

## References

- Abraham-Shrauner, B., Leach, P.G.L., Govinder, K.S., and Ratcliff, G. (1995). Hidden and contact symmetries of ordinary differential equations. *J. Phys.* **A28**, 6707–6716.
- Abramowitz, M. and Stegun, I.A. (eds.) (1970). *Handbook of Mathematical Functions*. Dover, New York, NY.
- Aguirre, M. and Krause, J. (1985). Infinitesimal symmetry transformations, II. Some one-dimensional nonlinear systems. *J. Math. Phys.* **26**, 593–600.
- Akhmatov, I.S., Gazizov, R.K., and Ibragimov, N.H. (1988). Bäcklund transformations and nonlocal symmetries. *Soviet Math. Dokl.* **36**, 393–395. See also *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. I, 1994, Chap. 13, pp. 258–291 (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.
- Aksenov, A.V. (1995). Symmetries of linear partial differential equations and fundamental solutions. *Dokl. Math.* **51**, 329–331.
- Ames, W.F., Lohner, R.J., and Adams, E. (1981). Group properties of  $u_{tt} = [f(u)u_x]_x$ . *Internat. J. Nonlinear Mech.* **16**, 439–447.
- Anco, S.C. and Bluman, G.W. (1996). Derivation of conservation laws from nonlocal symmetries of differential equations. *J. Math. Phys.* **37**, 2361–2375.
- Anco, S.C. and Bluman, G.W. (1997a). Direct construction of conservation laws from field equations. *Phys. Rev. Lett.* **78**, 2869–2873.
- Anco, S.C. and Bluman, G.W. (1997b). Nonlocal symmetries and nonlocal conservation laws of Maxwell's equations. *J. Math. Phys.* **38**, 3508–3532.
- Anco, S.C. and Bluman, G.W. (1998). Integrating factors and first integrals for ordinary differential equations. *European J. Appl. Math.* **9**, 245–259.
- Anco, S.C. and Bluman, G.W. (2002a). Direct construction method for conservation laws of partial differential equations. Part I: Examples of conservation law classifications. To appear in *European J. Appl. Math.*
- Anco, S.C. and Bluman, G.W. (2002b). Direct construction method for conservation laws of partial differential equations. Part II: General treatment. To appear in *European J. Appl. Math.*
- Anderson, I.M., Fels, M., and Torre, C.G. (2000). Group invariant solutions without transversality. *Comm. Math. Phys.* **212**, 653–686.
- Anderson, R.L., Kumei, S., and Wulfman, C.E. (1972). Generalization of the concept of invariance of differential equations. *Phys. Rev. Lett.* **28**, 988–991.
- Atkinson, F.V. and Peletier, L.A. (1974). Similarity solutions of the nonlinear diffusion equation. *Arch. Rat. Mech. Anal.* **54**, 373–392.

