A

The Green-Rees Local Structure Theory

The goal of this appendix is to give an admittedly terse review of the Green-Rees structure theory of stable semigroups (or what might be referred to as the local theory, in comparison with the semilocal theory of Section 4.6). More complete references for this material are [68, 139, 171].

A.1 Ideal Structure and Green's Relations

If S is a semigroup, then $S^I = S \cup \{I\}$, where I is a newly adjoined identity. If X, Y are subsets of S, then $XY = \{xy \mid x \in X, y \in Y\}$.

Definition A.1.1 (Ideals). *Let S be a semigroup. Then:*

1. *$\emptyset \neq R \subseteq S$ is a right ideal if $RS \subseteq R$;*
2. *$\emptyset \neq L \subseteq S$ is a left ideal if $SL \subseteq L$;*
3. *$\emptyset \neq I \subseteq S$ is an ideal if it is both a left ideal and a right ideal.*

If $s \in S$, then sS^I is the *principal right ideal* generated by s, $S^I s$ is the *principal left ideal* generated by s and $S^I s S^I$ is the *principal ideal* generated by s. If S is a monoid, then $S^I s = Ss$, $sS^I = sS$ and $S^I s S^I = SsS$.

A semigroup is called *left simple*, *right simple*, or *simple* if it has no proper, respectively, left ideal, right ideal, or ideal. The next proposition is straightforward; the proof is left to the reader.

Proposition A.1.2. *Let I, J be ideals of S. Then*

$$IJ = \{ij \mid i \in I,\ j \in J\}$$

is an ideal and $\emptyset \neq IJ \subseteq I \cap J$. Consequently, the set of ideals of S is closed under finite intersection.

Corollary A.1.3. *A finite semigroup S has a unique minimal ideal.*

Proof. Proposition A.1.2 implies that the intersection of all ideals of S is again an ideal; clearly it is the unique minimal ideal. □

Note that if a semigroup has a zero, then its minimal ideal is $\{0\}$. In this context, the notion of minimal ideal is not so useful, and so we introduce the notion of a 0-minimal ideal.

Definition A.1.4 (0-minimal ideal). *A minimal non-zero ideal of a semigroup is called a 0-minimal ideal. We also, by convention, consider the minimal ideal of the trivial semigroup to be a 0-minimal ideal.*

With this definition, the minimal ideal of a semigroup without zero is considered to be a 0-minimal ideal. This convention is somewhat non-standard, but is convenient for stating results uniformly for semigroups with 0 and without 0. Note that 0-minimal ideals do not have to be unique. Next we introduce Green's relations [68, 108, 171]. They are an essential ingredient in semigroup theory.

Definition A.1.5 (Green's relations). *Let S be a semigroup and $s, t \in S$. Green's equivalence relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ and $\mathscr{D}$ are defined as follows:*

- $s \mathrel{\mathscr{R}} t \iff sS^I = tS^I$;
- $s \mathrel{\mathscr{L}} t \iff S^I s = S^I t$;
- $s \mathrel{\mathscr{J}} t \iff S^I s S^I = S^I t S^I$;
- $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$;
- $\mathscr{D} = \mathscr{R} \vee \mathscr{L}$.

The $\mathscr{R}$-class of an element $s \in S$ is typically denoted by R_s. Similar meanings can be ascribed to L_s, J_s, H_s and D_s.

Exercise A.1.6. Show that $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ and deduce from this

$$\mathscr{L} \circ \mathscr{R} = \mathscr{D} = \mathscr{R} \circ \mathscr{L}.$$

All of Green's relations coincide for a commutative semigroup. Note that

$$\mathscr{H} \subseteq \mathscr{L}, \mathscr{R} \subseteq \mathscr{D} \subseteq \mathscr{J}.$$

We shall see $\mathscr{D} = \mathscr{J}$ for stable (in particular, finite) semigroups. Green's relations are more naturally derived from his preorders.

Definition A.1.7 (Green's preorders). *Green's preorders are defined by:*

- $s \leq_{\mathscr{R}} t \iff sS^I \subseteq tS^I$;
- $s \leq_{\mathscr{L}} t \iff S^I s \subseteq S^I t$;
- $s \leq_{\mathscr{J}} t \iff S^I s S^I \subseteq S^I t S^I$;
- $s \leq_{\mathscr{H}} t \iff s \leq_{\mathscr{L}} t$ *and* $s \leq_{\mathscr{R}} t$.

Green's relations are the equivalence relations associated to his preorders. Therefore, for instance, $\leq_{\mathscr{R}}$ induces a partial order on the set of $\mathscr{R}$-classes of S. An element e of a semigroup is *idempotent* if $e^2 = e$. Often, if e, f are idempotents, then $e \leq_{\mathscr{H}} f$ is abbreviated to $e \leq f$. Observe that $e \leq f$ if and only if $ef = e = fe$ and that this is a partial order on the set of idempotents.

Exercise A.1.8. Show that if e is an idempotent, then $s \leq_{\mathscr{L}} e$ if and only if $se = s$, and that $s \leq_{\mathscr{R}} e$ if and only if $es = s$. Conclude that if e, f are idempotents, then $e \leq f$ if and only if $ef = e = fe$.

Proposition A.1.9. *Green's relation $\mathscr{R}$ is a left congruence and $\mathscr{L}$ is a right congruence.*

Proof. Suppose $s \mathrel{\mathscr{R}} t$ and $x \in S$. Then $sS^I = tS^I$ and so $xsS^I = xtS^I$. Therefore $\mathscr{R}$ is a left congruence. The proof for $\mathscr{L}$ is dual. □

Similarly one can show that the preorder $\leq_{\mathscr{R}}$ is stable under left multiplication and the preorder $\leq_{\mathscr{L}}$ is stable under right multiplication.

Exercise A.1.10. Verify this last assertion.

The following notion, due to von Neumann in the context of ring theory [375], plays a fundamental role in semigroup theory.

Definition A.1.11 (Regular element). *An element s of a semigroup S is (von Neumann) regular if there exists $t \in S$ such that $sts = s$, i.e., $s \in sSs$. A semigroup is said to be regular if each of its elements is regular.*

It is immediate that idempotents are regular. In a group, all elements are regular.

Exercise A.1.12. Verify that if s is regular then $S^I s = Ss$, $sS^I = sS$ and $S^I sS^I = SsS$.

Definition A.1.13 (Inverse). *Two elements s, s' of a semigroup are said to be inverse to each other if $ss's = s$ and $s'ss' = s'$.*

Having an inverse is equivalent to being regular as the next proposition shows.

Proposition A.1.14. *Let $s \in S$ be a regular element. Then s has an inverse.*

Proof. Suppose $s = sts$. Setting $s' = tst$ yields

$$ss's = ststs = sts = s$$
$$s'ss' = tststst = tstst = tst = s'.$$

Thus s and s' are inverses. □

Inverse elements allow for a description of $\mathscr{D}$-equivalence of idempotents analogous to von Neumann-Murray equivalence of projections in operator algebras [31].

Proposition A.1.15. *Let S be a semigroup. Two idempotents $e, f \in S$ are $\mathscr{D}$-equivalent if and only if there exist inverse elements $x, x' \in S$ with $e = xx'$ and $f = x'x$. More precisely, if $e, f \in S$ are idempotents and $x \in S$, then $e \mathrel{\mathscr{R}} x \mathrel{\mathscr{L}} f$ if and only if there exists an inverse x' of x such that $xx' = e$ and $x'x = f$, in which case $e \mathrel{\mathscr{L}} x' \mathrel{\mathscr{R}} f$.*

Proof. We begin with the second statement, as the first is an immediate consequence of it. If x, x' are inverses, then it is straightforward to verify $x \mathrel{\mathscr{R}} xx' \mathrel{\mathscr{L}} x'$ and $x \mathrel{\mathscr{L}} x'x \mathrel{\mathscr{R}} x'$. This handles the if direction. For the only if direction, suppose $e \mathrel{\mathscr{R}} x \mathrel{\mathscr{L}} f$. Then we can find $u, v \in S^I$ with $xu = e$ and $vx = f$. Set $x' = fue$. Then

$$xx'x = x(fue)x = xux = ex = x$$
$$x'xx' = (fue)x(fue) = fuxue = fue = x'$$

and so x, x' are inverses. Moreover, $xx' = xfue = xue = e$ and $x'x = fuex = fux = vxux = vex = vx = f$ and so $e \mathrel{\mathscr{L}} x' \mathrel{\mathscr{R}} f$, completing the proof. □

A semigroup is called an *inverse semigroup* if each element has a unique inverse. A *block group* is a semigroup in which each element has at most one inverse, or equivalently a semigroup in which each regular element admits a unique inverse.

In general, Green's relations on a subsemigroup do not coincide with Green's relations on the ambient semigroup. For instance, if B_2 is the semigroup of 2×2 matrix units and the zero matrix, then $E_{11} \mathrel{\mathscr{J}} E_{22}$ in B_2, but in the subsemigroup $\{E_{11}, E_{22}, 0\}$ they are not $\mathscr{J}$-equivalent. However, the following is true.

Proposition A.1.16. *Let S be a semigroup and $T \leq S$ a subsemigroup. Suppose $t_1, t_2 \in T$ are regular. Then $t_1 \mathrel{\mathscr{K}} t_2$ in T if and only if $t_1 \mathrel{\mathscr{K}} t_2$ in S where $\mathscr{K}$ is any of $\mathscr{R}$, $\mathscr{L}$ or $\mathscr{H}$.*

Proof. We just handle $\mathscr{R}$ as the case of $\mathscr{L}$ is dual, and the result for $\mathscr{H}$ follows from those for $\mathscr{L}$ and $\mathscr{R}$. Clearly, $t_1 \mathrel{\mathscr{R}} t_2$ in T implies $t_1 \mathrel{\mathscr{R}} t_2$ in S. For the converse, suppose $t_1 \mathrel{\mathscr{R}} t_2$ in S and choose by regularity elements $x_1, x_2 \in T$ such that $t_i x_i t_i = t_i$, $i = 1, 2$. Then $e_i = t_i x_i$ is an idempotent of T with $e_i \mathrel{\mathscr{R}} t_i$ in T for $i = 1, 2$. It thus suffices to show $e_1 \mathrel{\mathscr{R}} e_2$ in T. But $e_1 \mathrel{\mathscr{R}} e_2$ in S implies $e_1 e_2 = e_2$ and $e_2 e_1 = e_1$. Thus $e_1 \mathrel{\mathscr{R}} e_2$ in T, as required. □

A.2 Stable Semigroups

For the rest of this appendix, we shall be interested in stable semigroups. This class of semigroups include finite semigroups, compact semigroups (Proposition 3.1.10) and commutative semigroups. Algebraic semigroups are also stable [250, 262]. Although at first sight the definition may seem bizarre, it is somehow the crucial property that makes finite semigroup theory "work."

