Kostant's Comments on Papers in Volume I

1. Holonomy and the Lie Algebra of Infinitesimal Motions of a Riemannian Manifold, *Trans. Amer. Math. Soc.*, **80** (1955), 528–542.

This paper develops a relation between the Lie algebra of Killing vector fields on a connected Riemannian manifold and the holonomy Lie algebra $\mathfrak{s}$ at a point $o \in M$.

If X is a vector field (v.f.), let A_X be the (1,1) tensor field on M defined so that $A_X Y = -\nabla_Y X$ and let $a_X = (A_X)_o$. If X is a Killing field, then $\exp t a_X$ tranforms, at time t, the effect of parallel transport of $T_o(M)$ along the trajectory of o, generated by X, to the flow of $T_o(M)$ along this trajectory. (The notation A_X seems to have been later retained by many authors). One result in [1] is

Theorem 1.1. *If M is compact, then $a_X \in \mathfrak{s}$.*

The proof of Theorem 1.1 depends on a novel use of Green's theorem (see MR0084825 (18,930a) of K. Yano). Theorem 1.1 is cited as Theorem 4.5, p. 247 in [Kobayashi–Nomizu, I].

Let $\mathfrak{h} \subset \operatorname{End} T_o(M)$ be the Lie algebra generated by all a_X for X a Killing field on M. The main result (Theorem 4.5) in [1] generalizing E. Cartan's theorem (for symmetric spaces) is

Theorem 1.2. *If M is a compact homogeneous space with an invariant Riemannian metric, then $\mathfrak{h} = \mathfrak{s}$.*

This result was cited as Theorem 4.7, p. 208 in [K-M, 2] and in K. Nomizu's Bourbaki Seminar No. 98. It also appears in [L] (see (13-1), p.110.) In general, in [L], Lichnérowicz refers to the group corresponding to $\mathfrak{h}$ as the *Groupe de Kostant*.

2 and 3. On the Conjugacy of Real Cartan Subalgebras I, *PNAS*, **41**, No. 11 (1955); and On the Conjugacy of Real Cartan Subalgebras II, (1955).

The problem of the classification of the conjugacy classes of Cartan subalgebras in a real simple Lie algebra was solved in these papers. Due to later work of Harish-Chandra the classification took on added significance since there is a "series" of group representations associated with each conjugacy class.

Both papers were submitted by NAS member Saunders Maclane for publication in PNAS. The second paper contains the tables of the conjugacy classes. However the PNAS said it could not print the second paper because of the complicated nature of the tables. A preprint of the second paper was widely circulated at the time. See e.g., the letter to the editor by A.J. Coleman in Volume 44 (1997), No. 4 of the *Notices* of the AMS.

At some later point M. Sugiwara independently published a list of the Cartan subalgebras. In a letter to me he verified that our lists were identical. The tables in Part II contain more than just a listing of the Cartan subalgebras. Any Cartan subalgebra decomposes into a direct sum of an elliptic part and a hyperbolic part. The tables list the centralizers of both of these parts.

The tables are printed in the present volume. We wish to thank Fokko du Cloux for recently checking the validity of the tables and making (mostly misprint) corrections.

4. On Invariant Skew-Tensors, *Proc. Nat. Acad. Sci. USA*, **42**, No. 3 (1956), 148–151.

This paper readily establishes that the holonomy group of n-sphere with an arbitrary Riemannian metric is necessarily the full rotation group $SO(n)$. I.M. Singer had previously proved this result if n is even. Let $o \in S^n$ and let G be the holonomy group at o. Any invariant of G operating on $\wedge T_o^*(S^n)$ defines (by parallel transport) a harmonic form on S^n. Since the Poincaré polynomial of S^n is $1+t^n$ it follows that G has no nontrivial invariants in $\wedge^k T_o^*(S^n)$ for $0 < k < n$. We show that this alone implies that $G = SO(n)$ (see Corollary 2.2 p.151) except for the possible special case where $n = 5$ and $G = SO(3)$. Although not shown in the paper this special case cannot occur since $G = SO(3)$ is not transitive on the unit sphere in $\mathbb{R}^5$. Indeed the non-transitivity implies, by M. Berger's theorem, that S^5, with the given metric must be locally symmetric with $SO(3)$ as holonomy group. But the simple connectivity of S^5 then implies $S^5 = SU(3)/SO(3)$ which is clearly false.