- Baikov, V.A., Gazizov, R.K., and Ibragimov, N.H. (1990). Classification of multi-dimensional wave equations with respect to exact and approximate symmetries. Preprint 51, Institute of Applied Mathematics, Academy of Sciences, USSR, Moscow. See also *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. I, 1994, Chap. 12, pp. 222–224 (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.
- Baikov, V.A., Gazizov, R.K., and Ibragimov, N.H. (1991). Approximate symmetries and conservation laws. In *Number Theory, Algebra, Analysis and Their Applications*. Proc. Steklov Math. Inst., Academy of Sciences, USSR, Vol. 200, Nauka, Moscow. See also *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. I, 1994, Chap. 12, pp. 222–224 (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.
- Barenblatt, G.I. (1979). *Similarity, Self-Similarity, and Intermediate Asymptotics*. Consultants Bureau, New York, NY.
- Barenblatt, G.I. (1987). *Dimensional Analysis*. Gordon and Breach, New York, NY.
- Barenblatt, G.I. (1996). *Scaling, Self-Similarity, and Intermediate Asymptotics*. Cambridge University Press, Cambridge, UK.
- Barenblatt, G.I. and Zel'dovich, Ya.B. (1972). Self-similar solutions as intermediate asymptotics. *Ann. Rev. Fluid Mech.* **4**, 285–312.
- Baumann, G. and Nonnenmacher, T.F. (1987). Lie transformations, similarity reduction, and solutions for the nonlinear Madelung fluid equations with external potential. *J. Math. Phys.* **28**, 1250–1260.
- Becker, H.A. (1976). *Dimensionless Parameters: Theory and Method*. Wiley, New York, NY.
- Bianchi, L. (1918). *Lezioni sulla Teoria dei Gruppi Continui Finiti di Trasformazioni*. Enrico Spoerri, Pisa, Italy.
- Birkhoff, G. (1950). *Hydrodynamics: A Study in Logic, Fact and Similitude*. Princeton University Press, Princeton, NJ.
- Bluman, G.W. (1967). Construction of Solutions to Partial Differential Equations by the Use of Transformation Groups. Ph.D. Thesis, California Institute of Technology.
- Bluman, G.W. (1971). Similarity solutions of the one-dimensional Fokker–Planck equation. *Internat. J. Nonlinear Mech.* **6**, 143–153.
- Bluman, G.W. (1974). Applications of the general similarity solution of the heat equation to boundary value problems. *Quart. Appl. Math.* **31**, 403–415.
- Bluman, G.W. (1980). On the transformation of diffusion processes into the Wiener process. *SIAM J. Appl. Math.* **39**, 238–247.
- Bluman, G.W. (1983a). Dimensional analysis, modelling, and symmetry. *Internat. J. Math. Ed. Sci. Tech.* **14**, 259–272.
- Bluman, G.W. (1983b). On mapping linear partial differential equations to constant coefficient equations. *SIAM J. Appl. Math.* **43**, 1259–1273.

- Bluman, G.W. (1990a). Simplifying the form of Lie groups admitted by a given differential equation. *J. Math. Anal. Appl.* **145**, 52–62.
- Bluman, G.W. (1990b). A reduction algorithm for an ordinary differential equation admitting a solvable Lie group. *SIAM J. Appl. Math.* **50**, 1689–1705.
- Bluman, G.W. (1990c). Invariant solutions for ordinary differential equations. *SIAM J. Appl. Math.* **50**, 1706–1715.
- Bluman, G.W. and Cole, J.D. (1969). The general similarity solution of the heat equation. *J. Math. Mech.* **18**, 1025–1042.
- Bluman, G.W. and Cole, J.D. (1974). *Similarity Methods for Differential Equations*. Applied Mathematical Sciences, No. 13. Springer-Verlag, New York, NY.
- Bluman, G.W. and Doran-Wu, P. (1995). The use of factors to discover potential systems or linearizations. *Acta Appl. Math.* **2**, 79–96.
- Bluman, G.W. and Gregory, R.D. (1985). On transformations of the biharmonic equation. *Mathematika* **32**, 118–130.
- Bluman, G.W. and Kumei, S. (1980). On the remarkable nonlinear diffusion equation
- $$\frac{\partial}{\partial x} \left[ a(u+b)^{-2} \frac{\partial u}{\partial x} \right] - \frac{\partial u}{\partial t} = 0.$$
- J. Math. Phys.* **21**, 1019–1023.
- Bluman, G.W. and Kumei, S. (1987). On invariance properties of the wave equation. *J. Math. Phys.* **28**, 307–318.
- Bluman, G.W. and Kumei, S. (1988). Exact solutions for wave equations of two-layered media with smooth transition. *J. Math. Phys.* **29**, 86–96.
- Bluman, G.W. and Kumei, S. (1989a). Use of group analysis in solving overdetermined systems of ordinary differential equations. *J. Math. Anal. Appl.* **138**, 95–105.
- Bluman, G.W. and Kumei, S. (1989b). *Symmetries and Differential Equations*. Applied Mathematical Sciences, No. 81. Springer-Verlag, New York, NY.
- Bluman, G.W. and Kumei, S. (1990a). Symmetry-based algorithms to relate partial differential equations, I. Local symmetries. *European J. Appl. Math.* **1**, 189–216.
- Bluman, G.W. and Kumei, S. (1990b). Symmetry-based algorithms to relate partial differential equations, II. Linearization by nonlocal symmetries. *European J. Appl. Math.* **1**, 217–223.
- Bluman, G.W., Kumei, S., and Reid, G.J. (1988). New classes of symmetries for partial differential equations. *J. Math. Phys.* **29**, 806–811; Erratum, *J. Math. Phys.* **29**, 2320.
- Bluman, G.W. and Shtelen, V.M. (1998). Preprint. Nonlocal transformations of diffusion processes to Wiener processes. Department of Mathematics, University of British Columbia, BC.