Definition A.2.1 (Stability). *A semigroup S is called stable if both*

$$s \mathscr{J} sx \iff s \mathscr{R} sx \text{ and also } s \mathscr{J} xs \iff s \mathscr{L} xs.$$

The archetypical example of an unstable semigroup is the bicyclic monoid. It is the monoid of transformations of $\mathbb{N}$ generated by $\sigma, \tau \colon \mathbb{N} \to \mathbb{N}$ given by $\sigma(x) = x + 1$ and

$$\tau(x) = \begin{cases} x-1 & x > 0 \\ 0 & x = 0. \end{cases}$$

Equivalently, it is the semigroup generated by a unilateral shift on a separable Hilbert space and its adjoint. The next exercise gives the reader a chance to become familiar with the notion of stability.

Exercise A.2.2. Suppose S is a stable semigroup.

1. Show that $xsx \mathscr{J} x$ if and only if $xsx \mathscr{H} x$.
2. Show that if $e \in S$ is an idempotent, then $eSe \cap J_e = H_e$.
3. Show that if S is a monoid with identity 1, then $J_1 = H_1$, which in turn is the group of units of S. Deduce that the non-units of S form an ideal.
4. Show that a congruence-free stable monoid is either a simple group or is isomorphic to $(\{0, 1\}, \cdot)$.
5. Show that if e, f are idempotents of S with $e \mathscr{J} f$ and $e \leq f$, then $e = f$.
6. Show that the bicyclic monoid is not stable.
7. Show that the bicyclic monoid does not embed in a stable semigroup.
8. Show that a regular semigroup is stable if and only if it does not contain an isomorphic copy of the bicyclic monoid.

Exercise A.2.3. Let S be a non-empty finite semigroup. Prove that S contains an idempotent. Hint: Let T be a minimal (non-empty) subsemigroup of S. Show that $tT = T = Tt$ for all $t \in T$. Deduce that if $tx = t$, then $x = x^2$.

The following result is a special case of Proposition 3.1.10.

Theorem A.2.4. *Finite semigroups are stable.*

Proof. Clearly, $s \mathscr{R} st$ implies $s \mathscr{J} st$. Suppose $s \mathscr{J} st$. Evidently $st \leq_{\mathscr{R}} s$, so we are left with establishing the reverse inequality. Because $s \mathscr{J} st$, we can find $x, y \in S^I$ such that $s = xsty$. One then has $s = x^n s(ty)^n$ for all $n > 0$. Let k be a positive integer such that $(ty)^k$ is idempotent. Then

$$s(ty)^k = x^k s(ty)^k (ty)^k = x^k s(ty)^k = s$$

so $st(y(ty)^{k-1}) = s(ty)^k = s$. Thus $s \leq_{\mathscr{R}} st$, as was required. The argument for $\mathscr{L}$ is dual. □

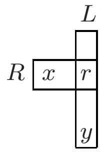

Fig. A.1. $\mathscr{L}$-classes and $\mathscr{R}$-classes of a $\mathscr{J}$-class intersect in stable semigroups

Stability implies Green's relations $\mathscr{J}$ and $\mathscr{D}$ coincide.

Corollary A.2.5. *Let S be a stable semigroup. Then $\mathscr{J} = \mathscr{D}$. More precisely, the following are equivalent for $s,\, t \in S$:*

1. $s \mathrel{\mathscr{J}} t$;
2. *there exists* $r \in S$ *such that* $s \mathrel{\mathscr{L}} r \mathrel{\mathscr{R}} t$;
3. *there exists* $z \in S$ *such that* $s \mathrel{\mathscr{R}} z \mathrel{\mathscr{L}} t$;
4. $s \mathrel{\mathscr{D}} t$.

Proof. Suppose first $s \mathrel{\mathscr{J}} t$. Then there exist $u, v \in S^I$ such that $usv = t$. Hence $us \mathrel{\mathscr{J}} s$, $sv \mathrel{\mathscr{J}} s$ and so $us \mathrel{\mathscr{L}} s$, $sv \mathrel{\mathscr{R}} s$ by stability. Because $\mathscr{R}$ is a left congruence, $s \mathrel{\mathscr{L}} us \mathrel{\mathscr{R}} usv = t$ and dually, because $\mathscr{L}$ is a right congruence, $s \mathrel{\mathscr{R}} sv \mathrel{\mathscr{L}} usv = t$ thereby establishing that 1 implies 2 and 3. Clearly 2 and 3 imply 4, and 4 implies 1. This completes the proof. □

What does all this mean about the structure of a stable semigroup? In any semigroup, each $\mathscr{J}$-class J is a disjoint union of $\mathscr{R}$-classes and also a disjoint union of $\mathscr{L}$-classes. If L is an $\mathscr{L}$-class and R is an $\mathscr{R}$-class of J, then either $L \cap R = \emptyset$ or $L \cap R$ is an $\mathscr{H}$-class. In a stable semigroup, this intersection is never empty. Indeed, suppose $x \in R$, $y \in L$. Then $x \mathrel{\mathscr{J}} y$ implies there exists r such that $x \mathrel{\mathscr{R}} r \mathrel{\mathscr{L}} y$. Thus $r \in L \cap R$, see Figure A.1.

The picture of a $\mathscr{J}$-class J of a stable semigroup is something like an eggbox where the rows represent the $\mathscr{R}$-classes of J, the columns represent the $\mathscr{L}$-classes and the boxes represent the $\mathscr{H}$-classes, see Figure A.2.

As an example, consider the full transformation monoid T_n of degree n. We view T_n as acting on the right of $\{0, \ldots, n-1\}$. If $f \in T_n$, define $\ker f$ to be the equivalence relation given by $(x, y) \in \ker f$ if $xf = yf$. Define the *rank* of f by $\mathrm{rk}(f) = |\operatorname{Im} f|$.

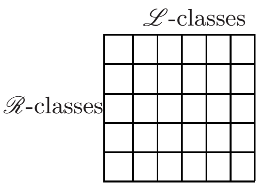

Fig. A.2. Eggbox picture of a $\mathscr{J}$-class of a stable semigroup

Exercise A.2.6. Let $f, g \in T_n$. Show:

1. $f \mathscr{L} g$ if and only if $\operatorname{Im} f = \operatorname{Im} g$;
2. $f \mathscr{R} g$ if and only if $\ker f = \ker g$;
3. $f \mathscr{J} g$ if and only if $\operatorname{rk}(f) = \operatorname{rk}(g)$.

A.3 Green's Lemma and Maximal Subgroups

The following is known as Green's Lemma [108] and is used throughout this book, often without comment.

Lemma A.3.1 (Green's Lemma). *Let S be a semigroup.*

1. *Let $s \mathscr{L} t$ and $u, v \in S^I$ be such that $us = t$, $vt = s$. Then $\varphi\colon R_s \to R_t$ defined by $x\varphi = ux$ and $\psi\colon R_t \to R_s$ defined by $y\psi = vy$ are inverse bijections. Moreover, if $x, x' \in R_s$ with $x \mathscr{H} x'$, then $x\varphi \mathscr{H} x'\varphi$.*
2. *Let $s \mathscr{R} t$ and $u, v \in S^I$ be such that $su = t$, $tv = s$. Then $\varphi\colon L_s \to L_t$ defined by $x\varphi = xu$ and $\psi\colon L_t \to L_s$ defined by $y\psi = yv$ are inverse bijections. Moreover, if $x, x' \in L_s$ with $x \mathscr{H} x'$, then $x\varphi \mathscr{H} x'\varphi$.*
3. *If $s \mathscr{L} r \mathscr{R} t$ and $u, v, w, z \in S^I$ are such that $us = r$, $vr = s$, $rw = t$ and $tz = r$, then $\varphi\colon H_s \to H_t$ given by $x\varphi = uxw$ and $\psi\colon H_t \to H_s$ given by $y\psi = vyz$ are inverse bijections.*

Proof. We prove only 1, as 2 is dual and 3 follows directly from 1 and 2. Let $x \in R_s$. Because $\mathscr{R}$ is a left congruence, $x \mathscr{R} s$ implies $ux \mathscr{R} us = t$. So $x\varphi = ux \in R_t$, establishing that φ is well defined. Similarly, ψ is well defined. Let us show that $\varphi\psi = 1_{R_s}$. If $x \in R_s$ then $x = sz$, for some $z \in S^I$. So

$$x\varphi\psi = ux\psi = vux = vusz = vtz = sz = x.$$

A similar verification shows $\psi\varphi = 1_{R_t}$.

Suppose now that $x, x' \in R_s$ and $x \mathscr{H} x'$. Then there exist $z, z' \in S^I$ such that $zx = x'$, and $z'x' = x$. So $(uzv)ux = uz(x\varphi\psi) = uzx = ux'$ and $(uz'v)ux' = uz'x' = ux$. Thus $x\varphi = ux \mathscr{L} ux' = x'\varphi$. But $x\varphi, x'\varphi \in R_t$, so in fact $x\varphi \mathscr{H} x'\varphi$, as required. □

In a stable semigroup, Green's Lemma basically says left multiplication bijectively maps rows to rows and right multiplication bijectively maps columns to columns in the eggbox picture (provided one does not leave the $\mathscr{J}$-class). See Figure A.3.

As a consequence of Green's Lemma, each row (respectively column) has the same number of boxes and all boxes are the same size.