5. On Differential Geometry and Homogeneous Spaces I, *Proc. Nat. Acad. Sci. USA*, **42** (1956), 258–261.

Assume G is a compact connected Lie group and $H \subset G$ a closed subgroup. Let $M = G/H$ be a given a G-invariant Riemannian metric and let $o \in M$ be the "point" H. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, H$. Let $\mathfrak{p}$ be an Ad H invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ so that we may identify $\mathfrak{p} = T_o(M)$. Let $\mathfrak{s}$ be the holonomy algebra at o so that $\mathfrak{s} \subset \mathrm{End}\,\mathfrak{p}$. Assume that the Levi-Civita connection on M is a canonical affine connection of the first kind, in the sense of Nomizu, with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and let $P : \mathfrak{g} \to \mathfrak{p}$ be the projection with respect to this decomposition. Let $\mathfrak{k}$ be the Lie subalgebra of $\mathrm{End}\,\mathfrak{p}$ generated by $P\,\mathrm{ad}\,x\,P$ for all $x \in \mathfrak{g}$. Then the results of paper #5 imply

Theorem 5.1. *One has*

$$\mathfrak{k} = \mathfrak{s}.$$

Let $\Psi \subset \operatorname{Aut}\mathfrak{p}$ be the full holonomy group at o and let Ψ_e be the identity component of Ψ so that $\operatorname{Lie}\Psi_e = \mathfrak{k}$ by Theorem 5.1. In general there are not many results about full holonomy groups like Ψ. The determination of Ψ is the main result (Theorem 2) of paper #5. It is stated here as Theorem 5.2 below. Let $\sigma : H \to \operatorname{Aut}\mathfrak{p}$ be the isotropy representation.

Theorem 5.2. *One has* $\sigma(H) \subset \Psi$ *and*

$$\Psi = \sigma(H)\,\Psi_e.$$

6. On Differential Geometry and Homogeneous Spaces II, *Proc. Nat. Acad. Sci. USA*, **42** (1956), 354–357.

Let the notation be as in comments in paper #5 except that there is no assumption of compactness for G. What is referred to as a canonical affine connection of the first kind in the sense of Nomizu in paper #5 is referred to as naturally reductive in [B], See 7.84 Definition, p. 196 in [B]. As referenced as Theorem 7.85 (Kostant) in [B], p. 196 we have established, in paper #6, a necessary and sufficient condition that the affine connection on M corresponding to an Ad H-invariant bilinear form B on $\mathfrak{p}$ be naturally reductive. The condition is that B should suitably extend to the $\mathfrak{g}$ ideal $\mathfrak{g}' = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ so as to be $\mathfrak{g}'$ invariant. This theorem plays a significant role in [D-Z]. See p. 4 in [D-Z].

7. On Holonomy and Homogeneous Spaces, *Nagoya Math. Jour.,* **12** (1957), 31–54.

In this paper we deal with the question of holonomic irreducibility for compact Riemannian homgeneous spaces. Let G be a compact connected Lie group and let H be a closed subgoup. Let $M = G/H$ and assume M is given an arbitrary G-invariant Riemmannian structure. First of all we establish a statement similar to that of Theorem 5.2 above without the assumption that the corresponding affine connection is a canonical affine connection of the first kind in the sense of Nomizu. The main result in paper # 7 is

Theorem 7.1. *Assume* $\chi(M) \neq 0$ *where* $\chi(M)$ *is the Euler characteristic. Assume also that* G *operates faithfully on* M*. Then* M *is holonomically irreducible if and only if* Lie G *is simple.*

This result extended a theorem of Matsushima and Hano in [M-H] which established only the "if" implication. Contrary to a conjecture of Nomizu, Theorem 7.1 is not true if the condition $\chi(M) \neq 0$ is omitted. A counterexample to this conjecture is given in §3 of paper #7. See also p. 377 in [K-M,2]. The example gives the existence of an interesting case where Lie G is simple but M is holonomically reducible (e.g., $M = SO(7)/SU(3)$). The example also proves that Theorem, parts 1^o and 3^o, p. 1413, in Lichnérowicz's paper [L,2] is false.