- Boisvert, R.E., Ames, W.F., and Srivastava, U.N. (1983). Group properties and new solutions of Navier-Stokes equations. *J. Engrg. Math.* **17**, 203–221.
- Boyer, T.H. (1967). Continuous symmetries and conserved currents. *Ann. Physics* **42**, 445–466.
- Bridgman, P.W. (1931). *Dimensional Analysis*, 2nd ed. Yale University Press, New Haven, CT.
- Buckingham, E. (1914). On physically similar systems; illustrations of the use of dimension equations. *Phys. Rev.* **4**, 345–376.
- Buckingham, E. (1915a). The principle of similitude. *Nature* **96**, 396–397.
- Buckingham, E. (1915b). Model experiments and the forms of empirical equations. *Trans. ASME* **37**, 263–296.
- Cantwell, B.J. (1978). Similarity transformations for the two-dimensional, unsteady, stream-function equation. *J. Fluid Mech.* **85**, 257–271.
- Champagne, B., Hereman, W., and Winternitz, P. (1991). The computer calculation of Lie point symmetries of large systems of differential equations. *Comput. Phys. Comm.* **66**, 319–340.
- Cheb-Terrab, E.S. and Roche, A.D. (1999). Integrating factors for second-order ODEs. *J. Symbolic Comput.* **27**, 501–519.
- Cicogna, G. and Vitali, D. (1990). Classification of the extended symmetries of Fokker–Planck equations. *J. Phys.* **A23**, L85–L88.
- Clarkson, P.A. and Mansfield, E.L. (1994a). Symmetry reductions and exact solutions of a class of nonlinear equations. *Physica* **D70**, 250–288.
- Clarkson, P.A. and Mansfield, E.L. (1994b). Algorithms for the nonclassical method of symmetry reductions. *SIAM J. Appl. Math.* **54**, 1693–1719.
- Coddington, E.A. (1961). *An Introduction to Ordinary Differential Equations*. Prentice-Hall, Englewood Cliffs, NJ. Reprinted (1989), Dover, New York, NY.
- Cohen, A. (1911). *An Introduction to the Lie Theory of One-Parameter Groups, with Applications to the Solution of Differential Equations*. D.C. Heath, New York, NY.
- Cohn, P.M. (1965). *Lie Groups*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 46. Cambridge University Press, Cambridge, UK.
- Cole, J.D. and Wagner, B. (1996). On self-similar solutions of Barenblatt’s nonlinear filtration equation. *European J. Appl. Math.* **7**, 151–167.
- Curtis, W.D., Logan, J.D., and Parker, W.A. (1982). Dimensional analysis and the pi theorem. *Linear Algebra Appl.* **47**, 117–126.
- de Jong, F.J. (1967). *Dimensional Analysis for Economists*. North-Holland, Amsterdam.

Dickson, L.E. (1924). Differential equations from the group standpoint. *Ann. of Math.* **25**, 287–378.

