

A

The Green-Rees Local Structure Theory

The goal of this appendix is to give an admittedly terse review of the Green-Rees structure theory of stable semigroups (or what might be referred to as the local theory, in comparison with the semilocal theory of Section 4.6). More complete references for this material are [68, 139, 171].

A.1 Ideal Structure and Green's Relations

If S is a semigroup, then $S^I = S \cup \{I\}$, where I is a newly adjoined identity. If X, Y are subsets of S , then $XY = \{xy \mid x \in X, y \in Y\}$.

Definition A.1.1 (Ideals). *Let S be a semigroup. Then:*

1. $\emptyset \neq R \subseteq S$ is a right ideal if $RS \subseteq R$;
2. $\emptyset \neq L \subseteq S$ is a left ideal if $SL \subseteq L$;
3. $\emptyset \neq I \subseteq S$ is an ideal if it is both a left ideal and a right ideal.

If $s \in S$, then sS^I is the *principal right ideal* generated by s , $S^I s$ is the *principal left ideal* generated by s and $S^I s S^I$ is the *principal ideal* generated by s . If S is a monoid, then $S^I s = Ss$, $sS^I = sS$ and $S^I s S^I = SsS$.

A semigroup is called *left simple*, *right simple*, or *simple* if it has no proper, respectively, left ideal, right ideal, or ideal. The next proposition is straightforward; the proof is left to the reader.

Proposition A.1.2. *Let I, J be ideals of S . Then*

$$IJ = \{ij \mid i \in I, j \in J\}$$

is an ideal and $\emptyset \neq IJ \subseteq I \cap J$. Consequently, the set of ideals of S is closed under finite intersection.

Corollary A.1.3. *A finite semigroup S has a unique minimal ideal.*

Proof. Proposition A.1.2 implies that the intersection of all ideals of S is again an ideal; clearly it is the unique minimal ideal. \square

Note that if a semigroup has a zero, then its minimal ideal is $\{0\}$. In this context, the notion of minimal ideal is not so useful, and so we introduce the notion of a 0-minimal ideal.

Definition A.1.4 (0-minimal ideal). *A minimal non-zero ideal of a semigroup is called a 0-minimal ideal. We also, by convention, consider the minimal ideal of the trivial semigroup to be a 0-minimal ideal.*

With this definition, the minimal ideal of a semigroup without zero is considered to be a 0-minimal ideal. This convention is somewhat non-standard, but is convenient for stating results uniformly for semigroups with 0 and without 0. Note that 0-minimal ideals do not have to be unique. Next we introduce Green’s relations [68, 108, 171]. They are an essential ingredient in semigroup theory.

Definition A.1.5 (Green’s relations). *Let S be a semigroup and $s, t \in S$. Green’s equivalence relations \mathcal{R} , \mathcal{L} , \mathcal{H} and \mathcal{D} are defined as follows:*

- $s \mathcal{R} t \iff sS^I = tS^I$;
- $s \mathcal{L} t \iff S^I s = S^I t$;
- $s \mathcal{J} t \iff S^I s S^I = S^I t S^I$;
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$;
- $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$.

The \mathcal{R} -class of an element $s \in S$ is typically denoted by R_s . Similar meanings can be ascribed to L_s , J_s , H_s and D_s .

Exercise A.1.6. Show that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ and deduce from this

$$\mathcal{L} \circ \mathcal{R} = \mathcal{D} = \mathcal{R} \circ \mathcal{L}.$$

All of Green’s relations coincide for a commutative semigroup. Note that

$$\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

We shall see $\mathcal{D} = \mathcal{J}$ for stable (in particular, finite) semigroups. Green’s relations are more naturally derived from his preorders.

Definition A.1.7 (Green’s preorders). *Green’s preorders are defined by:*

- $s \leq_{\mathcal{R}} t \iff sS^I \subseteq tS^I$;
- $s \leq_{\mathcal{L}} t \iff S^I s \subseteq S^I t$;
- $s \leq_{\mathcal{J}} t \iff S^I s S^I \subseteq S^I t S^I$;
- $s \leq_{\mathcal{H}} t \iff s \leq_{\mathcal{L}} t \text{ and } s \leq_{\mathcal{R}} t$.

Green's relations are the equivalence relations associated to his preorders. Therefore, for instance, $\leq_{\mathcal{R}}$ induces a partial order on the set of \mathcal{R} -classes of S . An element e of a semigroup is *idempotent* if $e^2 = e$. Often, if e, f are idempotents, then $e \leq_{\mathcal{H}} f$ is abbreviated to $e \leq f$. Observe that $e \leq f$ if and only if $ef = e = fe$ and that this is a partial order on the set of idempotents.

Exercise A.1.8. Show that if e is an idempotent, then $s \leq_{\mathcal{L}} e$ if and only if $se = s$, and that $s \leq_{\mathcal{R}} e$ if and only if $es = s$. Conclude that if e, f are idempotents, then $e \leq f$ if and only if $ef = e = fe$.

Proposition A.1.9. *Green's relation \mathcal{R} is a left congruence and \mathcal{L} is a right congruence.*

Proof. Suppose $s \mathcal{R} t$ and $x \in S$. Then $sS^I = tS^I$ and so $xsS^I = xtS^I$. Therefore \mathcal{R} is a left congruence. The proof for \mathcal{L} is dual. \square

Similarly one can show that the preorder $\leq_{\mathcal{R}}$ is stable under left multiplication and the preorder $\leq_{\mathcal{L}}$ is stable under right multiplication.

Exercise A.1.10. Verify this last assertion.

The following notion, due to von Neumann in the context of ring theory [375], plays a fundamental role in semigroup theory.

Definition A.1.11 (Regular element). *An element s of a semigroup S is (von Neumann) regular if there exists $t \in S$ such that $sts = s$, i.e., $s \in sSs$. A semigroup is said to be regular if each of its elements is regular.*

It is immediate that idempotents are regular. In a group, all elements are regular.

Exercise A.1.12. Verify that if s is regular then $S^I s = Ss$, $sS^I = sS$ and $S^I s S^I = SsS$.

Definition A.1.13 (Inverse). *Two elements s, s' of a semigroup are said to be inverse to each other if $ss's = s$ and $s'ss' = s'$.*

Having an inverse is equivalent to being regular as the next proposition shows.

Proposition A.1.14. *Let $s \in S$ be a regular element. Then s has an inverse.*

Proof. Suppose $s = sts$. Setting $s' = tst$ yields

$$\begin{aligned} ss's &= ststs = sts = s \\ s'ss' &= tststst = tstst = tst = s'. \end{aligned}$$

Thus s and s' are inverses. \square

Inverse elements allow for a description of \mathcal{D} -equivalence of idempotents analogous to von Neumann-Murray equivalence of projections in operator algebras [31].

Proposition A.1.15. *Let S be a semigroup. Two idempotents $e, f \in S$ are \mathcal{D} -equivalent if and only if there exist inverse elements $x, x' \in S$ with $e = xx'$ and $f = x'x$. More precisely, if $e, f \in S$ are idempotents and $x \in S$, then $e \mathcal{R} x \mathcal{L} f$ if and only if there exists an inverse x' of x such that $xx' = e$ and $x'x = f$, in which case $e \mathcal{L} x' \mathcal{R} f$.*

Proof. We begin with the second statement, as the first is an immediate consequence of it. If x, x' are inverses, then it is straightforward to verify $x \mathcal{R} xx' \mathcal{L} x'$ and $x \mathcal{L} x'x \mathcal{R} x'$. This handles the if direction. For the only if direction, suppose $e \mathcal{R} x \mathcal{L} f$. Then we can find $u, v \in S^I$ with $xu = e$ and $vx = f$. Set $x' = fue$. Then

$$\begin{aligned} xx'x &= x(fue)x = xux = ex = x \\ x'xx' &= (fue)x(fue) = fuxue = fue = x' \end{aligned}$$

and so x, x' are inverses. Moreover, $xx' = xfue = xue = e$ and $x'x = fuex = fux = vxux = v ex = vx = f$ and so $e \mathcal{L} x' \mathcal{R} f$, completing the proof. \square

A semigroup is called an *inverse semigroup* if each element has a unique inverse. A *block group* is a semigroup in which each element has at most one inverse, or equivalently a semigroup in which each regular element admits a unique inverse.

In general, Green's relations on a subsemigroup do not coincide with Green's relations on the ambient semigroup. For instance, if B_2 is the semigroup of 2×2 matrix units and the zero matrix, then $E_{11} \mathcal{J} E_{22}$ in B_2 , but in the subsemigroup $\{E_{11}, E_{22}, 0\}$ they are not \mathcal{J} -equivalent. However, the following is true.

Proposition A.1.16. *Let S be a semigroup and $T \leq S$ a subsemigroup. Suppose $t_1, t_2 \in T$ are regular. Then $t_1 \mathcal{H} t_2$ in T if and only if $t_1 \mathcal{H} t_2$ in S where \mathcal{H} is any of \mathcal{R}, \mathcal{L} or \mathcal{H} .*

Proof. We just handle \mathcal{R} as the case of \mathcal{L} is dual, and the result for \mathcal{H} follows from those for \mathcal{L} and \mathcal{R} . Clearly, $t_1 \mathcal{R} t_2$ in T implies $t_1 \mathcal{R} t_2$ in S . For the converse, suppose $t_1 \mathcal{R} t_2$ in S and choose by regularity elements $x_1, x_2 \in T$ such that $t_i x_i t_i = t_i, i = 1, 2$. Then $e_i = t_i x_i$ is an idempotent of T with $e_i \mathcal{R} t_i$ in T for $i = 1, 2$. It thus suffices to show $e_1 \mathcal{R} e_2$ in T . But $e_1 \mathcal{R} e_2$ in S implies $e_1 e_2 = e_2$ and $e_2 e_1 = e_1$. Thus $e_1 \mathcal{R} e_2$ in T , as required. \square

A.2 Stable Semigroups

For the rest of this appendix, we shall be interested in stable semigroups. This class of semigroups include finite semigroups, compact semigroups (Proposi-

tion 3.1.10) and commutative semigroups. Algebraic semigroups are also stable [250, 262]. Although at first sight the definition may seem bizarre, it is somehow the crucial property that makes finite semigroup theory “work.”

Definition A.2.1 (Stability). *A semigroup S is called stable if both*

$$s \mathcal{J} sx \iff s \mathcal{R} sx \text{ and also } s \mathcal{J} xs \iff s \mathcal{L} xs.$$

The archetypical example of an unstable semigroup is the bicyclic monoid. It is the monoid of transformations of \mathbb{N} generated by $\sigma, \tau: \mathbb{N} \rightarrow \mathbb{N}$ given by $\sigma(x) = x + 1$ and

$$\tau(x) = \begin{cases} x - 1 & x > 0 \\ 0 & x = 0. \end{cases}$$

Equivalently, it is the semigroup generated by a unilateral shift on a separable Hilbert space and its adjoint. The next exercise gives the reader a chance to become familiar with the notion of stability.

Exercise A.2.2. Suppose S is a stable semigroup.

1. Show that $xsx \mathcal{J} x$ if and only if $xsx \mathcal{H} x$.
2. Show that if $e \in S$ is an idempotent, then $eSe \cap J_e = H_e$.
3. Show that if S is a monoid with identity 1, then $J_1 = H_1$, which in turn is the group of units of S . Deduce that the non-units of S form an ideal.
4. Show that a congruence-free stable monoid is either a simple group or is isomorphic to $(\{0, 1\}, \cdot)$.
5. Show that if e, f are idempotents of S with $e \mathcal{J} f$ and $e \leq f$, then $e = f$.
6. Show that the bicyclic monoid is not stable.
7. Show that the bicyclic monoid does not embed in a stable semigroup.
8. Show that a regular semigroup is stable if and only if it does not contain an isomorphic copy of the bicyclic monoid.

Exercise A.2.3. Let S be a non-empty finite semigroup. Prove that S contains an idempotent. Hint: Let T be a minimal (non-empty) subsemigroup of S . Show that $tT = T = Tt$ for all $t \in T$. Deduce that if $tx = t$, then $x = x^2$.

The following result is a special case of Proposition 3.1.10.

Theorem A.2.4. *Finite semigroups are stable.*

Proof. Clearly, $s \mathcal{R} st$ implies $s \mathcal{J} st$. Suppose $s \mathcal{J} st$. Evidently $st \leq_{\mathcal{R}} s$, so we are left with establishing the reverse inequality. Because $s \mathcal{J} st$, we can find $x, y \in S^I$ such that $s = xsty$. One then has $s = x^n s (ty)^n$ for all $n > 0$. Let k be a positive integer such that $(ty)^k$ is idempotent. Then

$$s(ty)^k = x^k s (ty)^k (ty)^k = x^k s (ty)^k = s$$

so $st(y(ty)^{k-1}) = s(ty)^k = s$. Thus $s \leq_{\mathcal{R}} st$, as was required. The argument for \mathcal{L} is dual. □

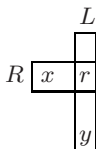


Fig. A.1. \mathcal{L} -classes and \mathcal{R} -classes of a \mathcal{J} -class intersect in stable semigroups

Stability implies Green’s relations \mathcal{J} and \mathcal{D} coincide.

Corollary A.2.5. *Let S be a stable semigroup. Then $\mathcal{J} = \mathcal{D}$. More precisely, the following are equivalent for $s, t \in S$:*

1. $s \mathcal{J} t$;
2. there exists $r \in S$ such that $s \mathcal{L} r \mathcal{R} t$;
3. there exists $z \in S$ such that $s \mathcal{R} z \mathcal{L} t$;
4. $s \mathcal{D} t$.

Proof. Suppose first $s \mathcal{J} t$. Then there exist $u, v \in S^I$ such that $usv = t$. Hence $us \mathcal{J} s, sv \mathcal{J} s$ and so $us \mathcal{L} s, sv \mathcal{R} s$ by stability. Because \mathcal{R} is a left congruence, $s \mathcal{R} us \mathcal{R} usv = t$ and dually, because \mathcal{L} is a right congruence, $s \mathcal{R} sv \mathcal{L} usv = t$ thereby establishing that 1 implies 2 and 3. Clearly 2 and 3 imply 4, and 4 implies 1. This completes the proof. \square

What does all this mean about the structure of a stable semigroup? In any semigroup, each \mathcal{J} -class J is a disjoint union of \mathcal{R} -classes and also a disjoint union of \mathcal{L} -classes. If L is an \mathcal{L} -class and R is an \mathcal{R} -class of J , then either $L \cap R = \emptyset$ or $L \cap R$ is an \mathcal{H} -class. In a stable semigroup, this intersection is never empty. Indeed, suppose $x \in R, y \in L$. Then $x \mathcal{J} y$ implies there exists r such that $x \mathcal{R} r \mathcal{L} y$. Thus $r \in L \cap R$, see Figure A.1.

The picture of a \mathcal{J} -class J of a stable semigroup is something like an eggbox where the rows represent the \mathcal{R} -classes of J , the columns represent the \mathcal{L} -classes and the boxes represent the \mathcal{H} -classes, see Figure A.2.

As an example, consider the full transformation monoid T_n of degree n . We view T_n as acting on the right of $\{0, \dots, n - 1\}$. If $f \in T_n$, define $\ker f$ to be the equivalence relation given by $(x, y) \in \ker f$ if $xf = yf$. Define the rank of f by $\text{rk}(f) = |\text{Im } f|$.

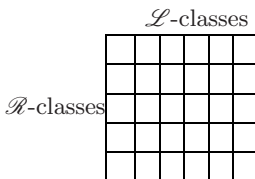


Fig. A.2. Eggbox picture of a \mathcal{J} -class of a stable semigroup

Exercise A.2.6. Let $f, g \in T_n$. Show:

1. $f \mathcal{L} g$ if and only if $\text{Im } f = \text{Im } g$;
2. $f \mathcal{R} g$ if and only if $\ker f = \ker g$;
3. $f \mathcal{J} g$ if and only if $\text{rk}(f) = \text{rk}(g)$.

A.3 Green's Lemma and Maximal Subgroups

The following is known as Green's Lemma [108] and is used throughout this book, often without comment.

Lemma A.3.1 (Green's Lemma). *Let S be a semigroup.*

1. *Let $s \mathcal{L} t$ and $u, v \in S^I$ be such that $us = t, vt = s$. Then $\varphi: R_s \rightarrow R_t$ defined by $x\varphi = ux$ and $\psi: R_t \rightarrow R_s$ defined by $y\psi = vy$ are inverse bijections. Moreover, if $x, x' \in R_s$ with $x \mathcal{H} x'$, then $x\varphi \mathcal{H} x'\varphi$.*
2. *Let $s \mathcal{R} t$ and $u, v \in S^I$ be such that $su = t, tv = s$. Then $\varphi: L_s \rightarrow L_t$ defined by $x\varphi = xu$ and $\psi: L_t \rightarrow L_s$ defined by $y\psi = yv$ are inverse bijections. Moreover, if $x, x' \in L_s$ with $x \mathcal{H} x'$, then $x\varphi \mathcal{H} x'\varphi$.*
3. *If $s \mathcal{L} r \mathcal{R} t$ and $u, v, w, z \in S^I$ are such that $us = r, vr = s, rw = t$ and $tz = r$, then $\varphi: H_s \rightarrow H_t$ given by $x\varphi = uxw$ and $\psi: H_t \rightarrow H_s$ given by $y\psi = vyz$ are inverse bijections.*

Proof. We prove only 1, as 2 is dual and 3 follows directly from 1 and 2. Let $x \in R_s$. Because \mathcal{R} is a left congruence, $x \mathcal{R} s$ implies $ux \mathcal{R} us = t$. So $x\varphi = ux \in R_t$, establishing that φ is well defined. Similarly, ψ is well defined. Let us show that $\varphi\psi = 1_{R_s}$. If $x \in R_s$ then $x = sz$, for some $z \in S^I$. So

$$x\varphi\psi = ux\psi = vux = vusz = vtz = sz = x.$$

A similar verification shows $\psi\varphi = 1_{R_t}$.

Suppose now that $x, x' \in R_s$ and $x \mathcal{H} x'$. Then there exist $z, z' \in S^I$ such that $zx = x'$, and $z'x' = x$. So $(uzv)ux = uz(x\varphi\psi) = uzx = ux'$ and $(uz'v)ux' = uz'x' = ux$. Thus $x\varphi = ux \mathcal{L} ux' = x'\varphi$. But $x\varphi, x'\varphi \in R_t$, so in fact $x\varphi \mathcal{H} x'\varphi$, as required. \square

In a stable semigroup, Green's Lemma basically says left multiplication bijectively maps rows to rows and right multiplication bijectively maps columns to columns in the eggbox picture (provided one does not leave the \mathcal{J} -class). See Figure A.3.

As a consequence of Green's Lemma, each row (respectively column) has the same number of boxes and all boxes are the same size.

Corollary A.3.2. *All \mathcal{H} -classes of a \mathcal{J} -class J of a finite semigroup have the same cardinality. Similarly, all \mathcal{L} -classes of J have the same size and all \mathcal{R} -classes of J have the same size. All \mathcal{L} -classes of J contain the same number of \mathcal{H} -classes and all \mathcal{R} -classes of J contain the same number of \mathcal{H} -classes.*

		vt, sw			s, vtz	
		t, usw			r	

Fig. A.3. Item 3 of Green’s Lemma

The next lemma is classical.

Lemma A.3.3. *A non-empty semigroup is a group if and only if it is both left and right simple.*

Proof. Clearly, a group is both left and right simple. For the converse, suppose S is left and right simple. Let $t \in S$. Choose $e, f \in S$ such that $te = t$ and $ft = t$. Then if $s \in S$, $s = zt$ for some $z \in S$. So $se = zte = zt = s$. Similarly $fs = s$. Thus e is a right identity and f is a left identity. We conclude $f = fe = e$ and so S is a monoid with identity e .

If $s \in S$, then there exist $z, z' \in S$ such that $zs = e$ and $sz' = e$. Thus z is a left inverse for s and z' is right inverse for s . Therefore, $z = zsz' = z'$ and so s is invertible. Hence S is a group. \square

Notice if G is a subgroup (with identity e) of a semigroup S , then $G \subseteq H_e$. We are now ready to prove a result of Green, which in particular identifies the maximal subgroups of a semigroup as the \mathcal{H} -classes of idempotents [68, 108, 171].

Theorem A.3.4 (Green). *Let S be a semigroup and $s \in S$. Then the following are equivalent:*

1. H_s is a group;
2. H_s contains an idempotent;
3. $s^2 \in H_s$;
4. there exist $x, y \in H_s$ such that $xy \in H_s$.

If S is stable, these are in addition equivalent to:

5. $s^2 \mathcal{J} s$;
6. there exist $x, y \in H_s$ such that $xy \in J_s$;
7. there exist $x \in L_s, y \in R_s$ such that $xy \in J_s$;
8. for all $x \in L_s, y \in R_s$, one has $xy \in J_s$.

Proof. Clearly, 1 implies 2 as the identity of a group is an idempotent. To deduce 3 from 2, let $s \mathcal{H} e$ with e an idempotent. Then $se = s$ by Exercise A.1.8. Let $t \in S^I$ with $st = e$. Then $s = se = sst = s^2t$. Thus $s^2 \mathcal{R} s$. Similarly, $s^2 \mathcal{L} s$ and so $s^2 \in H_s$, as required.

For 3 implies 4, take $x = y = s$. To establish 4 implies 1 we use Lemma A.3.3. Suppose $x, y \in H_s$ are such that $xy \in H_s$. We first prove

H_s is a subsemigroup. If $a, b \in H_s$, then we can find $u, w, z \in S^I$ such that $a = ux$, $xyw = x$ and $bz = y$. Then $abz = ayw = uxyw = ux = a$, so $ab \mathcal{R} a$. Similarly, $ab \mathcal{L} b$. So $ab \in R_a \cap L_b = R_s \cap L_s = H_s$. Thus H_s is a subsemigroup.

To prove H_s is left simple, let $a \in H_s$. Because H_s is a subsemigroup, $a^2 \in H_s$. As $aa = a^2$, Green's Lemma implies $\varphi: H_a \rightarrow H_{a^2}$ given by $x\varphi = ax$ is a bijection. As $H_a = H_s = H_{a^2}$, we obtain $aH_s = H_s\varphi = H_s$. A dual argument shows that H_s is right simple. Therefore, H_s is a group.

The implications 3 implies 5 and 4 implies 6, which in turn implies 7, are trivial. For 5 implies 3, observe that $s^2 \mathcal{J} s$ yields $s^2 \mathcal{L} s$ and $s^2 \mathcal{R} s$ by stability. For 7 implies 4, suppose $x \in L_s$, $y \in R_s$ and $xy \in J_s$. By stability, $xy \mathcal{R} x$ so $x = xyt$ for some $t \in S^I$. Also $xy \mathcal{L} y$ by stability, and hence $yt \mathcal{L} xyt = x$ as \mathcal{L} is a right congruence. By stability $yt \mathcal{R} y$ so $yt \in R_y \cap L_x = H_s$. Because $s \mathcal{L} x$, we can write $s = ux$ with $u \in S^I$. Then $s(yt) = uxyt = ux = s$. Therefore, $s(yt) \in H_s$ yielding 4.