It is also proved in paper #7 that if $\chi(M) \neq 0$ and Lie G is simple, then the affine connection uniquely determines the metric up to a scalar multiple.

8. A Theorem of Frobenius, a Theorem of Amitsur–Levitski and Cohomology Theory, *J. Math. and Mech.*, **7**, No. 2 (1958), 237–264.

I came upon the identity known as the Amitsur–Levitski theorem independently while at Berkeley. The following is a brief sketch of my connection with this result.

In the late 1950s at Berkeley I became interested in Lie algebra cohomology. My interest in this subject was very much stimulated by Koszul's beautiful thesis. I also became aware of work of Dynkin on formulas for the primitive cohomology classes for the unitary group. I was also becoming familiar with Weyl's book, involving Young diagrams, on the classical groups. Frobenius, having determined the characters of the symmetric group, published a paper in 1899 giving the characters of the alternating group. Playing a key role here are the self-dual Young diagrams of size k. The number of such diagrams is equal to the number of partitions of k whose parts are odd and distinct. As established in paper #8 it is no accident that this number is also the k-th Betti number of $U(n)$ for sufficiently large n.

Paper # 8 is mainly devoted to establishing the equivalence of the following 3 results: (I) the Amitsur–Levitski theorem, (II) the formula of Dynkin, and (III) a (special case) of a character formula of Frobenius for the alternating group. The failure of $SO(2n)$ to have a primitive cohomology class of degree $4n - 1$ gave rise to a new standard identity for $2n \times 2n$ skew-symmetric matrices. Having been encouraged by N. Jacobson, L. Rowen found a simple proof of this new identity in 1982.

Paper # 8 also introduced some new trace identities for $n \times n$ matrices. These were later rediscovered by C. Procesi. See the comments in the last paragraph on p. 581 in the Review in *Bull. Amer. Math. Soc.*,Volume 43, No.4, October 2006, by Edward Formanek of a book by Kanel–Belov and L. Rowen. Paper #8 was also the subject of Bourbaki Seminar No. 243 given by J. Dieudonné.

9. A Characterization of the Classical Groups, *Duke Math. Jour.*, **25**, No. 1 (1958), 107–124.

The classical groups are given in terms of what might be described as a defining representation. In this paper we deal with the question of characterizing the defining representation from the general point of view of Cartan–Weyl theory. The main theorems assert, in effect, that if $\mathfrak{g}$ is a reductive Lie algebra and π is an irreducible representation, then $\mathfrak{g}$ is classical and π is defining if and only if there exists $x \in \mathfrak{g}$ such that the rank of $\pi(x)$ is sufficiently small (actually 2 or 1 depending on the classical group). Applications are given to the sectional curvature of symmetric spaces. Another theorem algebraically chacterizes the curvature tensor of a symmetric space X in terms of the operator it defines on $\wedge^2 T_p(X)$ for $p \in X$. This result was used by J. Simons in his tour de force classification-independent proof (Ph.D. thesis) of M. Berger's theorem that the holonomy group of a non-symmetric irreducible Riemannian manifold is transitive on the unit tangent space at a point.

10. A Formula for the Multiplicity of a Weight, *Trans. Amer. Math. Soc.*, **93**, No. 1 (1959), 53–73.

Paper #10 gives a highly referenced solution to a problem in representation theory. Introducing what is now generally referred to as the Kostant Partition Function P, a formula for the multiplicity of a weight of a compact Lie group was written in this paper in terms of P. This initiated a large number of works by many authors to write branching laws in terms of partition functions (e.g., Blattner formula, Steinberg formula, by Kac for symmetrizable Kac–Moody groups and its extension by Shrawan Kumar and Olivier Mathieu to arbitrary Kac–Moody groups). A tensor product formula, in terms of a left ideal of the enveloping algebra of the nilradical of a Borel subalgebra, was also introduced in paper #10.