Dresner, L. (1983). *Similarity Solutions of Nonlinear Partial Differential Equations*. Research Notes in Mathematics, No. 88. Pitman, Boston, MA.

Dresner, L. (1999). *Applications of Lie's Theory of Ordinary and Partial Differential Equations*. Institute of Physics, Bristol, UK.

Drew, M.S., Kloster, S.C., and Gegenberg, J.D. (1989). Lie group analysis and similarity solutions for the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 (e^u)}{\partial z^2} = 0.$$

*Nonlinear Anal., Theory, Methods Appl.* **13**, 489–505.

Edwards, M.P. and Broadbridge, P. (1995). Exceptional symmetry reductions of Burgers' equation in two and three spatial dimensions. *Z. Angew. Math. Phys.* **46**, 595–622.

Eisenhart, L.P. (1933). *Continuous Groups of Transformations*. Princeton University Press, Princeton, NJ.

Friedman, A. and Kamin, S. (1980). The asymptotic behavior of gas in an  $n$ -dimensional porous medium. *Trans. Amer. Math. Soc.* **262**, 551–563.

Galaktionov, V.A., Dorodnitsyn, V.A., Elenin, G.G., Kurdyumov, S.P., and Samarskii, A.A. (1988). A quasilinear heat equation with a source: Peaking, localization, symmetry, exact solutions, asymptotics, structures. *J. Soviet Math.* **41**, 1222–1292.

Galaktionov, V.A. and Samarskii, A.A. (1984). Methods of constructing approximate self-similar solutions of nonlinear heat equations, IV. *Math. USSR-Sb.* **49**, 125–150.

Gandarias, M.L. (1996). Classical point symmetries of a porous medium equation. *J. Phys.* **A29**, 607–633.

Gilmore, R. (1974). *Lie Groups, Lie Algebras, and Some of Their Applications*. Wiley, New York, NY.

Goldenfeld, N. (1992). *Lectures on Phase Transitions and the Renormalization Group*. Addison-Wesley, Reading, MA.

Gonzalez-Gascon, F. and Gonzalez-Lopez, A. (1983). Symmetries of differential equations, IV. *J. Math. Phys.* **24**, 2006–2021.

Gordon, T.J. (1986). On the symmetries and invariants of the harmonic oscillator. *J. Phys.* **A19**, 183–189.

Görtler, H. (1975). Zur Geschichte des  $\pi$ -Theorems. *Z. Angew. Math. Mech.* **55**, 3–8.

Greub, W.H. (1967). *Multilinear Algebra*. Springer-Verlag, New York, NY.

Hansen, A.G. (1964). *Similarity Analyses of Boundary Value Problems in Engineering*. Prentice-Hall, Englewood Cliffs, NJ.

Haynes, R. (1982). *An Introduction to Dimensional Analysis for Geographers*. Institute of British Geographers, London, UK.

Head, A.K. (1992). LIE: A muMATH Program for the Calculation of the LIE Algebra of Differential Equations. CSIRO Division of Material Sciences, Clayton, Australia.

Herederó, R.H., and Olver, P.J. (1996). Classification of invariant wave equations, *J. Math. Phys.* **37**, 6414–6438.

Hereman, W. (1996). *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 3: *New Trends in Theoretical Developments and Computational Methods*, Chap. 13, pp. 367–413. (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.

Holmes, M.H. (1984). Comparison theorems and similarity solution approximations for a nonlinear diffusion equation arising in the study of soft tissue. *SIAM J. Appl. Math.* **44**, 545–556.

Hydon, P.E. (2000). *Symmetry Methods for Differential Equations. A Beginner's Guide*. Cambridge University Press, Cambridge, UK.

Ibragimov, N.H. (1985). *Transformation Groups Applied to Mathematical Physics*. Reidel, Boston, MA. See also *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2, Chap. 11, pp. 272–273 (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.

Ibragimov, N.H. (1995). *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2, Chap. 11, pp. 269–293 (N.H. Ibragimov, ed.). CRC Press, Boca Raton, FL.