Clearly 8 implies 7. We prove 3 implies 8. Indeed, if $x \in L_s$, $y \in R_s$, then there exist $u, v \in S^I$ with $s = ux$, $s = yv$. Then $uxyv = s^2 \in H_s$ and so $s \leq \mathcal{J} xy \leq \mathcal{J} x \leq \mathcal{J} s$. Therefore, $xy \in J_s$, as required. \square

Definition A.3.5 (Maximal subgroup). *If $e \in S$ is an idempotent, then H_e is called the maximal subgroup of S at e . It is the largest subgroup of S with identity e .*

Exercise A.3.6. Show that if e is an idempotent of a semigroup S , then H_e is the group of units of the monoid eSe .

The next theorem is one of the principal results in the structure theory of stable semigroups [68, 171].

Theorem A.3.7. *Let J be a \mathcal{J} -class of a stable semigroup S . Then the following are equivalent:*

1. J contains a subgroup;
2. J contains an idempotent;
3. J contains a regular element;
4. All elements of J are regular;
5. The \mathcal{R} -class of each element of J contains an idempotent;
6. The \mathcal{L} -class of each element of J contains an idempotent;
7. $J^2 \cap J \neq \emptyset$.

Proof. It is clear that 1 implies 2, which in turn implies 3 because idempotents are regular. To see that 3 implies 4, we first show that if $s \in J$ is regular, then L_s consists entirely of regular elements. Choose $t \in S$ such that $s = sts$. Let $y \in L_s$. There exist $u, v \in S^I$ such that $uy = s$, $vs = y$. Hence $ytuy = yts = vsts = vs = y$. As $tu \in S$, we conclude y is regular. A similar argument establishes each element of R_s is regular.

Now suppose $y \mathcal{J} s$ with s regular. Then there exists $r \in S$ such that $s \mathcal{L} r \mathcal{R} y$. From $s \mathcal{L} r$, we conclude r is regular. Hence $r \mathcal{R} y$ implies y is regular.

For 4 implies 5, let $s \in J$ be regular and choose $t \in S$ with $sts = s$. Then $stst = st$ and so st is an idempotent. Clearly $st \mathcal{R} s$, as $(st)s = s$. A dual argument shows that 4 implies 6.

The implications 5 implies 7 and 6 implies 7 are trivial as any idempotent $e \in J$ belongs to $J^2 \cap J$. For 7 implies 1, let $x, y \in J$ with $xy \in J$. Theorem A.3.4 shows that the \mathcal{H} -class $L_x \cap R_y$ is a group. □

Definition A.3.8 (Regular \mathcal{J} -class). A \mathcal{J} -class satisfying the equivalent conditions of Theorem A.3.7 is called a regular \mathcal{J} -class. A non-regular \mathcal{J} -class is called a null \mathcal{J} -class.

For example, every element of T_n is regular because there is an idempotent of every possible rank. The next theorem shows that all maximal subgroups in a regular \mathcal{J} -class of a stable semigroup are isomorphic.

Theorem A.3.9. Let S be a semigroup and let $e, f \in S$ be \mathcal{D} -equivalent idempotents. Then the maximal subgroups H_e and H_f are isomorphic. More precisely, if $e \mathcal{R} x \mathcal{L} f$ and x' is an inverse to x with $xx' = e, x'x = f$ as per Proposition A.1.15, then $\varphi: H_e \rightarrow H_f$ given by $s\varphi = x'sx$ is an isomorphism of groups with inverse $\psi: H_f \rightarrow H_e$ given by $t\psi = xt'x'$.

Proof. Green’s Lemma implies that φ and ψ are well-defined inverse bijections. It therefore suffices to verify that φ is a homomorphism. Indeed, $s\varphi s'\varphi = x'sxx's'x = x'ses's'x = x'ss's'x = (ss')\varphi$ because $s \mathcal{H} e$. □

Corollary A.3.10. The maximal subgroups at \mathcal{J} -equivalent idempotents of a stable semigroup are isomorphic.

A.3.1 The Schützenberger group

For a null \mathcal{J} -class, the role of the maximal subgroup of a regular \mathcal{J} -class is played by a “phantom” group, called the *Schützenberger group*. This group is the subject of [68, Section 2.4] and [171, Chapter 7, Prop. 2.8], but we quickly review some facts that do not appear in these references in the precise form that we need them.

For a subset X of a semigroup S , set $\text{Stab}(X) = \{s \in S^I \mid Xs \subseteq X\}$. Let H be an \mathcal{H} -class of a semigroup S . Then $\text{Stab}(H)$ acts on the right of H . Denote by $\Gamma_R(H)$ the quotient of $\text{Stab}(H)$ by the kernel of the action. Recall that a right permutation group (X, G) is called *regular* if $xg = x$ implies $g = 1$, for $x \in X$ and $g \in G$. In this case one also says G acts *freely* on X .

Theorem A.3.11. Let H be an \mathcal{H} -class of S contained in the \mathcal{L} -class L . Then the following hold:

1. If H' is an \mathcal{H} -class of L , then $\text{Stab}(H) = \text{Stab}(H')$;
2. $\text{Stab}(H) \leq \text{Stab}(L)$ and hence $\text{Stab}(H)$ acts on the right of L ;
3. The quotient of $\text{Stab}(H)$ by the kernel of its action on L is $\Gamma_R(H)$; consequently, if H and H' are \mathcal{L} -equivalent \mathcal{H} -classes, then $\Gamma_R(H) \cong \Gamma_R(H')$;
4. $(L, \Gamma_R(H))$ is a regular permutation group and the orbits are precisely the \mathcal{H} -classes of L and in particular $\Gamma_R(H)$ acts transitively on H ;
5. If S is a stable semigroup, then the action of $\Gamma_R(H)$ is by endomorphisms of the left action of S on L by partial transformations defined by

$$s \cdot x = \begin{cases} sx & sx \in L \\ \text{undefined} & \text{else} \end{cases}$$

for $s \in S$ and $x \in L$.

Proof. Suppose $s \in \text{Stab}(H)$ and let $h_0 \in H$ be fixed. Then $h_0s \in H$ and so $h_0st = h_0$ for some $t \in S^I$. According to Green's Lemma, right translation by s and by t induce mutually inverse permutations of L preserving \mathcal{H} . In particular, $t \in \text{Stab}(H)$ and $\text{Stab}(H)$ is contained in the stabilizer of every other \mathcal{H} -class of L . This implies immediately 1 and 2. The quotient of $\text{Stab}(H)$ by its action on L yields a permutation group as we saw above that $s \in \text{Stab}(H)$ acts as a permutation with inverse $t \in \text{Stab}(H)$. Let us verify that this permutation group is regular. Suppose $x \in L$ and $xs = x$ with $s \in \text{Stab}(H)$. If $y \in L$, then $y = ux$ some $u \in S^I$ and hence $ys = uxs = ux = y$. This establishes regularity. Moreover, if $s, s' \in \text{Stab}(H)$, then s and s' induce the same permutation of L if and only if $h_0s = h_0s'$ and so two elements act the same on L if and only if they act the same on H . Thus the quotient of $\text{Stab}(H)$ by the kernel of its action on L is precisely $\Gamma_R(H)$.

As we already saw that the action of each element of $\text{Stab}(H)$ preserves \mathcal{H} , each orbit is clearly contained in an \mathcal{H} -classes of L . Suppose $l, l' \in L$ are \mathcal{H} -equivalent. Then we can find $s, s' \in S^I$ so that $ls = l'$ and $l's' = l$. Green's Lemma implies that $s, s' \in \text{Stab}(H)$ and so the orbits are precisely the \mathcal{H} -classes.

For the final statement, we ask the reader to verify that we have indeed defined an action. Notice if $g \in \text{Stab}(H)$ and $x \in L$, then $x \mathcal{L} xg$ and so $sx \mathcal{L} sxg$ for any $s \in S$. Thus $s \cdot x$ is defined if and only if $s \cdot (xg)$ is defined in which case $(s \cdot x)g = s \cdot (xg)$ by associativity. This completes the proof. \square

The group $\Gamma_R(H)$ is called the *right Schützenberger group* of H . Because $(H, \Gamma_R(H))$ is a transitive regular permutation group, it follows easily that H and $\Gamma_R(H)$ have the same cardinality. One can define dually the *left Schützenberger group* $\Gamma_L(H)$ and the dual of Theorem A.3.11 holds. Now we wish to show that $\Gamma_R(H) \cong \Gamma_L(H)$. Because the left action of $\Gamma_L(H)$ on H clearly commutes with the right action of $\Gamma_R(H)$, this is an immediate consequence of the following lemma, whose proof we leave as an exercise.

Lemma A.3.12. *Let G' and G be groups with transitive regular permutation actions on the left and right of a set X , respectively, which commute. Let $x_0 \in X$ be a fixed element. Then G' and G are anti-isomorphic via the map γ sending $g \in G$ to the unique element $g\gamma \in G'$ such that $g\gamma x_0 = x_0 g$.*

Exercise A.3.13. Prove Lemma A.3.12.

Corollary A.3.14. *Let H be an \mathcal{H} -class of a \mathcal{D} -class D of a semigroup S . Then $\Gamma_R(H) \cong \Gamma_L(H)$ and if H' is any \mathcal{H} -class of D , then $\Gamma_R(H) \cong \Gamma_R(H')$.*

Proof. The first assertion is immediate from Lemma A.3.12 because (inner) left and right translations commute with each other. The second assertion follows from the first, together with Theorem A.3.11(3) and its dual. \square

So if J is a \mathcal{J} -class of a stable semigroup, then there is a well-defined (up to isomorphism) Schützenberger group of J . It remains to show that if J is a regular \mathcal{J} -class, then the Schützenberger group is the maximal subgroup.

Proposition A.3.15. *Let H be a maximal subgroup of S . Then $\Gamma_R(H) \cong H$.*

Proof. Let e be the identity of H . Clearly, $H \leq \text{Stab}(H)$ and if $h \in H$, then in $(H, \Gamma_R(H))$ we have $eh = h$. It follows that H can be identified with its image in $\Gamma_R(H)$. If $s \in \Gamma_R(H)$ and $es = h$ with $h \in H$, then $es = h = eh$ and so $s = h$ by regularity. Thus $H \cong \Gamma_R(H)$, as required. \square

A.4 Rees’s Theorem

Rees’s Theorem [258], characterizing stable 0-simple semigroups, can be considered the first major theorem in semigroup theory. See also [350] for the special case of a finite simple semigroup.

Definition A.4.1 (0-simple). *A semigroup S with zero is called 0-simple if it has no non-zero proper ideals and also $S^2 \neq 0$. A semigroup S for which $S^2 = 0$ is called null.*

The following proposition is straightforward so we omit the proof.

Proposition A.4.2. *Let $s \in S$. Then $S^I s S^I \setminus J_s$ is an ideal of $S^I s S^I$ unless it is empty.*

Remark A.4.3. If K is the minimal ideal of a semigroup S and $s \in K$, then $S^I s S^I \subseteq K$ and hence $S^I s S^I = K$, for all $s \in K$. In particular K is a \mathcal{J} -class and $S^I s S^I \setminus K = \emptyset$. In fact, the only time $S^I s S^I \setminus J_s = \emptyset$ is when J_s is the minimal ideal of S .

The class of 0-simple semigroups admits the following alternative characterization.

Proposition A.4.4. *A semigroup S is 0-simple if and only if $SaS = S$, for all $0 \neq a \in S$.*

Proof. If S is 0-simple, then $S^2 \neq \{0\}$. Let $a \in S \setminus \{0\}$. Then $S^I a S^I$ is an ideal containing a , so $S^I a S^I = S$ by 0-simplicity. Also S^2 is a non-zero ideal, so $S^2 = S$. Therefore, $S = S^2 = S^3$ and hence

$$0 \neq SSS = (S^I a S^I)(S^I a S^I)(S^I a S^I) \subseteq SaS.$$

Thus SaS is a non-zero ideal of S , whence $SaS = S$.

Next assume $SaS = S$, for all non-zero elements $a \in S$. In particular, if $a \neq 0$, then $S = SaS \subseteq S^2$ and so $S^2 \neq 0$. Also if $I \subseteq S$ is a non-zero ideal and $0 \neq a \in I$, then $S = SaS \subseteq SIS \subseteq I$ and thus $I = S$. We conclude S is 0-simple. □

As a consequence, $S \setminus \{0\}$ is a \mathcal{J} -class in a 0-simple semigroup S . Part of the importance of 0-simple semigroups stems from the following fact.

Proposition A.4.5. *The minimal ideal of a semigroup is simple. A regular 0-minimal ideal of a semigroup with zero is 0-simple.*

Proof. Let I be the minimal ideal of a semigroup S . Suppose J is an ideal of I . Then IJI is an ideal of S , so by minimality $I \subseteq IJI \subseteq J$. We conclude $I = J$. Next assume I is a regular 0-minimal ideal of S . Then $I^2 \neq 0$ by regularity. Let $0 \neq J \subseteq I$ be an ideal. Any ideal in a regular semigroup is regular and so $0 \neq J = J^3 \subseteq IJI$. Because IJI is an ideal of S , by 0-minimality $I \subseteq IJI \subseteq J$. This completes the proof. □

The semigroups appearing in the next definition are classically known as the principal factors [68], although we shall not make extensive use of this terminology.

Definition A.4.6 (Principal factor). *If $J = J_s$ is a \mathcal{J} -class, the principal factor associated to J is the semigroup*

$$J^0 = \begin{cases} S^I s S^I / (S^I s S^I \setminus J_s) & \text{if } J_s \text{ is not the minimal ideal} \\ J_s \cup \{0\} & \text{else.} \end{cases}$$

The principal factor J^0 can alternatively be described as the semigroup with underlying set $J \cup \{0\}$ and with multiplication given by

$$x \cdot y = \begin{cases} xy & \text{if } xy \in J \\ 0 & \text{else.} \end{cases}$$

The next proposition explains the terminology null \mathcal{J} -class and shows that principal factors associated to regular \mathcal{J} -classes are 0-simple.

Proposition A.4.7. *Let S be a stable semigroup and let J be a \mathcal{J} -class of S . Then J is a regular \mathcal{J} -class if and only if J^0 is 0-simple. If J is a null \mathcal{J} -class, then J^0 is null.*

Proof. Because $(J^0)^2 \neq \emptyset$ is equivalent to $J^2 \cap J \neq \emptyset$, Theorem A.3.7 shows that J^0 is null if and only if J is a null \mathcal{J} -class. In particular, if J^0 is 0-simple, then J is regular.

Now suppose J is regular. To show J^0 is 0-simple, it suffices by Proposition A.4.4 to show that, for all $x \in J$, $J^0 x J^0 = J^0$. Let $t \in J$. Then we can find $u, v \in S^I$ with $uxv = t$. Because J is regular, x is regular, so there exists $s \in S$ such that $x = xsx$. Hence $t = uxv = uxsxv = (uxs)x(sxv)$ and $uxs, sxv \in S$. Now $t \leq_{\mathcal{J}} uxs, sxv \leq_{\mathcal{J}} x$. So $uxs, sxv \in J$. Thus $t \in J^0 x J^0$, as required. \square

The previous proposition shows that “locally” (meaning in a \mathcal{J} -class) a semigroup is either null or 0-simple.

Corollary A.4.8. *A stable 0-simple semigroup is regular.*

Proof. Indeed, $S = J^0$ where $J = S \setminus \{0\}$ is a \mathcal{J} -class of S , so the previous proposition applies to show that S is regular. \square

Our next goal is to prove Rees’s Theorem [68, 171, 258], which describes stable 0-simple semigroups up to isomorphism. The key notion is that of a Rees matrix semigroup.

Definition A.4.9 (Rees matrix). *A Rees matrix is a map $C: B \times A \rightarrow G^0$ where A and B are non-empty sets, G is a group and $G^0 = G \cup \{0\}$ with 0 an adjoined zero. We adopt the symmetric notation bCa for the value of C on (b, a) .*

Denote by E_{ab} , for $a \in A$ and $b \in B$, the $A \times B$ elementary matrix unit with 1 in position (a, b) and 0 elsewhere, where 1 is the identity of G .

Definition A.4.10 (Rees Matrix Semigroup). *Let G be a group and let $C: B \times A \rightarrow G^0$ be a Rees matrix. The Rees matrix semigroup with sandwich matrix C is the set*

$$\mathcal{M}^0(G, A, B, C) = \{gE_{ab} \mid g \in G^0, a \in A, b \in B\}$$

with multiplication, for $X, Y \in \mathcal{M}^0(G, A, B, C)$, given by $X \diamond Y = XCY$.

The underlying set of $\mathcal{M}^0(G, A, B, C)$ consists of all $A \times B$ matrices over G^0 with at most one non-zero entry. Associativity follows from the associativity of matrix multiplication. Notice that

$$g_1 E_{a_1 b_1} \diamond g_2 E_{a_2 b_2} = (g_1 (b_1 C a_2) g_2) E_{a_1 b_2}$$

and so $\mathcal{M}^0(G, A, B, C)$ is indeed closed under the multiplication.

Oftentimes it is convenient to identify $\mathcal{M}^0(G, A, B, C)$ with $(A \times G \times B) \cup \{0\}$ via the correspondence sending gE_{ab} , with $g \in G$, to (a, g, b) and 0 to 0. Under this bijection, the multiplication rule becomes

$$(a_1, g_1, b_1)(a_2, g_2, b_2) = \begin{cases} (a_1, g_1(b_1Ca_2)g_2, b_2) & b_1Ca_2 \neq 0 \\ 0 & \text{else.} \end{cases} \tag{A.1}$$

Sometimes the matrix C is called the *structure matrix* or the *sandwich matrix* of $\mathcal{M}^0(G, A, B, C)$. If $C: B \times A \rightarrow G$, then $\mathcal{M}^0(G, A, B, C) \setminus \{0\}$ is a subsemigroup, denoted by $\mathcal{M}(G, A, B, C)$.

Definition A.4.11 (Regular Rees matrix). *A Rees matrix is called regular if it has no zero rows or columns.*

Proposition A.4.12. *Let $C: B \times A \rightarrow G^0$ be a Rees matrix. Then the following are equivalent for $S = \mathcal{M}^0(G, A, B, C)$:*

1. S is regular;
2. C is regular;
3. S is 0-simple.

Moreover, if 1–3 hold, then:

- (a) $S/\mathcal{R} = \{a \times G \times B \mid a \in A\}$;
- (b) $S/\mathcal{L} = \{A \times G \times b \mid b \in B\}$;
- (c) $S/\mathcal{H} = \{a \times G \times b \mid a \in A, b \in B\}$;
- (d) The \mathcal{H} -class $H = a \times G \times b$ is a group if and only if $bCa \neq 0$; if H is a group, its identity is $(a, (bCa)^{-1}, b)$ and it is isomorphic to G via

$$g \mapsto (a, (bCa)^{-1}g, b).$$

Proof. Assume S is regular and suppose $a \in A, b \in B$. Let $s = (a, 1, b)$ and choose $t = (a', g', b')$ such that $sts = s$. Then

$$(a, 1, b)(a', g', b')(a, 1, b) = s \neq 0$$

and so $bCa' \neq 0 \neq b'Ca$. We conclude C is regular.

Next assume C is regular. By Proposition A.4.4, to prove S is 0-simple, we need to show that $S(a, g, b)S = S$ for all $(a, g, b) \in S \setminus 0$. So let $(a', g', b') \in S \setminus 0$ and choose $a'' \in A, b'' \in B$ such that $bCa'' \neq 0 \neq b''Ca$. Then

$$(a', g'(b''Ca)^{-1}, b'')(a, g, b)(a'', (bCa'')^{-1}g^{-1}, b') = (a', g', b')$$

and hence $S(a, g, b)S = S$. Thus S is 0-simple.

Suppose S is 0-simple. We show that C is regular. Let $a \in A$ and $b \in B$. Because S is 0-simple, $S(a, 1, b)S = S$ and so

$$(a, 1, b) = (a', g', b')(a, 1, b)(a'', g'', b'')$$

for appropriate choices of a', a'', b', b'', g' and g'' . But then $b'Ca \neq 0 \neq bCa''$, and so C is regular.

Suppose now C is regular; we show S is regular. Let $(a, g, b) \in S$ and choose $a' \in A, b' \in B$ with $bCa' \neq 0 \neq b'Ca$. Then

$$(a, g, b)(a', (bCa')^{-1}g^{-1}(b'Ca)^{-1}, b')(a, g, b) = (a, g, b)$$

and so $\mathcal{M}^0(G, A, B, C)$ is regular, as required.

We now turn to (a)–(d). If $s = (a, g, b)$, then it follows directly from (A.1) that $R_s \subseteq a \times G \times B$. For the converse, suppose $(a, g', b') \in a \times G \times B$. Choose a'' such that $bCa'' \neq 0$. Then

$$(a, g, b)(a'', (bCa'')^{-1}g^{-1}g', b') = (a, g', b').$$

Hence $R_s = a \times G \times B$. The descriptions of S/\mathcal{L} and S/\mathcal{R} are handled similarly.

Next we turn to the case of when $H = a \times G \times b$ is a group. By Theorem A.3.4, H is a group if and only if it contains an idempotent. But

$$(a, g, b)(a, g, b) = \begin{cases} (a, g(bCa)g, b) & bCa \neq 0 \\ 0 & \text{else.} \end{cases}$$

So H contains an idempotent if and only if $bCa \neq 0$, in which case the idempotent is $(a, (bCa)^{-1}, b)$. Clearly, $g \mapsto (a, (bCa)^{-1}g, b)$ is a bijection from G to H . It is a homomorphism as the computation

$$(a, (bCa)^{-1}g_1, b)(a, (bCa)^{-1}g_2, b) = (a, (bCa)^{-1}g_1g_2, b)$$

shows. □

Corollary A.4.13. *Let S be a regular Rees matrix semigroup. Then S is stable.*

Exercise A.4.14. Prove Corollary A.4.13.

We now prove Rees’s Theorem [68, 171, 258].