As mentioned by the authors, the proof of Theorem 2.1 in the fundamental paper (PRV), representations of complex semisimple Lie groups and Lie algebras by K. Parthasarathy, R. Rao and V. Varadarajan were based on this idea. Using weights of the form $\sigma\rho - \rho$ where ρ has its usual meaning and σ is the Weyl group, another function Q was introduced in paper #10. The function Q seems to have been ignored by most mathematicians. Nevertheless Hans Freudenthal in his review of this paper described the establishment of the equality $P = Q$ as being "really marvelous."

11. The Principal Three Dimensional Subgroup and the Betti Numbers of a Complex Simple Lie Group, *Amer. Jour. of Math.*, **81** (1959), 973–1032.

Paper #11 was my first really incisive penetration into the structure of a simple complex Lie algebra $\mathfrak{g}$. At the time # 11 was written, there was a growing literature on the topology of compact Lie groups (Chevalley, Borel, Bott, Koszul, etc). Much centered on the Poincaré polynomial $\prod_{i=1}^{\ell}(1+t^{2m_i+1})$ of a corresponding compact simple group. Here ℓ is the rank and the m_i are called the exponents. I was particularly interested in determining how the root structure encodes the exponents. One ingredient in establishing some of the main results was the existence of a special three-dimensional simple Lie subalgebra (TDS) $\mathfrak{a}$ of $\mathfrak{g}$, called the principal TDS, already introduced by Dynkin and de Siebenthal. Another ingredient was the special element of the Weyl group (unique up to conjugacy) which I called the Coxeter–Killing transformation but it is now referred to as a Coxeter element. Decomposing $\mathfrak{g}$ into irreducible components under the adjoint action of a principal TDS it is proved in this paper that

$$\mathfrak{g} = \oplus_{i=1}^{\ell}\mathfrak{g}_i \tag{11.1}$$

where $\dim \mathfrak{g}_i = 2m_i + 1$. This result gave a general proof of a case-by-case observation of Arnold Shapiro. An element a in the adjoint group G of $\mathfrak{g}$ which induces the Coxeter element on some Cartan subalgebra is called principal. It is proved that the set of principal elements is a single conjugacy class and that principal elements are regular and have order equal to the Coxeter number h. Furthermore it is proved that any regular element g in $\operatorname{Ad}\mathfrak{g}$ has order $\geq h$ and that equality occurs if and only if g is principal.

Dynkin classified the (finite) set $\mathcal{S}$ of conjugacy classes of TDS in $\mathfrak{g}$. In #11 we proved that the map

$$\mathcal{C} \to \mathcal{S} \tag{11.2}$$

of the set $\mathcal{C}$ of conjugacy classes of nilpotent elements into $\mathcal{S}$, defined by Jacobson–Morosov, is in fact a bijection — proving incidentally that $\mathcal{C}$ is finite. This result is often mistakenly attributed to Dynkin. In fact Dynkin did not deal with $\mathcal{C}$. The elements of the nilpotent conjugacy class $\mathcal{O}$ which corresponds, via (11.2), to a principal TDS are called principal nilpotent. It is proved in #11 that $\mathcal{O}$ is open and Zariski dense in the nilpotent cone of $\mathfrak{g}$. Moreover it is shown that $\mathcal{O}$ has codimension ℓ in $\mathfrak{g}$ and the centralizer $\mathfrak{g}^e$ of any $e \in \mathcal{O}$ is an abelian Lie subalgebra of dimension ℓ whose elements are nilpotent. Moreover if $\mathfrak{n}$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$ it is shown that $e \in \mathcal{O} \cap \mathfrak{n}$ if and only if the coefficients of e corresponding to simple root vectors in $\mathfrak{n}$ are all nonzero.

