{d^2u_1}{dx_0^2}$$

etc.

where the first equation has been used to eliminate the mapping in the second. This determines all of the functions $u_1,\ u_2,\cdots$ present in the expansion of u.

Then, going to the water–ice interface, the equation $u(x = X,\ t) = 0$ requires

$$u_0(x_0 = X_0) = 0$$

$$u_1(x_0 = X_0) + X_1 \frac{du_0}{dx_0}(x_0 = X_0) = 0$$

etc.

These formulas can be used to determine $X_1,\ X_2,\ \cdots$, but they make X_1, for instance, depend on $\dfrac{d^2u_0}{dx_0^2}(x_0 = X_0)$. The remaining equation at the water–ice interface, namely

$$\frac{\partial u}{\partial x}(x = X,\ t) = -\frac{dX}{dt}(t)$$

requires

$$\frac{du_0}{dx_0}(x_0 = X_0) = -X_1$$

$$\frac{du_1}{dx_0}(x_0 = X_0) + X_1 \frac{d^2u_0}{dx_0^2}(x_0 = X_0) = -X_2$$

etc.

which also determine $X_1,\ X_2,\ \cdots$. By this, the region occupied by the water is determined to order ϵ^2 by an order ϵ calculation. But as $X_1,\ X_2,\ \cdots$ can all be determined in two ways, a consistency condition on the data, u_0, presents itself.

Due to the fact that it is at the boundary of the reference domain, and only there, that the mapping into current domain can be determined, by continuing the calculations, u and all its derivatives, $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\cdots$, can be estimated at the end points of the present domain (viz., at $x = 0$ and $x = X$). But this is sufficient, as indicated in the essay, to determine u everywhere on the present domain.

Now, the reader may be skeptical upon learning that a well-posed nonlinear problem turns out to be overdetermined when its solution is advanced in time, as above. What, then, is going on? How can two formulas be produced, where both can be used to predict X_1, but where each comes from one or the other of two independent equations which the solution to the problem must satisfy? Is it possible that the two formulas predict the same value for X_1? Our view is this: If u_0 were assigned arbitrarily at $t_0 = 0$, as it could be if it were an initial condition, then the two formulas would, most likely, not be compatible. But if, instead, u_0 is determined by taking it to be the solution at some time t_0 after an arbitrary initial condition has already been assigned, then the two formulas must be compatible. Notice that the two formulas for X_1 come, on the one hand, out of the phase-equilibrium equation at the interface, while, on the other hand, out of the heat balance equation across the interface. Then, notice that u_0 denotes u after time t_0, where u, the solution to the problem, has been satisfying both equations over a finite interval of time. By this, predictions beyond t_0 based on u_0 using one equation ought to be consistent with predictions using the other.

To illustrate this, take a nonlinear problem whose solution can be found, namely

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < X(t), \quad t > 0$$

$$u(x = 0) = -1, \quad t > 0$$

$$u(x = X) = 0, \quad t > 0$$

and

$$\frac{\partial u}{\partial x}(x = X) = \frac{dX}{dt}, \quad t > 0$$

where

$$X(t = 0) = 0$$

This corresponds to water, held at $u = 0$ at $t = 0$, being frozen by reducing its temperature at $x = 0$ to $u = -1$ and holding it there. The new sign in the heat balance equation across the interface is accounted for by the fact that now ice lies to the left, water to the right, instead of water to the left, ice to the right as before.[2] Also, the adiabatic boundary condition at $x = 0$ has been replaced by an isothermal boundary condition, but this does not alter the two formulas for X_1.

This problem can be solved by taking u and X to be

$$u = -1 + \frac{\text{erf}\left(\frac{x}{\sqrt{4t}}\right)}{\text{erf } c}$$

and

$$X = c\sqrt{4t}$$

where c must satisfy

$$c = \frac{1}{\sqrt{\pi}} \frac{e^{-c^2}}{\text{erf } c}$$

but the value of c is not important in what is to come.

[2]Putting ice in place of water, water in place of ice, all else remaining the same, changes the sign of the term which accounts for the latent heat.

Now, let $t_0 > 0$ be assigned; then, u_0 and X_0 turn out to be

$$u_0 = -1 + \frac{\text{erf}\left(\dfrac{x_0}{\sqrt{4t_0}}\right)}{\text{erf } c}$$

and

$$X_0 = c\sqrt{4t_0}$$

whereupon our two formulas for X_1, namely

$$X_1 = -\frac{u_1}{\dfrac{\partial u_0}{\partial x_0}} = -\frac{\dfrac{\partial^2 u_0}{\partial x_0^2}}{\dfrac{\partial u_0}{\partial x_0}}$$

and

$$X_1 = \frac{\partial u_0}{\partial x_0}$$

predict

$$X_1 = \frac{\dfrac{1}{\sqrt{\pi}} \dfrac{e^{-c^2}}{\text{erf } c} c \dfrac{1}{\sqrt{t_0}} \dfrac{1}{\sqrt{t_0}}}{\dfrac{1}{\sqrt{\pi}} \dfrac{e^{-c^2}}{\text{erf } c} \dfrac{1}{\sqrt{t_0}}}$$

and

$$X_1 = \frac{1}{\sqrt{\pi}} \frac{e^{-c^2}}{\text{erf } c} \frac{1}{\sqrt{t_0}}$$

That these are the same, illustrates that the two formulas for X_1 are consistent. Again, this consistency is thought to reside in the fact that u_0 satisfies the nonlinear equations and is the current value of a function, u, that has been doing so for some time.

Still, why did two formulas turn up? The reason seems to be this: To estimate u to order 1 via

$$u(x_0,\ t_0 + \epsilon) = u_0(x_0) + \epsilon u_1(x_0)$$

by carrying points of the present domain straight back to the reference domain requires only that u_1 be determined. Surprisingly, this can be obtained at order 0, i.e., before order 1, via

$$u_1 = \frac{d^2 u_0}{dx_0^2}$$

The calculation may then be said to be getting ahead of itself when ϵ is a time difference, thereby requiring fewer equations for its completion.

Another way to think about this is to notice that the phase-equilibrium equation, namely

$$u(X,\ t) = 0$$

can be differentiated to produce

$$\frac{\partial u}{\partial x}\frac{dX}{dt} + \frac{\partial u}{\partial t} = 0$$

whereupon there obtains

$$\frac{dX}{dt} = -\frac{\dfrac{\partial u}{\partial t}}{\dfrac{\partial u}{\partial x}} = -\frac{\dfrac{\partial^2 u}{\partial x^2}}{\dfrac{\partial u}{\partial x}}$$

and this is just the formula for X_1 that comes out of the heat equation across the interface.

A.4 Laminar Flow Through an Off-centered Annulus

To introduce our fourth example, let a rod of radius R_0 be placed inside a pipe of radius κR_0, $\kappa > 1$. Let a fixed pressure gradient, denoted $\dfrac{dp}{dz}$, be imposed on the fluid lying between the rod and the pipe, causing it to flow at a volumetric rate Q. Our job is to find out by how much Q differs from its base value Q_0, in case the axis of the rod is displaced a small distance, parallel to itself, from its on-centered position in the reference configuration. This is not a problem where the domain must be discovered. In fact, the domain is specified in advance, but it is not a symmetric domain and this leads us to a domain perturbation, just as surely as do the problems presented earlier in this appendix and in the essays.