Theorem A.4.15 (Rees). *Let S be a stable semigroup. Then S is 0-simple if and only if there exist a group G , sets A, B and a regular Rees matrix $C: B \times A \rightarrow G^0$ such that $S \cong \mathcal{M}^0(G, A, B, C)$. Similarly, a stable semigroup S is simple if and only if $S \cong \mathcal{M}(G, A, B, C)$ where $C: B \times A \rightarrow G$.*

Proof. We just handle the 0-simple case, as the simple case is a consequence. Corollary A.4.13 provides one direction. For the other, Corollary A.4.8 tells us S is regular. Let J be the non-zero \mathcal{J} -class of S and let $e \in J$ be an idempotent (guaranteed by Theorem A.3.7). Let $G = H_e, A = S/\mathcal{R}$ and $B = S/\mathcal{L}$. Theorem A.3.4 says that G is a group.

e, g				ℓ_b		
r_a				$r_a g \ell_b$		

Fig. A.4. Rees coordinates

Choose for each $a \in A$ and $b \in B$, $r_a \in a \cap L_e$ and $\ell_b \in b \cap R_e$. Notice $e\ell_b = \ell_b$ and $r_a e = r_a$. Green's Lemma tells us $x \mapsto r_a x \ell_b$ is a bijection from $G = H_e$ to $a \cap b$. Hence each non-zero element of S can be written uniquely in the form $r_a x \ell_b$ with $x \in G$. See Figure A.4.

To motivate the definition of the sandwich matrix C , notice for $g, g' \in G$

$$(r_a g \ell_b)(r_{a'} g' \ell_{b'}) = r_a (g \ell_b r_{a'} g') \ell_{b'}. \tag{A.2}$$

Now if $\ell_b r_{a'} \neq 0$, then $\ell_b r_{a'} \in J$. Stability then yields $\ell_b r_{a'} \mathcal{R} \ell_b \mathcal{R} e$ and $\ell_b r_{a'} \mathcal{L} r_{a'} \mathcal{L} e$, i.e., $\ell_b r_{a'} \in H_e = G$. In any event $\ell_b r_{a'} \in G^0$.

From (A.2) and the ensuing discussion, it is clear we should define $C: B \times A \rightarrow G^0$ by $bCa = \ell_b r_a \in G^0$. We then define an isomorphism $\psi: \mathcal{M}^0(G, A, B, C) \rightarrow S$ by $0 \mapsto 0$ and $(a, g, b) \mapsto r_a g \ell_b$. The discussion above shows that ψ is a bijection. To see that it is a homomorphism, we compute

$$((a, g, b)(a', g', b'))\psi = r_a (g \ell_b r_{a'} g') \ell_{b'}.$$

A comparison with (A.2) yields ψ is a homomorphism. Proposition A.4.12 now implies C is a regular Rees matrix, completing the proof of the theorem. \square

Let us show how Rees's Theorem determines the structure of stable left simple semigroups.

Lemma A.4.16. *Let S be a stable simple semigroup whose idempotents form a subsemigroup. Then $S \cong \mathcal{M}(G, A, B, C)$ where C is a $B \times A$ matrix all of whose entries are the identity of G .*

Proof. Each \mathcal{H} -class of a stable simple semigroup must be a group, so we can choose r_a and ℓ_b to be idempotents in the proof of Rees's Theorem. Then $bCa = \ell_b r_a$ is an idempotent of H_e , and hence is e . This proves the lemma. \square

Corollary A.4.17. *Let S be a stable semigroup. Then S is left simple if and only if $S \cong L \times G$ with L a left zero semigroup and G a group. Dually, S is right simple if and only if $S \cong G \times R$ with G a group and R a right zero semigroup.*

Proof. Trivially, if L is a left zero semigroup and G is a group, then $L \times G$ is left simple. For the converse, we first show the idempotents of S form a subsemigroup. Indeed, if $e, f \in S$ are idempotents, then $e \mathcal{L} f$ and so

$ef = e$, $fe = f$ by Exercise A.1.8. So Rees's Theorem, Lemma A.4.16 and Proposition A.4.12 show that $S \cong \mathcal{M}(G, A, \{b\}, C)$ where C is a matrix all of whose entries are the identity of G . Let L have underlying set A with the left zero multiplication. Then $(a, g, b) \mapsto (a, g)$ gives an isomorphism $S \rightarrow L \times G$. The right simple case is dual. \square

Definition A.4.18 (Rees coordinatization). *If S is a stable semigroup and J is a regular \mathcal{J} -class, then a Rees coordinatization of J is an isomorphism $\psi: J^0 \rightarrow \mathcal{M}^0(G, A, B, C)$.*

Optimal Rees coordinatizations are studied in Section 4.13. This brings to an end our brief introduction to the Green-Rees local structure theory for stable semigroups.

Remark A.4.19. A simple compact semigroup is always isomorphic to a Rees matrix semigroup $\mathcal{M}(G, A, B, C)$ where G is a compact group, A, B are compact sets and $C: B \times A \rightarrow G$ is continuous [135]. The situation for 0-simple semigroups is more complicated as 0 might not be an isolated point.

B

Tables on Preservation of Sups and Infs

The reader is referred to Section 1.1.2 for nomenclature on types of maps between posets. In the following table, \mathbf{V} is a pseudovariety of semigroups.

Table B.1. Preservation properties of products on \mathbf{PV}

OPERATOR	ORDER PROPERTIES	TYPE OF OPERATOR
$\mathbf{V} \cap (-)$	continuous, \mathbf{inf}_\top	$\mathbf{GMC}(\mathbf{PV})^-$
$\mathbf{V} \vee (-)$	\mathbf{sup}_B	$\mathbf{GMC}(\mathbf{PV})^+, \mathbf{GMC}(\mathbf{PV})_1^\rho$
$\mathbf{V} * (-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^+$
$\mathbf{V} ** (-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^+$
$\mathbf{V} \textcircled{\cap} (-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^+$
$(-) \cap \mathbf{V}$	continuous, \mathbf{inf}_\top	$\mathbf{GMC}(\mathbf{PV})^-$
$(-) \vee \mathbf{V}$	\mathbf{sup}_B	$\mathbf{GMC}(\mathbf{PV})^+, \mathbf{GMC}(\mathbf{PV})_1^\rho$
$(-) * \mathbf{V}$	\mathbf{sup}_B	$\mathbf{GMC}(\mathbf{PV})_{\perp 1}^\rho$
$(-) ** \mathbf{V}$	\mathbf{sup}_B	$\mathbf{GMC}(\mathbf{PV})_{\perp 1}^\rho$
$(-) \textcircled{\cap} \mathbf{V}$	continuous, \mathbf{inf}	$\mathbf{GMC}(\mathbf{PV})_{\perp 1}^\rho$

In the next table, \mathbf{V} denotes a pseudovariety of relational morphisms.

Table B.2. Preservation properties of products on \mathbf{PVRM}

OPERATOR	ORDER PROPERTIES
$\mathbf{V} \cap (-)$	continuous, \mathbf{inf}_\top
$\mathbf{V} \vee (-)$	\mathbf{sup}_B
$\mathbf{V} \odot (-)$	continuous
$(-) \cap \mathbf{V}$	continuous, \mathbf{inf}_\top
$(-) \vee \mathbf{V}$	\mathbf{sup}_B
$(-) \odot \mathbf{V}$	continuous, \wedge -map

In the following table, \mathbf{V} represents a continuously closed class.

Table B.3. Preservation properties of products on \mathbf{CC}

OPERATOR	ORDER PROPERTIES
$\mathbf{V} \cap (-)$	continuous, \mathbf{inf}_{\top}
$\mathbf{V} \vee (-)$	$\mathbf{sup}_{\mathbf{B}}$
$\mathbf{V} \odot (-)$	continuous
$(-) \cap \mathbf{V}$	continuous, \mathbf{inf}_{\top}
$(-) \vee \mathbf{V}$	$\mathbf{sup}_{\mathbf{B}}$
$(-) \odot \mathbf{V}$	continuous

In our final table, α denotes a continuous operator. Because all infs and sups of $\mathbf{GMC}(\mathbf{PV})$ coincide with infs and sups taken in $\mathbf{Cnt}(\mathbf{PV})$, the corresponding table for $\mathbf{GMC}(\mathbf{PV})$ is identical. We use here \circ for composition of operators.

Table B.4. Preservation properties of products on $\mathbf{Cnt}(\mathbf{PV})$

OPERATOR	ORDER PROPERTIES
$\alpha \wedge (-)$	continuous, \mathbf{inf}_{\top}
$\alpha \vee (-)$	$\mathbf{sup}_{\mathbf{B}}$
$\alpha \circ (-)$	continuous
$(-) \wedge \alpha$	continuous, \mathbf{inf}_{\top}
$(-) \vee \alpha$	$\mathbf{sup}_{\mathbf{B}}$
$(-) \circ \alpha$	\mathbf{sup} , \wedge -map

Exercise B.0.1. Prove all assertions in the tables and find counterexamples to properties that are not claimed to be preserved.

List of Problems

Some, but not all, of these problems appear in the text as questions.

Problem 1. What is $\mathbf{V} * \mathbf{1}$? It is easy to see that if S is a semigroup in $\mathbf{V} * \mathbf{1}$, then the quotient of S by the kernel of its action on the right of itself belongs to \mathbf{V} . The converse holds if \mathbf{gV} is definable by pseudoidentities over strongly connected graphs. Is the converse always valid?

Problem 2. What is $\mathbf{V} ** \mathbf{1}$? Let S be a semigroup and define a congruence on S by $s \equiv s'$ if, for all $s_L, s_R \in S$, one has $s_L s s_R = s_L s' s_R$. It is easy to see that if $S \in \mathbf{V} ** \mathbf{1}$, then $S/\equiv \in \mathbf{V}$. The converse holds if \mathbf{gV} is definable by pseudoidentities over strongly connected graphs. Does it hold in general?

Problem 3. Give an element of \mathbf{PVRM}^+ analogous to \mathbf{V}_D or \mathbf{V}_K yielding $\diamond(-)$ under \mathfrak{q} and find a basis of pseudoidentities for it.

Problem 4 (Tilson Question). Is $\mathbf{V}_D = \min(\mathbf{V} * (-))$?

Problem 5. Is it true that a pseudovariety of semigroups \mathbf{V} is decidable if and only if \mathbf{gV} is decidable?

Problem 6. Find a natural axiom to add to the definition of a pseudovariety of semigroupoids to ensure definability by path pseudoidentities.

Problem 7 (Tilson Question Version 2). Is $\mathbf{V}_K = \min(\mathbf{V} ** (-))$?

Problem 8. Is it true that if \mathbf{H}, \mathbf{K} are decidable pseudovarieties of groups, then $\mathbf{H} \vee \mathbf{K}$ is decidable?

Problem 9. Find a natural lattice in \mathbf{CC}^+ mapping to $\mathbf{GMC}_{\mathbb{L}1}^{\mathfrak{q}}(\mathbf{PV})$ under \mathfrak{q} .

Problem 10. Compute $\min(\alpha)$ for any operator α that is not a constant operator, or of the form $\mathbf{V} \cap (-)$.

Problem 11. Suppose \mathbf{K} and \mathbf{L} are equational continuously closed classes. Is $\mathbf{K} \odot \mathbf{L}$ equational?

Problem 12. Suppose \mathbf{K} and \mathbf{L} are Birkhoff continuously closed classes. Is $\mathbf{K} \odot \mathbf{L}$ Birkhoff?

Problem 13. Give an example showing that meets are not pointwise in the lattice $\mathbf{GMC}(\mathbf{PV})^+$.

Problem 14. Find a projective basis for \mathbf{V}_D .

Problem 15 (Almeida-Weil Basis Question). Find an example of pseudovarieties \mathbf{V} and \mathbf{W} so that pseudoidentities in Theorem 3.7.15 (i.e., [27, Thms. 5.2 and 5.3]) do *not* define $\mathbf{V} * \mathbf{W}$. For such an example, give an explicit example of a member of the basis from Theorem 3.8.18 that is not a consequence of the pseudoidentities in Theorem 3.7.15.

Problem 16. Is it decidable whether a finite semigroup is projective?

Problem 17 (Complexity Problem). Is the group complexity function c computable? More precisely, is there a Turing machine that given a finite semigroup S by its multiplication table as input can output $c(S)$?

Problem 18. Determine the pointwise meet of all local upper bounds to complexity. Find more local upper bounds.

Problem 19. A finite semigroup S is critical if each of its proper divisors has strictly smaller complexity. What are the possible posets for the \mathcal{J} -order of a critical semigroup? By [115], any finite poset with minimum is the \mathcal{J} -order of a finite semigroup. We ask the same question for critical semigroups with respect to two-sided complexity C or dot-depth.

Problem 20 (Henckell-Rhodes). Is it true that the problem of deciding whether a semigroup has complexity one reduces to the case of a semigroup with at most 3 non-zero \mathcal{J} -classes? More generally, is it true that the problem of deciding whether a semigroup has complexity n reduces to the case of a semigroup with at most $n + 2$ non-zero \mathcal{J} -classes?

Problem 21 (Schützenberger's Question). Does the pseudovariety $\mathbf{A} \vee \mathbf{G}$ have decidable membership?

Problem 22. Given a finite semigroup S , is the semidirect product closure of (S) decidable? How about the closure under the two-sided semidirect product?

Problem 23. Study the complexity function κ associated to the operators $\alpha = \mathbf{A} \textcircled{\mathbb{M}} (-)$ and $\beta = \mathbf{G}^{\mathbf{N}} \textcircled{\mathbb{M}} (-)$.

Problem 24. Is the complexity function for \mathbf{A} associated to the operators $\alpha = \mathbf{S1} ** (-)$ and $\beta = \mathbf{L1} \textcircled{\mathbb{M}} (-)$ computable? How exactly does it compare to dot-depth?

Problem 25. Find a proof of the classical Prime Decomposition Theorem using the Derived Category Theorem and some factorization theorem for relational morphisms.

Problem 26. Is the operator $\alpha = \mathbf{A} ** (-)$ idempotent? We suspect that it is not. This would be equivalent to $\mathbf{A} \textcircled{\text{m}} (-) = \mathbf{A} ** (-)$. It is known $\alpha^2 = \alpha^3$. If PT_n is the semigroup of partial transformations of an n element set and I is the aperiodic ideal of maps of rank at most 1, then we suspect $PT_n \notin \mathbf{A} ** (PT_n/I)$ although the projection $PT_n \rightarrow PT_n/I$ is an aperiodic morphism.

Problem 27. Find an easier proof than the one in [362] that $c(\widehat{S}^{\mathcal{L}}) = c(S)$.

Problem 28. Suppose that S is a semigroup such that the longest chain of non-zero \mathcal{J} -classes of S has length at most 2. Must $C(S) \leq 1$?

Problem 29. Is the two-sided complexity function C computable? Start with a semigroup with at most 3 non-zero \mathcal{J} -classes.

Problem 30. Is there some sort of Presentation Lemma to describe the existence of a cross section with respect to $\mathbb{L}\mathbf{G} \textcircled{\text{m}} \mathbf{V}$?

Problem 31. Are there arbitrarily large numbers n such that there is a semigroup S with $c(S) = n = C(S)$? Probably there are.

Problem 32. What is the two-sided complexity of the semigroup of binary relations on an n element set?

Problem 33. A lattice is said to be *semi-distributive* at an element ℓ if $\ell \wedge x = \ell \wedge y$ implies $\ell \wedge x = \ell \wedge (x \vee y)$. Because meets and joins are continuous in \mathbf{PV} , if \mathbf{PV} is semi-distributive at \mathbf{V} and $\mathbf{W} \leq \mathbf{V}$, then there is a largest pseudovariety $\widehat{\mathbf{W}}$ such that $\mathbf{V} \cap \widehat{\mathbf{W}} = \mathbf{W}$. For instance, \mathbf{PV} is semi-distributive at \mathbf{G} and if \mathbf{H} is a pseudovariety of groups, then $\overline{\mathbf{H}}$ is the maximum pseudovariety with $\mathbf{G} \cap \overline{\mathbf{H}} = \mathbf{H}$. Reilly and Zhang proved that \mathbf{PV} is semi-distributive at the pseudovariety of bands \mathbf{B} [260]. It follows from a result of G. Higman [217, 54.24] that \mathbf{PV} is not semi-distributive at certain pseudovarieties of groups. At which pseudovarieties \mathbf{V} is \mathbf{PV} semi-distributive?

Problem 34. Is a subspace of \mathbf{PV} sequentially compact in the strong topology if and only if it is closed and Noetherian?

Problem 35. What are the atoms of $\mathbf{Cnt}(\mathbf{PV})^+$, $\mathbf{GMC}(\mathbf{PV})^+$, \mathbf{PVRM}^+ and \mathbf{CC}^+ ?

Problem 36. Give more examples of *smi*, *fmi*, *sfmi* pseudovarieties that are not *mi*. In particular, can a compact pseudovariety be *smi*? This is related to the existence of atoms in $\mathbf{Cnt}(\mathbf{PV})^+$.

Problem 37. Classify the \times -prime finite semigroups. Is the property of being \times -prime decidable?

Problem 38. Is I_n , the symmetric inverse monoid, fji for all $n \geq 2$? A positive answer would imply **ESI** is fji, in light of Ash's Theorem [32] and Lemma 6.1.13. How about sfji? This may be easier.

Problem 39. If (\mathbf{n}, G) is a permutation group, let $S_{(\mathbf{n}, G)}$ be the action semigroup of the partial transformation inverse semigroup $(\mathbf{n}, G \cup B_n)$ where we identify B_n with the rank one partial bijections of \mathbf{n} . Theorem 4.7.21 implies $S_{(\mathbf{n}, G)}$ is subdirectly indecomposable. Of course, $BZ_p = S_{(\mathbf{p}, \mathbb{Z}_p)}$. The question is: when is $S_{(\mathbf{n}, G)}$ fji? One needs at least that the permutation group is subdirectly indecomposable as a permutation group.

Problem 40. Is the full transformation monoid T_n an fji semigroup for all $n \geq 2$? If so this would give a new proof that **FSgp** is fji by Lemma 6.1.13. How about sfji? This may be easier.

Problem 41. Is there an increasing sequence of compact fji aperiodic pseudovarieties $(S_1) \leq (S_2) \leq \dots$ with $\bigcup(S_i) = \mathbf{A}$? If so, then \mathbf{A} would be fji by Lemma 6.1.13.

Problem 42. Given a finite set of computable pseudoidentities (say identities if you like), is it decidable whether $\llbracket E \rrbracket$ is sfji? A similar question can be asked for fji or for any of our other lattice theoretic adjectives.

Problem 43. Are the complexity pseudovarieties \mathbf{C}_n fji or sfji? We ask the same question for $\mathbf{C}_n \cap \mathbf{DS}$ and $\mathbf{C}_n \cap \mathbf{CR}$.

Problem 44. Study the topological spaces $\text{Spec } \mathbf{PV}$ and $\text{Spec } \mathbf{PV}^{op}$.

Problem 45. Give a proof of Theorem 7.3.32 that does not use pseudoidentities.

Problem 46. Find a proof of Theorem 7.3.39 that does not use pseudoidentities.

Problem 47. Is it true that $\overline{\mathbf{H}}$ is sfji for every pseudovariety of groups \mathbf{H} ? Consider the same question for fji.

Problem 48. Let S be a KN semigroup with KN maximal subgroup G in its minimal ideal. Describe $\text{Excl}(S)$ in terms of $\text{Excl}(G)$. In particular, if $\text{Excl}(G) = \llbracket u = v \rrbracket$, then give a pseudoidentity defining $\text{Excl}(S)$.

Problem 49. Describe the semigroup structure of $(\mathbf{PV}, *)$. It is known that the semigroup of Birkhoff varieties of groups with the semidirect product as multiplication is a free monoid [217]. But this is not the case for \mathbf{PV} , which has many idempotents. Some results can be found in [369].

Problem 50. Determine the s^*i , sf^*i and f^*i pseudovarieties.

Problem 51. Classify the irreducible relational morphisms. The results of Chapter 5 imply that an irreducible relational morphism $\varphi: S \rightarrow T$ must either have a non-trivial congruence-free monoid M in the Mal'cev kernel and belong to $(M)_K$, or belong to $\ell\mathbf{1}_K$. Are the irreducible relational morphisms precisely the relations of this sort?

Problem 52. Study the abstract spectral theory of $\mathbf{GMC}(\mathbf{PV})$. What are the mi , smi , $sfmi$, fmi , ji , fji , sji , $sfmi$ elements of $\mathbf{GMC}(\mathbf{PV})$?

Problem 53. What does the \mathcal{J} -order look like in $\mathbf{Cnt}(\mathbf{PV})^+$? Give an example of when $\alpha \leq \beta$ but $\beta \not\leq_{\mathcal{J}} \alpha$ (or prove there is no such example). Consider the analogous questions for $\mathbf{GMC}(\mathbf{PV})^+$, $\mathbf{Cnt}(\mathbf{PV})^-$ and $\mathbf{GMC}(\mathbf{PV})^-$.

Problem 54. Suppose $R \in \mathbf{CC}$ has decidable membership. Must Rq send compact pseudovarieties to decidable pseudovarieties? Notice that if $R \in \mathbf{BCC}$ has decidable membership, then $Rq(\mathbf{V})$ is decidable for each compact pseudovariety \mathbf{V} . Indeed, if $\mathbf{V} = (T)$, then $S \in Rq(\mathbf{V})$ if and only if the canonical relational morphism $\rho: S \rightarrow F_{(T)}(S \times T)$ belongs to R .

Problem 55. Study the counital bialgebras $\mathbf{Rec}(A^*)$, $\mathbf{A}(A^*)$ and $\mathbf{J}(A^*)$. Also study the Hopf algebra of group languages $\mathbf{G}(A^*)$.

Problem 56. Is it decidable whether a finite semigroup divides a semigroup of lower triangular Boolean matrices? In other words, is dot-depth two decidable?

Problem 57. In [241–244], Polák sets up a correspondence between pseudovarieties of idempotent semirings and certain classes of languages. What is the effect of the triangular product on languages? It should be similar to the effect of the Schützenberger product in the theory of varieties of languages.

Problem 58. What is the smallest pseudovariety of idempotent semirings closed under triangular product and containing \mathbb{B} ? One would guess it should consist of all idempotent semirings whose underlying semigroup is aperiodic.

Problem 59. Classify all Δ -irreducible idempotent semirings.

Problem 60 (Margolis). Is it true that if S is Δ -irreducible, then so is $M_n(S)$?

Problem 61. Classify the basic Δ -irreducible semirings.

Problem 62. Compute the exclusion class of $M_n(Q)$ where Q is one of \mathbb{B} , G^{\natural} with G non-trivial monolithic or $P(G)$ with G a non-trivial group.

Problem 63. Given an oracle deciding group complexity, is c_q computable? We ask the same question for C_q .

Problem 64. Study Δ -irreducibility over more general semirings.

Problem 65 (Fox-Rhodes). Let $S = \mathcal{M}^0(G, A, B, C)$ be a 0-simple semigroup.