Kamin, S. (1975). Similarity solutions and the asymptotics of infiltration equations. *Arch. Rational Mech. Anal.* **60**, 171–183.

Kamke, E. (1943). *Differentialgleichungen, Lösungsmethoden und Lösungen*, Vol. 1, 2nd ed. Akademische Verlagsgesellschaft, Leipzig.

Kaplan, W. (1958). *Ordinary Differential Equations*. Addison-Wesley, Reading, MA.

Kersten, P.H.M. (1987). *Infinitesimal Symmetries: a Computational Approach*. CWI Tract No. 34. Centrum voor Wiskunde en Informatica, Amsterdam.

King, J.R. (1989). Exact solutions to some nonlinear diffusion equations. *Quart. J. Mech. Appl. Math.* **42**, 537–552.

King, J.R. (1991). Exact results for the nonlinear diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( u^{-4/3} \frac{\partial u}{\partial x} \right) \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( u^{-2/3} \frac{\partial u}{\partial x} \right).$$

*J. Phys.* **A24**, 5721–5745.

Klamkin, M.S. (1962). On the transformation of a class of boundary value problems into initial value problems for ordinary differential equations. *SIAM Rev.* **4**, 43–47.

- Krasil'shchik, I.S. and Vinogradov, A.M. (1989). Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations. *Acta Appl. Math.* **15**, 161–209.
- Kumei, S. and Bluman, G.W. (1982). When nonlinear differential equations are equivalent to linear differential equations. *SIAM J. Appl. Math.* **42**, 1157–1173.
- Kurth, K. (1972). *Dimensional Analysis and Group Theory in Astrophysics*. Pergamon Press, Oxford, UK.
- Lefschetz, S. (1963). *Differential Equations: Geometric Theory*, 2nd ed. Interscience, New York, NY.
- Levi, D. and Winternitz, P. (1989). Non-classical symmetry reduction: example of the Boussinesq equation. *J. Phys.* **A22**, 2915–2924.
- Lie, S. (1881). Über die Integration durch bestimmte Integrale von einer Klasse linearer partieller Differentialgleichungen. *Arch. Math.* **6**, 328–368; also *Gesammelte Abhandlungen*, Vol. III, pp. 492–523. B.G. Teubner, Leipzig, 1922.
- Lie, S. (1893). *Theorie der Transformationsgruppen*, Vol. III. B.G. Teubner, Leipzig.
- Lighthill, M.J (1958). *Introduction to Fourier Analysis and Generalised Functions*, Cambridge University Press, Cambridge, UK.
- Lisle, I. (1992). Equivalence Transformations for Classes of Differential Equations. Ph.D. Thesis, University of British Columbia, BC.
- Liu, Q. and Fang, F. (1986). Symmetry and invariant solution of the Schlögl model. *Physica* **139A**, 543–552.
- Mandelbrot, B.B. (1977). *Fractals: Form, Chance and Dimension*. W.H. Freeman, San Francisco, CA.
- Mandelbrot, B.B. (1982). *The Fractal Geometry of Nature*. W.H. Freeman, New York, NY.
- Mansfield, E.L. (1996). The differential algebra package diffgrob2. *Maple Tech.* **3**, 33–37.
- Mansfield, E.L. and Clarkson, P.A. (1997). Applications of the differential algebra package diffgrob2 to classical symmetries of differential equations. *J. Symbolic Comput.* **23**, 517–533.
- Marsden, J.E. and Ratiu, T.S. (1999). *Introduction to Mechanics and Symmetry*. Texts in Applied Mathematics, No. 17. Springer, New York, NY.
- Mikhailov, A.V., Shabat, A.B., and Sokolov, V.V. (1991). The symmetry approach to classification of integrable equations. In *What Is Integrability?* (V.E. Zakharov, ed.). Springer-Verlag, Berlin, pp. 115–184.
- Milinzazzo, F. (1974). Numerical Algorithms for the Solution of a Single Phase One-Dimensional Stefan Problem. Ph.D. Thesis, University of British Columbia, BC.