The important geometric variables are indicated in Figure A.6 which illustrates a cross section perpendicular to the z direction, the direction in which the fluid is flowing. The axis of the pipe lies at $x = 0,\ y = 0$, while the axis of the rod is displaced to $x = \epsilon,\ y = 0$.

The pipe presents no problem; it is the rod whose surface is displaced. Let this surface be denoted

$$r = R(\theta,\ \epsilon)$$

Then, if X and Y denote the Cartesian coordinates of the point $(R,\ \theta)$ on the surface of the rod, there obtains

$$[X - \epsilon]^2 + Y^2 = R_0^2$$

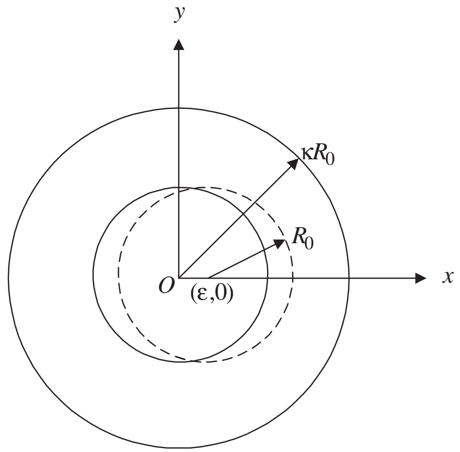

Figure A.6. *The Cross Section of an Off-centered Annulus.*

whereupon R, as a function of θ and ϵ, must satisfy

$$R^2 - \epsilon 2R\cos\theta = R_0^2 - \epsilon^2$$

This fourth problem differs in an important way from the three earlier problems. Here, the present configuration is known in advance and so too the reference configuration which we can take to be the on-centered configuration. Due to this, the mapping of the displaced surface can be determined at the outset. It is not part of the solution. Hence, by writing the mapping of the surface of the rod in the form

$$\theta = \theta_0$$

and

$$R = R(\theta, \epsilon) = R_0 + \epsilon R_1(\theta_0) + \frac{1}{2}\epsilon^2 R_2(\theta_0) + \cdots$$

there obtains

$$R_1 = \frac{dR}{d\epsilon}(\epsilon = 0) = \cos\theta_0$$

and

$$R_2 = \frac{d^2R}{d\epsilon^2}(\epsilon = 0) = -\frac{\sin^2\theta_0}{R_0}$$

Now, the volumetric flow rate is determined by the speed of the fluid via the formula

$$Q = \int_0^{2\pi} \int_{R(\theta,\ \epsilon)}^{\kappa R_0} v_z(r,\ \theta,\ \epsilon) r\ dr\ d\theta$$

where the integral is over the cross section of the displaced annulus. Carrying out the integration over the cross section of the reference annulus turns this formula into

$$Q = \int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_z r \frac{\partial(r,\ \theta)}{\partial(r_0,\ \theta_0)}\ dr_0\ d\theta_0$$

whereupon substituting

$$v_z = v_{z_0}(r_0) + \epsilon \left[v_{z_1}(r_0,\ \theta_0) + r_1(r_0,\ \theta_0) \frac{dv_{z_0}}{dr_0}(r_0) \right] + \cdots$$

$$r = r_0 + \epsilon r_1(r_0,\ \theta_0) + \cdots$$

and

$$\theta = \theta_0$$

leads to

$$Q = Q_0 + \epsilon Q_1 + \frac{1}{2}\epsilon^2 Q_2 + \cdots$$

The readers can work out the formulas for Q_1 and Q_2 and thereby assure themselves that only the mapping of the surface of the rod appears in these formulas, and the surface of the rod is specified. The second endnote to the second essay will help, but it does not go far enough.

Our aim is to carry the calculation of Q to second order in ϵ and this requires v_z to second order in ϵ. The problem to be solved for $v_z(r, \theta)$ on the present domain is

$$\frac{1}{\mu}\frac{dp}{dz} = \nabla^2 v_z, \quad R(\theta,\ \epsilon) < r < \kappa R_0, \quad 0 < \theta < 2\pi$$

where v_z must vanish at the surface of the rod [i.e., at $r = R(\theta,\ \epsilon)$, $0 < \theta < 2\pi$] and at the wall of the pipe [i.e., at $r = \kappa R_0$, $0 < \theta < 2\pi$].

The base problem on the reference domain is then

$$\frac{1}{\mu}\frac{dp}{dz} = \frac{d^2 v_{z_0}}{dr_0^2} + \frac{1}{r_0}\frac{dv_{z_0}}{dr_0}, \quad R_0 < r_0 < \kappa R_0$$

where v_{z_0} must vanish at $r = R_0$ and at $r = \kappa R_0$. Its solution is

$$v_{z_0} = \frac{1}{2\mu}\frac{dp}{dz}\left[\frac{1}{2}r_0^2 - \frac{1}{2}R_0^2 \frac{\kappa^2 - 1}{\ln\kappa} \ln\frac{r_0}{R_0} - \frac{1}{2}R_0^2\right]$$

and this can be used to find Q_0. To find v_{z_1} and v_{z_2} and, hence, Q_1 and Q_2, two derivatives of v_{z_0} will be needed. They are given by

$$\frac{dv_{z_0}}{dr_0}(r_0 = R_0) = \frac{1}{2\mu}\frac{dp}{dz} R_0 \left[1 - \frac{1}{2}\frac{\kappa^2 - 1}{\ln\kappa}\right]$$

and

$$\frac{d^2 v_{z_0}}{dr_0^2}(r_0 = R_0) = \frac{1}{2\mu}\frac{dp}{dz}\left[1 + \frac{1}{2}\frac{\kappa^2 - 1}{\ln\kappa}\right]$$

The problem to be solved for v_{z_1}, on the reference domain, is

$$0 = \frac{\partial^2 v_{z_1}}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial v_{z_1}}{\partial r_0} + \frac{1}{r_0^2}\frac{\partial^2 v_{z_1}}{\partial \theta_0^2},\ R_0 < r_0 < \kappa R_0,\ 0 < \theta_0 < 2\pi \quad (A.14)$$

where

$$v_{z_1}(r_0 = \kappa R_0) = 0 \quad (A.15)$$

and

$$v_{z_1}(r_0 = R_0) = -R_1 \frac{dv_{z_0}}{dr_0}(r_0 = R_0) \quad (A.16)$$

while the problem for v_{z_2}, again on the reference domain, is

$$0 = \frac{\partial^2 v_{z_2}}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial v_{z_2}}{\partial r_0} + \frac{1}{r_0^2}\frac{\partial^2 v_{z_2}}{\partial \theta_0^2},\ R_0 < r_0 < \kappa R_0,\ 0 < \theta_0 < 2\pi \quad (A.17)$$

where

$$v_{z_2}(r_0 = \kappa R_0) = 0 \quad (A.18)$$

and

$$v_{z_2}(r_0 = R_0) = -2R_1 \frac{\partial v_{z_1}}{\partial r_0}(r_0 = R_0) - R_1^2 \frac{d^2 v_{z_0}}{dr_0^2}(r_0 = R_0)$$

$$-R_2 \frac{dv_{z_0}}{dr_0}(r_0 = R_0) \quad (A.19)$$

Equations (A.16) and (A.19), satisfied by v_{z_1} and v_{z_2} at $r_0 = R_0$, are determined by the expansion of v_z at $r = R(\theta,\ \epsilon)$, namely by

$$v_z(R,\ \theta,\ \epsilon) = v_{z_0}(R_0,\ \theta_0) + \epsilon \left[v_{z_1}(R_0,\ \theta_0) + R_1(\theta_0)\frac{dv_{z_0}}{dr_0}(R_0) \right]$$

$$+\frac{1}{2}\epsilon^2 \left[v_{z_2}(R_0,\ \theta_0) + 2R_1(\theta_0)\frac{\partial v_{z_1}}{\partial r_0}(R_0,\ \theta_0) \right.$$

$$\left. +R_1^2(\theta_0)\frac{d^2 v_{z_0}}{dr_0^2}(R_0) + R_2(\theta_0)\frac{dv_{z_0}}{dr_0}(R_0) \right] + \cdots$$

and by the fact that $v_z(R,\ \theta,\ \epsilon)$ must vanish for all values of ϵ.