Let $\mathfrak{h}$ be a Cartan subalgebra and let $z \in \mathfrak{g}$ be given by

$$z = c_{-\psi}e_{-\psi} + \sum_{i=1}^{\ell} c_i\, e_{\alpha_i} \tag{11.3}$$

where the coefficients c_i and $c_{-\psi}$ are nonzero scalars, the e_{α_i} are root vectors corresponding to a choice of simple positive roots and $e_{-\psi}$ is a root vector corresponding to the negative of the highest root ψ. An element $x \in \mathfrak{g}$ is called cyclic if x is not nilpotent and $p(x) = 0$ for all G-invariant polynomial functions p on $\mathfrak{g}$ where $\deg p < h$. It is proved in #11 that any cyclic element is regular semisimple and, up to a scalar multiple, any two cyclic elements are conjugate. Furthermore it is proved that $x \in \mathfrak{g}$ is cyclic if and only if x is conjugate to an element z of the form (11.3). Let $\omega = e^{2\pi i/h}$. The significance of cyclic elements in #11 goes back to a result of A.J. Coleman which asserts that a Coxeter element has a unique eigenvector with eigenvalue ω and the eigenvector is regular. It is established in #11 that the eigenvector is cyclic and that any cyclic element x is such an eigenvector for a unique Coxeter element σ operating in the unique Cartan subalgebra $\mathfrak{h}_x$ which contains x. For $x = z$, given by (11.3), a σ-eigenbasis of $\mathfrak{h}_z$ is given (see Theorem 6.7 in No.11) using natural bases of $\mathfrak{g}^e$ and $\mathfrak{g}^{\tilde{e}}$, where e and $\tilde{e}$ are certain "dual" principal nilpotent elements associated to $\mathfrak{h}$.

Paper #11 contains a number of other results, some which have to do with the ℓ orbits of a Coxeter element on the set of roots and cross sections to these orbits.

It was presented by J-L. Koszul in the Bourbaki Seminar No. 191.

12. A Characterization of Invariant Affine Connections, *Nagoya Math. Jour.*, **16** (1960), 35–50.

It is a result of Nomizu that if $M = G/H$ is a reductive homogeneous space, then there exists a G-invariant affine connection on M. I was curious about the opposite direction. Given, say, a simply-connected manifold M with an affine connection A, what is a geometric condition on A so that M admits a transitive Lie group G whose action preserves A. W. Ambrose and I. Singer in a 1950 paper in the *Duke Math. Journ.* gave a necessary and sufficient condition in the Riemannian case. In this paper we give a generalization of the Ambrose–Singer theorem. Our main result introduces a new idea about a pair of affine connections. If A and B are affine connections on a manifold M then, as one knows, A and B differ by a tensor field S of type (1,2). We then say that A is rigid with respect to B if S is covariant constant with respect to B. Our main theorem then asserts that if M is a simply-connected manifold with an affine connection A, then M is a reductive homogeneous space with respect to a connected Lie group G whose action leaves A invariant if and only if there exists an affine connection B on M such that (1) B is invariant under parallelism (i.e., its curvature tensor R^B and torsion tensor T^B are B-covariant constant), (2) A is rigid with respect to B, and (3) M is complete with respect to B (i.e., every B-geodesic may be extended for arbitrary large values of its canonical parameter).

This result was cited in Volume II of S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, 1969. See p. 376.

13. Lie Algebra Cohomology and the Generalized Borel–Weil Theorem, *Ann. of Math.*, **74**, No. 2 (1961), 329–387.

Let G be a complex semisimple Lie group and $B \subset G$ a Borel subgroup. Let $X = G/B$ and let E be a G-homogeneous complex line bundle over X, and let SE be the sheaf of germs of holomorphic sections of E. Motivated by Hirzebruch's Riemann–Roch theorem, Raoul Bott determined the sheaf cohomology $H(X, SE)$ and explicitly described its structure as a G-module. This result, now referred to as the Bott–Borel–Weil theorem (BBW), is called the Generalized Borel Theorem in paper #13. Let $\mathfrak{b} = \text{Lie}\, B$ and let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. Let W be the Weyl group regarded as acting on $\mathfrak{b}/\mathfrak{n}$. Then Bott drew, as a consequence of BBW, the equality

$$\dim H^q(\mathfrak{n}, V_\lambda) = \text{Number of elements of } W \text{ of length } q. \tag{13.1}$$

Let V_λ be an irreducible G-module with highest weight λ. Bott's paper [B] was published in the *Ann. of Math.* **66** (1957), 203–248. In Remark 1 of [B], p. 247, Bott says that his proof of (13.1) is obviously unsatisfactory but that he knows of no direct argument to establish (13.1).