1. Is $c(P(S)) = \iota(C) - 1 + c(G)$ where $\iota(C)$ is the largest size of an identity submatrix of C (up to reordering rows and columns and renormalizing)? The right-hand side is a lower bound by the argument in Theorem 9.5.11. In [91], it is shown that an upper bound is $\tau(C) - 1 + c(G)$ where $\tau(C)$ is the largest size of a triangular submatrix of C .
2. Recall that the action of S on the right of $G \times B$ extends to an action on the free left $P(G)$ -module generated by B and hence yields a representation $\rho: S \rightarrow M_B(P(G))$ (which concretely speaking is the classical Schützenberger representation by row monomial matrices). This in turn yields a representation $\tilde{\rho}: P(S) \rightarrow M_B(P(G))$. What is the relationship between $c(P(S))$ and $c(P(S)\tilde{\rho})$?

Problem 66 (Fox-Rhodes). What is the largest size of an identity submatrix of the structure matrix for the rank k \mathcal{J} -class of T_n ?

Problem 67. If S is a semigroup and I is an ideal, what is the relationship between $c(P(S))$ and $c(P(I))$, $c(P_0(S/I))$? Proposition 9.5.14 gives some indication, but $c(P(S)_{P(I)})$ and $P(I)$ do not seem so clearly related. In [91], it is shown that if S is a monoid and $I = S \setminus G$ where G is the group of units, then $c(P(S)) \leq c(P(I)) + c(P_0(S/I))$.

Problem 68. Is $c(P(S)) = \max\{c(P_0(J^0)) \mid J \in S/\mathcal{J}\}$?

Problem 69. Develop homological algebra for idempotent semirings. Find a homological definition of c_q and C_q . Relate this to dot-depth.

Problem 70. Develop a derived/kernel category theory for the triangular product.

Problem 71. Let S be a finite semigroup. Is

$$c(S) = \max\{c(T) \mid T \in (S), T \text{ is fji}\}?$$

Consider the analogous problem for other hierarchical complexity functions such as two-sided complexity or dot-depth. This question is essentially asking if c passes to $\text{Spec } \mathbf{PV}^{op}$.

Problem 72. Transfer the results of this book to Formal Language Theory via the Eilenberg Correspondence [85]. Is there a quantized version of the Eilenberg Correspondence? We have a notion of an A -generated relational morphism. Perhaps each pseudovariety of relational morphisms is generated by canonical relational morphisms of A -generated syntactic semigroups where A is any finite alphabet? What objects are recognized by relational morphisms? Is there a connection with Straubing's \mathcal{C} -pseudovarieties [347]?

Problem 73. Let S be a finite semigroup and let $P = [\mathbf{1}, (S)] \cap K(\mathbf{P}\mathbf{V})$ be the join semilattice of compact subpseudovarieties of (S) . This is a countable poset. What countable linear orders can occur as chains in P ? Both (\mathbb{N}, \leq) and (\mathbb{N}, \geq) can occur [309].

Problem 74. Relate two-sided complexity to composition of bimachines [84, 284].

References

1. M. Aguiar and S. Mahajan. *Coxeter groups and Hopf algebras*, volume 23 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2006. With a foreword by Nantel Bergeron.
2. D. Albert, R. Baldinger, and J. Rhodes. Undecidability of the identity problem for finite semigroups. *The Journal of Symbolic Logic*, 57(1):179–192, 1992.
3. D. Allen, Jr. and J. Rhodes. Synthesis of classical and modern theory of finite semigroups. *Advances in Mathematics*, 11(2):238–266, 1973.
4. J. Almeida. Implicit operations on finite \mathcal{J} -trivial semigroups and a conjecture of I. Simon. *Journal of Pure and Applied Algebra*, 69(3):205–218, 1991.
5. J. Almeida. On direct product decompositions of finite \mathcal{J} -trivial semigroups. *International Journal of Algebra and Computation*, 1(3):329–337, 1991.
6. J. Almeida. On finite simple semigroups. *Proceedings of the Edinburgh Mathematical Society. Series II*, 34(2):205–215, 1991.
7. J. Almeida. *Finite semigroups and universal algebra*, volume 3 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.
8. J. Almeida. A syntactical proof of locality of \mathbf{DA} . *International Journal of Algebra and Computation*, 6(2):165–177, 1996.
9. J. Almeida. Hyperdecidable pseudovarieties and the calculation of semidirect products. *International Journal of Algebra and Computation*, 9(3-4):241–261, 1999. Dedicated to the memory of Marcel-Paul Schützenberger.
10. J. Almeida. Dynamics of implicit operations and tameness of pseudovarieties of groups. *Transactions of the American Mathematical Society*, 354(1):387–411 (electronic), 2002.
11. J. Almeida. Profinite groups associated with weakly primitive substitutions. *Fundamental'naya i Prikladnaya Matematika*, 11(3):13–48, 2005. Translation in *J. Math. Sci. (N. Y.)* 144(2):3881–3903, 2007.
12. J. Almeida, A. Azevedo, and L. Teixeira. On finitely based pseudovarieties of the form $\mathbf{V} * \mathbf{D}$ and $\mathbf{V} * \mathbf{D}_n$. *Journal of Pure and Applied Algebra*, 146(1):1–15, 2000.
13. J. Almeida, A. Azevedo, and M. Zeitoun. Pseudovariety joins involving \mathcal{J} -trivial semigroups. *International Journal of Algebra and Computation*, 9(1):99–112, 1999.

14. J. Almeida, J. C. Costa, and M. Zeitoun. Tameness of pseudovariety joins involving R . *Monatshefte für Mathematik*, 146(2):89–111, 2005.
15. J. Almeida and M. Delgado. Tameness of the pseudovariety of abelian groups. *International Journal of Algebra and Computation*, 15(2):327–338, 2005.
16. J. Almeida and A. Escada. On the equation $\mathbf{V} * \mathbf{G} = \mathcal{E} \mathbf{V}$. *Journal of Pure and Applied Algebra*, 166(1-2):1–28, 2002.
17. J. Almeida, S. W. Margolis, B. Steinberg, and M. V. Volkov. Characterization of group radicals with an application to Mal'cev products. Work in progress, 2006.
18. J. Almeida, S. W. Margolis, B. Steinberg, and M. V. Volkov. Representation theory of finite semigroups, semigroup radicals and formal language theory. *Transactions of the American Mathematical Society*, to appear.
19. J. Almeida and B. Steinberg. On the decidability of iterated semidirect products with applications to complexity. *Proceedings of the London Mathematical Society. Third Series*, 80(1):50–74, 2000.
20. J. Almeida and B. Steinberg. Syntactic and global semigroup theory: a synthesis approach. In *Algorithmic problems in groups and semigroups (Lincoln, NE, 1998)*, Trends Math., pages 1–23. Birkhäuser Boston, Boston, MA, 2000.
21. J. Almeida and B. Steinberg. Rational codes and free profinite monoids. Preprint, 2008.
22. J. Almeida and M. V. Volkov. Profinite identities for finite semigroups whose subgroups belong to a given pseudovariety. *Journal of Algebra and its Applications*, 2(2):137–163, 2003.
23. J. Almeida and M. V. Volkov. Subword complexity of profinite words and subgroups of free profinite semigroups. *International Journal of Algebra and Computation*, 16(2):221–258, 2006.
24. J. Almeida and P. Weil. Free profinite semigroups over semidirect products. *Izvestiya Vysshikh Uchebnykh Zavedeniï. Matematika*, 39(1):3–31, 1995.
25. J. Almeida and P. Weil. Relatively free profinite monoids: an introduction and examples. In *Semigroups, formal languages and groups (York, 1993)*, volume 466 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 73–117. Kluwer Acad. Publ., Dordrecht, 1995.
26. J. Almeida and P. Weil. Free profinite \mathcal{R} -trivial monoids. *International Journal of Algebra and Computation*, 7(5):625–671, 1997.
27. J. Almeida and P. Weil. Profinite categories and semidirect products. *Journal of Pure and Applied Algebra*, 123(1-3):1–50, 1998.
28. J. Almeida and M. Zeitoun. The pseudovariety \mathbf{J} is hyperdecidable. *RAIRO Informatique Théorique et Applications. Theoretical Informatics and Applications*, 31(5):457–482, 1997.
29. J. Almeida and M. Zeitoun. Tameness of some locally trivial pseudovarieties. *Communications in Algebra*, 31(1):61–77, 2003.
30. J. Almeida and M. Zeitoun. The equational theory of ω -terms for finite \mathcal{R} -trivial semigroups. In *Semigroups and languages*, pages 1–22. World Sci. Publ., River Edge, NJ, 2004.
31. W. Arveson. *An invitation to C^* -algebras*. Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 39.
32. C. J. Ash. Finite semigroups with commuting idempotents. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 43(1):81–90, 1987.

33. C. J. Ash. Inevitable graphs: a proof of the type II conjecture and some related decision procedures. *International Journal of Algebra and Computation*, 1(1):127–146, 1991.
34. K. Auinger. A new proof of the Rhodes type II conjecture. *International Journal of Algebra and Computation*, 14(5-6):551–568, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.
35. K. Auinger, T. E. Hall, N. R. Reilly, and S. Zhang. Congruences on the lattice of pseudovarieties of finite semigroups. *International Journal of Algebra and Computation*, 7(4):433–455, 1997.
36. K. Auinger and B. Steinberg. On the extension problem for partial permutations. *Proceedings of the American Mathematical Society*, 131(9):2693–2703 (electronic), 2003.
37. K. Auinger and B. Steinberg. The geometry of profinite graphs with applications to free groups and finite monoids. *Transactions of the American Mathematical Society*, 356(2):805–851 (electronic), 2004.
38. K. Auinger and B. Steinberg. Constructing divisions into power groups. *Theoretical Computer Science*, 341(1-3):1–21, 2005.
39. K. Auinger and B. Steinberg. A constructive version of the Ribes-Zaleskiĭ product theorem. *Mathematische Zeitschrift*, 250(2):287–297, 2005.
40. K. Auinger and B. Steinberg. On power groups and embedding theorems for relatively free profinite monoids. *Mathematical Proceedings of the Cambridge Philosophical Society*, 138(2):211–232, 2005.
41. K. Auinger and B. Steinberg. Varieties of finite supersolvable groups with the M. Hall property. *Mathematische Annalen*, 335(4):853–877, 2006.
42. K. Auinger and P. G. Trotter. Pseudovarieties, regular semigroups and semidirect products. *Journal of the London Mathematical Society. Second Series*, 58(2):284–296, 1998.
43. B. Austin, K. Henckell, C. Nehaniv, and J. Rhodes. Subsemigroups and complexity via the presentation lemma. *Journal of Pure and Applied Algebra*, 101(3):245–289, 1995.
44. B. Banaschewski. The Birkhoff theorem for varieties of finite algebras. *Algebra Universalis*, 17(3):360–368, 1983.
45. T. Bandman, G.-M. Greuel, F. Grunewald, B. Kunyavskiĭ, G. Pfister, and E. Plotkin. Identities for finite solvable groups and equations in finite simple groups. *Compositio Mathematica*, 142(3):734–764, 2006.
46. G. M. Bergman. *An invitation to general algebra and universal constructions*. Henry Helson, Berkeley, CA, 1998.
47. G. M. Bergman. Every finite semigroup is embeddable in a finite relatively free semigroup. *Journal of Pure and Applied Algebra*, 186(1):1–19, 2004.
48. J. Berstel and C. Reutenauer. *Rational series and their languages*, volume 12 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1988.
49. P. Bidigare, P. Hanlon, and D. Rockmore. A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements. *Duke Mathematical Journal*, 99(1):135–174, 1999.
50. J.-C. Birget. Arbitrary vs. regular semigroups. *Journal of Pure and Applied Algebra*, 34(1):57–115, 1984.
51. J.-C. Birget. Iteration of expansions—unambiguous semigroups. *Journal of Pure and Applied Algebra*, 34(1):1–55, 1984.

52. J.-C. Birget. The synthesis theorem for finite regular semigroups, and its generalization. *Journal of Pure and Applied Algebra*, 55(1-2):1–79, 1988.
53. J.-C. Birget, S. Margolis, and J. Rhodes. Semigroups whose idempotents form a subsemigroup. *Bulletin of the Australian Mathematical Society*, 41(2):161–184, 1990.
54. J.-C. Birget and J. Rhodes. Almost finite expansions of arbitrary semigroups. *Journal of Pure and Applied Algebra*, 32(3):239–287, 1984.
55. M. J. J. Branco. The kernel category and variants of the concatenation product. *International Journal of Algebra and Computation*, 7(4):487–509, 1997.
56. J. N. Bray, J. S. Wilson, and R. A. Wilson. A characterization of finite soluble groups by laws in two variables. *The Bulletin of the London Mathematical Society*, 37(2):179–186, 2005.
57. K. S. Brown. Semigroups, rings, and Markov chains. *Journal of Theoretical Probability*, 13(3):871–938, 2000.
58. K. S. Brown. Semigroup and ring theoretical methods in probability. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, volume 40 of *Fields Inst. Commun.*, pages 3–26. Amer. Math. Soc., Providence, RI, 2004.
59. T. C. Brown. An interesting combinatorial method in the theory of locally finite semigroups. *Pacific Journal of Mathematics*, 36:285–289, 1971.
60. J. A. Brzozowski and I. Simon. Characterizations of locally testable events. *Discrete Mathematics*, 4:243–271, 1973.
61. S. Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.
62. B. Carré. *Graphs and networks*. The Clarendon Press Oxford University Press, New York, 1979. Oxford Applied Mathematics and Computing Science Series.
63. J. H. Carruth, J. A. Hildebrant, and R. J. Koch. *The theory of topological semigroups*, volume 75 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1983.
64. J. H. Carruth, J. A. Hildebrant, and R. J. Koch. *The theory of topological semigroups. Vol. 2*, volume 100 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1986.
65. N. Chomsky and M. P. Schützenberger. The algebraic theory of context-free languages. In *Computer programming and formal systems*, pages 118–161. North-Holland, Amsterdam, 1963.
66. A. H. Clifford. Matrix representations of completely simple semigroups. *American Journal of Mathematics*, 64:327–342, 1942.
67. A. H. Clifford. Basic representations of completely simple semigroups. *American Journal of Mathematics*, 82:430–434, 1960.
68. A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. I*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, RI, 1961.
69. A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups. Vol. II*. Mathematical Surveys, No. 7. American Mathematical Society, Providence, RI, 1967.
70. E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *Journal für die Reine und Angewandte Mathematik*, 391:85–99, 1988.
71. R. S. Cohen and J. A. Brzozowski. Dot-depth of star-free events. *Journal of Computer and System Sciences*, 5:1–16, 1971.

72. J. H. Conway. *Regular algebra and finite machines*. Chapman & Hall, London, 1971.
73. J. C. Costa. Free profinite \mathcal{R} -trivial, locally idempotent and locally commutative semigroups. *Semigroup Forum*, 58(3):423–444, 1999.
74. J. C. Costa. Free profinite semigroups over some classes of semigroups locally in $\mathcal{D}\mathbf{G}$. *International Journal of Algebra and Computation*, 10(4):491–537, 2000.
75. J. C. Costa. Free profinite locally idempotent and locally commutative semigroups. *Journal of Pure and Applied Algebra*, 163(1):19–47, 2001.
76. T. Coulbois. Free product, profinite topology and finitely generated subgroups. *International Journal of Algebra and Computation*, 11(2):171–184, 2001.
77. D. F. Cowan, N. R. Reilly, P. G. Trotter, and M. V. Volkov. The finite basis problem for quasivarieties and pseudovarieties generated by regular semigroups. I. Quasivarieties generated by regular semigroups. *Journal of Algebra*, 267(2):635–653, 2003.
78. R. A. Dean and T. Evans. A remark on varieties of lattices and semigroups. *Proceedings of the American Mathematical Society*, 21:394–396, 1969.
79. M. Delgado. Abelian pointlikes of a monoid. *Semigroup Forum*, 56(3):339–361, 1998.
80. M. Delgado, V. H. Fernandes, S. Margolis, and B. Steinberg. On semigroups whose idempotent-generated subsemigroup is aperiodic. *International Journal of Algebra and Computation*, 14(5-6):655–665, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.
81. M. Delgado, S. Margolis, and B. Steinberg. Combinatorial group theory, inverse monoids, automata, and global semigroup theory. *International Journal of Algebra and Computation*, 12(1-2):179–211, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).
82. M. Delgado, A. Masuda, and B. Steinberg. Solving systems of equations modulo pseudovarieties of abelian groups and hyperdecidability. In J. André, V. H. Fernandes, M. J. J. Branco, G. Gomes, J. Fountain, and J. C. Meakin, editors, *Semigroups and formal languages*, pages 57–65. World Sci. Publ., Hackensack, NJ, 2007.
83. P. M. Edwards. Eventually regular semigroups. *Bulletin of the Australian Mathematical Society*, 28(1):23–38, 1983.
84. S. Eilenberg. *Automata, languages, and machines. Vol. A*. Academic Press, New York, 1974. Pure and Applied Mathematics, Vol. 58.
85. S. Eilenberg. *Automata, languages, and machines. Vol. B*. Academic Press, New York, 1976. With two chapters (“Depth decomposition theorem” and “Complexity of semigroups and morphisms”) by Bret Tilson, Pure and Applied Mathematics, Vol. 59.
86. S. Eilenberg and M. P. Schützenberger. On pseudovarieties. *Advances in Mathematics*, 19(3):413–418, 1976.
87. G. Z. Elston. Semigroup expansions using the derived category, kernel, and Malcev products. *Journal of Pure and Applied Algebra*, 136(3):231–265, 1999.
88. G. Z. Elston and C. L. Nehaniv. Holonomy embedding of arbitrary stable semigroups. *International Journal of Algebra and Computation*, 12(6):791–810, 2002.
89. J. A. Erdos. On products of idempotent matrices. *Glasgow Mathematical Journal*, 8:118–122, 1967.

90. D. G. Fitz-Gerald. On inverses of products of idempotents in regular semigroups. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 13:335–337, 1972.
91. C. Fox and J. Rhodes. The complexity of the power set of a semigroup. Technical Report PAM-217, Center for Pure and Applied Mathematics, Math. Dept., Univ. of California at Berkeley, Berkeley, California, 1984.
92. P. Gabriel. Unzerlegbare Darstellungen. I. *Manuscripta Mathematica*, 6:71–103; correction, *ibid.* 6 (1972), 309, 1972.
93. P. Gabriel. Indecomposable representations. II. In *Symposia Mathematica, Vol. XI (Convegno di Algebra Commutativa, INDAM, Rome, 1971)*, pages 81–104. Academic Press, London, 1973.
94. P. Gabriel and A. V. Roiter. *Representations of finite-dimensional algebras*. Springer-Verlag, Berlin, 1997. Translated from the Russian, with a chapter by B. Keller, Reprint of the 1992 English translation.
95. O. Ganyushkin, V. Mazorchuk, and B. Steinberg. On the irreducible representations of a finite semigroup. Preprint, 2008.
96. M. Gehrke, S. Grigorieff, and J.-E. Pin. Duality and equational theory of regular languages. In *Automata, languages and programming*, volume 5216 of *Lecture Notes in Comput. Sci.*, pages 246–257. Springer, Berlin, 2008.
97. G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.
98. R. Gitik, S. W. Margolis, and B. Steinberg. On the Kurosh theorem and separability properties. *Journal of Pure and Applied Algebra*, 179(1-2):87–97, 2003.
99. R. Gitik and E. Rips. On separability properties of groups. *International Journal of Algebra and Computation*, 5(6):703–717, 1995.
100. K. Głazek. *A guide to the literature on semirings and their applications in mathematics and information sciences*. Kluwer Academic Publishers, Dordrecht, 2002. With complete bibliography.
101. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1994.
102. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 2. Part I. Chapter G*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1996. General group theory.
103. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998. Almost simple K -groups.
104. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 4. Part II. Chapters 1–4*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999. Uniqueness theorems, With errata: *The classification of the finite simple groups. Number 3. Part I. Chapter A* [Amer. Math. Soc., Providence, RI, 1998; MR1490581 (98j:20011)].
105. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 5. Part III. Chapters 1–6*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

- and *Monographs*. American Mathematical Society, Providence, RI, 2002. The generic case, stages 1–3a.
106. D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 6. Part IV*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. The special odd case.
 107. R. L. Graham. On finite 0-simple semigroups and graph theory. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 2:325–339, 1968.
 108. J. A. Green. On the structure of semigroups. *Annals of Mathematics. Second Series*, 54:163–172, 1951.
 109. R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ. Automata, dynamical systems, and groups. *Proceedings of the Steklov Institute of Mathematics*, 231(4):128–203, 2000.
 110. P.-A. Grillet. *Semigroups*, volume 193 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1995. An introduction to the structure theory.
 111. D. Groves and M. Vaughan-Lee. Finite groups of bounded exponent. *The Bulletin of the London Mathematical Society*, 35(1):37–40, 2003.
 112. M. Hall, Jr. A topology for free groups and related groups. *Annals of Mathematics. Second Series*, 52:127–139, 1950.
 113. M. Hall, Jr. *The theory of groups*. The Macmillan Co., New York, 1959.
 114. T. E. Hall. On regular semigroups. *Journal of Algebra*, 24:1–24, 1973.
 115. T. E. Hall. The partially ordered set of all J -classes of a finite semigroup. *Semigroup Forum*, 6(3):263–264, 1973.
 116. T. E. Hall, S. I. Kublanovskii, S. Margolis, M. V. Sapir, and P. G. Trotter. Algorithmic problems for finite groups and finite 0-simple semigroups. *Journal of Pure and Applied Algebra*, 119(1):75–96, 1997.
 117. P. R. Halmos. *Lectures on Boolean algebras*. Van Nostrand Mathematical Studies, No. 1. D. Van Nostrand Co., Inc., Princeton, NJ, 1963.
 118. J. Hartmanis. Loop-free structure of sequential machines. *Information and Computation*, 5:25–43, 1962.
 119. J. Hartmanis. Further results on the structure of sequential machines. *Journal of the Association for Computing Machinery*, 10:78–88, 1963.
 120. U. Hebisch and H. J. Weinert. *Semirings: algebraic theory and applications in computer science*, volume 5 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1998. Translated from the 1993 German original.
 121. K. Henckell. Pointlike sets: the finest aperiodic cover of a finite semigroup. *Journal of Pure and Applied Algebra*, 55(1-2):85–126, 1988.
 122. K. Henckell. Blockgroups = powergroups: a consequence of Ash’s proof of the Rhodes type II conjecture. In *Monash Conference on Semigroup Theory (Melbourne, 1990)*, pages 117–134. World Sci. Publ., River Edge, NJ, 1991.
 123. K. Henckell. Idempotent pointlike sets. *International Journal of Algebra and Computation*, 14(5-6):703–717, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.
 124. K. Henckell. Stable pairs. *International Journal of Algebra and Computation*, to appear.
 125. K. Henckell, S. Lazarus, and J. Rhodes. Prime decomposition theorem for arbitrary semigroups: general holonomy decomposition and synthesis theorem. *Journal of Pure and Applied Algebra*, 55(1-2):127–172, 1988.