The solution to the v_{z_1} problem can be written

$$v_{z_1} = \left[A_1 r_0 + \frac{B_1}{r_0} \right] \cos\theta_0$$

where A_1 and B_1 must satisfy

$$A_1 \kappa R_0 + \frac{B_1}{\kappa R_0} = 0$$

and

$$A_1 R_0 + \frac{B_1}{R_0} = -\frac{dv_{z_0}}{dr_0}(r_0 = R_0)$$

whence v_{z_1} is given by

$$v_{z_1} = -\frac{dv_{z_0}}{dr_0}(r_0 = R_0)\ \frac{r_0 - \kappa^2 R_0^2/r_0}{R_0[1-\kappa^2]} \cos\theta_0$$

and by this, there obtains

$$\frac{\partial v_{z_1}}{\partial r_0}(r_0 = R_0) = \frac{dv_{z_0}}{dr_0}(r_0 = R_0)\frac{1}{R_0}\frac{\kappa^2+1}{\kappa^2-1}\cos\theta_0$$

which is needed in the v_{z_2} problem.

To solve the v_{z_2} problem, expand the right-hand side of equation (A.19) in terms of the functions 1, $\cos\theta_0$, $\cos 2\theta_0$, etc. All the information required to do this is available and there obtains

$$v_{z_2}(r_0 = R_0)$$
$$= \frac{1}{2\mu}\frac{dp}{dz}\left[\left[\frac{1+\kappa^2}{1-\kappa^2} + \frac{1}{\ln\kappa}\right] + \left[\frac{2\kappa^2}{1-\kappa^2} + \frac{1}{2}\frac{1+\kappa^2}{\ln\kappa}\right]\cos 2\theta_0\right] \tag{A.20}$$

In view of this, the solution to the v_{z_2} problem can be written

$$v_{z_2} = A_0 + B_0 \ln\frac{r_0}{R_0} + \left[A_2 r_0^2 + \frac{B_2}{r_0^2}\right]\cos 2\theta_0$$

where the constants A_0, B_0, A_2 and B_2 must satisfy

$$A_0 + B_0 \ln\kappa = 0 \tag{A.21}$$

$$A_0 = \frac{1}{2\mu}\frac{dp}{dz}\left[\frac{1+\kappa^2}{1-\kappa^2} + \frac{1}{\ln\kappa}\right] \tag{A.22}$$

$$A_2\kappa^2 R_0^2 + \frac{B_2}{\kappa^2 R_0^2} = 0 \tag{A.23}$$

and

$$A_2 R_0^2 + \frac{B_2}{R_0^2} = \frac{1}{2\mu}\frac{dp}{dz}\left[\frac{2\kappa^2}{1-\kappa^2} + \frac{1}{2}\frac{1+\kappa^2}{\ln\kappa}\right] \tag{A.24}$$

where equations (A.21) and (A.23) come from equation (A.18), and where equations (A.22) and (A.24) come from equation (A.19), rewritten as equation (A.20).

Now, A_2 and B_2 are not required in order to obtain Q_2. Our solution, then, dropping the terms in A_2 and B_2, is

$$v_{z_2} = \frac{1}{2\mu}\frac{dp}{dz}\left[\frac{1+\kappa^2}{1-\kappa^2} + \frac{1}{\ln\kappa}\right]\left[1 - \frac{\ln(r_0/R_0)}{\ln\kappa}\right]$$

Again, this is not all of v_{z_2}.

Let us turn now to the estimation of Q to second order in ϵ. The formulas for Q_1 and Q_2 will be needed. Hopefully, the reader has derived these formulas and found them to be[3]

$$Q_1 = \int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_{z_1} r_0 \; dr_0 \; d\theta_0 - \int_0^{2\pi} R_1 v_{z_0}(r_0 = R_0) R_0 \; d\theta_0$$

and

$$Q_2 = \int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_{z_2} r_0 \; dr_0 \; d\theta_0 - \int_0^{2\pi} R_2 v_{z_0}(r_0 = R_0) R_0 \; d\theta_0$$

$$-2\int_0^{2\pi} R_1 v_{z_1}(r_0 = R_0) R_0 \; d\theta_0 - \int_0^{2\pi} R_1^2 v_{z_0}(r_0 = R_0) \; d\theta_0$$

$$-\int_0^{2\pi} R_1^2 \frac{dv_{z_0}}{dr_0}(r_0 = R_0) R_0 \; d\theta_0$$

These formulas include all of the terms due to the displacement of the surface of the rod, even those that turn out to be zero in this problem. Then, as v_{z_1} and R_1 are multiples of $\cos\theta_0$ and $v_{z_0}(r_0 = R_0)$ is zero, Q_1 turns out to be zero and Q_2 is simply given by

$$\int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_{z_2} r_0 \; dr_0 \; d\theta_0 - \int_0^{2\pi} 2R_1 v_{z_1}(r_0 = R_0) R_0 \; d\theta_0$$

$$-\int_0^{2\pi} R_1^2 \frac{dv_{z_0}}{dr_0}(r_0 = R_0) R_0 \; d\theta_0$$

whereupon Q_2 turns out to be

$$\left[\frac{1}{2\mu}\frac{dp}{dz}\pi R_0^2\right]\left[1 - \frac{1}{2}\frac{\kappa^2 - 1}{\ln\kappa}\right]\left[\frac{2\kappa^2}{\kappa^2 - 1} - \frac{1}{\ln\kappa}\right]$$

The first and second factors are negative for all $\kappa > 1$ and the last factor is positive, whence Q_2 is positive.

[3]Should the rule have been used, Q_1 and Q_2 would have been

$$Q_1 = \int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_{z_1} r_0 \; dr_0 \; d\theta_0$$

and

$$Q_2 = \int_0^{2\pi} \int_{R_0}^{\kappa R_0} v_{z_2} r_0 \; dr_0 \; d\theta_0$$

but these formulas account only in part for the displacement of the rod. Of course, nobody would use the rule to obtain Q_1 and Q_2!

Two results [viz., $Q_1 = 0$ and $Q_2 > 0$], could have been anticipated and the readers might wish to satisfy themselves about this. Then, the reader might redo this calculation setting dp/dz to zero and requiring, instead, $v_z(r = R) = V$, where V is specified and held fixed. Again, Q_1 must be zero.

Three additional calculations can be suggested where some of the above may be helpful.