- Milinzazzo, F. and Bluman, G.W. (1975). Numerical similarity solutions to Stefan problems. *Z. Angew. Math. Mech.* **55**, 423–429.
- Miller, W., Jr. (1977). *Symmetry and Separation of Variables*. Addison-Wesley, Reading, MA.
- Mimura, F. and Nôno, T. (1994). A new conservation law for a system of second-order differential equations. *Bull. Kyushu Inst. Tech.* **41**, 1–10.
- Murota, K. (1985). Use of the concept of physical dimensions in the structural approach to systems analysis. *Japan. J. Appl. Math.* **2**, 471–494.
- Na, T.Y. (1967). Transforming boundary conditions to initial conditions for ordinary differential equations. *SIAM Rev.* **9**, 204–210.
- Na, T.Y. (1979). *Computational Methods in Engineering Boundary Value Problems*. Academic Press, New York, NY.
- Neuringer, J.L. (1968). Green's function for an instantaneous line particle source diffusing in a gravitational field and under the influence of a linear shear wind. *SIAM J. Appl. Math.* **16**, 834–842.
- Newman, W.I. (1984). A Lyapunov functional for the evolution of solutions to the porous medium equation to self-similarity, I. *J. Math. Phys.* **25**, 3120–3123.
- Noether, E. (1918). Invariante Variationsprobleme. *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.*, pp. 235–257.
- Nucci, M.C. and Clarkson, P.A. (1992). The nonclassical method is more general than the direct method for symmetry reductions. An example of the Fitzhugh–Nagumo equation. *Phys. Lett.* **A164**, 49–56.
- Olver, P.J. (1977). Evolution equations possessing infinitely many symmetries. *J. Math. Phys.* **18**, 1212–1215.
- Olver, P.J. (1986). *Applications of Lie Groups to Differential Equations*. Graduate Texts in Mathematics, No. 107. Springer-Verlag, New York, NY.
- Ovsianikov, L.V. (1959). Group properties of the nonlinear heat conduction equation. *Dokl. Akad. Nauk. USSR* **125**, 492–495 (in Russian).
- Ovsianikov, L.V. (1962). *Group Properties of Differential Equations*. Nauka, Novosibirsk (in Russian).
- Ovsianikov, L.V. (1982). *Group Analysis of Differential Equations*. Academic Press, New York, NY.
- Page, J.M. (1896). Note on singular solutions. *Amer. J. Math.* **XVIII**, 95–97.
- Page, J.M. (1897). *Ordinary Differential Equations with an Introduction to Lie's Theory of the Group of One Parameter*. Macmillan, London, UK.