126. K. Henckell, S. W. Margolis, J.-E. Pin, and J. Rhodes. Ash's type II theorem, profinite topology and Mal'cev products. I. *International Journal of Algebra and Computation*, 1(4):411–436, 1991.
127. K. Henckell and J. Rhodes. The theorem of Knast, the $PG = BG$ and type-II conjectures. In *Monoids and semigroups with applications (Berkeley, CA, 1989)*, pages 453–463. World Sci. Publ., River Edge, NJ, 1991.
128. K. Henckell, J. Rhodes, and B. Steinberg. Complexity is decidable: the lower bound. Work in progress.
129. K. Henckell, J. Rhodes, and B. Steinberg. Aperiodic pointlikes and beyond. *International Journal of Algebra and Computation*, to appear.
130. K. Henckell, J. Rhodes, and B. Steinberg. A profinite approach to stable pairs. *International Journal of Algebra and Computation*, to appear.
131. B. Herwig and D. Lascar. Extending partial automorphisms and the profinite topology on free groups. *Transactions of the American Mathematical Society*, 352(5):1985–2021, 2000.
132. P. J. Higgins. Categories and groupoids. *Reprints in Theory and Applications of Categories*, (7):1–178 (electronic), 2005. Reprint of the 1971 original [*Notes on categories and groupoids*, Van Nostrand Reinhold, London] with a new preface by the author.
133. P. M. Higgins. *Techniques of semigroup theory*. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992. With a foreword by G. B. Preston.
134. K. H. Hofmann, M. Mislove, and A. Stralka. *The Pontryagin duality of compact 0-dimensional semilattices and its applications*. Springer-Verlag, Berlin, 1974. Lecture Notes in Mathematics, Vol. 396.
135. K. H. Hofmann and P. S. Mostert. *Elements of compact semigroups*. Charles E. Merril Books, Inc., Columbus, Ohio, 1966.
136. C. H. Houghton. Completely 0-simple semigroups and their associated graphs and groups. *Semigroup Forum*, 14(1):41–67, 1977.
137. J. M. Howie. The subsemigroup generated by the idempotents of a full transformation semigroup. *Journal of the London Mathematical Society. Second Series*, 41:707–716, 1966.
138. J. M. Howie. Idempotents in completely 0-simple semigroups. *Glasgow Mathematical Journal*, 19(2):109–113, 1978.
139. J. M. Howie. *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1995. Oxford Science Publications.
140. R. P. Hunter. Certain finitely generated compact zero-dimensional semigroups. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 44(2):265–270, 1988.
141. B. Huppert. *Endliche Gruppen. I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, 1967.
142. M. Jackson. Finite semigroups whose varieties have uncountably many subvarieties. *Journal of Algebra*, 228(2):512–535, 2000.
143. M. Jackson. Small inherently nonfinitely based finite semigroups. *Semigroup Forum*, 64(2):297–324, 2002.
144. M. Jackson. Finite semigroups with infinite irredundant identity bases. *International Journal of Algebra and Computation*, 15(3):405–422, 2005.

145. M. Jackson and R. McKenzie. Interpreting graph colorability in finite semigroups. *International Journal of Algebra and Computation*, 16(1):119–140, 2006.
146. M. Jackson and O. Sapir. Finitely based, finite sets of words. *International Journal of Algebra and Computation*, 10(6):683–708, 2000.
147. P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition.
148. P. R. Jones. Monoid varieties defined by $x^{n+1} = x$ are local. *Semigroup Forum*, 47(3):318–326, 1993.
149. P. R. Jones. Profinite categories, implicit operations and pseudovarieties of categories. *Journal of Pure and Applied Algebra*, 109(1):61–95, 1996.
150. P. R. Jones and S. Pustejovsky. A kernel for relational morphisms of categories. In *Semigroups with applications (Oberwolfach, 1991)*, pages 152–161. World Sci. Publ., River Edge, NJ, 1992.
151. P. R. Jones and M. B. Szendrei. Local varieties of completely regular monoids. *Journal of Algebra*, 150(1):1–27, 1992.
152. P. R. Jones and P. G. Trotter. Locality of **DS** and associated varieties. *Journal of Pure and Applied Algebra*, 104(3):275–301, 1995.
153. M. Kambites. On the Krohn-Rhodes complexity of semigroups of upper triangular matrices. *International Journal of Algebra and Computation*, 17(1):187–201, 2007.
154. M. Kambites and B. Steinberg. Wreath product decompositions for triangular matrix semigroups. In J. André, V. H. Fernandes, M. J. J. Branco, G. Gomes, J. Fountain, and J. C. Meakin, editors, *Semigroups and formal languages*, pages 129–144. World Sci. Publ., Hackensack, NJ, 2007.
155. I. Kapovich and A. Myasnikov. Stallings foldings and subgroups of free groups. *Journal of Algebra*, 248(2):608–668, 2002.
156. J. Karnofsky and J. Rhodes. Decidability of complexity one-half for finite semigroups. *Semigroup Forum*, 24(1):55–66, 1982.
157. Y. Katsov. Toward homological characterization of semirings: Serre’s conjecture and Bass’s perfectness in a semiring context. *Algebra Universalis*, 52(2-3):197–214, 2004.
158. A. S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
159. O. G. Kharlampovich and M. V. Sapir. Algorithmic problems in varieties. *International Journal of Algebra and Computation*, 5(4-5):379–602, 1995.
160. M. Kilp, U. Knauer, and A. V. Mikhalev. *Monoids, acts and categories*, volume 29 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 2000. With applications to wreath products and graphs, A handbook for students and researchers.
161. S. C. Kleene. Representation of events in nerve nets and finite automata. In *Automata studies*, Annals of mathematics studies, no. 34, pages 3–41. Princeton University Press, Princeton, NJ, 1956.
162. D. J. Kleitman, B. R. Rothschild, and J. H. Spencer. The number of semigroups of order n . *Proceedings of the American Mathematical Society*, 55(1):227–232, 1976.
163. R. Knast. Some theorems on graph congruences. *RAIRO Informatique Théorique*, 17(4):331–342, 1983.

164. L. G. Kovács and M. F. Newman. Cross varieties of groups. *Proceedings of the Royal Society Series A*, 292:530–536, 1966.
165. L. G. Kovács and M. F. Newman. Minimal verbal subgroups. *Proceedings of the Cambridge Philosophical Society*, 62:347–350, 1966.
166. L. G. Kovács and M. F. Newman. On critical groups. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 6:237–250, 1966.
167. M. Krasner and L. Kaloujnine. Produit complet des groupes de permutations et problème d’extension de groupes. I. *Acta Universitatis Szegediensis. Acta Scientiarum Mathematicarum*, 13:208–230, 1950.
168. K. Krohn, R. Mateosian, and J. Rhodes. Methods of the algebraic theory of machines. I. Decomposition theorem for generalized machines; properties preserved under series and parallel compositions of machines. *Journal of Computer and System Sciences*, 1:55–85, 1967.
169. K. Krohn and J. Rhodes. Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines. *Transactions of the American Mathematical Society*, 116:450–464, 1965.
170. K. Krohn and J. Rhodes. Complexity of finite semigroups. *Annals of Mathematics. Second Series*, 88:128–160, 1968.
171. K. Krohn, J. Rhodes, and B. Tilson. *Algebraic theory of machines, languages, and semigroups*. Edited by Michael A. Arbib. With a major contribution by Kenneth Krohn and John L. Rhodes. Academic Press, New York, 1968. Chapters 1, 5–9.
172. D. Kruml, J. W. Pelletier, P. Resende, and J. Rosický. On quantales and spectra of C^* -algebras. *Applied Categorical Structures. A Journal Devoted to Applications of Categorical Methods in Algebra, Analysis, Order, Topology and Computer Science*, 11(6):543–560, 2003.
173. D. Kruml and P. Resende. On quantales that classify C^* -algebras. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 45(4):287–296, 2004.
174. G. Lallement. *Semigroups and combinatorial applications*. John Wiley & Sons, New York-Chichester-Brisbane, 1979. Pure and Applied Mathematics, A Wiley-Interscience Publication.
175. G. Lallement. Augmentations and wreath products of monoids. *Semigroup Forum*, 21(1):89–90, 1980.
176. M. V. Lawson. *Inverse semigroups*. World Scientific Publishing Co. Inc., River Edge, NJ, 1998. The theory of partial symmetries.
177. M. V. Lawson. *Finite automata*. Chapman & Hall/CRC, Boca Raton, FL, 2004.
178. M. V. Lawson, S. W. Margolis, and B. Steinberg. Expansions of inverse semigroups. *Journal of the Australian Mathematical Society*, 80(2):205–228, 2006.
179. B. Le Saëc, J.-E. Pin, and P. Weil. Semigroups with idempotent stabilizers and applications to automata theory. *International Journal of Algebra and Computation*, 1(3):291–314, 1991.
180. E. W. H. Lee. On a simpler basis for the pseudovariety **EDS**. *Semigroup Forum*, 75(2):477–479, 2007.
181. J. Leech. \mathcal{H} -coextensions of monoids. *Memoirs of the American Mathematical Society*, 1(issue 2, 157):1–66, 1975.
182. J. Leech. The structure of a band of groups. *Memoirs of the American Mathematical Society*, 1(issue 2, 157):67–95, 1975.
183. M. Loganathan. Cohomology of inverse semigroups. *Journal of Algebra*, 70(2):375–393, 1981.

184. R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.
185. S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.
186. S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.
187. S. W. Margolis. k -transformation semigroups and a conjecture of Tilson. *Journal of Pure and Applied Algebra*, 17(3):313–322, 1980.
188. S. W. Margolis and J. C. Meakin. E -unitary inverse monoids and the Cayley graph of a group presentation. *Journal of Pure and Applied Algebra*, 58(1):45–76, 1989.
189. S. W. Margolis and J. C. Meakin. Free inverse monoids and graph immersions. *International Journal of Algebra and Computation*, 3(1):79–99, 1993.
190. S. W. Margolis and J.-E. Pin. Power monoids and finite \mathcal{J} -trivial monoids. *Semigroup Forum*, 29(1-2):99–108, 1984.
191. S. W. Margolis and J.-E. Pin. Varieties of finite monoids and topology for the free monoid. In *Proceedings of the 1984 Marquette conference on semigroups (Milwaukee, Wis., 1984)*, pages 113–129, Milwaukee, WI, 1985. Marquette University.
192. S. W. Margolis and J.-E. Pin. Expansions, free inverse semigroups, and Schützenberger product. *Journal of Algebra*, 110(2):298–305, 1987.
193. S. W. Margolis and J.-E. Pin. Inverse semigroups and extensions of groups by semilattices. *Journal of Algebra*, 110(2):277–297, 1987.
194. S. W. Margolis and J.-E. Pin. Inverse semigroups and varieties of finite semigroups. *Journal of Algebra*, 110(2):306–323, 1987.
195. S. W. Margolis, M. Sapir, and P. Weil. Irreducibility of certain pseudovarieties. *Communications in Algebra*, 26(3):779–792, 1998.
196. S. W. Margolis, M. Sapir, and P. Weil. Closed subgroups in pro- \mathbf{V} topologies and the extension problem for inverse automata. *International Journal of Algebra and Computation*, 11(4):405–445, 2001.
197. S. W. Margolis and B. Tilson. An upper bound for the complexity of transformation semigroups. *Journal of Algebra*, 73(2):518–537, 1981.
198. D. B. McAlister. Representations of semigroups by linear transformations. I, II. *Semigroup Forum*, 2(4):283–320, 1971.
199. D. B. McAlister. Groups, semilattices and inverse semigroups. I, II. *Transactions of the American Mathematical Society*, 192:227–244; *ibid.* 196 (1974), 351–370, 1974.
200. D. B. McAlister. Regular semigroups, fundamental semigroups and groups. *Australian Mathematical Society. Journal. Series A*, 29(4):475–503, 1980.
201. D. B. McAlister. Semigroups generated by a group and an idempotent. *Communications in Algebra*, 26(2):515–547, 1998.
202. J. McCammond and J. Rhodes. Geometric semigroup theory. *International Journal of Algebra and Computation*, to appear.
203. R. N. McKenzie, G. F. McNulty, and W. F. Taylor. *Algebras, lattices, varieties. Vol. I*. The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987.
204. R. McNaughton. Algebraic decision procedures for local testability. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 8(1):60–76, 1974.

205. R. McNaughton and S. Papert. *Counter-free automata*. The M.I.T. Press, Cambridge, Mass.–London, 1971. With an appendix by William Henneman, M.I.T. Research Monograph, No. 65.
206. R. McNaughton and H. Yamada. Regular expressions and state graphs for automata. *IEEE Transactions on Electronic Computers*, 9:39–47, 1960.
207. A. R. Meyer. A note on star-free events. *Journal of the Association for Computing Machinery*, 16:220–225, 1969.
208. A. Minasyan. Separable subsets of GFERF negatively curved groups. *Journal of Algebra*, 304(2):1090–1100, 2006.
209. E. F. Moore. Gedanken-experiments on sequential machines. In *Automata studies*, Annals of mathematics studies, no. 34, pages 129–153. Princeton University Press, Princeton, NJ, 1956.
210. W. D. Munn. On semigroup algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51:1–15, 1955.
211. W. D. Munn. Matrix representations of semigroups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 53:5–12, 1957.
212. K. S. S. Nambooripad. Structure of regular semigroups. I. *Memoirs of the American Mathematical Society*, 22(224):vii+119, 1979.
213. K. S. S. Nambooripad. The natural partial order on a regular semigroup. *Proceedings of the Edinburgh Mathematical Society. Series II*, 23(3):249–260, 1980.
214. C. L. Nehaniv. Cascade decomposition of arbitrary semigroups. In *Semigroups, formal languages and groups (York, 1993)*, volume 466 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 391–425. Kluwer Acad. Publ., Dordrecht, 1995.
215. C. L. Nehaniv. Monoids and groups acting on trees: characterizations, gluing, and applications of the depth preserving actions. *International Journal of Algebra and Computation*, 5(2):137–172, 1995.
216. V. Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.
217. H. Neumann. *Varieties of groups*. Springer-Verlag New York, 1967.
218. G. A. Niblo. Separability properties of free groups and surface groups. *Journal of Pure and Applied Algebra*, 78(1):77–84, 1992.
219. W. R. Nico. Wreath products and extensions. *Houston Journal of Mathematics*, 9(1):71–99, 1983.
220. N. Nikolov and D. Segal. On finitely generated profinite groups. I. Strong completeness and uniform bounds. *Annals of Mathematics. Second Series*, 165(1):171–238, 2007.
221. N. Nikolov and D. Segal. On finitely generated profinite groups. II. Products in quasisimple groups. *Annals of Mathematics. Second Series*, 165(1):239–273, 2007.
222. K. Numakura. Theorems on compact totally disconnected semigroups and lattices. *Proceedings of the American Mathematical Society*, 8:623–626, 1957.
223. L. O’Carroll. Inverse semigroups as extensions of semilattices. *Glasgow Mathematical Journal*, 16(1):12–21, 1975.
224. J.-E. Pin. *Varieties of formal languages*. Foundations of Computer Science. Plenum Publishing Corp., New York, 1986. With a preface by M.-P. Schützenberger, Translated from the French by A. Howie.
225. J.-E. Pin. A topological approach to a conjecture of Rhodes. *Bulletin of the Australian Mathematical Society*, 38(3):421–431, 1988.

226. J.-E. Pin. Topologies for the free monoid. *Journal of Algebra*, 137(2):297–337, 1991.
227. J.-E. Pin. **BG** = **PG**: a success story. In *Semigroups, formal languages and groups (York, 1993)*, volume 466 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 33–47. Kluwer Acad. Publ., Dordrecht, 1995.
228. J.-E. Pin. Eilenberg’s theorem for positive varieties of languages. *Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika*, 39(1):80–90, 1995.
229. J.-E. Pin. Syntactic semigroups. In *Handbook of formal languages, Vol. 1*, pages 679–746. Springer, Berlin, 1997.
230. J.-E. Pin. Algebraic tools for the concatenation product. *Theoretical Computer Science*, 292(1):317–342, 2003. Selected papers in honor of Jean Berstel.
231. J.-E. Pin, A. Pinguet, and P. Weil. Ordered categories and ordered semigroups. *Communications in Algebra*, 30(12):5651–5675, 2002.
232. J.-E. Pin and C. Reutenauer. A conjecture on the Hall topology for the free group. *The Bulletin of the London Mathematical Society*, 23(4):356–362, 1991.
233. J.-E. Pin and H. Straubing. Monoids of upper triangular matrices. In *Semigroups (Szeged, 1981)*, volume 39 of *Colloq. Math. Soc. János Bolyai*, pages 259–272. North-Holland, Amsterdam, 1985.
234. J.-E. Pin, H. Straubing, and D. Thérien. Locally trivial categories and unambiguous concatenation. *Journal of Pure and Applied Algebra*, 52(3):297–311, 1988.
235. J.-E. Pin and P. Weil. Profinite semigroups, Mal’cev products, and identities. *Journal of Algebra*, 182(3):604–626, 1996.
236. J.-E. Pin and P. Weil. Polynomial closure and unambiguous product. *Theory of Computing Systems*, 30(4):383–422, 1997.
237. J.-E. Pin and P. Weil. Semidirect products of ordered semigroups. *Communications in Algebra*, 30(1):149–169, 2002.
238. J.-E. Pin and P. Weil. The wreath product principle for ordered semigroups. *Communications in Algebra*, 30(12):5677–5713, 2002.
239. B. I. Plotkin. Triangular products of pairs. *Latvijas Valsts Univ. Zinātn. Raksti*, 151:140–170, 1971. Certain Questions of Group Theory (Proc. Algebra Sem. No. 2, Riga, 1969/1970).
240. B. I. Plotkin, L. J. Greenglaz, and A. A. Gvaramija. *Algebraic structures in automata and databases theory*. World Scientific Publishing Co. Inc., River Edge, NJ, 1992.
241. L. Polák. Syntactic semiring of a language (extended abstract). In *Mathematical foundations of computer science, 2001 (Mariánské Lázně)*, volume 2136 of *Lecture Notes in Comput. Sci.*, pages 611–620. Springer, Berlin, 2001.
242. L. Polák. Syntactic semiring and language equations. In *Implementation and application of automata*, volume 2608 of *Lecture Notes in Comput. Sci.*, pages 182–193. Springer, Berlin, 2003.
243. L. Polák. Syntactic semiring and universal automaton. In *Developments in language theory*, volume 2710 of *Lecture Notes in Comput. Sci.*, pages 411–422. Springer, Berlin, 2003.
244. L. Polák. A classification of rational languages by semilattice-ordered monoids. *Universitatis Masarykianae Brunensis. Facultas Scientiarum Naturalium. Archivum Mathematicum*, 40(4):395–406, 2004.
245. I. S. Ponzovskii. On matrix representations of associative systems. *Matematicheskii Sbornik. Novaya Seriya*, 38(80):241–260, 1956.

246. R. Pöschel, M. V. Sapir, N. W. Sauer, M. G. Stone, and M. V. Volkov. Identities in full transformation semigroups. *Algebra Universalis*, 31(4):580–588, 1994.
247. H. A. Priestley. Representation of distributive lattices by means of ordered stone spaces. *The Bulletin of the London Mathematical Society*, 2:186–190, 1970.
248. M. S. Putcha. Semilattice decompositions of semigroups. *Semigroup Forum*, 6(1):12–34, 1973.
249. M. S. Putcha. Algebraic monoids whose nonunits are products of idempotents. *Proceedings of the American Mathematical Society*, 103(1):38–40, 1988.
250. M. S. Putcha. *Linear algebraic monoids*, volume 133 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1988.
251. M. S. Putcha. Complex representations of finite monoids. *Proceedings of the London Mathematical Society. Third Series*, 73(3):623–641, 1996.
252. M. S. Putcha. Monoid Hecke algebras. *Transactions of the American Mathematical Society*, 349(9):3517–3534, 1997.
253. M. S. Putcha. Complex representations of finite monoids. II. Highest weight categories and quivers. *Journal of Algebra*, 205(1):53–76, 1998.
254. M. S. Putcha. Hecke algebras and semisimplicity of monoid algebras. *Journal of Algebra*, 218(2):488–508, 1999.
255. M. S. Putcha. Semigroups and weights for group representations. *Proceedings of the American Mathematical Society*, 128(10):2835–2842, 2000.
256. M. S. Putcha. Products of idempotents in algebraic monoids. *Journal of the Australian Mathematical Society*, 80(2):193–203, 2006.
257. D. Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
258. D. Rees. On semi-groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 36:387–400, 1940.
259. N. R. Reilly and S. Zhang. Operators on the lattice of pseudovarieties of finite semigroups. *Semigroup Forum*, 57(2):208–239, 1998.
260. N. R. Reilly and S. Zhang. Decomposition of the lattice of pseudovarieties of finite semigroups induced by bands. *Algebra Universalis*, 44(3-4):217–239, 2000.
261. J. Reiterman. The Birkhoff theorem for finite algebras. *Algebra Universalis*, 14(1):1–10, 1982.
262. L. E. Renner. *Linear algebraic monoids*, volume 134 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, V.
263. P. Resende. Quantaes, finite observations and strong bisimulation. *Theoretical Computer Science*, 254(1-2):95–149, 2001.
264. P. Resende. Étale groupoids and their quantaes. *Advances in Mathematics*, 208(1):147–209, 2007.
265. C. Reutenauer. \mathbf{N} -rationality of zeta functions. *Advances in Applied Mathematics*, 18(1):1–17, 1997.
266. J. Rhodes. Some results on finite semigroups. *Journal of Algebra*, 4:471–504, 1966.
267. J. Rhodes. A homomorphism theorem for finite semigroups. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 1:289–304, 1967.