In the first, a fluid lies between two circular cylinders, spinning at certain specified angular speeds. Take the case where the inner cylinder is spinning while the outer cylinder is held fixed, and where it is required to determine the torque required to spin the inner cylinder as a function of its angular speed. Let the axes of the cylinders be parallel, but let the axis of the inner cylinder be displaced slightly from its concentric configuration.

In the second calculation, let a fluid be flowing under an assigned pressure gradient in a pipe of elliptical cross section. It is required to find the volumetric flow rate in case the cross section is nearly a circle.

The third is a heat conduction calculation for which many of the above formulas are useful. Let a hot rod lose heat by conduction to the cold wall of a pipe containing the rod. This can be worked out more or less along the lines of our example, with the result that by moving the rod off-center, the rate of heat loss is increased, but not at first order. The reader should do the calculation the hard way, by integrating the heat flow over the surface of the displaced rod. Letting Q denote the heat loss per unit length divided by the product of the thermal conductivity and the temperature difference, the reader should find

$$Q_1 = 0$$

and

$$Q_2 = \frac{1}{R_0^2} \frac{\pi}{[\kappa^2 - 1] \ln \kappa} \left[-[\kappa^2 + 1] \frac{1 - \ln \kappa}{\ln \kappa} + 6\kappa^2 \right]$$

Appendix B
The Curvature of Surfaces

B.1 Background

Let x, y and z denote the Cartesian coordinates of a point P whose position vector is denoted by $\vec{r}$. Then, let x, y and z be smooth functions of two variables u^1 and u^2. Under ordinary conditions, such a mapping of a region of the u^1, u^2- plane, given by

$$\vec{r}(u^1,\ u^2) = x(u^1,\ u^2)\vec{\imath} + y(u^1,\ u^2)\vec{\jmath} + z(u^1,\ u^2)\vec{k}$$

defines a surface, denoted S, in $x, y, z-$ space.

The variables u^1 and u^2 are called surface coordinates on S. Holding one coordinate fixed and letting the other run through its range of variation defines a coordinate curve on the surface. The vectors $\vec{r}_1 = \dfrac{\partial \vec{r}}{\partial u^1}$ and $\vec{r}_2 = \dfrac{\partial \vec{r}}{\partial u^2}$ lie tangent to these coordinate curves and, at each point P of the surface, $\vec{r}_1$ and $\vec{r}_2$ must be independent. This requirement on the mapping is satisfied if and only if its Jacobian matrix, namely

$$\begin{pmatrix} \dfrac{\partial x}{\partial u^1} & \dfrac{\partial y}{\partial u^1} & \dfrac{\partial z}{\partial u^1} \\ \\ \dfrac{\partial x}{\partial u^2} & \dfrac{\partial y}{\partial u^2} & \dfrac{\partial z}{\partial u^2} \end{pmatrix}$$

is everywhere of full rank (i.e., if and only if $\vec{r}_1 \times \vec{r}_2$ is never zero).

Then, $\vec{r}_1$ and $\vec{r}_2$ span the tangent plane to S at P and $\vec{r}_1 \times \vec{r}_2$ is perpendicular to the surface there. This is illustrated in Figure B.1. The vectors

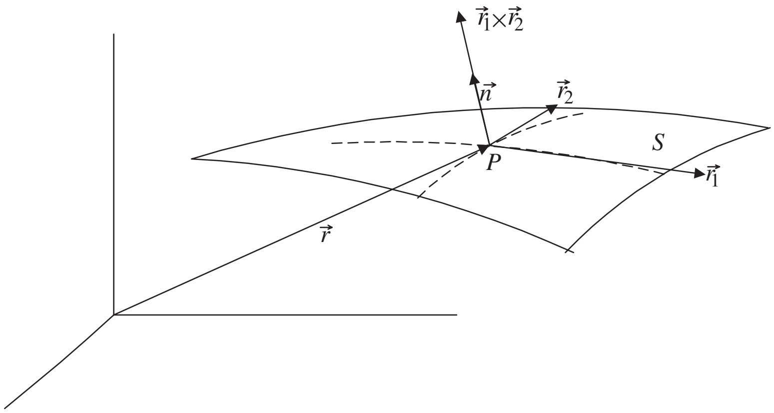

Figure B.1. *The Vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{n}$ at a Point of a Surface*

$\vec{r}_1$, $\vec{r}_2$ and $\vec{n}$, where

$$\vec{n} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

form a right-handed set of independent vectors at every point P of S.

Introduce the notation

$$g_{\alpha\beta} = \vec{r}_\alpha \cdot \vec{r}_\beta, \quad \alpha,\ \beta = 1,\ 2$$

and

$$g = g_{11}\ g_{22} - g_{12}^2 = [\vec{r}_1 \cdot \vec{r}_1][\vec{r}_2 \cdot \vec{r}_2] - [\vec{r}_1 \cdot \vec{r}_2]^2$$

whence there obtains

$$|\vec{r}_1 \times \vec{r}_2|^2 = [\vec{r}_1 \times \vec{r}_2] \cdot [\vec{r}_1 \times \vec{r}_2] = g > 0$$

and

$$[\vec{r}_1 \times \vec{r}_2] \cdot \vec{n} = |\vec{r}_1 \times \vec{r}_2| \vec{n} \cdot \vec{n} = \sqrt{g}$$

The coefficients $g_{\alpha\beta}$ determine lengths and areas on the surface. Indeed, the square of the length of a displacement $d\vec{r}$, where

$$d\vec{r} = \vec{r}_1\ du^1 + \vec{r}_2\ du^2$$

is given by

$$ds^2 = d\vec{r} \cdot d\vec{r} = \sum \sum g_{\alpha\beta}\ du^\alpha\ du^\beta$$

while the vector area of the parallelogram defined by the vectors $\vec{r}_1\ du^1$ and $\vec{r}_2\ du^2$ is given by

$$d\vec{A} = \vec{r}_1 \times \vec{r}_2\ du^1\ du^2 = \vec{n}|\vec{r}_1 \times \vec{r}_2|\ du^1\ du^2 = \vec{n}\sqrt{g}\ du^1\ du^2$$

Let a function be defined at the points of the surface S. Let it be smooth. It may be defined off the surface as well, but that is not important, and it may be scalar-valued or vector-valued, and again that is not important. Denote its values $f(u^1,\ u^2)$. To determine its directional derivative in the direction of the tangent to a curve C lying on the surface, let the curve be specified in terms of its arc length s by $u^1 = u^1(s)$ and $u^2 = u^2(s)$. The unit tangent to this curve is denoted by $\vec{t}$, where

$$\vec{t} = \frac{d\vec{r}}{ds} = \vec{r}_1 \frac{du^1}{ds} + \vec{r}_2 \frac{du^2}{ds}$$

Let the vectors $\vec{r}^{\,1}$ and $\vec{r}^{\,2}$, lying in the tangent plane at P, be defined by

$$\vec{r}^{\,1} = \frac{\vec{r}_2 \times \vec{n}}{\vec{r}_1 \times \vec{r}_2 \cdot \vec{n}} = \frac{\vec{r}_2 \times \vec{n}}{\sqrt{g}}$$

and

$$\vec{r}^{\,2} = \frac{\vec{n} \times \vec{r}_1}{\vec{r}_1 \times \vec{r}_2 \cdot \vec{n}} = \frac{\vec{n} \times \vec{r}_1}{\sqrt{g}}$$