Corollary A.3.2. *All $\mathscr{H}$-classes of a $\mathscr{J}$-class J of a finite semigroup have the same cardinality. Similarly, all $\mathscr{L}$-classes of J have the same size and all $\mathscr{R}$-classes of J have the same size. All $\mathscr{L}$-classes of J contain the same number of $\mathscr{H}$-classes and all $\mathscr{R}$-classes of J contain the same number of $\mathscr{H}$-classes.*

		vt, sw			s, vtz	
		t, usw			r	

Fig. A.3. Item 3 of Green's Lemma

The next lemma is classical.

Lemma A.3.3. *A non-empty semigroup is a group if and only if it is both left and right simple.*

Proof. Clearly, a group is both left and right simple. For the converse, suppose S is left and right simple. Let $t \in S$. Choose $e, f \in S$ such that $te = t$ and $ft = t$. Then if $s \in S$, $s = zt$ for some $z \in S$. So $se = zte = zt = s$. Similarly $fs = s$. Thus e is a right identity and f is a left identity. We conclude $f = fe = e$ and so S is a monoid with identity e.

If $s \in S$, then there exist $z, z' \in S$ such that $zs = e$ and $sz' = e$. Thus z is a left inverse for s and z' is right inverse for s. Therefore, $z = zsz' = z'$ and so s is invertible. Hence S is a group. □

Notice if G is a subgroup (with identity e) of a semigroup S, then $G \subseteq H_e$. We are now ready to prove a result of Green, which in particular identifies the maximal subgroups of a semigroup as the $\mathscr{H}$-classes of idempotents [68, 108, 171].

Theorem A.3.4 (Green). *Let S be a semigroup and $s \in S$. Then the following are equivalent:*

1. *H_s is a group;*
2. *H_s contains an idempotent;*
3. *$s^2 \in H_s$;*
4. *there exist $x, y \in H_s$ such that $xy \in H_s$.*

If S is stable, these are in addition equivalent to:

5. *$s^2 \mathrel{\mathscr{J}} s$;*
6. *there exist $x, y \in H_s$ such that $xy \in J_s$;*
7. *there exist $x \in L_s$, $y \in R_s$ such that $xy \in J_s$;*
8. *for all $x \in L_s$, $y \in R_s$, one has $xy \in J_s$.*

Proof. Clearly, 1 implies 2 as the identity of a group is an idempotent. To deduce 3 from 2, let $s \mathrel{\mathscr{H}} e$ with e an idempotent. Then $se = s$ by Exercise A.1.8. Let $t \in S^I$ with $st = e$. Then $s = se = sst = s^2t$. Thus $s^2 \mathrel{\mathscr{R}} s$. Similarly, $s^2 \mathrel{\mathscr{L}} s$ and so $s^2 \in H_s$, as required.

For 3 implies 4, take $x = y = s$. To establish 4 implies 1 we use Lemma A.3.3. Suppose $x, y \in H_s$ are such that $xy \in H_s$. We first prove

H_s is a subsemigroup. If $a, b \in H_s$, then we can find $u, w, z \in S^I$ such that $a = ux$, $xyw = x$ and $bz = y$. Then $abzw = ayw = uxyw = ux = a$, so $ab \mathscr{R} a$. Similarly, $ab \mathscr{L} b$. So $ab \in R_a \cap L_b = R_s \cap L_s = H_s$. Thus H_s is a subsemigroup.

To prove H_s is left simple, let $a \in H_s$. Because H_s is a subsemigroup, $a^2 \in H_s$. As $aa = a^2$, Green's Lemma implies $\varphi\colon H_a \to H_{a^2}$ given by $x\varphi = ax$ is a bijection. As $H_a = H_s = H_{a^2}$, we obtain $aH_s = H_s\varphi = H_s$. A dual argument shows that H_s is right simple. Therefore, H_s is a group.

The implications 3 implies 5 and 4 implies 6, which in turn implies 7, are trivial. For 5 implies 3, observe that $s^2 \mathscr{J} s$ yields $s^2 \mathscr{L} s$ and $s^2 \mathscr{R} s$ by stability. For 7 implies 4, suppose $x \in L_s$, $y \in R_s$ and $xy \in J_s$. By stability, $xy \mathscr{R} x$ so $x = xyt$ for some $t \in S^I$. Also $xy \mathscr{L} y$ by stability, and hence $yt \mathscr{L} xyt = x$ as $\mathscr{L}$ is a right congruence. By stability $yt \mathscr{R} y$ so $yt \in R_y \cap L_x = H_s$. Because $s \mathscr{L} x$, we can write $s = ux$ with $u \in S^I$. Then $s(yt) = uxyt = ux = s$. Therefore, $s(yt) \in H_s$ yielding 4.

Clearly 8 implies 7. We prove 3 implies 8. Indeed, if $x \in L_s$, $y \in R_s$, then there exist $u, v \in S^I$ with $s = ux$, $s = yv$. Then $uxyv = s^2 \in H_s$ and so $s \leq_{\mathscr{J}} xy \leq_{\mathscr{J}} x \leq_{\mathscr{J}} s$. Therefore, $xy \in J_s$, as required. □

Definition A.3.5 (Maximal subgroup). *If $e \in S$ is an idempotent, then H_e is called the maximal subgroup of S at e. It is the largest subgroup of S with identity e.*

Exercise A.3.6. Show that if e is an idempotent of a semigroup S, then H_e is the group of units of the monoid eSe.

The next theorem is one of the principal results in the structure theory of stable semigroups [68, 171].

Theorem A.3.7. *Let J be a $\mathscr{J}$-class of a stable semigroup S. Then the following are equivalent:*

1. *J contains a subgroup;*
2. *J contains an idempotent;*
3. *J contains a regular element;*
4. *All elements of J are regular;*
5. *The $\mathscr{R}$-class of each element of J contains an idempotent;*
6. *The $\mathscr{L}$-class of each element of J contains an idempotent;*
7. *$J^2 \cap J \neq \emptyset$.*

Proof. It is clear that 1 implies 2, which in turn implies 3 because idempotents are regular. To see that 3 implies 4, we first show that if $s \in J$ is regular, then L_s consists entirely of regular elements. Choose $t \in S$ such that $s = sts$. Let $y \in L_s$. There exist $u, v \in S^I$ such that $uy = s$, $vs = y$. Hence $ytuy = yts = vsts = vs = y$. As $tu \in S$, we conclude y is regular. A similar argument establishes each element of R_s is regular.

Now suppose $y \mathrel{\mathscr{J}} s$ with s regular. Then there exists $r \in S$ such that $s \mathrel{\mathscr{L}} r \mathrel{\mathscr{R}} y$. From $s \mathrel{\mathscr{L}} r$, we conclude r is regular. Hence $r \mathrel{\mathscr{R}} y$ implies y is regular.

For 4 implies 5, let $s \in J$ be regular and choose $t \in S$ with $sts = s$. Then $stst = st$ and so st is an idempotent. Clearly $st \mathrel{\mathscr{R}} s$, as $(st)s = s$. A dual argument shows that 4 implies 6.

The implications 5 implies 7 and 6 implies 7 are trivial as any idempotent $e \in J$ belongs to $J^2 \cap J$. For 7 implies 1, let $x, y \in J$ with $xy \in J$. Theorem A.3.4 shows that the $\mathscr{H}$-class $L_x \cap R_y$ is a group. □

Definition A.3.8 (Regular $\mathscr{J}$-class). *A $\mathscr{J}$-class satisfying the equivalent conditions of Theorem A.3.7 is called a regular $\mathscr{J}$-class. A non-regular $\mathscr{J}$-class is called a null $\mathscr{J}$-class.*

For example, every element of T_n is regular because there is an idempotent of every possible rank. The next theorem shows that all maximal subgroups in a regular $\mathscr{J}$-class of a stable semigroup are isomorphic.

Theorem A.3.9. *Let S be a semigroup and let $e, f \in S$ be $\mathscr{D}$-equivalent idempotents. Then the maximal subgroups H_e and H_f are isomorphic. More precisely, if $e \mathrel{\mathscr{R}} x \mathrel{\mathscr{L}} f$ and x' is an inverse to x with $xx' = e$, $x'x = f$ as per Proposition A.1.15, then $\varphi\colon H_e \to H_f$ given by $s\varphi = x'sx$ is an isomorphism of groups with inverse $\psi\colon H_f \to H_e$ given by $t\psi = xtx'$.*

Proof. Green's Lemma implies that φ and ψ are well-defined inverse bijections. It therefore suffices to verify that φ is a homomorphism. Indeed, $s\varphi s'\varphi = x'sxx's'x = x'ses'x = x'ss'x = (ss')\varphi$ because $s \mathrel{\mathscr{H}} e$. □

Corollary A.3.10. *The maximal subgroups at $\mathscr{J}$-equivalent idempotents of a stable semigroup are isomorphic.*

A.3.1 The Schützenberger group

For a null $\mathscr{J}$-class, the role of the maximal subgroup of a regular $\mathscr{J}$-class is played by a "phantom" group, called the *Schützenberger group*. This group is the subject of [68, Section 2.4] and [171, Chapter 7, Prop. 2.8], but we quickly review some facts that do not appear in these references in the precise form that we need them.

For a subset X of a semigroup S, set $\mathrm{Stab}(X) = \{s \in S^I \mid Xs \subseteq X\}$. Let H be an $\mathscr{H}$-class of a semigroup S. Then $\mathrm{Stab}(H)$ acts on the right of H. Denote by $\Gamma_R(H)$ the quotient of $\mathrm{Stab}(H)$ by the kernel of the action. Recall that a right permutation group (X, G) is called *regular* if $xg = x$ implies $g = 1$, for $x \in X$ and $g \in G$. In this case one also says G acts *freely* on X.