268. J. Rhodes. The fundamental lemma of complexity for arbitrary finite semigroups. *Bulletin of the American Mathematical Society*, 74:1104–1109, 1968.
269. J. Rhodes. Algebraic theory of finite semigroups. Structure numbers and structure theorems for finite semigroups. In K. Folley, editor, *Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968)*, pages 125–162. Academic Press, New York, 1969.
270. J. Rhodes. Characters and complexity of finite semigroups. *Journal of Combinatorial Theory*, 6:67–85, 1969.
271. J. Rhodes. Proof of the fundamental lemma of complexity (weak version) for arbitrary finite semigroups. *Journal of Combinatorial Theory. Series A*, 10:22–73, 1971.
272. J. Rhodes. Axioms for complexity for all finite semigroups. *Advances in Mathematics*, 11(2):210–214, 1973.
273. J. Rhodes. Finite binary relations have no more complexity than finite functions. *Semigroup Forum*, 7(1-4):92–103, 1974. Collection of articles dedicated to Alfred Hobilzelle Clifford on the occasion of his 65th birthday and to Alexander Doniphan Wallace on the occasion of his 68th birthday.
274. J. Rhodes. Proof of the fundamental lemma of complexity (strong version) for arbitrary finite semigroups. *Journal of Combinatorial Theory. Series A*, 16:209–214, 1974.
275. J. Rhodes. Kernel systems—a global study of homomorphisms on finite semigroups. *Journal of Algebra*, 49(1):1–45, 1977.
276. J. Rhodes. Infinite iteration of matrix semigroups. I. Structure theorem for torsion semigroups. *Journal of Algebra*, 98(2):422–451, 1986.
277. J. Rhodes. Infinite iteration of matrix semigroups. II. Structure theorem for arbitrary semigroups up to aperiodic morphism. *Journal of Algebra*, 100(1):25–137, 1986. With an appendix by Jerrold R. Goodwin.
278. J. Rhodes. Survey of global semigroup theory. In *Lattices, semigroups, and universal algebra (Lisbon, 1988)*, pages 243–269. Plenum, New York, 1990.
279. J. Rhodes. Monoids acting on trees: elliptic and wreath products and the holonomy theorem for arbitrary monoids with applications to infinite groups. *International Journal of Algebra and Computation*, 1(2):253–279, 1991.
280. J. Rhodes. Flows on automata. Preprint, 1995.
281. J. Rhodes. Undecidability, automata, and pseudovarieties of finite semigroups. *International Journal of Algebra and Computation*, 9(3-4):455–473, 1999. Dedicated to the memory of Marcel-Paul Schützenberger.
282. J. Rhodes. c is decidable. Preprint, 2000.
283. J. Rhodes. *Applications of automata theory and algebra via the mathematical theory of complexity to biology, physics, psychology, philosophy, and games*. World Scientific Press, in press. With a foreword by Morris W. Hirsch.
284. J. Rhodes and P. Silva. Turing machines and bimachines. *Theoretical Computer Science*, 400(1):182–224, 2008.
285. J. Rhodes and B. Steinberg. Pointlike sets, hyperdecidability and the identity problem for finite semigroups. *International Journal of Algebra and Computation*, 9(3-4):475–481, 1999. Dedicated to the memory of Marcel-Paul Schützenberger.
286. J. Rhodes and B. Steinberg. Profinite semigroups, varieties, expansions and the structure of relatively free profinite semigroups. *International Journal of Algebra and Computation*, 11(6):627–672, 2001.

287. J. Rhodes and B. Steinberg. Join irreducible pseudovarieties, group mapping, and Kovács-Newman semigroups. In *LATIN 2004: Theoretical informatics*, volume 2976 of *Lecture Notes in Comput. Sci.*, pages 279–291. Springer, Berlin, 2004.
288. J. Rhodes and B. Steinberg. Krohn-Rhodes complexity pseudovarieties are not finitely based. *Theoretical Informatics and Applications. Informatique Théorique et Applications*, 39(1):279–296, 2005.
289. J. Rhodes and B. Steinberg. Complexity pseudovarieties are not local; type II subsemigroups can fall arbitrarily in complexity. *International Journal of Algebra and Computation*, 16(4):739–748, 2006.
290. J. Rhodes and B. Steinberg. Closed subgroups of free profinite monoids are projective profinite groups. *The Bulletin of the London Mathematical Society*, 40(3):375–383, 2008.
291. J. Rhodes and B. Tilson. A reduction theorem for complexity of finite semigroups. *Semigroup Forum*, 10(2):96–114, 1975.
292. J. Rhodes and B. Tilson. Local complexity of finite semigroups. In *Algebra, topology, and category theory (collection of papers in honor of Samuel Eilenberg)*, pages 149–168. Academic Press, New York, 1976.
293. J. Rhodes and B. Tilson. The kernel of monoid morphisms. *Journal of Pure and Applied Algebra*, 62(3):227–268, 1989.
294. J. Rhodes and B. R. Tilson. Lower bounds for complexity of finite semigroups. *Journal of Pure and Applied Algebra*, 1(1):79–95, 1971.
295. J. Rhodes and B. R. Tilson. Improved lower bounds for the complexity of finite semigroups. *Journal of Pure and Applied Algebra*, 2:13–71, 1972.
296. J. Rhodes and P. Weil. Decomposition techniques for finite semigroups, using categories. I, II. *Journal of Pure and Applied Algebra*, 62(3):269–284, 285–312, 1989.
297. J. Rhodes and Y. Zalcstein. Elementary representation and character theory of finite semigroups and its application. In *Monoids and semigroups with applications (Berkeley, CA, 1989)*, pages 334–367. World Sci. Publ., River Edge, NJ, 1991.
298. L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.
299. L. Ribes and P. Zalesskii. Profinite topologies in free products of groups. *International Journal of Algebra and Computation*, 14(5-6):751–772, 2004. International Conference on Semigroups and Groups in honor of the 65th birthday of Prof. John Rhodes.
300. L. Ribes and P. A. Zalesskii. On the profinite topology on a free group. *The Bulletin of the London Mathematical Society*, 25(1):37–43, 1993.
301. L. Ribes and P. A. Zalesskii. The pro- p topology of a free group and algorithmic problems in semigroups. *International Journal of Algebra and Computation*, 4(3):359–374, 1994.
302. J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In *Idempotent mathematics and mathematical physics*, volume 377 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 2005.
303. K. I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

304. K. I. Rosenthal. *The theory of quantaloids*, volume 348 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, 1996.
305. A. Salomaa and M. Soittola. *Automata-theoretic aspects of formal power series*. Springer-Verlag, New York, 1978. Texts and Monographs in Computer Science.
306. M. V. Sapir. Inherently non-finitely based finite semigroups. *Matematicheskii Sbornik. Novaya Seriya*, 133(175)(2):154–166, 270, 1987.
307. M. V. Sapir. Problems of Burnside type and the finite basis property in varieties of semigroups. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 51(2):319–340, 447, 1987.
308. M. V. Sapir. Sur la propriété de base finie pour les pseudovariétés de semi-groupes finis. *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, 306(20):795–797, 1988.
309. M. V. Sapir. On Cross semigroup varieties and related questions. *Semigroup Forum*, 42(3):345–364, 1991.
310. S. Satoh, K. Yama, and M. Tokizawa. Semigroups of order 8. *Semigroup Forum*, 49(1):7–29, 1994.
311. M. P. Schützenberger. \mathcal{T} représentation des demi-groupes. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 244:1994–1996, 1957.
312. M.-P. Schützenberger. Sur la représentation monomiale des demi-groupes. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 246:865–867, 1958.
313. M. P. Schützenberger. On finite monoids having only trivial subgroups. *Information and Control*, 8:190–194, 1965.
314. M. P. Schützenberger. Sur le produit de concaténation non ambigu. *Semigroup Forum*, 13(1):47–75, 1976/77.
315. D. Scott. Continuous lattices. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*, pages 97–136. Lecture Notes in Math., Vol. 274. Springer, Berlin, 1972.
316. J.-P. Serre. Groupes proalgébriques. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (7):67, 1960.
317. J.-P. Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.
318. I. Simon. Piecewise testable events. In *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975)*, pages 214–222. Lecture Notes in Comput. Sci., Vol. 33. Springer, Berlin, 1975.
319. I. Simon. Locally finite semigroups and limited subsets of a free monoid. Unpublished manuscript, 1978.
320. I. Simon. A short proof of the factorization forest theorem. In *Tree automata and languages (Le Touquet, 1990)*, volume 10 of *Stud. Comput. Sci. Artificial Intelligence*, pages 433–438. North-Holland, Amsterdam, 1992.
321. J. R. Stallings. Topology of finite graphs. *Inventiones Mathematicae*, 71(3):551–565, 1983.
322. B. Steinberg. On pointlike sets and joins of pseudovarieties. *International Journal of Algebra and Computation*, 8(2):203–234, 1998. With an addendum by the author.
323. B. Steinberg. Monoid kernels and profinite topologies on the free abelian group. *Bulletin of the Australian Mathematical Society*, 60(3):391–402, 1999.
324. B. Steinberg. Semidirect products of categories and applications. *Journal of Pure and Applied Algebra*, 142(2):153–182, 1999.

325. B. Steinberg. Polynomial closure and topology. *International Journal of Algebra and Computation*, 10(5):603–624, 2000.
326. B. Steinberg. $PG = BG$: redux. In *Semigroups (Braga, 1999)*, pages 181–190. World Sci. Publ., River Edge, NJ, 2000.
327. B. Steinberg. A delay theorem for pointlikes. *Semigroup Forum*, 63(3):281–304, 2001.
328. B. Steinberg. Finite state automata: a geometric approach. *Transactions of the American Mathematical Society*, 353(9):3409–3464 (electronic), 2001.
329. B. Steinberg. Inevitable graphs and profinite topologies: some solutions to algorithmic problems in monoid and automata theory, stemming from group theory. *International Journal of Algebra and Computation*, 11(1):25–71, 2001.
330. B. Steinberg. On algorithmic problems for joins of pseudovarieties. *Semigroup Forum*, 62(1):1–40, 2001.
331. B. Steinberg. Inverse automata and profinite topologies on a free group. *Journal of Pure and Applied Algebra*, 167(2-3):341–359, 2002.
332. B. Steinberg. A modern approach to some results of Stiffler. In *Semigroups and languages*, pages 240–249. World Sci. Publ., River Edge, NJ, 2004.
333. B. Steinberg. On an assertion of J. Rhodes and the finite basis and finite vertex rank problems for pseudovarieties. *Journal of Pure and Applied Algebra*, 186(1):91–107, 2004.
334. B. Steinberg. On aperiodic relational morphisms. *Semigroup Forum*, 70(1):1–43, 2005.
335. B. Steinberg. Möbius functions and semigroup representation theory. *Journal of Combinatorial Theory. Series A*, 113(5):866–881, 2006.
336. B. Steinberg. Möbius functions and semigroup representation theory. II. Character formulas and multiplities. *Advances in Mathematics*, 217(4):1521–1557, 2008.
337. B. Steinberg. Maximal subgroups of the minimal ideal of a free profinite monoid are free. *Israel Journal of Mathematics*, to appear.
338. B. Steinberg. A structural approach to the locality of pseudovarieties of the form $\mathbf{LH} \textcircled{m} \mathbf{V}$. *International Journal of Algebra and Computation*, to appear.
339. B. Steinberg and B. Tilson. Categories as algebra. II. *International Journal of Algebra and Computation*, 13(6):627–703, 2003.
340. P. Stiffler, Jr. Extension of the fundamental theorem of finite semigroups. *Advances in Mathematics*, 11(2):159–209, 1973.
341. M. H. Stone. The theory of representations for Boolean algebras. *Transactions of the American Mathematical Society*, 40(1):37–111, 1936.
342. M. H. Stone. Applications of the theory of Boolean rings to general topology. *Transactions of the American Mathematical Society*, 41(3):375–481, 1937.
343. H. Straubing. Aperiodic homomorphisms and the concatenation product of recognizable sets. *Journal of Pure and Applied Algebra*, 15(3):319–327, 1979.
344. H. Straubing. Families of recognizable sets corresponding to certain varieties of finite monoids. *Journal of Pure and Applied Algebra*, 15(3):305–318, 1979.
345. H. Straubing. Finite semigroup varieties of the form $\mathbf{V} * \mathbf{D}$. *Journal of Pure and Applied Algebra*, 36(1):53–94, 1985.
346. H. Straubing. *Finite automata, formal logic, and circuit complexity*. Progress in Theoretical Computer Science. Birkhäuser Boston, Boston, MA, 1994.
347. H. Straubing. On logical descriptions of regular languages. In *LATIN 2002: Theoretical informatics (Cancun)*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 528–538. Springer, Berlin, 2002.

348. H. Straubing and D. Thérien. Partially ordered finite monoids and a theorem of I. Simon. *Journal of Algebra*, 119(2):393–399, 1988.
349. H. Straubing and D. Thérien. Weakly iterated block products of finite monoids. In *LATIN 2002: Theoretical informatics (Cancun)*, volume 2286 of *Lecture Notes in Comput. Sci.*, pages 91–104. Springer, Berlin, 2002.
350. A. Suschkewitsch. Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit. *Mathematische Annalen*, 99(1):30–50, 1928.
351. M. E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.
352. S. Talwar. Morita equivalence for semigroups. *Australian Mathematical Society. Journal. Series A. Pure Mathematics and Statistics*, 59(1):81–111, 1995.
353. S. Talwar. Strong Morita equivalence and a generalisation of the Rees theorem. *Journal of Algebra*, 181(2):371–394, 1996.
354. S. Talwar. Strong Morita equivalence and the synthesis theorem. *International Journal of Algebra and Computation*, 6(2):123–141, 1996.
355. M. L. Teixeira. On semidirectly closed pseudovarieties of aperiodic semigroups. *Journal of Pure and Applied Algebra*, 160(2-3):229–248, 2001.
356. D. Thérien. On the equation $x^t = x^{t+q}$ in categories. *Semigroup Forum*, 37(3):265–271, 1988.
357. D. Thérien. Two-sided wreath product of categories. *Journal of Pure and Applied Algebra*, 74(3):307–315, 1991.
358. D. Thérien and A. Weiss. Graph congruences and wreath products. *Journal of Pure and Applied Algebra*, 36(2):205–215, 1985.
359. B. Tilson. Decomposition and complexity of finite semigroups. *Semigroup Forum*, 3(3):189–250, 1971/72.
360. B. Tilson. Complexity of two- \mathcal{J} class semigroups. *Advances in Mathematics*, 11(2):215–237, 1973.
361. B. Tilson. On the complexity of finite semigroups. *Journal of Pure and Applied Algebra*, 5:187–208, 1974.
362. B. Tilson. *Complexity of semigroups and morphisms*, chapter XII, pages 313–384. In Eilenberg [85], 1976.
363. B. Tilson. *Depth decomposition theorem*, chapter XI, pages 287–312. In Eilenberg [85], 1976.
364. B. Tilson. Categories as algebra: an essential ingredient in the theory of monoids. *Journal of Pure and Applied Algebra*, 48(1-2):83–198, 1987.
365. B. Tilson. Type II redux. In *Semigroups and their applications (Chico, Calif., 1986)*, pages 201–205. Reidel, Dordrecht, 1987.
366. B. Tilson. Presentation lemma... the short form. Unpublished manuscript, 1995.
367. B. Tilson. Modules. Unpublished manuscript, 2005.
368. B. R. Tilson. Appendix to “Algebraic theory of finite semigroups.” On the p -length of p -solvable semigroups: Preliminary results. In K. Folley, editor, *Semigroups (Proc. Sympos., Wayne State Univ., Detroit, Mich., 1968)*, pages 163–208. Academic Press, New York, 1969.
369. A. V. Tishchenko. The ordered monoid of semigroup varieties with respect to a wreath product. *Fundamental'naya i Prikladnaya Matematika*, 5(1):283–305, 1999.
370. P. G. Trotter and M. V. Volkov. The finite basis problem in the pseudovariety joins of aperiodic semigroups with groups. *Semigroup Forum*, 52(1):83–91,

1996. Dedicated to the memory of Alfred Hobilitzelle Clifford (New Orleans, LA, 1994).
371. M. Vaughan-Lee. *The restricted Burnside problem*, volume 8 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, second edition, 1993.
372. M. V. Volkov. The finite basis property of varieties of semigroups. *Akademiya Nauk SSSR. Matematicheskie Zametki*, 45(3):12–23, 127, 1989.
373. M. V. Volkov. On a class of semigroup pseudovarieties without finite pseudoidentity basis. *International Journal of Algebra and Computation*, 5(2):127–135, 1995.
374. M. V. Volkov. The finite basis problem for finite semigroups. *Scientiae Mathematicae Japonicae*, 53(1):171–199, 2001.
375. J. von Neumann. *Continuous geometry*. Foreword by Israel Halperin. Princeton Mathematical Series, No. 25. Princeton University Press, Princeton, NJ, 1960.
376. S. M. Vovsi. *Triangular products of group representations and their applications*, volume 17 of *Progress in Mathematics*. Birkhäuser Boston, MA, 1981.
377. J. H. M. Wedderburn. Homomorphism of groups. *Annals of Mathematics. Second Series*, 42:486–487, 1941.
378. P. Weil. Products of languages with counter. *Theoretical Computer Science*, 76(2-3):251–260, 1990.
379. P. Weil. Closure of varieties of languages under products with counter. *Journal of Computer and System Sciences*, 45(3):316–339, 1992.
380. P. Weil. Profinite methods in semigroup theory. *International Journal of Algebra and Computation*, 12(1-2):137–178, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).
381. H. Wielandt and B. Huppert. Arithmetical and normal structure of finite groups. In *Proc. Sympos. Pure Math., Vol. VI*, pages 17–38. American Mathematical Society, Providence, RI, 1962.
382. S. You. The product separability of the generalized free product of cyclic groups. *Journal of the London Mathematical Society. Second Series*, 56(1):91–103, 1997.
383. Y. Zalcstein. Locally testable languages. *Journal of Computer and System Sciences*, 6:151–167, 1972.
384. Y. Zalcstein. Locally testable semigroups. *Semigroup Forum*, 5:216–227, 1972/73.
385. Y. Zalcstein. Group-complexity and reversals of finite semigroups. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 8(3):235–242, 1974/75.
386. P. Zeiger. Yet another proof of the cascade decomposition theorem for finite automata. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 1(3):225–228, 1967.
387. P. Zeiger. Yet another proof of the cascade decomposition theorem for finite automata: Correction. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 2(4):381, 1968.
388. E. I. Zel'manov. Solution of the restricted Burnside problem for groups of odd exponent. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 54(1):42–59, 221, 1990.
389. E. I. Zel'manov. Solution of the restricted Burnside problem for 2-groups. *Matematicheskii Sbornik*, 182(4):568–592, 1991.

Table of Pseudovarieties

NOTATION	NAME	PSEUDOIDENTITIES
1	Trivial pseudovariety	$\llbracket x = y \rrbracket$
A	Aperiodic semigroups	$\llbracket x^{\omega+1} = x^\omega \rrbracket$
Ab	Abelian groups	$\llbracket x^\omega = 1, xy = yx \rrbracket$
Ab(n)	Abelian groups of exponent n	$\llbracket x^n = 1, xy = yx \rrbracket$
ACom	Aperiodic commutative	$\llbracket x^{\omega+1} = x^\omega, xy = yx \rrbracket$
B	Bands	$\llbracket x^2 = x \rrbracket$
C_n	Level n complexity	
Com	Commutative	$\llbracket xy = yx \rrbracket$
CR	Completely regular	$\llbracket x^{\omega+1} = x \rrbracket$
CS	Completely simple	$\llbracket (xy)^\omega x = x \rrbracket$
D	Delay	$\llbracket xy^\omega = y^\omega \rrbracket$
DS	Regular \mathcal{J} -classes are ssgps.	$\llbracket ((xy)^\omega (yx)^\omega (xy)^\omega)^\omega = (xy)^\omega \rrbracket$
ER	Idem.-gen. ssgp. \mathcal{R} -trivial	$\llbracket (x^\omega y^\omega)^\omega x^\omega = (x^\omega y^\omega)^\omega \rrbracket$
G	Groups	$\llbracket x^\omega = 1 \rrbracket$
G_{nil}	Nilpotent groups	See [10]
G_p	p -groups	$\llbracket x^{p^\omega} = 1 \rrbracket$
G_π	π -groups	$\llbracket x^{\pi^\omega} = 1 \rrbracket$
$G_{\pi'}$	π' -groups	$\llbracket x^{(\pi')^\omega} = 1 \rrbracket$
G_{sol}	Solvable groups	See [45, 56]
G^N	Nilpotent extensions of groups	$\llbracket x^\omega = y^\omega \rrbracket$
J	\mathcal{J} -trivial	$\llbracket (xy)^\omega = (yx)^\omega, x^{\omega+1} = x^\omega \rrbracket$
K	Reverse delay	$\llbracket x^\omega y = x^\omega \rrbracket$
K_n	Level n two-sided complexity	
L	\mathcal{L} -trivial	$\llbracket y(xy)^\omega = (xy)^\omega \rrbracket$
L1	Locally trivial	$\llbracket x^\omega y x^\omega = x^\omega \rrbracket$
LG	Local groups	$\llbracket (x^\omega y x^\omega)^\omega = x^\omega \rrbracket$
LRB	Left regular (= \mathcal{R} -trivial) band	$\llbracket x^2 = x, xyx = xy \rrbracket$
LS	Left simple	$\llbracket xy^\omega = x \rrbracket$

LS^N	Nilpotent exts. of left simple	$\llbracket x^\omega y^\omega = x^\omega \rrbracket$
LSI	Local semilattices	$\llbracket (x^\omega y x^\omega)^2 = x^\omega y x^\omega, \\ x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega \rrbracket$
LZ	Left zero	$\llbracket xy = x \rrbracket$
N	Nilpotent semigroups	$\llbracket x^\omega = 0 \rrbracket$
R	\mathcal{R} -trivial	$\llbracket (xy)^\omega x = (xy)^\omega \rrbracket$
RB	Rectangular bands	$\llbracket x^2 = x, xyx = x \rrbracket$
RRB	Right regular (= \mathcal{L} -trivial) band	$\llbracket x^2 = x, yxy = xy \rrbracket$
RS	Right simple	$\llbracket x^\omega y = y \rrbracket$
RS^N	Nilpotent exts. of right simple	$\llbracket x^\omega y^\omega = y^\omega \rrbracket$
RZ	Right zero	$\llbracket xy = y \rrbracket$
SI	Semilattices	$\llbracket x^2 = x, xy = yx \rrbracket$

Table of Operators and Products

NAME	NOTATION	REFERENCE
Semidirect product	*	Example 2.4.9
Two-sided semidirect product	**	Example 2.4.14
Mal'cev product	$\textcircled{\text{m}}$	Example 2.4.4
Generalized Mal'cev product	$(\mathbf{V}, \mathbf{W}) \textcircled{\text{m}} (-)$	Example 2.4.5
Join	\vee	Example 2.4.7
Intersection	\cap or \wedge	Example 2.4.17
Constants	$q_{\mathbf{V}}$	Example 2.4.2
Identity	$1_{\mathbf{P}\mathbf{V}}$	Example 2.4.1
Old wreath product	\circ	Example 2.4.24
Local operator	\mathbb{L}	Example 2.4.27
Idempotents	\mathbf{E}	Example 2.4.28
Regular \mathcal{D} -classes	\mathbf{D}	Example 2.4.26
Power operator	\mathbf{P}	Example 2.4.25
Schützenberger product	\diamond	Example 2.4.18
Regular elements	\mathbf{R}	Example 2.4.29
Semidirect closure	$()^{\omega}$	Example 2.4.31