Then, $\{\vec{r}_1,\ \vec{r}_2,\ \vec{n}\}$ and $\{\vec{r}^{\,1},\ \vec{r}^{\,2},\ \vec{n}\}$ are biorthogonal sets of vectors and $\vec{r}^{\,1}$ and $\vec{r}^{\,2}$ can be used to write $\dfrac{du^1}{ds}$ and $\dfrac{du^2}{ds}$ in terms of $\vec{t}$ via

$$\vec{r}^{\,1} \cdot \vec{t} = \frac{du^1}{ds}$$

and

$$\vec{r}^{\,2} \cdot \vec{t} = \frac{du^2}{ds}$$

Now, the directional derivative of f along the curve C, denoted df/ds, can be determined by using the chain rule to carry out the differentiation. The result is

$$\frac{df}{ds} = \frac{\partial f}{\partial u^1}\frac{du^1}{ds} + \frac{\partial f}{\partial u^2}\frac{du^2}{ds} = f_1 \frac{du^1}{ds} + f_2 \frac{du^2}{ds}$$

and using $du^1/ds = \vec{t} \cdot \vec{r}^{\,1}$, etc., it becomes

$$\frac{df}{ds} = \vec{t} \cdot [\vec{r}^{\,1} f_1 + \vec{r}^{\,2} f_2]$$

The second factor on the right-hand side of the formula for df/ds (viz., $\vec{r}^{\,1} f_1 + \vec{r}^{\,2} f_2$), is defined at each point of S, and at a point P of S, it determines the directional derivative of f in any direction tangent to the surface at P. Denote it by $\nabla_s f$ and write

$$\frac{df}{ds} = \vec{t} \cdot \nabla_s f$$

where

$$\nabla_s f = \vec{r}^{\,1} \frac{\partial f}{\partial u^1} + \vec{r}^{\,2} \frac{\partial f}{\partial u^2}$$

Then, the directional derivative of f along the tangent to a curve C at P depends on the curve only via $\vec{t}$.

If, at a point P of the surface, df/ds_1 and df/ds_2 denote the directional derivatives of f in the directions $\vec{t}_1$ and $\vec{t}_2$, then

$$\nabla_s f = \vec{a}^{\,1} \frac{df}{ds_1} + \vec{a}^{\,2} \frac{df}{ds_2}$$

where $\vec{t}_1$, $\vec{t}_2$, $\vec{n}$ and $\vec{a}^{\,1}$, $\vec{a}^{\,2}$, $\vec{n}$ are biorthogonal sets of vectors. This frees $\nabla_s f$ of the surface coordinates u^1 and u^2.

If f is vector-valued, the scalar and vector invariants of $\nabla_s f$ are given by

$$\nabla_s \cdot f = \vec{r}^{\,1} \cdot f_1 + \vec{r}^{\,2} \cdot f_2$$

and

$$\nabla_s \times f = \vec{r}^{\,1} \times f_1 + \vec{r}^{\,2} \times f_2$$

The unit normal $\vec{n}$ is a vector-valued function defined over the surface and its surface gradient is given by

$$\nabla_s \vec{n} = \vec{r}^{\,1} \vec{n}_1 + \vec{r}^{\,2} \vec{n}_2$$

This tensor is important. Its vector invariant is zero and, hence, it is symmetric. To determine that this is so, write

$$\nabla_s \times \vec{n} = \vec{r}^{\,1} \times \vec{n}_1 + \vec{r}^{\,2} \times \vec{n}_2 = \frac{[\vec{r}_2 \times \vec{n}] \times \vec{n}_1}{\sqrt{g}} + \frac{[\vec{n} \times \vec{r}_1] \times \vec{n}_2}{\sqrt{g}}$$

and work out the right-hand side using $[\vec{a} \times \vec{b}] \times \vec{c} = \vec{b}[\vec{c} \cdot \vec{a}] - \vec{a}[\vec{b} \cdot \vec{c}]$. Then, use

$$\vec{n} \cdot \vec{n} = 1, \quad \vec{r}_1 \cdot \vec{n} = 0, \quad \vec{r}_2 \cdot \vec{n} = 0$$

to determine that

$$\vec{n} \cdot \vec{n}_1 = 0 = \vec{n} \cdot \vec{n}_2$$

and

$$\vec{r}_1 \cdot \vec{n}_2 = -\vec{n} \cdot \vec{r}_{12} = \vec{r}_2 \cdot \vec{n}_1$$

and, hence, that

$$\nabla_s \times \vec{n} = \vec{0}$$

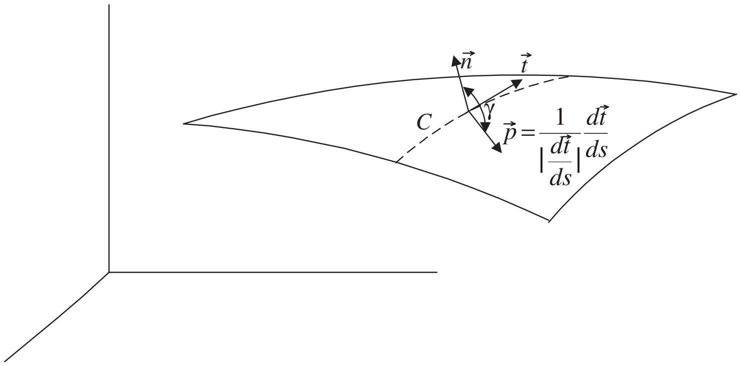

Figure B.2. *The Angle Between the Principle Normal to C and the Normal to S*

B.2 Curvature

Again, let C be a curve lying on the surface S. Let γ denote the angle between its principal normal, $\vec{p}$, and the normal to the surface, $\vec{n}$. This is illustrated in Figure B.2, where

$$\cos\gamma = \vec{p}\cdot\vec{n}$$

and where $\vec{p}$ is defined in terms of $\vec{t}$ by

$$\kappa\vec{p} = \frac{d\vec{t}}{ds}$$

This introduces κ via $\kappa = |\frac{d\vec{t}}{ds}|$, where κ, the rate at which the tangent is turning, is called the curvature of C.

On substituting for $\vec{p}$, the formula for $\cos\gamma$ can be written

$$\kappa\cos\gamma = \kappa\vec{p}\cdot\vec{n} = \frac{d\vec{t}}{ds}\cdot\vec{n}$$

and then, using $\vec{t}\cdot\vec{n} = 0$, it can be written in terms of $\nabla_s\vec{n}$ as

$$\kappa\cos\gamma = -\vec{t}\cdot\frac{d\vec{n}}{ds} = -\vec{t}\cdot[t\cdot\nabla_s\vec{n}] = -\vec{t}\,\vec{t} : \nabla_s\vec{n}$$

The tensor $\nabla_s\vec{n}$ is called the curvature tensor. It is defined at each point P of S. The right-hand side of this formula depends on the point P of the surface S via $\nabla_s\vec{n}$ and on the direction of the curve C lying on S, and passing through P, via $\vec{t}$.