Theorem A.3.11. *Let H be an $\mathscr{H}$-class of S contained in the $\mathscr{L}$-class L. Then the following hold:*

1. *If H' is an $\mathscr{H}$-class of L, then* $\mathrm{Stab}(H) = \mathrm{Stab}(H')$*;*
2. $\mathrm{Stab}(H) \leq \mathrm{Stab}(L)$ *and hence* $\mathrm{Stab}(H)$ *acts on the right of L;*
3. *The quotient of* $\mathrm{Stab}(H)$ *by the kernel of its action on L is $\Gamma_R(H)$; consequently, if H and H' are $\mathscr{L}$-equivalent $\mathscr{H}$-classes, then $\Gamma_R(H) \cong \Gamma_R(H')$;*
4. *$(L, \Gamma_R(H))$ is a regular permutation group and the orbits are precisely the $\mathscr{H}$-classes of L and in particular $\Gamma_R(H)$ acts transitively on H;*
5. *If S is a stable semigroup, then the action of $\Gamma_R(H)$ is by endomorphisms of the left action of S on L by partial transformations defined by*

$$s \cdot x = \begin{cases} sx & sx \in L \\ \textit{undefined} & \textit{else} \end{cases}$$

for $s \in S$ and $x \in L$.

Proof. Suppose $s \in \mathrm{Stab}(H)$ and let $h_0 \in H$ be fixed. Then $h_0 s \in H$ and so $h_0 st = h_0$ for some $t \in S^I$. According to Green's Lemma, right translation by s and by t induce mutually inverse permutations of L preserving $\mathscr{H}$. In particular, $t \in \mathrm{Stab}(H)$ and $\mathrm{Stab}(H)$ is contained in the stabilizer of every other $\mathscr{H}$-class of L. This implies immediately 1 and 2. The quotient of $\mathrm{Stab}(H)$ by its action on L yields a permutation group as we saw above that $s \in \mathrm{Stab}(H)$ acts as a permutation with inverse $t \in \mathrm{Stab}(H)$. Let us verify that this permutation group is regular. Suppose $x \in L$ and $xs = x$ with $s \in \mathrm{Stab}(H)$. If $y \in L$, then $y = ux$ some $u \in S^I$ and hence $ys = uxs = ux = y$. This establishes regularity. Moreover, if $s, s' \in \mathrm{Stab}(H)$, then s and s' induce the same permutation of L if and only if $h_0 s = h_0 s'$ and so two elements act the same on L if and only if they act the same on H. Thus the quotient of $\mathrm{Stab}(H)$ by the kernel of its action on L is precisely $\Gamma_R(H)$.

As we already saw that the action of each element of $\mathrm{Stab}(H)$ preserves $\mathscr{H}$, each orbit is clearly contained in an $\mathscr{H}$-classes of L. Suppose $\ell, \ell' \in L$ are $\mathscr{H}$-equivalent. Then we can find $s, s' \in S^I$ so that $\ell s = \ell'$ and $\ell' s' = \ell$. Green's Lemma implies that $s, s' \in \mathrm{Stab}(H)$ and so the orbits are precisely the $\mathscr{H}$-classes.

For the final statement, we ask the reader to verify that we have indeed defined an action. Notice if $g \in \mathrm{Stab}(H)$ and $x \in L$, then $x \mathrel{\mathscr{L}} xg$ and so $sx \mathrel{\mathscr{L}} sxg$ for any $s \in S$. Thus $s \cdot x$ is defined if and only if $s \cdot (xg)$ is defined in which case $(s \cdot x)g = s \cdot (xg)$ by associativity. This completes the proof. □

The group $\Gamma_R(H)$ is called the *right Schützenberger group* of H. Because $(H, \Gamma_R(H))$ is a transitive regular permutation group, it follows easily that H and $\Gamma_R(H)$ have the same cardinality. One can define dually the *left Schützenberger group* $\Gamma_L(H)$ and the dual of Theorem A.3.11 holds. Now we wish to show that $\Gamma_R(H) \cong \Gamma_L(H)$. Because the left action of $\Gamma_L(H)$ on H clearly commutes with the right action of $\Gamma_R(H)$, this is an immediate consequence of the following lemma, whose proof we leave as an exercise.

Lemma A.3.12. *Let G' and G be groups with transitive regular permutation actions on the left and right of a set X, respectively, which commute. Let $x_0 \in X$ be a fixed element. Then G' and G are anti-isomorphic via the map γ sending $g \in G$ to the unique element $g\gamma \in G'$ such that $g\gamma x_0 = x_0 g$.*

Exercise A.3.13. Prove Lemma A.3.12.

Corollary A.3.14. *Let H be an $\mathscr{H}$-class of a $\mathscr{D}$-class D of a semigroup S. Then $\Gamma_R(H) \cong \Gamma_L(H)$ and if H' is any $\mathscr{H}$-class of D, then $\Gamma_R(H) \cong \Gamma_R(H')$.*

Proof. The first assertion is immediate from Lemma A.3.12 because (inner) left and right translations commute with each other. The second assertion follows from the first, together with Theorem A.3.11(3) and its dual. □

So if J is a $\mathscr{J}$-class of a stable semigroup, then there is a well-defined (up to isomorphism) Schützenberger group of J. It remains to show that if J is a regular $\mathscr{J}$-class, then the Schützenberger group is the maximal subgroup.

Proposition A.3.15. *Let H be a maximal subgroup of S. Then $\Gamma_R(H) \cong H$.*

Proof. Let e be the identity of H. Clearly, $H \leq \mathrm{Stab}(H)$ and if $h \in H$, then in $(H, \Gamma_R(H))$ we have $eh = h$. It follows that H can be identified with its image in $\Gamma_R(H)$. If $s \in \Gamma_R(H)$ and $es = h$ with $h \in H$, then $es = h = eh$ and so $s = h$ by regularity. Thus $H \cong \Gamma_R(H)$, as required. □

A.4 Rees's Theorem

Rees's Theorem [258], characterizing stable 0-simple semigroups, can be considered the first major theorem in semigroup theory. See also [350] for the special case of a finite simple semigroup.

Definition A.4.1 (0-simple). *A semigroup S with zero is called* 0*-simple if it has no non-zero proper ideals and also $S^2 \neq 0$. A semigroup S for which $S^2 = 0$ is called null.*

The following proposition is straightforward so we omit the proof.

Proposition A.4.2. *Let $s \in S$. Then $S^I sS^I \setminus J_s$ is an ideal of $S^I sS^I$ unless it is empty.*

Remark A.4.3. If K is the minimal ideal of a semigroup S and $s \in K$, then $S^I sS^I \subseteq K$ and hence $S^I sS^I = K$, for all $s \in K$. In particular K is a $\mathscr{J}$-class and $S^I sS^I \setminus K = \emptyset$. In fact, the only time $S^I sS^I \setminus J_s = \emptyset$ is when J_s is the minimal ideal of S.

The class of 0-simple semigroups admits the following alternative characterization.

Proposition A.4.4. *A semigroup S is 0-simple if and only if $SaS = S$, for all $0 \neq a \in S$.*

Proof. If S is 0-simple, then $S^2 \neq \{0\}$. Let $a \in S \setminus \{0\}$. Then S^IaS^I is an ideal containing a, so $S^IaS^I = S$ by 0-simplicity. Also S^2 is a non-zero ideal, so $S^2 = S$. Therefore, $S = S^2 = S^3$ and hence

$$0 \neq SSS = (S^IaS^I)(S^IaS^I)(S^IaS^I) \subseteq SaS.$$

Thus SaS is a non-zero ideal of S, whence $SaS = S$.

Next assume $SaS = S$, for all non-zero elements $a \in S$. In particular, if $a \neq 0$, then $S = SaS \subseteq S^2$ and so $S^2 \neq 0$. Also if $I \subseteq S$ is a non-zero ideal and $0 \neq a \in I$, then $S = SaS \subseteq SIS \subseteq I$ and thus $I = S$. We conclude S is 0-simple. □

As a consequence, $S \setminus \{0\}$ is a $\mathscr{J}$-class in a 0-simple semigroup S. Part of the importance of 0-simple semigroups stems from the following fact.

Proposition A.4.5. *The minimal ideal of a semigroup is simple. A regular 0-minimal ideal of a semigroup with zero is 0-simple.*

Proof. Let I be the minimal ideal of a semigroup S. Suppose J is an ideal of I. Then IJI is an ideal of S, so by minimality $I \subseteq IJI \subseteq J$. We conclude $I = J$. Next assume I is a regular 0-minimal ideal of S. Then $I^2 \neq 0$ by regularity. Let $0 \neq J \subseteq I$ be an ideal. Any ideal in a regular semigroup is regular and so $0 \neq J = J^3 \subseteq IJI$. Because IJI is an ideal of S, by 0-minimality $I \subseteq IJI \subseteq J$. This completes the proof. □

The semigroups appearing in the next definition are classically known as the principal factors [68], although we shall not make extensive use of this terminology.

Definition A.4.6 (Principal factor). *If $J = J_s$ is a $\mathscr{J}$-class, the principal factor associated to J is the semigroup*

$$J^0 = \begin{cases} S^IsS^I/(S^IsS^I \setminus J_s) & \text{if } J_s \text{ is not the minimal ideal} \\ J_s \cup \{0\} & \text{else.} \end{cases}$$

The principal factor J^0 can alternatively be described as the semigroup with underlying set $J \cup \{0\}$ and with multiplication given by

$$x \cdot y = \begin{cases} xy & \text{if } xy \in J \\ 0 & \text{else.} \end{cases}$$

The next proposition explains the terminology null $\mathscr{J}$-class and shows that principal factors associated to regular $\mathscr{J}$-classes are 0-simple.

Proposition A.4.7. *Let S be a stable semigroup and let J be a $\mathscr{J}$-class of S. Then J is a regular $\mathscr{J}$-class if and only if J^0 is 0-simple. If J is a null $\mathscr{J}$-class, then J^0 is null.*

Proof. Because $(J^0)^2 \neq 0$ is equivalent to $J^2 \cap J \neq \emptyset$, Theorem A.3.7 shows that J^0 is null if and only if J is a null $\mathscr{J}$-class. In particular, if J^0 is 0-simple, then J is regular.

Now suppose J is regular. To show J^0 is 0-simple, it suffices by Proposition A.4.4 to show that, for all $x \in J$, $J^0xJ^0 = J^0$. Let $t \in J$. Then we can find $u, v \in S^I$ with $uxv = t$. Because J is regular, x is regular, so there exists $s \in S$ such that $x = xsx$. Hence $t = uxv = uxsxv = (uxs)x(sxv)$ and uxs, $sxv \in S$. Now $t \leq_{\mathscr{J}} uxs, sxv \leq_{\mathscr{J}} x$. So uxs, $sxv \in J$. Thus $t \in J^0xJ^0$, as required. □

The previous proposition shows that "locally" (meaning in a $\mathscr{J}$-class) a semigroup is either null or 0-simple.