Table of Implicit Operations

SYMBOL	DESCRIPTION	DEFINING SEQUENCE
x^ω	Idempotent in $\overline{\langle x \rangle}$	$\lim x^{n!}$
$x^{\omega+1}$	Generator of the kernel of $\overline{\langle x \rangle}$	$\lim x^{n!+1}$
$x^{\omega-1}$	Inverse of $x^{\omega+1}$	$\lim x^{n!-1}$
x^{p^ω}	p' -component of $x^{\omega+1}$	$\lim x^{p^{n!}}$
x^{π^ω}	π' -component of $x^{\omega+1}$	$\lim x^{(p_1 \cdots p_n)^{n!}}, p_i \in \pi$
$x^{(\pi')^\omega}$	π -component of $x^{\omega+1}$	$\lim x^{(p_1 \cdots p_n)^{n!}}, p_i \in \pi'$

Index of Notation

(φ)	Relational morphism pseudovariety generated by φ , p. 75
(\mathbf{V}, \mathbf{W})	Relational morphisms with \mathbf{W} -preimages in \mathbf{V} , p. 56
(S)	Semigroup pseudovariety generated by S , p. 35
$\#(S)$	Type I–Type II bound for S , p. 298
$\#\varphi$	Graph of the relational morphism φ , p. 37
**	Two-sided semidirect product of pseudovarieties, p. 34
AGGM	Aperiodic generalized group mapping congruence, p. 262
*	Semidirect product of pseudovarieties, p. 34
M^\sharp	Augmented monoid of M , p. 222
\square	Block product, p. 31
\circ	Old wreath product, p. 88
$\mathbf{Cnt}(\mathbf{PV})$	Quantale of continuous operators on \mathbf{PV} , p. 57
$\mathbf{Cnt}(\mathbf{PV})^+$	Non-decreasing continuous operators, p. 57
$\mathbf{Cnt}(\mathbf{PV})^-$	Non-increasing continuous operators, p. 72
\mathcal{D}_Φ	Derived automaton of Φ , p. 328
Δ	Diagonal mapping, p. 31
$\delta(S, T)$	Compact continuous operator, p. 58
\mathbf{Der}	Derived semigroupoid pre-identifications, p. 93
ℓ_{1D}	Relational morphisms whose derived semigroupoid divides a locally trivial category, p. 99
ℓ_{1K}	Relational morphisms whose kernel semigroupoid divides a locally trivial category, p. 111
$\ell\mathbf{V}$	Semigroupoids with local semigroups in \mathbf{V} , p. 99
ℓ_C	Labeling of incidence graph associated to the Rees matrix C , p. 314
$\Gamma(M, A)$	Cayley graph of M with generators A , p. 349
Γ_J	Aperiodic generalized group mapping representation, p. 259
γ_J	Generalized group mapping representation, p. 257
GGM	Generalized group mapping congruence, p. 262
$\mathbf{GMC}(\mathbf{PV})$	Quantale of continuous operators on \mathbf{PV} satisfying the global Mal'cev condition, p. 72

$\mathbf{GMC}(\mathbf{PV})^+$	Non-decreasing global Mal'cev operators, p. 72
$\mathbf{GMC}(\mathbf{PV})^-$	Non-increasing global Mal'cev operators, p. 72
\mathcal{H}	Green's relation \mathcal{H} , p. 596
\mathcal{J}	Green's relation \mathcal{J} , p. 596
\rtimes	Two-sided semidirect product, p. 30
\mathbf{K}_{er}	Kernel semigroupoid pre-identifications, p. 105
\mathcal{L}	Green's relation \mathcal{L} , p. 596
λ_J	Left mapping representation, p. 257
\ll	Subdirect product, p. 34
LLM	Left letter mapping congruence, p. 262
LM	Left mapping congruence, p. 262
$\textcircled{\text{m}}$	Mal'cev product, p. 69
\mathbb{B}	Two-element Boolean algebra, p. 540
\mathbf{PV}	Lattice of semigroup pseudovarieties, p. 33
\mathfrak{q}	\mathfrak{q} -operator, p. 61
Im	Image of a relational morphism, p. 37
$\text{Reg}(S)$	Regular elements of S , p. 88
\mathcal{D}	Green's relation \mathcal{D} , p. 596
$\mathcal{C}_{\mathcal{E}}$	Systems of pseudoidentities associated to \mathcal{E} , p. 181
$\text{PL}_{\mathbf{V}}$	Pointlike subsets with respect to \mathbf{V} , p. 82
\mathcal{M}^0	Rees matrix semigroup, p. 608
\models	Satisfaction of pseudoidentities, p. 144
μ_J^L	Left letter mapping representation, p. 258
μ_J^R	Right letter mapping representation, p. 258
∇	Triple product, p. 558
$\overline{\mathbf{H}}$	Semigroups whose subgroups belong to \mathbf{H} , p. 233
\overline{n}	Right zero semigroup with n elements, p. 224
$\varphi_{\mathbf{V}, Q}$	Free A -generated pro- \mathbf{V} relational morphism to Q , p. 155
$\Pi_1(\Gamma)$	Fundamental groupoid, p. 310
$\pi_1(\Gamma, \ell, v)$	Fundamental group of a labeled graph, p. 311
$\pi_1(\Gamma, v)$	Fundamental group, p. 310
\prec	Division, p. 32
\prec_s	Strong division, p. 44
$\mathbf{V}(\mathbf{H})$	$\mathbf{V} \cap \overline{\mathbf{H}}$, p. 508
$\mathbf{V} \times^{\omega} \mathbf{W}$	$\mathbf{V} \times (\mathbf{V} \times (\dots (\mathbf{V} \times \mathbf{W}) \dots))$, p. 92
$\mathbf{V}^{\mathbf{N}}$	Nilpotent extensions of semigroups in \mathbf{V} , p. 250
\mathbf{V}_D	Derived semigroupoid in \mathbf{gV} , p. 83
\mathbf{V}_K	Kernel semigroupoid in \mathbf{gV} , p. 84
\mathbf{V}_W	Old derived semigroupoid in \mathbf{gV} , p. 88
\mathbf{V}_{\vee}	Relational morphisms satisfying \mathbf{V} -slice condition, p. 82
$\mathbf{V}^{\omega} \times \mathbf{W}$	$((\dots (\mathbf{V} \times \mathbf{W}) \dots) \times \mathbf{W}) \times \mathbf{W}$, p. 92
$\text{sup}_{\mathbf{B}}$	Preserves non-empty sups, p. 20
\mathbf{D}	Pseudovariety of divisions, p. 81
$\mathbf{K}_{\mathbf{G}}(S)$	Type II subsemigroup of S , p. 88
\mathbf{BoolA}	Category of Boolean algebras, p. 535

BoolR	Category of Boolean rings, p. 536
Cnt	Category of complete lattices with continuous maps, p. 20
FSgp	Category of finite semigroups, p. 16
GMC$_{\mathbf{U}}^g$	Operators satisfying reverse global Mal'cev condition relative to \mathbf{U} , p. 124
gV	Semigroupoids dividing semigroup in \mathbf{V} , p. 99
inf	Preserves infs, p. 20
inf$_{\top}$	Preserves non-empty infs, p. 20
PSgp	Category of profinite semigroups, p. 135
PSgp$_A$	Category of A -generated profinite semigroups, p. 136
Sup	Category of complete lattices with sup maps, p. 20
sup	Preserves sups, p. 19
\mathcal{R}	Green's relation \mathcal{R} , p. 596
ρ_J	Right mapping representation, p. 257
RLM	Right letter mapping congruence, p. 262
RM	Right mapping congruence, p. 262
\rtimes	Semidirect product, p. 24
Span X	Span of X , p. 415
Stab(X)	Stabilizer of X , p. 604
\subseteq_s	Containment of coterminial relational morphisms, p. 38
$\tau(S)$	Tilson number of S , p. 278
$\tilde{\mathbf{V}}$	Relational morphisms with domain in \mathbf{V} , p. 81
Δ	Triangular product, p. 559
\vdash	Satisfaction of relations, p. 157
V-map	Preserves finite sups, p. 20
\vee_{det}	Determined join, p. 428
\wedge-map	Preserves finite meets, p. 20
\wedge_{det}	Determined meet, p. 428
\widehat{A}^+	Free profinite semigroup generated by A , p. 135
$\widehat{F_{\mathbf{V}}}(A)$	Free pro- \mathbf{V} semigroup generated by A , p. 144
\wr	Wreath product, p. 25
op	Opposite object, p. 17
A^*	Free monoid on A , p. 135
A^+	Free semigroup on A , p. 135
A^ℓ	A with the left zero multiplication, p. 224
B_n	Brandt semigroup of order $n^2 + 1$, p. 59
$B_n(R)$	R -span of $n \times n$ matrix units, p. 552
$C(S)$	Two-sided complexity of the semigroup S , p. 391
$c(S)$	Group complexity of the semigroup S , p. 240
$c_{\mathbf{A}}$	U_2 -length of an aperiodic semigroup, p. 242
D_φ	Derived semigroupoid of φ , p. 94
$E(S)$	Idempotents of S , p. 88
$E_{\mathbf{W},n}$	Presentation of free n -generated pro- \mathbf{W} semigroup, p. 160
G^\natural	Quantale associated to group G with top ∞ and bottom 0, p. 526

$H(\Gamma, G)$	G -cohomology of Γ , p. 312
I_n	Symmetric inverse monoid of degree n , p. 220
k_0S	Contracted semigroup algebra, p. 554
K_φ	Kernel semigroupoid of φ , p. 105
kS	Semigroup algebra, p. 550
M_n	$n \times n$ matrices, p. 550
$P(S)$	Power set of S , p. 82
$P_0(S)$	Contracted power semigroup of semigroup with zero S , p. 554
$q_{\mathbf{V}}$	Constant map to the pseudovariety \mathbf{V} , p. 81
S^0	S with adjoined 0, p. 507
S^I	S with adjoined identity I , p. 22
S^\bullet	S with adjoined identity if not a monoid, p. 22
S_E	Idempotent splitting of S , p. 279
S_n	Symmetric group of degree n , p. 91
T_n	Full transformation monoid of degree n , p. 91
U_n	Semigroup with n right zeroes and an adjoined identity, p. 224
$w(L)$	Weight of a continuous lattice, p. 451
$Z(\Gamma, G)$	Set of G -labelings of Γ , p. 311
MPV	Lattice of monoid pseudovarieties, p. 33
OP	Category of posets with order preserving maps, p. 20

Author Index

- Aguiar, M. 2, 221
Albert, D. 36, 126, 164, 474, 476
Alexandrov, P. 456
Allen, D., Jr. viii, 4, 303, 383, 418
Almeida, J. vii, ix, 2, 7, 8, 11, 12, 24, 27, 30, 47, 49, 60, 63, 67, 68, 72, 88, 94, 95, 100, 102–104, 113, 124, 126–128, 135, 136, 143, 145, 147, 168, 171, 173, 177, 181, 187–192, 194–196, 198–202, 206, 212, 216, 222, 229, 255, 256, 276, 277, 292, 386–388, 390, 461, 465, 468, 473, 475, 476, 482, 487, 488, 492, 493, 496, 506, 539, 540, 566, 616, 643
Arveson, W. 435, 436, 598
Ash, C. J. ix, 7, 11, 28, 72, 88, 168, 171, 173, 187, 212, 215, 276, 297, 300, 310, 349, 352, 359, 360, 362, 364, 368, 385, 386, 427, 487, 618
Auinger, K. 7, 8, 36, 89, 102, 126, 212, 265, 321, 349, 357, 361–364, 368, 369, 383, 385, 386, 475, 494, 495, 524, 527
Austin, B. ix, 215, 324, 327–329, 335, 337, 385
Azevedo, A. 124, 386
Baldinger, R. 36, 126, 164, 474, 476
Banaschewski, B. 212
Bandman, T. 643
Bergman, G. M. 11, 102, 136, 167, 472
Berstel, J. 547, 550, 594
Bidigare, P. 221
Birget, J.-C. viii, 4, 5, 7, 18, 276, 361, 383, 407, 534
Birkhoff, G. 3, 32, 44, 153, 459, 462, 464
Branco, M. J. J. 6, 424
Bray, J. N. 643
Brown, K. S. 2, 221, 481
Brown, T. C. 5, 71, 215, 234, 236, 383
Brzozowski, J. A. 4, 11, 104, 161, 247, 360, 387, 398, 400, 424
Burriss, S. 11, 166, 461, 472, 535, 536
Carré, B. 524
Carruth, J. H. 128, 212
Chomsky, N. 550, 594
Clifford, A. H. viii, 1, 2, 21, 218, 220, 256, 261, 273, 305, 309, 381, 382, 409, 414, 424, 566, 570–572, 595, 596, 602–604, 607, 608, 610
Cline, E. 2
Cohen, R. S. 11, 387, 424
Conway, J. H. 523, 524, 547
Costa, J. C. 145, 212
Coulbois, T. 357, 386
Cowan, D. F. viii
Coxeter, H. M. S. 49
Dean, R. A. 492, 493
Delgado, M. 92, 115, 124, 187, 276, 350, 388
Dieudonné, J. 456
Dixon, J. 500
Edwards, P. M. 384
Eilenberg, S. vii, viii, 3–6, 10, 15, 24, 25, 27, 33, 36, 41, 47, 62, 86, 90, 93,

- 95, 115, 126, 140, 145, 161, 164, 216,
 217, 221, 222, 224, 225, 242, 243,
 245–247, 277, 278, 280, 281, 327, 328,
 344, 360, 362, 372, 383, 384, 387, 390,
 398, 410, 424, 444, 524, 534, 540, 547,
 551, 558, 574, 594, 620, 621
 Elston, G. Z. viii, 156, 212, 363, 364,
 383, 386, 406, 407
 Erdos, J. A. 306
 Escada, A. 276
 Evans, T. 492, 493

 Fernandes, V. H. 92, 124, 276, 388
 Fitz-Gerald, D. G. 308, 309, 317
 Fox, C. 12, 307, 385, 547, 570, 572, 588,
 590, 592–594, 620
 Freyd, P. 438
 Frobenius, G. 3

 Gabriel, P. 2, 594
 Ganyushkin, O. 412, 566
 Gehrke, M. 545
 Gierz, G. ix, 10, 11, 16, 19, 20, 34, 36,
 47, 57, 427, 431, 433–441, 443,
 445–452, 454, 456, 459–461, 518, 519,
 526, 532, 535, 536, 545
 Gitik, R. 356, 357, 362, 386
 Glazek, K. 523, 547, 594
 Goodwin, J. R. viii, 4, 345, 383, 418,
 429
 Gorenstein, D. 3
 Graham, R. L. ix, 215, 308, 309, 312,
 314, 385
 Green, J. A. 1, 596, 601, 602
 Greenglaz, L. J. x, 12, 547, 558, 559,
 570, 590, 594
 Greuel, G.-M. 643
 Grigorchuk, R. I. 371, 372
 Grigorieff, S. 545
 Grillet, P.-A. 216, 222, 236, 370, 383,
 408
 Grothendieck, A. 6, 26, 456
 Groves, D. 369
 Grunewald, F. 643
 Gvaramija, A. A. x, 12, 547, 558, 559,
 570, 590, 594

 Hajji, W. 460
 Hall, M., Jr. 3, 8, 219, 356

 Hall, T. E. 265, 321, 383–385, 483, 484,
 616
 Halmos, P. R. 11, 133, 135, 141, 212,
 461, 535, 536
 Hanlon, P. 221
 Hartmanis, J. 2
 Hebisch, U. 547, 594
 Henckell, K. viii, ix, 7, 10, 28, 36, 49,
 70, 88, 89, 124, 168, 171, 173, 177,
 179, 181, 187–189, 199, 205, 212, 215,
 229, 242, 276, 288, 298, 300–302, 324,
 327–329, 335, 337, 339, 345, 360–362,
 368, 369, 381, 383–386, 390, 392, 421,
 427, 439, 528, 534, 566
 Herwig, B. 357
 Higgins, P. J. 6, 17, 312, 313
 Higgins, P. M. viii, 304, 384
 Higman, G. 433, 617
 Hildebrandt, J. A. 128, 212
 Hofmann, K. H. ix, 10, 11, 16, 19, 20,
 34, 36, 47, 57, 128, 140, 212, 427, 431,
 433–441, 443, 445–452, 454, 456,
 459–461, 518, 519, 526, 532, 535, 536,
 545, 612
 Hölder, O. 3
 Hopf, H. 536
 Houghton, C. H. 308, 312, 314, 385
 Howie, J. M. viii, 91, 305, 385, 595
 Hunter, R. P. 212
 Huppert, B. 3, 219

 Jackson, M. viii, 45
 Johnstone, P. T. 11, 16, 133, 135, 139,
 212, 427, 431, 436, 456, 459–461, 518,
 526, 535, 536, 545
 Jones, P. R. 7, 8, 62, 100, 104, 361, 407,
 424, 484
 Jordan, C. 3

 Kaloujnine, L. 219
 Kambites, M. 292, 562
 Kapovich, I. 386
 Karnofsky, J. 187, 188, 254, 267, 277,
 292–295, 369, 384
 Katsov, Y. 549, 594
 Kechris, A. S. viii, 133, 139
 Keimel, K. ix, 10, 11, 16, 19, 20, 34, 36,
 47, 57, 427, 431, 433–441, 443,

- 445–452, 454, 456, 459–461, 518, 519,
526, 532, 535, 536, 545
- Kharlampovich, O. G. viii
- Kilp, M. 523, 547, 594
- Kleene, S. C. 2, 235, 243
- Kleitman, D. J. 2, 15
- Knast, R. 5, 162, 362
- Knauer, U. 523, 547, 594
- Koch, R. J. 128, 212
- Kovács, L. G. 497–499, 519
- Krasner, M. 219
- Krohn, K. vii–ix, 2–4, 6, 11, 15, 32, 47,
215, 216, 225–228, 232, 243, 249, 251,
252, 255, 256, 258, 259, 263–265, 267,
269, 270, 273, 287, 288, 291, 298, 305,
309, 381–384, 394, 402, 409, 424, 501,
519, 595, 596, 602–604, 608, 610
- Kruml, D. 11, 524
- Kublanovskii, S. I. 321, 483, 484
- Kunyavskii, B. 643
- Lallement, G. 216, 222, 338
- Lascar, D. 357
- Lawson, J. D. ix, 10, 11, 16, 19, 20, 34,
36, 47, 57, 427, 431, 433–441, 443,
445–452, 454, 456, 459–461, 518, 519,
526, 532, 535, 536, 545
- Lawson, M. V. viii, 21, 70, 220, 243, 530
- Lazarus, S. viii, 383, 386
- Le Saëc, B. 234, 235, 383
- Lee, E. W. H. 482
- Leech, J. 230, 231
- Loganathan, M. 6, 94
- Lyndon, R. C. viii, 310, 311, 350, 351,
474
- Lyons, R. 3
- Mac Lane, S. viii, 6, 11, 16, 18, 19, 21,
26, 28, 31, 38, 42, 47, 93–95, 133, 136,
137, 212, 428, 431, 437, 438, 460, 532,
574
- Mahajan, S. 2, 221
- Margolis, S. W. x, 2, 5–8, 12, 21, 28, 36,
49, 70, 88, 89, 92, 94, 124, 168, 171,
173, 177, 179, 181, 187, 188, 215, 221,
255, 256, 271, 276, 278, 280, 285, 288,
292, 298, 300, 301, 321, 350, 355, 357,
360–362, 384–386, 388, 390, 465, 468,
479, 483, 484, 493, 509, 510, 519, 527,
528, 566, 576, 619
- Masuda, A. 187
- Mateosian, R. 47, 402, 424
- Mazorchuk, V. 412, 566
- McAlister, D. B. 1, 6, 306, 369
- McCammond, J. 364
- McKenzie, R. N. 20, 45, 47, 429, 430,
443, 462, 518
- McNaughton, R. 4, 235, 243
- McNulty, G. F. 20, 47, 429, 430, 443,
462, 518
- Meakin, J. C. 350, 355, 386
- Meyer, A. R. 4, 242, 245, 383
- Mikhalev, A. V. 523, 547, 594
- Minasyan, A. 357, 386
- Mislove, M. ix, 10, 11, 16, 19, 20, 34, 36,
47, 57, 427, 431, 433–441, 443,
445–452, 454, 456, 459–461, 518, 519,
526, 532, 535, 536, 545
- Moerdijk, I. 6, 11, 26, 93, 94, 431, 460,
532
- Moore, E. F. 2
- Mostert, P. S. 128, 140, 212, 612
- Munn, W. D. x, 1, 566, 570
- Myasnikov, A. 386
- Nambooripad, K. S. S. 230, 308, 385
- Nehaniv, C. L. viii, ix, 215, 324,
327–329, 335, 337, 383, 385, 386
- Nekrashevych, V. 371, 372
- Neumann, H. 268, 433, 444, 474, 497,
499, 511, 519, 617, 618
- Newman, M. F. 497, 499, 519
- Niblo, G. A. 356
- Nico, W. R. 6, 94
- Nikolov, N. 140
- Numakura, K. 140, 212
- O'Carroll, L. 6
- Papert, S. 243
- Parshall, B. 2
- Pelletier, J. W. 11, 524
- Pfister, G. 643
- Pin, J.-E. viii, 5–7, 10–12, 28, 30, 33,
36, 47, 49, 62, 63, 70, 86, 88, 89, 94,
119, 124, 127, 145, 168, 171, 173, 177,
179, 181, 184, 187–193, 212, 215, 221,