Now, as $\cos\gamma$ is just $\vec{p}\cdot\vec{n}$, all curves lying on S passing through a point P of S and having the same tangent and principal normal must have the same curvature. This curvature then must be that of the plane curve passing through P defined by the intersection of the $\vec{t},\vec{p}-$ plane at P and the surface S. This tells us that the curvature of a surface can be described in terms of the curvature of its plane curves. The product $\kappa\cos\gamma$ is determined by the point P of the surface and the tangent direction $\vec{t}$ at that point. In terms of this product, the curvature of any plane curve passing through P in the direction $\vec{t}$ is determined only by $\cos\gamma$. The product $\kappa\cos\gamma$ is called the normal curvature and it is denoted κ_n. It is $\pm$ the curvature of the normal section of S at P in the direction $\vec{t}$. In particular, the curvature of the plane curve made by the intersection of the surface S and its normal plane at P in the direction $\vec{t}$ (i.e., the $\vec{n}, \vec{t}$– plane), is $\pm\kappa_n$ and it depends only on whether $\vec{p} = \vec{n}$ or $\vec{p} = -\vec{n}$.

The curvature tensor, $\nabla_s\vec{n}$, at a point P then tells us κ_n at that point as it depends on $\vec{t}$, namely

$$\kappa_n = -\vec{t}\,\vec{t} : \nabla_s\vec{n}$$

where, again, κ_n is $\pm$ the curvature of the normal section at P in the direction $\vec{t}$. If $\vec{n}$ is replaced by $-\vec{n}$ over the surface, then κ_n is replaced by $-\kappa_n$.

To produce a formula by which κ_n can be determined at a point P, as it depends on the direction of $\vec{t}$ there, write

$$\kappa_n = \kappa\cos\gamma = \frac{d\vec{t}}{ds}\cdot\vec{n} = \frac{d}{ds}\frac{d\vec{r}}{ds}\cdot\vec{n}$$

$$= \frac{d}{ds}\sum \vec{r}_\alpha\frac{du^\alpha}{ds}\cdot\vec{n} = \sum\sum \vec{r}_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\cdot\vec{n} + \sum \vec{r}_\alpha\frac{d^2u^\alpha}{ds^2}\cdot\vec{n}$$

$$= \sum\sum \vec{r}_{\alpha\beta}\cdot\vec{n}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

Then, defining $b_{\alpha\beta}$ by

$$b_{\alpha\beta} = \vec{r}_{\alpha\beta}\cdot\vec{n} = b_{\beta\alpha}$$

our formula for κ_n can be written

$$\kappa_n = \sum\sum b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}$$

where

$$\frac{du^\alpha}{ds} = \vec{r}^{\,\alpha}\cdot\vec{t}$$

and

$$b_{\alpha\beta} = \vec{r}_{\alpha\beta}\cdot\vec{n} = -\vec{r}_\alpha\cdot\vec{n}_\beta$$

$$= -\vec{r}_\alpha\vec{r}_\beta : \nabla_s\vec{n}$$

The normal curvature at a point P of S, in terms of the tangent direction, (viz., du^1, du^2) is then

$$\kappa_n = \frac{\sum\sum b_{\alpha\beta}du^\alpha du^\beta}{\sum\sum g_{\alpha\beta}du^\alpha du^\beta}$$

and the sign of κ_n is determined by the sign of $\sum b_{\alpha\beta} du^\alpha du^\beta$. If b, where $b = b_{11}b_{22} - b_{12}^2$, is positive, this sign is the same for all tangent directions. If b is equal to zero, there is one direction in which $\kappa_n = 0$, otherwise it is of one sign. If b is negative, there are two directions in which $\kappa_n = 0$ and κ_n takes both positive and negative values.

To discover the basic facts about the dependence of κ_n on $\vec{t}$, write

$$\kappa_n = -\vec{t}\,\vec{t} : \nabla_s \vec{n}$$

and let $\vec{t}$ make an angle θ with a fixed reference line in the tangent plane at P. Then, the dependence of κ_n on $\vec{t}$ is the dependence of κ_n on θ. If an arbitrary tangent $\vec{t}$ corresponds to the angle θ, let the tangent $\vec{t}^{\,\prime}$ correspond to the angle $\theta + \frac{1}{2}\pi$ and observe that

$$\vec{t} = -\vec{n} \times \vec{t}^{\,\prime}$$

and

$$\vec{t}^{\,\prime} = \vec{n} \times \vec{t}$$

and, hence, that

$$\frac{d\vec{t}}{d\theta} = \vec{n} \times \vec{t} = \vec{t}^{\,\prime}$$

and

$$\frac{d\vec{t}^{\,\prime}}{d\theta} = \vec{n} \times \vec{t}^{\,\prime} = -t$$

These results tell us that $[\vec{t}\,\vec{t} + \vec{t}^{\,\prime}\vec{t}^{\,\prime}]$ does not depend on θ, namely that

$$\frac{d}{d\theta}[\vec{t}\,\vec{t} + \vec{t}^{\,\prime}\vec{t}^{\,\prime}] = \vec{\vec{0}}$$

Now, $[\vec{t}\,\vec{t} + \vec{t}^{\,\prime}\vec{t}^{\,\prime}]$ determines the sum of the normal curvatures in the two perpendicular directions $\vec{t}$ and $\vec{t}^{\,\prime}$. This sum is given by

$$-[\vec{t}\,\vec{t} + \vec{t}^{\,\prime}\vec{t}^{\,\prime}] : \nabla_s \vec{n}$$

and, as it does not depend on θ, the sum of the normal curvatures in any two perpendicular directions must be fixed at each point of a surface. This defines what is called twice the mean curvature of the surface at that point. Denote the mean curvature by H; then,

$$2H = \kappa_n(\theta) + \kappa_n(\theta + \frac{1}{2}\pi)$$

and this is independent of θ.

The greatest, or the least, value of κ_n corresponds to a direction $\vec{t}$ such that κ_n, as it depends on θ, is stationary. This requirement is

$$\frac{d\kappa_n}{d\theta} = 0 = -[\vec{t}^{\,\prime}\vec{t} + \vec{t}\,\vec{t}^{\,\prime}] : \nabla_s \vec{n}$$

and, as $\nabla_s \vec{n}$ is symmetric, it reduces to

$$\vec{t}\,\vec{t}^{\,\prime} : \nabla_s \vec{n} = 0$$

If κ_n is greatest or least in the direction $\vec{t}$, it must be least or greatest in the direction $\vec{t}^{\,\prime}$.

Now, in any basis $\{\vec{t}, \vec{t}^{\,\prime}\}$, $\nabla_s \vec{n}$ can be expanded as

$$\nabla_s \vec{n} = [\vec{t}\,\vec{t} : \nabla_s \vec{n}]\vec{t}\,\vec{t} + [\vec{t}^{\,\prime}\,\vec{t} : \nabla_s \vec{n}]\vec{t}\,\vec{t}^{\,\prime} + [\vec{t}\,\vec{t}^{\,\prime} : \nabla_s \vec{n}]\vec{t}^{\,\prime}\,\vec{t} + [\vec{t}^{\,\prime}\,\vec{t}^{\,\prime} : \nabla_s \vec{n}]\vec{t}^{\,\prime}\,\vec{t}^{\,\prime}$$

Then, if the direction $\vec{t}$ is such that κ_n is stationary as a function of θ and if $\vec{t}^{\,\prime}$ is taken to be $\vec{n} \times \vec{t}$, this simplifies to

$$-\nabla_s \vec{n} = \kappa_n(\vec{t})\vec{t}\,\vec{t} + \kappa_n(\vec{t}^{\,\prime})\vec{t}^{\,\prime}\,\vec{t}^{\,\prime}$$

whence the greatest and the least values of κ_n turn out to be the eigenvalues of $-\nabla_s \vec{n}$. Denote these eigenvalues by κ_1 and κ_2. They can be determined as the roots of

$$\det(-\nabla_s \vec{n} - \kappa I_s) = 0$$

and, hence, of

$$\det(b_{\alpha\beta} - \kappa g_{\alpha\beta}) = 0$$

or of

$$\det(b^{\alpha}_{\beta} - \kappa \delta^{\alpha}_{\beta}) = 0$$

In view of this, $2H$ can be obtained as the sum of b^1_1 and b^2_2, namely

$$2H = \kappa_1 + \kappa_2 = b^1_1 + b^2_2$$

This is a formula that can be used to calculate the mean curvature of a surface at any point P of the surface, yet sometimes it is simpler to make use of the fact that the mean curvature at a point of a surface is one-half the sum of the normal curvatures in any two perpendicular directions at that point.