Corollary A.4.8. *A stable 0-simple semigroup is regular.*

Proof. Indeed, $S = J^0$ where $J = S \setminus \{0\}$ is a $\mathscr{J}$-class of S, so the previous proposition applies to show that S is regular. □

Our next goal is to prove Rees's Theorem [68, 171, 258], which describes stable 0-simple semigroups up to isomorphism. The key notion is that of a Rees matrix semigroup.

Definition A.4.9 (Rees matrix). *A Rees matrix is a map $C\colon B \times A \to G^0$ where A and B are non-empty sets, G is a group and $G^0 = G \cup \{0\}$ with 0 an adjoined zero. We adopt the symmetric notation bCa for the value of C on (b, a).*

Denote by E_{ab}, for $a \in A$ and $b \in B$, the $A \times B$ elementary matrix unit with 1 in position (a, b) and 0 elsewhere, where 1 is the identity of G.

Definition A.4.10 (Rees Matrix Semigroup). *Let G be a group and let $C\colon B \times A \to G^0$ be a Rees matrix. The Rees matrix semigroup with sandwich matrix C is the set*

$$\mathscr{M}^0(G, A, B, C) = \{gE_{ab} \mid g \in G^0,\ a \in A,\ b \in B\}$$

with multiplication, for $X, Y \in \mathscr{M}^0(G, A, B, C)$, given by $X \diamond Y = XCY$.

The underlying set of $\mathscr{M}^0(G, A, B, C)$ consists of all $A \times B$ matrices over G^0 with at most one non-zero entry. Associativity follows from the associativity of matrix multiplication. Notice that

$$g_1E_{a_1b_1} \diamond g_2E_{a_2b_2} = (g_1(b_1Ca_2)g_2)E_{a_1b_2}$$

and so $\mathscr{M}^0(G, A, B, C)$ is indeed closed under the multiplication.

Oftentimes it is convenient to identify $\mathscr{M}^0(G,A,B,C)$ with $(A\times G\times B)\cup\{0\}$ via the correspondence sending gE_{ab}, with $g\in G$, to (a,g,b) and 0 to 0. Under this bijection, the multiplication rule becomes

$$(a_1,g_1,b_1)(a_2,g_2,b_2)=\begin{cases}(a_1,g_1(b_1Ca_2)g_2,b_2) & b_1Ca_2\neq 0\\ 0 & \text{else.}\end{cases} \tag{A.1}$$

Sometimes the matrix C is called the *structure matrix* or the *sandwich matrix* of $\mathscr{M}^0(G,A,B,C)$. If $C\colon B\times A\to G$, then $\mathscr{M}^0(G,A,B,C)\setminus\{0\}$ is a subsemigroup, denoted by $\mathscr{M}(G,A,B,C)$.

Definition A.4.11 (Regular Rees matrix). *A Rees matrix is called regular if it has no zero rows or columns.*

Proposition A.4.12. *Let $C\colon B\times A\to G^0$ be a Rees matrix. Then the following are equivalent for $S=\mathscr{M}^0(G,A,B,C)$:*

1. *S is regular;*
2. *C is regular;*
3. *S is 0-simple.*

Moreover, if 1–3 hold, then:

(a) $S/\mathscr{R}=\{a\times G\times B\mid a\in A\}$;
(b) $S/\mathscr{L}=\{A\times G\times b\mid b\in B\}$;
(c) $S/\mathscr{H}=\{a\times G\times b\mid a\in A,\ b\in B\}$;
(d) The $\mathscr{H}$-class $H=a\times G\times b$ is a group if and only if $bCa\neq 0$; if H is a group, its identity is $(a,(bCa)^{-1},b)$ and it is isomorphic to G via

$$g\longmapsto(a,(bCa)^{-1}g,b).$$

Proof. Assume S is regular and suppose $a\in A$, $b\in B$. Let $s=(a,1,b)$ and choose $t=(a',g',b')$ such that $sts=s$. Then

$$(a,1,b)(a',g',b')(a,1,b)=s\neq 0$$

and so $bCa'\neq 0\neq b'Ca$. We conclude C is regular.

Next assume C is regular. By Proposition A.4.4, to prove S is 0-simple, we need to show that $S(a,g,b)S=S$ for all $(a,g,b)\in S\setminus 0$. So let $(a',g',b')\in S\setminus 0$ and choose $a''\in A$, $b''\in B$ such that $bCa''\neq 0\neq b''Ca$. Then

$$(a',g'(b''Ca)^{-1},b'')(a,g,b)(a'',(bCa'')^{-1}g^{-1},b')=(a',g',b')$$

and hence $S(a,g,b)S=S$. Thus S is 0-simple.

Suppose S is 0-simple. We show that C is regular. Let $a\in A$ and $b\in B$. Because S is 0-simple, $S(a,1,b)S=S$ and so

$$(a,1,b)=(a',g',b')(a,1,b)(a'',g'',b'')$$

for appropriate choices of a', a'', b', b'', g' and g''. But then $b'Ca \neq 0 \neq bCa''$, and so C is regular.

Suppose now C is regular; we show S is regular. Let $(a, g, b) \in S$ and choose $a' \in A$, $b' \in B$ with $bCa' \neq 0 \neq b'Ca$. Then

$$(a, g, b)(a', (bCa')^{-1}g^{-1}(b'Ca)^{-1}, b')(a, g, b) = (a, g, b)$$

and so $\mathscr{M}^0(G, A, B, C)$ is regular, as required.

We now turn to (a)–(d). If $s = (a, g, b)$, then it follows directly from (A.1) that $R_s \subseteq a \times G \times B$. For the converse, suppose $(a, g', b') \in a \times G \times B$. Choose a'' such that $bCa'' \neq 0$. Then

$$(a, g, b)(a'', (bCa'')^{-1}g^{-1}g', b') = (a, g', b').$$

Hence $R_s = a \times G \times B$. The descriptions of $S/\mathscr{L}$ and $S/\mathscr{H}$ are handled similarly.

Next we turn to the case of when $H = a \times G \times b$ is a group. By Theorem A.3.4, H is a group if and only if it contains an idempotent. But

$$(a, g, b)(a, g, b) = \begin{cases} (a, g(bCa)g, b) & bCa \neq 0 \\ 0 & \text{else.} \end{cases}$$

So H contains an idempotent if and only if $bCa \neq 0$, in which case the idempotent is $(a, (bCa)^{-1}, b)$. Clearly, $g \mapsto (a, (bCa)^{-1}g, b)$ is a bijection from G to H. It is a homomorphism as the computation

$$(a, (bCa)^{-1}g_1, b)(a, (bCa)^{-1}g_2, b) = (a, (bCa)^{-1}g_1g_2, b)$$

shows. □

Corollary A.4.13. *Let S be a regular Rees matrix semigroup. Then S is stable.*

Exercise A.4.14. Prove Corollary A.4.13.

We now prove Rees's Theorem [68, 171, 258].

Theorem A.4.15 (Rees). *Let S be a stable semigroup. Then S is 0-simple if and only if there exist a group G, sets A, B and a regular Rees matrix $C\colon B \times A \to G^0$ such that $S \cong \mathscr{M}^0(G, A, B, C)$. Similarly, a stable semigroup S is simple if and only if $S \cong \mathscr{M}(G, A, B, C)$ where $C\colon B \times A \to G$.*

Proof. We just handle the 0-simple case, as the simple case is a consequence. Corollary A.4.13 provides one direction. For the other, Corollary A.4.8 tells us S is regular. Let J be the non-zero $\mathscr{J}$-class of S and let $e \in J$ be an idempotent (guaranteed by Theorem A.3.7). Let $G = H_e$, $A = S/\mathscr{R}$ and $B = S/\mathscr{L}$. Theorem A.3.4 says that G is a group.

e, g				ℓ_b		
r_a				$r_a g \ell_b$		

Fig. A.4. Rees coordinates

Choose for each $a \in A$ and $b \in B$, $r_a \in a \cap L_e$ and $\ell_b \in b \cap R_e$. Notice $e\ell_b = \ell_b$ and $r_a e = r_a$. Green's Lemma tells us $x \mapsto r_a x \ell_b$ is a bijection from $G = H_e$ to $a \cap b$. Hence each non-zero element of S can be written uniquely in the form $r_a x \ell_b$ with $x \in G$. See Figure A.4.

To motivate the definition of the sandwich matrix C, notice for $g, g' \in G$

$$(r_a g \ell_b)(r_{a'} g' \ell_{b'}) = r_a (g \ell_b r_{a'} g') \ell_{b'}. \tag{A.2}$$

Now if $\ell_b r_{a'} \neq 0$, then $\ell_b r_{a'} \in J$. Stability then yields $\ell_b r_{a'} \mathrel{\mathscr{R}} \ell_b \mathrel{\mathscr{R}} e$ and $\ell_b r_{a'} \mathrel{\mathscr{L}} r_{a'} \mathrel{\mathscr{L}} e$, i.e., $\ell_b r_{a'} \in H_e = G$. In any event $\ell_b r_{a'} \in G^0$.

From (A.2) and the ensuing discussion, it is clear we should define $C\colon B \times A \to G^0$ by $bCa = \ell_b r_a \in G^0$. We then define an isomorphism $\psi\colon \mathscr{M}^0(G, A, B, C) \to S$ by $0 \mapsto 0$ and $(a, g, b) \mapsto r_a g \ell_b$. The discussion above shows that ψ is a bijection. To see that it is a homomorphism, we compute

$$\big((a, g, b)(a', g', b')\big)\psi = r_a (g \ell_b r_{a'} g') \ell_{b'}.$$

A comparison with (A.2) yields ψ is a homomorphism. Proposition A.4.12 now implies C is a regular Rees matrix, completing the proof of the theorem. $\square$

Let us show how Rees's Theorem determines the structure of stable left simple semigroups.