The main result of paper #13 is a direct algebraic proof of a generalization of (13.1). If $\mathfrak{a}$ is a complex finite-dimensional Lie algebra, $\mathfrak{a}'$ is its dual space, and V is a finite-dimensional complex $\mathfrak{a}$-module, then there is a coboundary operator of d_V on $C = \wedge\mathfrak{a}' \otimes V$ (making the pair (C, d_V) into a cochain complex) whose derived cohomology is the Lie algebra cohomology $H(\mathfrak{a}, V)$. The approach to the determination of $H(\mathfrak{a}, V)$ in this paper is via a "Laplacian" operator L_V. One introduces a Hilbert space structure on C, and then one defines a positive semidefinite operator L_V on C by puttting $L_V = d_V\, d_V^* + d_V^*\, d_V$ where d_V^* is the Hermitian adjoint of d_V. One then has a natural isomorphism

$$\text{Ker}\, L_V \cong H^*(\mathfrak{a}, V). \tag{13.2}$$

This of course transfers the problem to the determination of $\text{Ker}\, L_V$.

In paper #13 we introduce a special class of Lie subalgebras of $\mathfrak{g}$. A Lie subalgebra in this class is called a Lie summand. If $\mathfrak{a} \subset \mathfrak{g}$ is a Lie subalgebra, then $\mathfrak{a}$ is a Lie summand if its orthogonal complement $\mathfrak{a}^0$ with respect to the Killing form is again a Lie subalgebra.

For any Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and $\mathfrak{g}$ module V we write d_V as a difference of two simpler operators involving the full $\mathfrak{g}$-module structure on V and also involving a choice of a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ (see Proposition 3.13). The choice of $\mathfrak{k}$ also naturally defines a Hilbert space structure on C. But now if $\mathfrak{a}$ is a Lie summand, one has a neat formula for L_V involving a difference of two explicit positive semidefinite operators (see Theorem 4.4.) The nilradical of a parabolic subalgebra is a Lie summand. If $\mathfrak{u}$ is the nilradical of a standard parabolic subalgebra $\mathfrak{p}$ and we choose $\mathfrak{a} = \mathfrak{u}$ and $V = V_\lambda$, then Theorem 4.4 yields an explicit determination of the spectral resolution of L_V (see Theorem 5.7). This resolution involves the natural action of a Levi factor $\mathfrak{g}_1$ of $\mathfrak{p}$ on $H(\mathfrak{u}, V_\lambda)$. The choice of $\mathfrak{g}_1$ also defines a now familar decomposition $W = W_1 W^1$ of the Weyl group W. (This notational terminology, especially the superscripted W^1, introduced in this paper, is now widely used.) Restricting consideration to Ker L_V, one obtains Theorem 5.14 which describes $H(\mathfrak{u}, V_\lambda)$ as a $\mathfrak{g}_1$-module. For any $\sigma \in W^1$ let $\xi_\sigma = \sigma(\rho + \lambda) - \rho$, regarded as a $\mathfrak{g}_1$-dominant weight. If V_1 is a $\mathfrak{g}_1$-module, let $V_1^{\xi_\sigma}$ denote the primary $\mathfrak{g}_1$-submodule corresponding to ξ_σ as a highest weight. Also for $j \in \mathbb{Z}_+$ let $W^1(j)$ be the set of $\sigma \in W^1$ having length j. Then Theorem 5.14 asserts that $H(\mathfrak{u}, V_\lambda)^{\xi_\sigma}$ is nonzero and irreducible for any $\sigma \in W^1$ and in fact

$$\sigma \mapsto H(\mathfrak{u}, V_\lambda)^{\xi_\sigma}$$

is a bijection of W^1 onto the set of all $\mathfrak{g}_1$-irreducible components of $H(\mathfrak{u}, V_\lambda)$. In particular $H(\mathfrak{u}, V_\lambda)$ is multiplicity-free. Degree-wise for $j \in \mathbb{Z}_+$ one has the direct sum

$$H^j(\mathfrak{u}, V_\lambda) = \sum_{\sigma \in W^1(j)} H(\mathfrak{u}, V_\lambda)^{\xi_\sigma}. \qquad (13.3)$$