- 234, 235, 243, 276, 288, 298, 300, 301, 349, 356–358, 360–362, 383–388, 392, 396, 407, 424, 523, 524, 527, 528, 534, 545, 564, 572
- Pinguet, A. 524
- Plotkin, B. I. x, 12, 547, 558, 559, 569, 570, 590, 594
- Plotkin, E. 643
- Polák, L. 12, 523, 524, 547, 573, 575, 594, 619
- Ponizovskii, I. S. x, 1, 566, 570
- Pöschel, R. viii
- Preston, G. B. viii, 1, 2, 21, 218, 220, 256, 261, 273, 305, 309, 381, 382, 409, 414, 424, 566, 570–572, 595, 596, 602–604, 607, 608, 610
- Priestley, H. A. 545
- Pustejovsky, S. 8, 62
- Putcha, M. S. 2, 220, 265, 292, 306, 593, 599
- Quillen, D. 6, 26, 38, 94
- Rees, D. 1, 606, 608, 610
- Reilly, N. R. viii, 265, 383, 384, 433, 486, 487, 617
- Reiterman, J. 6, 127, 136, 142, 144, 146, 147, 153, 164, 210, 212, 494
- Renner, L. E. 2, 220, 599
- Resende, P. 11, 524, 545
- Reutenauer, C. 7, 234, 349, 356, 358, 386, 547, 550, 594
- Rhodes, J. vii–x, 1–8, 11, 12, 15, 18, 27, 28, 30, 32, 36, 47, 49, 51, 62, 63, 68, 70, 72, 84, 88, 89, 92, 93, 104, 105, 109, 110, 119, 120, 124, 126, 156, 164, 168, 171, 173, 177, 179, 181, 187–189, 199, 203, 205, 212, 215, 216, 220, 225–230, 232, 240–243, 249, 251, 252, 254–256, 258, 259, 263–265, 267, 269–271, 273, 276, 277, 280, 281, 285, 287, 288, 291–298, 300–303, 305–307, 309, 319, 321, 324, 327–329, 335, 337, 339, 343, 345, 360–362, 364, 368, 369, 381–388, 390–392, 394, 396–398, 400–402, 406, 407, 409, 411, 412, 418, 421, 424, 427, 439, 474, 476, 480, 497–499, 501, 502, 519, 534, 547, 566, 570, 572, 588, 590, 592–596, 602–604, 608, 610, 620, 621
- Ribes, L. ix, 7, 88, 128, 134, 136, 138, 141, 171, 212, 215, 297, 310, 349, 356, 357, 359, 362, 368, 386, 467, 494
- Richter-Gebert, J. 523, 550, 594
- Rips, E. 356, 362
- Rockmore, D. 221
- Roiter, A. V. 2, 594
- Rosenthal, K. I. 11, 16, 36, 436, 523–525, 528, 530, 532, 545, 547, 594
- Rosický, J. 11, 524
- Rothschild, B. R. 2, 15
- Salomaa, A. 547, 550, 594
- Sankappanavar, H. P. 11, 166, 461, 472, 535, 536
- Sapir, M. V. viii, x, 8, 321, 350, 386, 471, 479, 483, 484, 493, 509, 510, 519, 621
- Sapir, O. viii
- Satoh, S. 2
- Sauer, N. W. viii
- Schupp, P. E. viii, 310, 311, 350, 351, 474
- Schützenberger, M. P. 1, 4, 6, 15, 33, 47, 140, 215, 242, 243, 246, 257, 265, 383, 550, 594, 604
- Scott, D. S. ix, 10, 11, 16, 19, 20, 34, 36, 47, 57, 427, 431, 433–441, 443, 445–452, 454, 456, 459–461, 518, 519, 526, 532, 535, 536, 545
- Scott, L. 2
- Segal, D. 140
- Serre, J.-P. 100, 310, 311, 545
- Sierpínski, W. 462
- Silva, P. 390, 424, 621
- Simon, I. 4, 104, 161, 215, 234, 236, 247, 360, 362, 383, 398, 400
- Soittola, M. 547, 550, 594
- Solomon, R. 3
- Spencer, J. H. 2, 15
- Stallings, J. R. 8, 310, 311, 349–353, 356, 386
- Steinberg, B. ix, x, 2, 7–12, 21, 30, 36, 41, 44, 47, 49, 62, 63, 68, 72, 82, 83, 86, 92–100, 102, 104, 114–116, 121–124, 126, 127, 156, 164, 168, 171, 173, 181, 183, 187–190, 196, 198, 199,

- 205, 212, 215, 229, 242, 243, 255, 256,
269, 276, 277, 280, 292, 295, 302, 324,
327–329, 332, 335, 340, 343, 345, 346,
349, 350, 352, 357, 358, 362–364, 368,
369, 383–386, 388, 390, 392, 400, 407,
412, 421, 424, 427, 439, 453, 465, 468,
476, 480, 494, 495, 497–499, 502, 519,
524, 527, 534, 562, 566
- Stiffler, P., Jr. 92, 215, 221, 247, 248,
276–279, 383, 384, 400
- Stone, M. G. viii
- Stone, M. H. 11, 135, 212, 436, 461,
523, 536
- Stralka, A. 427, 456, 460
- Straubing, H. vii, viii, 4–6, 12, 30, 63,
86, 93, 113, 124, 126, 242–245, 267,
279, 383, 384, 396, 403, 407, 424, 527,
572, 620
- Sturmfels, B. 523, 550, 594
- Suschkewitsch, A. 1, 606
- Sushchanskiĭ, V. I. 371, 372
- Sweedler, M. E. 536
- Szendrei, M. B. 104, 361, 484
- Talwar, S. 303
- Taylor, W. F. 20, 47, 429, 430, 443, 462,
518
- Teixeira, L. 124, 388
- Theobald, T. 523, 550, 594
- Thérien, D. 5, 6, 30, 63, 93, 113, 124,
126, 162, 267, 361, 396, 407, 424, 527
- Tilson, B. vii–ix, 2–11, 17, 24–27, 30,
36, 41, 44, 47, 49, 50, 52, 62, 63, 72,
83, 84, 88, 92–100, 102–105, 109, 110,
116, 118–124, 126, 161, 162, 164, 165,
171, 179, 187, 188, 212, 215, 216, 220,
225, 226, 234, 243, 247, 249, 251, 252,
255, 256, 258, 259, 263–265, 267,
269–271, 273, 276–288, 291, 295–298,
300, 303, 305, 307–309, 319, 321,
327–329, 332, 338, 344, 352, 360–362,
369, 370, 373, 381–385, 387–389, 391,
393, 394, 396–398, 401, 405–409, 424,
427, 471, 496, 497, 501, 519, 588, 595,
596, 602–604, 608, 610, 617
- Tishchenko, A. V. 511, 618
- Tits, J. 49
- Tokizawa, M. 2
- Trotter, P. G. viii, 104, 320, 321, 361,
407, 424, 483, 484
- Vaughan-Lee, M. 115, 369
- Volkov, M. V. viii, 2, 12, 91, 212, 255,
256, 292, 320, 369, 390, 465, 468, 487,
566
- von Neumann, J. 597
- Vovsi, S. M. 12, 547, 558–560, 569, 594
- Wallace, A. D. 128, 212
- Wedderburn, J. H. M. 1, 47, 115
- Weil, P. ix, x, 6–8, 10–12, 24, 27, 30, 49,
62, 63, 67, 68, 70, 72, 86, 88, 93–95,
100, 102, 103, 113, 119, 126–128, 136,
145, 147, 168, 171, 177, 179, 181, 184,
187–196, 199, 201, 202, 206, 212, 234,
235, 270, 271, 281, 350, 383, 386, 387,
391, 392, 394, 396–398, 400, 401, 406,
407, 424, 479, 492, 493, 509, 510, 519,
524, 616
- Weinert, H. J. 547, 594
- Weiss, A. 5, 93, 162
- Wielandt, H. 3, 219
- Wilson, J. S. 643
- Wilson, R. A. 643
- Yama, K. 2
- Yamada, H. 235
- You, S. 357, 386
- Zalcstein, Y. 1, 4, 255, 256, 292, 345,
390, 411, 412, 566, 570
- Zalesskii, P. A. ix, 7, 88, 128, 134, 136,
138, 141, 171, 212, 215, 297, 310, 349,
356, 357, 359, 362, 368, 386, 467, 494
- Zeiger, P. 216, 281, 383
- Zeitoun, M. 145, 212, 386
- Zel'manov, E. I. 115, 369
- Zhang, S. 265, 383, 384, 433, 486, 487,
617

Index

- C^* -algebra 435
- E -solid 361
- U_2 -free 282
- U_2 -length 242
- \mathcal{H} -morphism 395
- \times -prime 403
- q -operator 61
- \mathcal{J} -singular 394
- \mathbf{A} - \mathbf{LG} chain 422
- \mathbf{V} -morphism 119
- Δ -irreducible 575
 - basic 576
- $*i$ 510
- $f*i$ 510
- $sf*i$ 510
- $s*i$ 510
- fji 430
 - semigroup 463
- fmi 430
- ji 430
- mi 430
- $sfji$ 430
- $sfmi$ 430
- sji 430
- smi 430

- absolute Type I semigroup 296
- acyclic 311
- adjoint
 - morphism 560
 - pair 18
 - translation 565
- adjunction 18
- admissible partition 329

- algebraic closure operator 443
- algebraic lattice 20
 - locally dually 477
- aperiodic 216
- aperiodic element 370
- aperiodic relational morphism 243
- aperiodic semigroup 89
- aperiodic string 370
- Aperiodicity Lemma 244
- Ash's Theorem 359
- associated homomorphism 413
 - small monoid 418
- associated partial homomorphism 414
- atom 467
- augmented
 - monoid 222
 - transformation semigroup 222
- automaton 327
- automaton congruence 328
 - injective 328
- axiom
 - corestriction 52
 - division 50
 - free 56
 - identity maps 50
 - pullback 51
 - range extension 50
 - strong containment 116
 - strong division 50
- Axiom (\prec) 50
- Axiom (\prec_s) 50
- Axiom (\subseteq_s) 116
- Axiom (\times) 50

- Axiom (co-re) 52
- Axiom (free) 56
- Axiom (id) 50
- Axiom (pb) 51
- Axiom (r-e) 50

- backtrack 310
- band 72, 361
 - rectangular 72
- base pseudovariety 389
- basis
 - continuous lattice 451
 - of pseudoidentities 146
- Basis Theorem for Semidirect Products 205
- Berkeley School 387, 415
- bialgebra 537
 - Boolean 537
 - counital 537
- bilateral consistency equations 161
- bilinear 548
- bimodule 548
- bipartite graph 310
- Birkhoff variety 472
- block group 361, 598
- block product 31
 - unitary 31
- blowup operator 377
- Boolean algebra 535
- Boolean ring 535
- Brandt semigroup B_n 59
- Brown's Theorem 72, 236

- canonical factorization 37
- canonical relational morphism 45, 154
- category of elements 6, 94
- Cauchy completion 279
- chain 428
- characteristic subgroup 268
- CL-morphism 438
- classifying topos 94
- clopen 133
- clopen congruence 138
- closed congruence 130
- closed equivalence relation 129
- closed set of strong relational pseudoidentities 209
- closure operator 18, 436
- co-compact 430
 - strictly 430
- coboundary 312
- coherent space 461
- cohomologous 312, 318
- cohomology 312
- column monomial 409
- compact 20, 428, 430, 434
 - strictly 430
- companion relation 327
- compatible 318
- completely computable 187
- completely regular 288
- complexity
 - function
 - group complexity 240
 - hierarchical 238
 - idempotent semirings 587
 - two-sided 391
 - hierarchy 238
 - of a pair 389
 - of one operator 387
 - local 287, 300
 - pseudovarieties 240
- complexity hierarchy
 - standard 389
 - two-sided 390
- computably finitely equivalent 178
- comultiplication 537
- cone 132
- congruence 440
 - A -graph 352
 - clopen 138
 - closed 130
 - injective 352
 - proper 268
 - trivial 268
- congruence-free 272
- connected
 - components 310
 - graph 310
- consequence 473
- consistency equations 160
- consolidation 235, 279, 408
- continuous 20
- continuously closed class 51
 - Birkhoff 56
 - equational 51
 - positive 51
- core injective 95, 106

- counit 18, 537
- cover 349
- covering 351
- covers 429
- critical semigroup 337
- cross section 325

- Delay Theorem 279
- depth 284
- derived automaton 328
- derived category 94
- derived semigroupoid 95
- Derived Semigroupoid Theorem 98
 - Pseudovarieties 102
- determined join 428
- determined meet 428
- directed 20
 - downwards 20
 - inversely 20
- directed path 358
- discrete fibration 94
- distinguished \mathcal{J} -class 256
- distributive lattice 431
- division 40
 - diagram 165
 - strong 211
 - of idempotent semirings 573
 - of profinite semigroups 148
 - of relational morphisms 42
 - of semigroupoids 95
 - of semigroups 32
- divisionally equivalent 47
- domain 434
- dot-depth hierarchy 387
- downset 456, 555
- dual continuous 20
- dual lattice 471
- dual module 560
- dually algebraic lattice 472

- elementary reduction 370
- elementary tensor 549
- embedding
 - of modules 551
 - of transformation semigroups 223
- enough points 433
- equation 472
- equational pseudovariety 475
- equationally closed 473

- essential \mathcal{J} -class 284
- Euler totient function 468
- expansion 407
- explicit operation 136
- extension-closed 233

- factorizable 220
- filter 445
- finite trace 475
- finite vertex rank 188
- finitely equivalent collection 178
- finitely generated 444
- fixed-point set 448
- flag 370
- flip-flop 224
- flow 200
 - configuration 201
 - inevitable 202
- fold 353
- forest 311
- frame 431, 532
- frame morphism 431
- free action 604
- free algebra 550
- free monoid 242
- free pro- \mathbf{V} semigroup 144
- free profinite semigroup 135
- full
 - linear monoid 306
 - transformation monoid 304
- functional 560
- fundamental group 310
 - of a G -labeled graph 311
- fundamental groupoid 310
- Fundamental Lemma of Complexity 281

- Galois connection 17, 437
- gap 429
- generalized group mapping 255, 501
- generalized Mal'cev product 69
- geometric edge 310
- global Mal'cev condition 71
 - reverse 124
- GMC 71
- Graham normalization 312, 321
- Graham normalized 313
- Graham's Theorem 321
- graph 100

- A- 349
- bipartite 310
- Cayley 349
- of a relational morphism 37
- sense of Serre 310
- Green's Lemma 601
- Green's preorders 596
- Green's relations 596
- group
 - monolithic 268
 - group complexity function 240
 - group mapping 256
 - group mapping-right letter mapping
 - chain 286
- Henckell's Theorem 368
- Heyting algebra 431
- hierarchical complexity function 238
- homomorphism of semirings 548
- homotopic 310
- homotopic paths 311
- Hopf algebra 538
- hyperdecidable 171
- ideal 595
 - 0-minimal 255, 596
 - lattice 445, 555
 - semiring 553
- Ideal Decomposition Theorem 570
- ideal extension 413, 507
 - total 413
- ideal quotient 554
- Ideal Theorem 285
 - two-sided version 408
- idempotent 597
- idempotent pointlike 171, 172
- idempotent semiring 549
- idempotent splitting 279
- identity 472
- implicit operation 136
- incidence graph 314
- inevitable
 - flow configuration 202
 - graphs 171, 173
 - substitution 172
- injective automaton 328
 - congruence 328
- injective congruence 352
- inner translation 409
- inverse 597
- inverse A-graph 349
- inverse limit 132
- inverse semigroup 21, 220, 598
- inverse system 132
 - quotient 134
- irreducible
 - finite semidirect 510
 - join 430
 - finite 430
 - strictly 430
 - strictly finite 430
 - meet 430
 - finite 430
 - strictly 430
 - strictly finite 430
 - relational morphism 512
 - finite 512
 - strictly 512
 - strictly finite 512
 - semidirect 510
 - strictly 510
 - strictly finite 510
 - topological space 517
- join 428
- join-continuous lattice 431
- Karoubi envelope 279
- kernel category 105
- kernel operator 18, 437
- kernel semigroupoid 105
- Kernel Semigroupoid Theorem 109
 - Pseudovarieties 111
- Kovács-Newman
 - group 497
 - semigroup 501
- Krohn-Rhodes Theorem 228
- label 93, 105
- labeled graph 311
- labeling 171, 200
 - commutes 171, 200
- lattice 19, 428
 - algebraic 20, 428
 - complete 428
 - continuous 434
 - distributive 431
 - join-continuous 431
 - meet-continuous 431

- left adjoint 18
- left ideal 595
- left letter mapping 256
- left mapping 255
- left regular band 221
- left simple 226, 595
- left translation 409
- Lemma
 - $V \cup T$ 226
- linked 409
- linked equations 410
- local
 - sense of Eilenberg 90, 344
 - sense of Tilson 104
- local complexity 287, 300
- local monoid 34
- local upper bound 280
- locale 431
- locally dually algebraic lattice 477
- locally extensible 494
- locally finite
 - category 235
 - pseudovariety 44
 - pseudovariety of relational morphisms 156
 - semigroup 235
 - variety 476
- Mal'cev kernel 396
- Mal'cev product 70
 - generalized 69
- maximal proper surmorphism 393
- meet 428
- meet-continuous lattice 431
- modeling q by composition 120
- module 327, 548
 - free 549
- monodromy group 351
- monoidal pseudovariety 471
- monolith 268
- monolithic group 268
- monomial map 219
- Morita equivalence 303
- morphism 38
 - \mathcal{H} - 250
 - \mathcal{H}' - 250
 - \mathcal{I} - 250
 - \mathcal{I}' - 250
 - \mathcal{L} - 250
 - \mathcal{L}' - 250
 - \mathcal{R} - 250
 - \mathcal{R}' - 250
- MPS 393
 - Decomposition Theorem 397
 - factorization 394
- Noetherian poset 459
- non-degenerate 416
- null
 - \mathcal{I} -class 604
 - semigroup 606
 - type 269
- ordered semigroup 524
- orientation 310
- parameterized relational morphism 328
- partial constant matrix 416
- partial homomorphism 413
- partition
 - admissible 329
- path 310
- permutation automaton 350
- permutation group 22
 - regular 604
- pointed transformation semigroup 22
- pointlike 82, 171, 172
 - idempotent 171, 172
- poset 428
- positive operator 57
- positively oriented 310
- preblowup operator 376
- prehomomorphism 21
 - dual 21, 530
- presentation 329
 - of profinite semigroups 139
 - over a pseudovariety 329
 - partition of 329
 - sets of 329
- Presentation Lemma 334
- primary component 467
- prime
 - \times - 228, 480
 - \times - 482
- Prime Decomposition Theorem 228
 - idempotent semirings 573
 - quantales 573

- Relational Morphisms 404
 - Two-Sided 402
- prime ideal 461
- primes 430
- principal
 - factor 607
 - filter 454
 - ideal 445, 595
 - left ideal 595
 - right ideal 595
 - series 570
- pro- V 149
- product 39, 42
 - block 31
 - unitary 31
- Mal'cev 70
 - generalized 69
- Schützenberger 85, 564
- semidirect 24, 25
 - of pseudovarieties 34
 - reverse 24
 - two-sided 30
- subdirect 34
- tensor 548
- triangular 559
 - of pseudovarieties 573
- triple 558
- two-sided semidirect
 - of pseudovarieties 34
- weak 38
- wreath
 - partial transformation semigroups 217
 - semigroups 27
 - transformation semigroups 25
 - unitary 26
- profinite
 - congruence 139
 - semigroup 135
 - free 135
 - space 134
- projection 436
- projective basis 197
- projective limit 132
- projective semigroup 229
- proper homomorphism 393
- pseudoidentities
 - collection of systems of 176
 - system of 172
- pseudoidentity 144, 157, 474
 - relational 157
- pseudovariety
 - idempotent semirings 573
 - relational morphisms 51
 - positive 51
 - semigroupoids 99
 - semigroups 33
- pullback 39, 40
- quantale 524
 - left positive 529
 - left-sided 529
 - right positive 529
 - right-sided 529
 - two-sided 529
 - two-sided positive 529
- quantic co-nucleus 530
- quantic nucleus 530
- quasivariety 166
- quotient automaton 328
- quotient module 557
- ramification 414
- rank 305, 600
- rational set 243
- recognizable 539
- recognizable set 243
- reduced path 310
- reduction 370
- Reduction Theorem 285
- Rees coordinates 611
- Rees coordinatization 612
- Rees's Theorem 610
- regular 539
 - element 597
- regular \mathcal{J} -class 604
- regular set 243
- Reiterman's Theorem 146
- relational morphism 36
 - free pro- V 155, 208
 - irreducible 512
 - of modules 327
 - of partial transformation semigroups 327
 - of profinite semigroups 148
 - of relational morphisms 41
 - of semigroupoids 95
 - pro- V 149

- strong 42
- relational pseudoidentity 157
- renormalization 318
- retract 450
- reverse global Mal'cev condition 124
- reverse semidirect product 24
- Rhodes expansion 370, 408
- Ribes and Zaleskii Theorem 357
- right adjoint 18
- right ideal 595
- right letter mapping 256
- right mapping 255
- right multiplier 377
- right regular band 480
- right simple 595
- right stabilizer 408
- row monomial 409

- sandwich matrix 608, 609
- satisfies 472
- Schützenberger group 381, 571
 - of a \mathcal{J} -class 606
- Schützenberger product 85, 564
- Schützenberger representation 257
- Schützenberger's Theorem 246
- semidirect product 24, 25
 - of pseudovarieties 34
- semigroup
 - 0-simple 606
 - \mathcal{T}_1 298
 - fji 463
 - sfji 464
 - completely regular 288
 - congruence-free 272
 - free pro- \mathbf{V} 144
 - pro- \mathbf{V} 142
- semigroup algebra 550
 - contracted 554
- semigroupoid 26, 95
- semiring 547, 548
 - unital 548
- semisimple type 269
- separative 560
- shift 388
- short exact sequence 557
- simple 595
- simplicial 310
- singular 305
- singular square 230

- sink 217
- Slice Theorem 114
- sober space 532
- socle 268
- space
 - coherent 461
 - irreducible 517
 - profinite 134
 - sober 532
- span 415
- spectrum 433, 435, 461
 - of a lattice 461
- split 567
- splitting maps 567
- stable 130, 216, 599
- star-free set 243
- Stiffler's Theorem 247, 400
- Stone duality 212, 461, 536
- strongly connected 358
- structure matrix 609
- subalgebra 438
- subcommutative diagram 41
- Subdirect Representation Theorem
 - joins 477
 - meets 464
- subdirectly indecomposable 267
- subquantale 532
- subsemigroup
 - \mathbf{V} -like 296
- subset
 - \mathbf{V} -like 179
- symmetric inverse monoid 220
- syntactic monoid 243
- syntactic morphism 243

- Tall Fork 324
- Tarski's Fixed Point Theorem 448
- tensor product 548
- Tilson number 278
- Tilson's Lemma 54
- toolbox 249
- topological semigroup 128
- topologically generated 135
- topology
 - Alexandrov 457, 517
 - hull-kernel 433, 461
 - patch 454
 - profinite 355
 - Scott 456

- Sierpinski 462
- strong 457
- zero-dimensional 454
- totally disconnected 133
- transformation
 - group 22
 - monoid 22
 - semigroup 22
- transition edge 358
- transition group 351
- transition semigroup 328
- translational hull 409
- tree 311
- Triangular Decomposition Theorem
 - 569
- triangular product 559
 - of pseudovarieties 573
- triple product 558
- trivial labeling 313
- two-sided π -length 392
- two-sided semidirect product 30
 - of pseudovarieties 34
- Type I 296
- Type II 296
- Type II Conjecture 297
- Type II Theorem 359
- unit 18, 537
- unitary wreath product 217
- universal relation 38
- variety
 - Birkhoff 472
 - semigroups 472
- way below 433
- weak conjugate 296
- weak inverse 296
- weak product 38
- weak pullback 39
- weight 451
- witness 173, 176
 - flow configurations 202
- wreath product 25, 27
 - partial transformation semigroups
 - 217
 - unitary 26
- Zeiger property 373
- zero-dimensional 133