Lemma A.4.16. *Let S be a stable simple semigroup whose idempotents form a subsemigroup. Then $S \cong \mathscr{M}(G, A, B, C)$ where C is a $B \times A$ matrix all of whose entries are the identity of G.*

Proof. Each $\mathscr{H}$-class of a stable simple semigroup must be a group, so we can choose r_a and ℓ_b to be idempotents in the proof of Rees's Theorem. Then $bCa = \ell_b r_a$ is an idempotent of H_e, and hence is e. This proves the lemma. $\square$

Corollary A.4.17. *Let S be a stable semigroup. Then S is left simple if and only if $S \cong L \times G$ with L a left zero semigroup and G a group. Dually, S is right simple if and only if $S \cong G \times R$ with G a group and R a right zero semigroup.*

Proof. Trivially, if L is a left zero semigroup and G is a group, then $L \times G$ is left simple. For the converse, we first show the idempotents of S form a subsemigroup. Indeed, if $e, f \in S$ are idempotents, then $e \mathrel{\mathscr{L}} f$ and so

$ef = e$, $fe = f$ by Exercise A.1.8. So Rees's Theorem, Lemma A.4.16 and Proposition A.4.12 show that $S \cong \mathscr{M}(G, A, \{b\}, C)$ where C is a matrix all of whose entries are the identity of G. Let L have underlying set A with the left zero multiplication. Then $(a, g, b) \mapsto (a, g)$ gives an isomorphism $S \to L \times G$. The right simple case is dual. □

Definition A.4.18 (Rees coordinatization). *If S is a stable semigroup and J is a regular $\mathscr{J}$-class, then a Rees coordinatization of J is an isomorphism $\psi\colon J^0 \to \mathscr{M}^0(G, A, B, C)$.*

Optimal Rees coordinatizations are studied in Section 4.13. This brings to an end our brief introduction to the Green-Rees local structure theory for stable semigroups.

Remark A.4.19. A simple compact semigroup is always isomorphic to a Rees matrix semigroup $\mathscr{M}(G, A, B, C)$ where G is a compact group, A, B are compact sets and $C\colon B \times A \to G$ is continuous [135]. The situation for 0-simple semigroups is more complicated as 0 might not be an isolated point.

B

Tables on Preservation of Sups and Infs

The reader is referred to Section 1.1.2 for nomenclature on types of maps between posets. In the following table, $\mathbf{V}$ is a pseudovariety of semigroups.

Table B.1. Preservation properties of products on **PV**

Operator	Order Properties	Type of Operator
$\mathbf{V} \cap (-)$	continuous, $\mathbf{inf}_{\mathsf{T}}$	$\mathbf{GMC}(\mathbf{PV})^{-}$
$\mathbf{V} \vee (-)$	$\mathbf{sup}_{\mathsf{B}}$	$\mathbf{GMC}(\mathbf{PV})^{+}$, $\mathbf{GMC}(\mathbf{PV})^{\rho}_{\mathbf{1}}$
$\mathbf{V} * (-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^{+}$
$\mathbf{V} ** (-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^{+}$
$\mathbf{V}$ ⓜ $(-)$	continuous	$\mathbf{GMC}(\mathbf{PV})^{+}$
$(-) \cap \mathbf{V}$	continuous, $\mathbf{inf}_{\mathsf{T}}$	$\mathbf{GMC}(\mathbf{PV})^{-}$
$(-) \vee \mathbf{V}$	$\mathbf{sup}_{\mathsf{B}}$	$\mathbf{GMC}(\mathbf{PV})^{+}$, $\mathbf{GMC}(\mathbf{PV})^{\rho}_{\mathbf{1}}$
$(-) * \mathbf{V}$	$\mathbf{sup}_{\mathsf{B}}$	$\mathbf{GMC}(\mathbf{PV})^{\rho}_{\mathbb{L}\mathbf{1}}$
$(-) ** \mathbf{V}$	$\mathbf{sup}_{\mathsf{B}}$	$\mathbf{GMC}(\mathbf{PV})^{\rho}_{\mathbb{L}\mathbf{1}}$
$(-)$ ⓜ $\mathbf{V}$	continuous, $\mathbf{inf}$	$\mathbf{GMC}(\mathbf{PV})^{\rho}_{\mathbb{L}\mathbf{1}}$

In the next table, V denotes a pseudovariety of relational morphisms.

Table B.2. Preservation properties of products on **PVRM**

Operator	Order Properties
$\mathsf{V} \cap (-)$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$\mathsf{V} \vee (-)$	$\mathbf{sup}_{\mathsf{B}}$
$\mathsf{V} \odot (-)$	continuous
$(-) \cap \mathsf{V}$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$(-) \vee \mathsf{V}$	$\mathbf{sup}_{\mathsf{B}}$
$(-) \odot \mathsf{V}$	continuous, $\wedge$-map

In the following table, V represents a continuously closed class.

Table B.3. Preservation properties of products on **CC**

OPERATOR	ORDER PROPERTIES
$\mathsf{V} \cap (-)$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$\mathsf{V} \vee (-)$	$\mathbf{sup}_{\mathsf{B}}$
$\mathsf{V} \odot (-)$	continuous
$(-) \cap \mathsf{V}$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$(-) \vee \mathsf{V}$	$\mathbf{sup}_{\mathsf{B}}$
$(-) \odot \mathsf{V}$	continuous

In our final table, α denotes a continuous operator. Because all infs and sups of **GMC**(**PV**) coincide with infs and sups taken in **Cnt**(**PV**), the corresponding table for **GMC**(**PV**) is identical. We use here $\circ$ for composition of operators.

Table B.4. Preservation properties of products on **Cnt**(**PV**)

OPERATOR	ORDER PROPERTIES
$\alpha \wedge (-)$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$\alpha \vee (-)$	$\mathbf{sup}_{\mathsf{B}}$
$\alpha \circ (-)$	continuous
$(-) \wedge \alpha$	continuous, $\mathbf{inf}_{\mathsf{T}}$
$(-) \vee \alpha$	$\mathbf{sup}_{\mathsf{B}}$
$(-) \circ \alpha$	$\mathbf{sup}$, $\wedge$-map

Exercise B.0.1. Prove all assertions in the tables and find counterexamples to properties that are not claimed to be preserved.

List of Problems

Some, but not all, of these problems appear in the text as questions.

Problem 1. What is $\mathbf{V} * \mathbf{1}$? It is easy to see that if S is a semigroup in $\mathbf{V} * \mathbf{1}$, then the quotient of S by the kernel of its action on the right of itself belongs to $\mathbf{V}$. The converse holds if $\mathbf{gV}$ is definable by pseudoidentities over strongly connected graphs. Is the converse always valid?

Problem 2. What is $\mathbf{V} ** \mathbf{1}$? Let S be a semigroup and define a congruence on S by $s \equiv s'$ if, for all $s_L, s_R \in S$, one has $s_L s s_R = s_L s' s_R$. It is easy to see that if $S \in \mathbf{V} ** \mathbf{1}$, then $S/{\equiv} \in \mathbf{V}$. The converse holds if $\mathbf{gV}$ is definable by pseudoidentities over strongly connected graphs. Does it hold in general?

Problem 3. Give an element of $\mathbf{PVRM}^+$ analogous to $\mathbf{V}_D$ or $\mathbf{V}_K$ yielding $\Diamond(-)$ under $\mathfrak{q}$ and find a basis of pseudoidentities for it.

Problem 4 (Tilson Question). Is $\mathbf{V}_D = \min(\mathbf{V} * (-))$?

Problem 5. Is it true that a pseudovariety of semigroups $\mathbf{V}$ is decidable if and only if $\mathbf{gV}$ is decidable?

Problem 6. Find a natural axiom to add to the definition of a pseudovariety of semigroupoids to ensure definability by path pseudoidentities.

Problem 7 (Tilson Question Version 2). Is $\mathbf{V}_K = \min(\mathbf{V} ** (-))$?

Problem 8. Is it true that if $\mathbf{H}, \mathbf{K}$ are decidable pseudovarieties of groups, then $\mathbf{H} \vee \mathbf{K}$ is decidable?

Problem 9. Find a natural lattice in $\mathbf{CC}^+$ mapping to $\mathbf{GMC}^{\varrho}_{\mathbb{L}\mathbf{1}}(\mathbf{PV})$ under $\mathfrak{q}$.

Problem 10. Compute $\min(\alpha)$ for any operator α that is not a constant operator, or of the form $\mathbf{V} \cap (-)$.

Problem 11. Suppose K and L are equational continuously closed classes. Is $\mathsf{K} \odot \mathsf{L}$ equational?

Problem 12. Suppose K and L are Birkhoff continuously closed classes. Is $\mathsf{K} \odot \mathsf{L}$ Birkhoff?

Problem 13. Give an example showing that meets are not pointwise in the lattice $\mathbf{GMC}(\mathbf{PV})^+$.

Problem 14. Find a projective basis for $\mathbf{V}_D$.

Problem 15 (Almeida-Weil Basis Question). Find an example of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ so that pseudoidentities in Theorem 3.7.15 (i.e., [27, Thms. 5.2 and 5.3]) do *not* define $\mathbf{V} * \mathbf{W}$. For such an example, give an explicit example of a member of the basis from Theorem 3.8.18 that is not a consequence of the pseudoidentities in Theorem 3.7.15.

Problem 16. Is it decidable whether a finite semigroup is projective?

Problem 17 (Complexity Problem). Is the group complexity function c computable? More precisely, is there a Turing machine that given a finite semigroup S by its multiplication table as input can output $c(S)$?

Problem 18. Determine the pointwise meet of all local upper bounds to complexity. Find more local upper bounds.

Problem 19. A finite semigroup S is critical if each of its proper divisors has strictly smaller complexity. What are the possible posets for the $\mathscr{J}$-order of a critical semigroup? By [115], any finite poset with minimum is the $\mathscr{J}$-order of a finite semigroup. We ask the same question for critical semigroups with respect to two-sided complexity C or dot-depth.

Problem 20 (Henckell-Rhodes). Is it true that the problem of deciding whether a semigroup has complexity one reduces to the case of a semigroup with at most 3 non-zero $\mathscr{J}$-classes? More generally, is it true that the problem of deciding whether a semigroup has complexity n reduces to the case of a semigroup with at most $n+2$ non-zero $\mathscr{J}$-classes?