Finally the harmonic (Ker L_V) representative of the highest weight vector in $H(\mathfrak{u}, V_\lambda)^{\xi_\sigma}$ is explicitly determined and shown to be a completely decomposable element of C.

The results above generalize Bott's formula (13.1) and the latter is derived, by Bott, as a consequence of BBW. However, as pointed out by Wilfried Schmid in his article, The Mathematical Legacy of the Paper "Homogeneous Vector Bundles," p. 40 in Volume I of Bott's , *Collected Papers*, R. MacPherson Editor, Birkhäuser, 1994, [BC], formula (13.1) is equivalent to BBW. See p. 42 in [BC]. In fact Schmid goes on to point out on p. 42 that my computation of L_V "carries over directly to a computation of the Laplace operator on the Dolbeault complex of a homogeneous line bundle $E \longrightarrow X$."

Much of the remainder of paper #13 is devoted to applications of Theorem 5.14. One of the applications is a generalization of a theorem of Ehresmann. Consider the case where $\mathfrak{u}$ is commutative (i.e., $Y = G/P$ is a Hermitian symmetric space where $P \subset G$ corresponds to $\mathfrak{p}$). Here, as in all cases, W^1 parameterizes the Schubert classes in Y. Take $\lambda = 0$ so that $C = \wedge\mathfrak{u}'$ and $\xi_\sigma = \sigma\,\rho - \rho$ for $\sigma \in W^1$. We may identify $\mathfrak{u}'$ with the nilradical of the parabolic subalgebra $\mathfrak{p}'$ "opposite" to $\mathfrak{p}$ noting that $\mathfrak{g}_1$ is also a Levi factor of $\mathfrak{p}'$. In this case $d_V = 0$ so that $H(\mathfrak{u}, V_\lambda) = \wedge\mathfrak{u}'$. Theorem 5.14, among other things, then implies that $\wedge\mathfrak{u}'$ is a multiplicity-free $\mathfrak{g}_1$-module and

$$\sigma \mapsto (\wedge\mathfrak{u}')^{\xi_\sigma}$$

is a bijection of W^1 onto the set of all $\mathfrak{g}_1$-irreducible components of $\wedge\mathfrak{u}$. When $\mathfrak{g}$ is classical this is then a result of Ehresmann.

Another application is an extension of Weyl's character formula to disconnected complex reductive Lie groups.

Howard Garland extended the Laplacian idea of paper #13 to the affine Kac–Moody case and consequently gave a beautiful proof of the Kac–MacDonald denominator formula. More than that, the positive semidefinite nature of the Laplacian established remarkable inequalites. Garland's result is a special case of a much more general result by Shrawan Kumar. In particular Kumar obtained a far-reaching infinite-dimensional analogue of Theorem 5.7 in paper #13.

14. (with G. Hochschild and A. Rosenberg), Differential Forms on Regular Affine Algebras, *Trans. Amer. Math. Soc.*, **102**, No. 3 (1962), 383–408.

Assume R is a commutative ring over a field K. Let $S = R \otimes_K R$. Then the Hochschild homology $H_*(R)$ of R is $\mathrm{Tor}^S(R, R)$ (using terminology of Cartan–Eilenberg). One then knows that $H_1(R)$ is the R-module of formal differentials of R (the span of $f\,dg$, $f, g \in R$, with $dc = 0$ if $c \in K$). Also $H_*(R)$ has a skew-commutative R-algebra structure. Under Noetherian and regularity assumptions, it is shown in Theorem 3.1 that $H_*(R)$ is projective, finitely generated over R and

$$H_*(R) \text{ is isomorphic to the exterior algebra over } H_1(R). \tag{14.1}$$