B.3 Some Examples

Some mean curvature formulas needed in our essays are worked out in this section as examples of the use of the formula

$$2H = b^1_1 + b^2_2 = g^{11}b_{11} + 2g^{12}b_{12} + g^{22}b_{22}$$

Example 1

Let a surface be given by the formula

$$z = f(x, y)$$

Then, using x and y as surface coordinates (viz., $u^1 = x,\ u^2 = y$) and writing

$$\vec{r} = x\vec{i} + y\vec{j} + f(x,y)\vec{k}$$

the required calculations produce

$$\vec{r}_1 = \vec{i} + f_x\vec{k}, \quad \vec{r}_2 = \vec{j} + f_y\vec{k}$$

$$g_{11} = 1 + f_x^2, \quad g_{12} = f_x f_y, \quad g_{22} = 1 + f_y^2$$

$$g = 1 + f_x^2 + f_y^2$$

$$\vec{n} = \frac{-f_x\vec{i} - f_y\vec{j} + \vec{k}}{\sqrt{g}}$$

$$\vec{r}_{11} = f_{xx}\vec{k}, \quad \vec{r}_{12} = f_{xy}\vec{k}, \quad \vec{r}_{22} = f_{yy}\vec{k}$$

$$b_{11} = \frac{f_{xx}}{\sqrt{g}}, \quad b_{12} = \frac{f_{xy}}{\sqrt{g}}, \quad b_{22} = \frac{f_{yy}}{\sqrt{g}}$$

and

$$g^{11} = \frac{1 + f_y^2}{g}, \quad g^{12} = -\frac{f_x f_y}{g}, \quad g^{22} = \frac{1 + f_x^2}{g}$$

whence

$$2H = \frac{[1 + f_y^2]f_{xx} - 2f_x f_y f_{xy} + [1 + f_x^2]f_{yy}}{g^{\frac{3}{2}}}$$

If f_x, f_y, etc., are all small, the right-hand side of this formula can be approximated by

$$f_{xx} + f_{yy} = \nabla^2 f$$

If f is independent of y, the formula simplifies to

$$\frac{f_{xx}}{[1 + f_x^2]^{3/2}}$$

Example 2

Let a surface be given by the formula

$$r = R + f(\theta, z)$$

where R is fixed. Then, using θ and z as surface coordinates (viz., $u^1 = \theta,\ u^2 = z$) and writing

$$\vec{r} = [R + f(\theta, z)]\ \vec{i}_r(\theta) + z\vec{i}_z$$

the required calculations produce

$$\vec{r}_1 = [R + f]\ \vec{i}_\theta + f_\theta \vec{i}_r, \quad \vec{r}_2 = f_z \vec{i}_r + \vec{i}_z$$

$$g_{11} = [R + f]^2 + f_\theta^2, \quad g_{12} = f_\theta f_z, \quad g_{22} = f_z^2 + 1$$

$$g = [R + f]^2 [1 + f_z^2] + f_\theta^2$$

$$\vec{n} = \frac{[R + f]\ \vec{i}_r - f_\theta \vec{i}_\theta - [R + f] f_z \vec{i}_z}{\sqrt{g}}$$

$$\vec{r}_{11} = [-[R + f] + f_{\theta\theta}]\, \vec{i}_r + 2 f_\theta \vec{i}_\theta, \quad \vec{r}_{12} = f_{z\theta} \vec{i}_r + f_z \vec{i}_\theta, \quad \vec{r}_{22} = f_{zz} \vec{i}_r$$

$$b_{11} = \frac{-[R + f]^2 + [R + f] f_{\theta\theta} - 2 f_\theta^2}{\sqrt{g}}$$

$$b_{12} = \frac{[R + f] f_{z\theta} - f_\theta f_z}{\sqrt{g}}$$

$$b_{22} = \frac{[R + f] f_{zz}}{\sqrt{g}}$$

and

$$g^{11} = \frac{f_z^2 + 1}{g}, \quad g^{12} = -\frac{f_\theta f_z}{g}, \quad g^{22} = \frac{[R + f]^2 + f_\theta^2}{g}$$

whence

$$2H = \left[\frac{1 + f_z^2}{g}\right] \left[\frac{-[R + f]^2 - 2 f_\theta^2 + [R + f] f_{\theta\theta}}{\sqrt{g}}\right]$$

$$+2 \left[\frac{-f_\theta f_z}{g}\right] \left[\frac{[R + f] f_{z\theta} - f_\theta f_z}{\sqrt{g}}\right] + \left[\frac{[R + f]^2 + f_\theta^2}{g}\right] \left[\frac{[R + f] f_{zz}}{\sqrt{g}}\right]$$

If f, f_θ, f_z, etc. are all small, some approximations can be introduced. First, write

$$2H = \frac{1}{g^{\frac{3}{2}}} [-R^2 - 2Rf + R f_{\theta\theta}] + 0 + \frac{1}{g^{\frac{3}{2}}} [R^2 + 2Rf][R f_{zz}]$$

and

$$g = R^2 + 2Rf = R^2 \left[1 + 2\frac{f}{R}\right]$$

Then, use

$$g^{\frac{3}{2}} = R^3 \left[1 + 2\frac{f}{R}\right]^{\frac{3}{2}}$$

to get

$$2H = \left[-\frac{1}{R} + \frac{f}{R^2} + \frac{f_{\theta\theta}}{R^2} + f_{zz} \right]$$

In this formula, $-\frac{1}{R} + \frac{f}{R^2}$ is an approximation to $-\frac{1}{R+f}$, which is the curvature of a circle of radius $R + f$. The remaining terms account for the circumferential and the longitudinal ripples on the surface.