Problem 21 (Schützenberger's Question). Does the pseudovariety $\mathbf{A} \vee \mathbf{G}$ have decidable membership?

Problem 22. Given a finite semigroup S, is the semidirect product closure of (S) decidable? How about the closure under the two-sided semidirect product?

Problem 23. Study the complexity function κ associated to the operators $\alpha = \mathbf{A} \circledm (-)$ and $\beta = \mathbf{G}^{\mathbf{N}} \circledm (-)$.

Problem 24. Is the complexity function for $\mathbf{A}$ associated to the operators $\alpha = \mathbf{Sl} ** (-)$ and $\beta = \mathbb{L}\mathbf{1} \circledm (-)$ computable? How exactly does it compare to dot-depth?

Problem 25. Find a proof of the classical Prime Decomposition Theorem using the Derived Category Theorem and some factorization theorem for relational morphisms.

Problem 26. Is the operator $\alpha = \mathbf{A} ** (-)$ idempotent? We suspect that it is not. This would be equivalent to $\mathbf{A} \circledm (-) = \mathbf{A} ** (-)$. It is known $\alpha^2 = \alpha^3$. If PT_n is the semigroup of partial transformations of an n element set and I is the aperiodic ideal of maps of rank at most 1, then we suspect $PT_n \notin \mathbf{A} ** (PT_n/I)$ although the projection $PT_n \to PT_n/I$ is an aperiodic morphism.

Problem 27. Find an easier proof than the one in [362] that $c(\widehat{S}^{\mathscr{L}}) = c(S)$.

Problem 28. Suppose that S is a semigroup such that the longest chain of non-zero $\mathscr{J}$-classes of S has length at most 2. Must $C(S) \leq 1$?

Problem 29. Is the two-sided complexity function C computable? Start with a semigroup with at most 3 non-zero $\mathscr{J}$-classes.

Problem 30. Is there some sort of Presentation Lemma to describe the existence of a cross section with respect to $\mathbb{L}\mathbf{G} \circledm \mathbf{V}$?

Problem 31. Are there arbitrarily large numbers n such that there is a semigroup S with $c(S) = n = C(S)$? Probably there are.

Problem 32. What is the two-sided complexity of the semigroup of binary relations on an n element set?

Problem 33. A lattice is said to be *semi-distributive* at an element ℓ if $\ell \wedge x = \ell \wedge y$ implies $\ell \wedge x = \ell \wedge (x \vee y)$. Because meets and joins are continuous in $\mathbf{PV}$, if $\mathbf{PV}$ is semi-distributive at $\mathbf{V}$ and $\mathbf{W} \leq \mathbf{V}$, then there is a largest pseudovariety $\widehat{\mathbf{W}}$ such that $\mathbf{V} \cap \widehat{\mathbf{W}} = \mathbf{W}$. For instance, $\mathbf{PV}$ is semi-distributive at $\mathbf{G}$ and if $\mathbf{H}$ is a pseudovariety of groups, then $\overline{\mathbf{H}}$ is the maximum pseudovariety with $\mathbf{G} \cap \overline{\mathbf{H}} = \mathbf{H}$. Reilly and Zhang proved that $\mathbf{PV}$ is semi-distributive at the pseudovariety of bands $\mathbf{B}$ [260]. It follows from a result of G. Higman [217, 54.24] that $\mathbf{PV}$ is not semi-distributive at certain pseudovarieties of groups. At which pseudovarieties $\mathbf{V}$ is $\mathbf{PV}$ semi-distributive?

Problem 34. Is a subspace of $\mathbf{PV}$ sequentially compact in the strong topology if and only if it is closed and Noetherian?

Problem 35. What are the atoms of $\mathbf{Cnt}(\mathbf{PV})^+$, $\mathbf{GMC}(\mathbf{PV})^+$, $\mathbf{PVRM}^+$ and $\mathbf{CC}^+$?

Problem 36. Give more examples of smi, fmi, sfmi pseudovarieties that are not mi. In particular, can a compact pseudovariety be smi? This is related to the existence of atoms in $\mathbf{Cnt}(\mathbf{PV})^+$.

Problem 37. Classify the $\times$-prime finite semigroups. Is the property of being $\times$-prime decidable?

Problem 38. Is I_n, the symmetric inverse monoid, fji for all $n \geq 2$? A positive answer would imply **ESl** is fji, in light of Ash's Theorem [32] and Lemma 6.1.13. How about sfji? This may be easier.

Problem 39. If $(\mathbf{n}, G)$ is a permutation group, let $S_{(\mathbf{n},G)}$ be the action semigroup of the partial transformation inverse semigroup $(\mathbf{n}, G \cup B_n)$ where we identify B_n with the rank one partial bijections of $\mathbf{n}$. Theorem 4.7.21 implies $S_{(\mathbf{n},G)}$ is subdirectly indecomposable. Of course, $BZ_p = S_{(\mathbf{p},\mathbb{Z}_p)}$. The question is: when is $S_{(\mathbf{n},G)}$ fji? One needs at least that the permutation group is subdirectly indecomposable as a permutation group.

Problem 40. Is the full transformation monoid T_n an fji semigroup for all $n \geq 2$? If so this would give a new proof that **FSgp** is fji by Lemma 6.1.13. How about sfji? This may be easier.

Problem 41. Is there an increasing sequence of compact fji aperiodic pseudovarieties $(S_1) \leq (S_2) \leq \cdots$ with $\bigcup(S_i) = \mathbf{A}$? If so, then $\mathbf{A}$ would be fji by Lemma 6.1.13.

Problem 42. Given a finite set of computable pseudoidentities (say identities if you like), is it decidable whether $[\![E]\!]$ is sfji? A similar question can be asked for fji or for any of our other lattice theoretic adjectives.

Problem 43. Are the complexity pseudovarieties $\mathbf{C}_n$ fji or sfji? We ask the same question for $\mathbf{C}_n \cap \mathbf{DS}$ and $\mathbf{C}_n \cap \mathbf{CR}$.

Problem 44. Study the topological spaces $\operatorname{Spec} \mathbf{PV}$ and $\operatorname{Spec} \mathbf{PV}^{op}$.

Problem 45. Give a proof of Theorem 7.3.32 that does not use pseudoidentities.

Problem 46. Find a proof of Theorem 7.3.39 that does not use pseudoidentities.

Problem 47. Is it true that $\overline{\mathbf{H}}$ is sfji for every pseudovariety of groups $\mathbf{H}$? Consider the same question for fji.

Problem 48. Let S be a KN semigroup with KN maximal subgroup G in its minimal ideal. Describe $\mathsf{Excl}(S)$ in terms of $\mathsf{Excl}(G)$. In particular, if $\mathsf{Excl}(G) = [\![u = v]\!]$, then give a pseudoidentity defining $\mathsf{Excl}(S)$.

Problem 49. Describe the semigroup structure of $(\mathbf{PV}, *)$. It is known that the semigroup of Birkhoff varieties of groups with the semidirect product as multiplication is a free monoid [217]. But this is not the case for **PV**, which has many idempotents. Some results can be found in [369].

Problem 50. Determine the $\mathsf{s^*i}$, $\mathsf{sf^*i}$ and $\mathsf{f^*i}$ pseudovarieties.

Problem 51. Classify the irreducible relational morphisms. The results of Chapter 5 imply that an irreducible relational morphism $\varphi\colon S \to T$ must either have a non-trivial congruence-free monoid M in the Mal'cev kernel and belong to $(M)_K$, or belong to $\ell\mathbf{1}_K$. Are the irreducible relational morphisms precisely the relations of this sort?

Problem 52. Study the abstract spectral theory of $\mathbf{GMC}(\mathbf{PV})$. What are the mi, smi, sfmi, fmi, ji, fji, sji, sfmi elements of $\mathbf{GMC}(\mathbf{PV})$?

Problem 53. What does the $\mathscr{J}$-order look like in $\mathbf{Cnt}(\mathbf{PV})^+$? Give an example of when $\alpha \leq \beta$ but $\beta \nleq_{\mathscr{J}} \alpha$ (or prove there is no such example). Consider the analogous questions for $\mathbf{GMC}(\mathbf{PV})^+$, $\mathbf{Cnt}(\mathbf{PV})^-$ and $\mathbf{GMC}(\mathbf{PV})^-$.

Problem 54. Suppose $\mathsf{R} \in \mathbf{CC}$ has decidable membership. Must Rq send compact pseudovarieties to decidable pseudovarieties? Notice that if $\mathsf{R} \in \mathbf{BCC}$ has decidable membership, then $\mathsf{Rq}(\mathbf{V})$ is decidable for each compact pseudovariety $\mathbf{V}$. Indeed, if $\mathbf{V} = (T)$, then $S \in \mathsf{Rq}(\mathbf{V})$ if and only if the canonical relational morphism $\rho\colon S \to F_{(T)}(S \times T)$ belongs to R.

Problem 55. Study the counital bialgebras $\mathrm{Rec}(A^*)$, $\mathbf{A}(A^*)$ and $\mathbf{J}(A^*)$. Also study the Hopf algebra of group languages $\mathbf{G}(A^*)$.

Problem 56. Is it decidable whether a finite semigroup divides a semigroup of lower triangular Boolean matrices? In other words, is dot-depth two decidable?

Problem 57. In [241–244], Polák sets up a correspondence between pseudovarieties of idempotent semirings and certain classes of languages. What is the effect of the triangular product on languages? It should be similar to the effect of the Schützenberger product in the theory of varieties of languages.

Problem 58. What is the smallest pseudovariety of idempotent semirings closed under triangular product and containing $\mathbb{B}$? One would guess it should consist of all idempotent semirings whose underlying semigroup is aperiodic.

Problem 59. Classify all $\triangle$-irreducible idempotent semirings.

Problem 60 (Margolis). Is it true that if S is $\triangle$-irreducible, then so is $M_n(S)$?

Problem 61. Classify the basic $\triangle$-irreducible semirings.