- Reid, G.J. (1990). A triangularization algorithm which determines the Lie symmetry algebra of any system of PDEs. *J. Phys.* **A23**, L853–L859.
- Reid, G.J. (1991). Algorithms for reducing a system of PDEs to standard form, determining the dimension of its solution space and calculating its Taylor series solution. *European J. Appl. Math.* **2**, 293–318.
- Rosinger, E.E. and Walus, Y.E. (1994). Group invariance of generalized solutions obtained through the algebraic method. *Nonlinearity* **7**, 837–859.
- Rudra, P. (1990). Symmetry classes of Fokker–Planck-type equations. *J. Phys.* **A23**, 1663–1670.
- Sarlet, W., Cantrijn, F., and Crampin, M. (1987). Pseudo-symmetries, Noether's theorem and the adjoint equations. *J. Phys.* **A20**, 1365–1376.
- Sarlet, W., Prince, G.E., and Crampin, M. (1990). Adjoint symmetries for time-independent second-order equations. *J. Phys.* **A23**, 1335–1347.
- Schepartz, B. (1980). *Dimensional Analysis in the Biomedical Sciences*. C.C. Thomas, Springfield, IL.
- Schlichting, H. (1955). *Boundary Layer Theory*. McGraw-Hill, New York, NY.
- Schwarz, F. (1985). Automatically determining symmetries of partial differential equations. *Computing* **34**, 91–106.
- Schwarz, F. (1988). Symmetries of differential equations: From Sophus Lie to computer algebra. *SIAM Rev.* **30**, 450–481.
- Sedov, L.I. (1982). *Similarity and Dimensional Methods in Mechanics*, 9th ed. Mir, Moscow.
- Senthilvelan, M. and Lakshmanan, M. (1995). Lie symmetries and infinite-dimensional Lie algebras of certain nonlinear dissipative systems. *J. Phys.* **A28**, 1929–1942.
- Seshadri, R. and Na, T.Y. (1985). *Group Invariance in Engineering Boundary Value Problems*. Springer-Verlag, New York, NY.
- Sheftel, M.B. (1997). *A Course on Group Analysis of Differential Equations*. Part II. *Ordinary Differential Equations*. Nauka, St. Petersburg (in Russian).
- Sophocleous, C. (1992). On symmetries of radially symmetric nonlinear diffusion equations. *J. Math. Phys.* **33**, 3687–3693.
- Stephani, H. (1989). *Differential Equations: Their Solution Using Symmetries*. Cambridge University Press, Cambridge, UK.
- Stognii, V.I. and Shtelen, V.M. (1991). Symmetries and particular solutions of some Fokker–Planck equations. *Ukrainian Math. J.* **43**, 456–460 (in Russian).
- Tajiri, M. (1983). Similarity reductions of the one- and two-dimensional nonlinear Schrödinger equations. *J. Phys. Soc. Japan* **52**, 1908–1917.

- Tajiri, M. and Hagiwara, M. (1983). Similarity solutions for the two-dimensional coupled nonlinear Schrödinger equation. *J. Phys. Soc. Japan* **52**, 3727–3734.
- Taylor, E.S. (1974). *Dimensional Analysis for Engineers*. Clarendon Press, Oxford, UK.
- Taylor, Sir G.I. (1950). The formation of a blast wave by a very intense explosion, II. The atomic explosion of 1945. *Proc. Roy. Soc.* **A201**, 175–186.
- Torrise, M. and Valenti, A. (1985). Group properties and invariant solutions for infinitesimal transformations of a nonlinear wave equation. *Internat. J. Nonlinear Mech.* **20**, 135–144.
- Venikov, V.A. (1969). *Theory of Similarity and Simulation*. MacDonald Technical and Scientific, London, UK.
- Warner, F.J. (1983). *Foundations of Differentiable Manifolds and Lie Groups*. Graduate Texts in Mathematics, No. 94. Springer-Verlag, New York, NY.
- Watson, G.N. (1922). *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, UK.
- Wolf, T. (2002a). Investigating differential equations with CRACK, LiePDE, Applysimm and ConLaw. To appear in *Handbook of Computer Algebra, Foundations, Applications, Systems* (J. Grabmeier, E. Kaltofen, and V. Weispfenning, eds.). Springer, New York, NY.
- Wolf, T. (2002b). A comparison of four approaches to the calculation of conservation laws. To appear in *European J. Appl. Math.*
- Wolf, T. and Brand, A. (1992). The computer algebra package CRACK for investigating PDEs. In: *Proceedings of ERCIM Advanced Course on Partial Differential Equations and Group Theory*, Bonn.
- Wulfman, C.E. (1979). Limit cycles as invariant functions of Lie groups. *J. Phys.* **A12**, L73–L75.
- Zel'dovich, Ya.B. (1956). The motion of a gas under the action of a short term pressure (shock). *Akust. Zh.* **2**, 28–38 (in Russian).
- Zierer, J. (1971). *Similarity Laws and Modeling*. Marcel Dekker, New York, NY.

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