Example 3

Let an axisymmetric surface be given by the formula

$$z = f(r)$$

Then, using r and θ as surface coordinates (viz., $u^1 = r$, $u^2 = \theta$) and writing

$$\vec{r} = r\vec{i}_r(\theta) + f(r)\vec{i}_z$$

the required calculations lead to

$$\vec{r}_1 = \vec{i}_r + f^{'}\vec{i}_z, \quad \vec{r}_2 = r\vec{i}_\theta$$

$$g_{11} = 1 + [f^{'}]^2, \quad g_{12} = 0, \quad g_{22} = r^2$$

$$g = r^2 \left[1 + [f^{'}]^2 \right]$$

$$\vec{n} = \frac{r\vec{i}_z - rf^{'}\vec{i}_r}{\sqrt{g}}$$

$$\vec{r}_{11} = f^{''}\vec{i}_z, \quad \vec{r}_{12} = \vec{i}_\theta, \quad \vec{r}_{22} = -r\vec{i}_r$$

$$b_{11} = \frac{rf^{''}}{\sqrt{g}}, \quad b_{12} = 0, \quad b_{22} = \frac{r^2 f^{'}}{\sqrt{g}}$$

and

$$g^{11} = \frac{1}{1 + [f^{'}]^2}, \quad g^{12} = 0, \quad g^{22} = \frac{1}{r^2}$$

Here, the surface coordinates are orthogonal and $2H$ can be determined easily by adding the normal curvatures in the r and θ directions, but continuing in the usual way, there obtains

$$2H = b_1^1 + b_2^2 = g^{11}b_{11} + 2g^{12}b_{12} + g^{22}b_{22} = \frac{f^{''} + \frac{1}{r}f^{'} \left[1 + [f^{'}]^2 \right]}{\left[1 + [f^{'}]^2 \right]^{\frac{3}{2}}}$$

If $f^{'}$ is small, the right-hand side of this formula can be approximated by

$$f^{''} + \frac{1}{r} f^{'} = \nabla^2 f$$

Now, let f depend on θ as well as on r; then, write

$$\vec{r} = r\vec{i}_r(\theta) + f(r,\ \theta)\vec{i}_z$$

Again, take $u^1 = r$ and $u^2 = \theta$, whereupon

$$\vec{r}_1 = \vec{i}_r + f_r\vec{i}_z$$

$$\vec{r}_2 = r\vec{i}_\theta + f_\theta\vec{i}_z$$

and

$$\vec{n} = \frac{-rf_r\vec{i}_r - f_\theta\vec{i}_\theta + r\vec{i}_z}{\sqrt{g}}$$

By these formulas, there obtain

$$g_{11} = 1 + f_r^2$$

$$g_{12} = f_r f_\theta$$

$$g_{22} = r^2 + f_\theta^2$$

and

$$g = r^2 + r^2 f_r^2 + f_\theta^2$$

as well as

$$\vec{r}_{11} = f_{rr}\vec{i}_z$$

$$\vec{r}_{12} = \vec{i}_\theta + f_{r\theta}\vec{i}_z$$

and

$$\vec{r}_{22} = -r\vec{i}_r + f_{\theta\theta}\vec{i}_z$$

These lead to

$$b_{11} = \frac{rf_{rr}}{\sqrt{g}}$$

$$b_{12} = \frac{-f_\theta + rf_{r\theta}}{\sqrt{g}}$$

and

$$b_{22} = \frac{r^2 f_r + rf_{\theta\theta}}{\sqrt{g}}$$

which along with

$$g^{11} = \frac{r^2 f_\theta^2}{g}$$

$$g^{12} = -\frac{f_r f_\theta}{g}$$

and

$$g^{22} = \frac{1 + f_r^2}{g}$$

produce the result

$$2H = \frac{[r^2 + f_\theta^2][r f_{rr}] - 2 f_r f_\theta [-f_\theta + r f_{r\theta}] + [1 + f_r^2][r^2 f_r + r f_{\theta\theta}]}{g^{3/2}}$$

where

$$g^{3/2} = r^3 \left[1 + \frac{1}{r^2} \, f_\theta^2 + f_r^2 \right]^{3/2}$$

This simplifies to the earlier result when f is independent of θ. If f_r, f_θ, etc. are all small, the right-hand side can be approximated by

$$f_{rr} + \frac{1}{r} \, f_r + \frac{1}{r^2} \, f_{\theta\theta} = \nabla^2 f$$

Appendix C
The Normal Speed of a Surface

Let a surface be denoted by

$$f(\vec{r},\ t) = 0$$

Then, f is positive on one side of $f = 0$, negative on the other, and the normal pointing into the region where f is positive is given by

$$\vec{n} = \frac{\nabla f}{|\nabla f|}$$

Let the surface move a small distance Δs along this normal in time Δt. Then, $f(\vec{r} \pm \Delta s\ \vec{n},\ t + \Delta t)$ is given by

$$f(\vec{r} \pm \Delta s\ \vec{n}, t + \Delta t) = f(\vec{r}, t) \pm \Delta s\ \vec{n} \cdot \nabla f(\vec{r}, t) + \Delta t\ f_t(\vec{r}, t) + \ \cdots$$

whence $f(\vec{r} \pm \Delta s\ \vec{n},\ t + \Delta t) = 0 = f(\vec{r},\ t)$ requires

$$\pm\Delta s\ \vec{n} \cdot \nabla f + \Delta t\ f_t = 0$$

The normal speed, denoted u, is then given by

$$u = \pm\frac{\Delta s}{\Delta t} = -\frac{f_t}{\vec{n} \cdot \nabla f} = -\frac{f_t}{|\nabla f|}$$

where u is positive at points of the surface $f = 0$ which are moving into the region where $f > 0$. This is illustrated in Figure C.1.

It is often useful to introduce the surface velocity. Denote this by $\vec{u}$ and take $\vec{u}$ to be $u\vec{n}$. Then, we have $\vec{n} \cdot \vec{u} = u$.

The special case where

$$r = R(\theta,\ z,\ t)$$

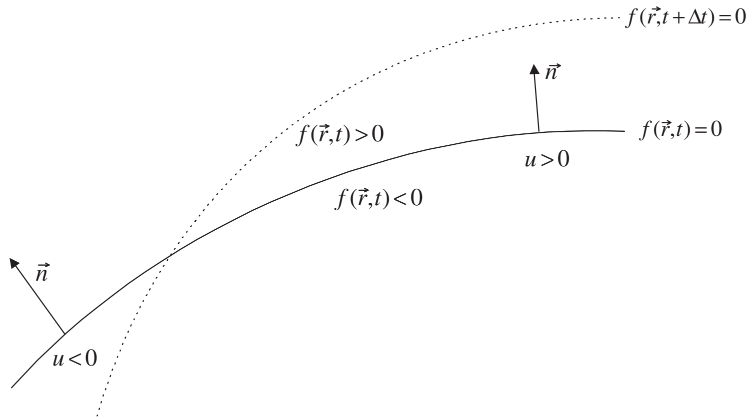

Figure C.1. *The Speed of a Moving Surface*

and, hence, where

$$f(\vec{r},\ t) = r - R(\theta,\ z,\ t)$$

is of interest in the first, third, fourth and fifth essays. There, $f(\vec{r},\ t) = 0$ denotes the surface dividing a liquid in the form of a jet from the surrounding fluid. The normal speed of this surface is given by

$$u = \frac{\dfrac{\partial R}{\partial t}}{\left[1 + \left[\dfrac{1}{R}\dfrac{\partial R}{\partial \theta}\right]^2 + \left[\dfrac{\partial R}{\partial z}\right]^2\right]^{1/2}}$$

The equation $\vec{n} \cdot \vec{v} = u$ is also of interest. It specifies no flow across a moving surface. Using $\vec{n} = \dfrac{\nabla f}{|\nabla f|}$ and $u = \dfrac{-f_t}{|\nabla f|}$, it can be written

$$\vec{v} \cdot \nabla f = -f_t$$

and, hence, if the surface is denoted by $f(x,\ y,\ z,\ t) = 0$,

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + v_z \frac{\partial f}{\partial z} = -f_t$$

whence the no-flow surface is seen to be material.

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