Problem 62. Compute the exclusion class of $M_n(Q)$ where Q is one of $\mathbb{B}$, $G^\natural$ with G non-trivial monolithic or $P(G)$ with G a non-trivial group.

Problem 63. Given an oracle deciding group complexity, is c_q computable? We ask the same question for C_q.

Problem 64. Study $\triangle$-irreducibility over more general semirings.

Problem 65 (Fox-Rhodes). Let $S = \mathscr{M}^0(G, A, B, C)$ be a 0-simple semigroup.

1. Is $c(P(S)) = \imath(C) - 1 + c(G)$ where $\imath(C)$ is the largest size of an identity submatrix of C (up to reordering rows and columns and renormalizing)? The right-hand side is a lower bound by the argument in Theorem 9.5.11. In [91], it is shown that an upper bound is $\tau(C) - 1 + c(G)$ where $\tau(C)$ is the largest size of a triangular submatrix of C.
2. Recall that the action of S on the right of $G \times B$ extends to an action on the free left $P(G)$-module generated by B and hence yields a representation $\rho\colon S \to M_B(P(G))$ (which concretely speaking is the classical Schützenberger representation by row monomial matrices). This in turn yields a representation $\widetilde{\rho}\colon P(S) \to M_B(P(G))$. What is the relationship between $c(P(S))$ and $c(P(S)\widetilde{\rho})$?

Problem 66 (Fox-Rhodes). What is the largest size of an identity submatrix of the structure matrix for the rank k $\mathscr{J}$-class of T_n?

Problem 67. If S is a semigroup and I is an ideal, what is the relationship between $c(P(S))$ and $c(P(I))$, $c(P_0(S/I))$? Proposition 9.5.14 gives some indication, but $c(P(S)_{P(I)})$ and $P(I)$ do not seem so clearly related. In [91], it is shown that if S is a monoid and $I = S \setminus G$ where G is the group of units, then $c(P(S)) \leq c(P(I)) + c(P_0(S/I))$.

Problem 68. Is $c(P(S)) = \max\{c(P_0(J^0)) \mid J \in S/\mathscr{J}\}$?

Problem 69. Develop homological algebra for idempotent semirings. Find a homological definition of c_q and C_q. Relate this to dot-depth.

Problem 70. Develop a derived/kernel category theory for the triangular product.

Problem 71. Let S be a finite semigroup. Is

$$c(S) = \max\{c(T) \mid T \in (S),\ T \text{ is } \mathsf{fji}\}?$$

Consider the analogous problem for other hierarchical complexity functions such as two-sided complexity or dot-depth. This question is essentially asking if c passes to $\operatorname{Spec} \mathbf{PV}^{op}$.

Problem 72. Transfer the results of this book to Formal Language Theory via the Eilenberg Correspondence [85]. Is there a quantized version of the Eilenberg Correspondence? We have a notion of an A-generated relational morphism. Perhaps each pseudovariety of relational morphisms is generated by canonical relational morphisms of A-generated syntactic semigroups where A is any finite alphabet? What objects are recognized by relational morphisms? Is there a connection with Straubing's $\mathscr{C}$-pseudovarieties [347]?

Problem 73. Let S be a finite semigroup and let $P = [\mathbf{1}, (S)] \cap K(\mathbf{PV})$ be the join semilattice of compact subpseudovarieties of (S). This is a countable poset. What countable linear orders can occur as chains in P? Both $(\mathbb{N}, \leq)$ and $(\mathbb{N}, \geq)$ can occur [309].

Problem 74. Relate two-sided complexity to composition of bimachines [84, 284].

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Table of Pseudovarieties

Notation	Name	Pseudoidentities
$\mathbf{1}$	Trivial pseudovariety	$[\![x = y]\!]$
$\mathbf{A}$	Aperiodic semigroups	$[\![x^{\omega+1} = x^{\omega}]\!]$
$\mathbf{Ab}$	Abelian groups	$[\![x^{\omega} = 1, xy = yx]\!]$
$\mathbf{Ab}(n)$	Abelian groups of exponent n	$[\![x^{n} = 1, xy = yx]\!]$
$\mathbf{ACom}$	Aperiodic commutative	$[\![x^{\omega+1} = x^{\omega}, xy = yx]\!]$
$\mathbf{B}$	Bands	$[\![x^2 = x]\!]$
$\mathbf{C}_n$	Level n complexity	
$\mathbf{Com}$	Commutative	$[\![xy = yx]\!]$
$\mathbf{CR}$	Completely regular	$[\![x^{\omega+1} = x]\!]$
$\mathbf{CS}$	Completely simple	$[\![(xy)^{\omega}x = x]\!]$
$\mathbf{D}$	Delay	$[\![xy^{\omega} = y^{\omega}]\!]$
$\mathbf{DS}$	Regular $\mathscr{J}$-classes are ssgps.	$[\![((xy)^{\omega}(yx)^{\omega}(xy)^{\omega})^{\omega} = (xy)^{\omega}]\!]$
$\mathbf{ER}$	Idem.-gen. ssgp. $\mathscr{R}$-trivial	$[\![(x^{\omega}y^{\omega})^{\omega}x^{\omega} = (x^{\omega}y^{\omega})^{\omega}]\!]$
$\mathbf{G}$	Groups	$[\![x^{\omega} = 1]\!]$
$\mathbf{G}_{nil}$	Nilpotent groups	See [10]
$\mathbf{G}_p$	p-groups	$[\![x^{p^{\omega}} = 1]\!]$
$\mathbf{G}_{\pi}$	π-groups	$[\![x^{\pi^{\omega}} = 1]\!]$
$\mathbf{G}_{\pi'}$	π'-groups	$[\![x^{(\pi')^{\omega}} = 1]\!]$
$\mathbf{G}_{sol}$	Solvable groups	See [45, 56]
$\mathbf{G}^{\mathbf{N}}$	Nilpotent extensions of groups	$[\![x^{\omega} = y^{\omega}]\!]$
$\mathbf{J}$	$\mathscr{J}$-trivial	$[\![(xy)^{\omega} = (yx)^{\omega}, x^{\omega+1} = x^{\omega}]\!]$
$\mathbf{K}$	Reverse delay	$[\![x^{\omega}y = x^{\omega}]\!]$
$\mathbf{K}_n$	Level n two-sided complexity	
$\mathbf{L}$	$\mathscr{L}$-trivial	$[\![y(xy)^{\omega} = (xy)^{\omega}]\!]$
$\mathbb{L}\mathbf{1}$	Locally trivial	$[\![x^{\omega}yx^{\omega} = x^{\omega}]\!]$
$\mathbb{L}\mathbf{G}$	Local groups	$[\![(x^{\omega}yx^{\omega})^{\omega} = x^{\omega}]\!]$
$\mathbf{LRB}$	Left regular (=$\mathscr{R}$-trivial) band	$[\![x^2 = x, xyx = xy]\!]$
$\mathbf{LS}$	Left simple	$[\![xy^{\omega} = x]\!]$

$\mathbf{LS^N}$	Nilpotent exts. of left simple	$[\![x^\omega y^\omega = x^\omega]\!]$
$\mathbb{L}\mathbf{Sl}$	Local semilattices	$[\![(x^\omega y x^\omega)^2 = x^\omega y x^\omega,$ $x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega]\!]$
$\mathbf{LZ}$	Left zero	$[\![xy = x]\!]$
$\mathbf{N}$	Nilpotent semigroups	$[\![x^\omega = 0]\!]$
$\mathbf{R}$	$\mathscr{R}$-trivial	$[\![(xy)^\omega x = (xy)^\omega]\!]$
$\mathbf{RB}$	Rectangular bands	$[\![x^2 = x, xyx = x]\!]$
$\mathbf{RRB}$	Right regular (=$\mathscr{L}$-trivial) band	$[\![x^2 = x, yxy = xy]\!]$
$\mathbf{RS}$	Right simple	$[\![x^\omega y = y]\!]$
$\mathbf{RS^N}$	Nilpotent exts. of right simple	$[\![x^\omega y^\omega = y^\omega]\!]$
$\mathbf{RZ}$	Right zero	$[\![xy = y]\!]$
$\mathbf{Sl}$	Semilattices	$[\![x^2 = x, xy = yx]\!]$

Table of Operators and Products

Name	Notation	Reference
Semidirect product	$*$	Example 2.4.9
Two-sided semidirect product	$**$	Example 2.4.14
Mal'cev product	ⓜ	Example 2.4.4
Generalized Mal'cev product	$(\mathbf{V}, \mathbf{W})$ ⓜ $(-)$	Example 2.4.5
Join	$\vee$	Example 2.4.7
Intersection	$\cap$ or $\wedge$	Example 2.4.17
Constants	$q_{\mathbf{V}}$	Example 2.4.2
Identity	$1_{\mathbf{PV}}$	Example 2.4.1
Old wreath product	$\circ$	Example 2.4.24
Local operator	$\mathbb{L}$	Example 2.4.27
Idempotents	$\mathbf{E}$	Example 2.4.28
Regular $\mathscr{D}$-classes	$\mathbf{D}$	Example 2.4.26
Power operator	$\mathbf{P}$	Example 2.4.25
Schützenberger product	$\Diamond$	Example 2.4.18
Regular elements	$\mathbf{R}$	Example 2.4.29
Semidirect closure	$()^{\omega}$	Example 2.4.31

Table of Implicit Operations

Symbol	Description	Defining Sequence
x^{ω}	Idempotent in $\overline{\langle x \rangle}$	$\lim x^{n!}$
$x^{\omega+1}$	Generator of the kernel of $\overline{\langle x \rangle}$	$\lim x^{n!+1}$
$x^{\omega-1}$	Inverse of $x^{\omega+1}$	$\lim x^{n!-1}$
$x^{p^{\omega}}$	p'-component of $x^{\omega+1}$	$\lim x^{p^{n!}}$
$x^{\pi^{\omega}}$	π'-component of $x^{\omega+1}$	$\lim x^{(p_1 \cdots p_n)^{n!}}$, $p_i \in \pi$
$x^{(\pi')^{\omega}}$	π-component of $x^{\omega+1}$	$\lim x^{(p_1 \cdots p_n)^{n!}}$, $p_i \in \pi'$

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