The cohomology $H^*(R)$ of R, by definition, is $\mathrm{Ext}_S^*(R, R)$. Also $H^1(R)$ is the R-module of K-derivations of R. If E is the exterior algebra of $H^1(R)$ and $\Omega(R) = \mathrm{Hom}_R(E, R)$, then

$$\Omega(R) \text{ is the algebra of differential forms defined by } R.$$

On the other hand, one has a natural isomorphism

$$h : H^1(R) \to \mathrm{Hom}_R(H_1(R), R). \tag{14.2}$$

Henceforth assume that the conditions imposed on R in Theorem 3.1 are satisfied. But then, by (14.1), this extends to an isomorphism

$$E \to \mathrm{Hom}_R(H_*(R), R). \tag{14.3}$$

Dualizing (14.3) one establishes the isomorphism

$$H_*(R) \to \Omega(R) \qquad (14.4).$$

See Theorem 5.20.

The isomorphism (14.4) has generally been referred to as the HKR theorem. Subsequently a number of mathematicians have established (14.4) with different assumptions about the ring R. Perhaps most notable is the result of A. Connes proving (14.4) for the case where R is the ring of smooth functions on a manifold. In any event (14.4) has become the focus of active research.

The ring $\Omega(R)$ (or $H_*(R)$) is itself a differential complex leading to the de Rham cohomology $H_{dR}(R)$. Section 7 in this paper is devoted to obtaining the de Rham cohomology as an Ext functor on the algebra of differential operators of R. See Corollory 7.1. The motivation for this comes from Lie algebra cohomology. However this result seems to have attracted less attention than 14.4. One reason no doubt for this is the neat relationship found by A. Connes between de Rham cohomology and cyclic (co)homology.

15. (with G. Hochschild), Differential Forms and Lie Algebra Cohomology for Algebraic Linear Groups, *Illinois Jour. Math.*, **6** (1962), 264–281.

Paper #15 extends earlier results of Hochschild on algebraic groups to the homogeneous case. Let R be the affine ring of an affine algebraic group over an algebraically closed field F. Assume that K is a reductive subgroup of G. Under right translations let R^K be the subring of K-invariant functions. It is proved in paper #15 that G/K is an affine variety and R^K is its affine ring. See Theorem 5.1.

If F has characteristic zero and G itself is reductive, then it is proved in this paper that the deRham cohomology of R^K is the same as the relative Lie algebra cohomology. See Theorem 3.2. This says in effect that if, say, $F = \mathbb{C}$, then the algebraic holomorphic differentials on the affine variety G/K "sees" the topological cohomology of G/K. One keeps in mind here that this is true in spite of the fact that there is no Poincaré lemma for algebraic holomorphic differentials. Later, a general theorem of Grothendieck established this fact for any nonsingular affine variety. Grothendieck cited this paper as an inspiring influence for his very general result.

16. Lie Group Representations on Polynomial Rings, *Bull. Amer. Math. Soc.*, **69**, No. 1 (1963), 518–526.

Paper #16 is a *Bull. Amer. Math. Soc.* announcement of some of the results in paper #17.

17. *Lie Group Representations on Polynomial Rings*, Amer. J. Math., **85** (1963), 327–404.

Paper #17 establishes a rather large number of results. We will take this opportunity to revisit some of the main results and present them from a more modern perspective.

Let $\mathfrak{g}$ be a complex reductive Lie algebra and let S be the ring of polynomial functions on $\mathfrak{g}$. Let G be the adjoint group of $\mathfrak{g}$. Using methods of algebraic geometry, this paper is a study of the orbit structure of G on $\mathfrak{g}$ and, contragrediently, the module structure of G on S. Let $n = \dim \mathfrak{g}$, $\ell = \operatorname{rank} \mathfrak{g}$ and let $J = S^G$ be the ring of polynomial G-invariants in S. One knows (Chevalley) that J is a polynomial ring $\mathbb{C}[u_1, \ldots, u_\ell]$ where the u_i are homogeneous algebraically independent polynomials, and if we write $\deg u_i = m_i + 1$, then the m_i are the exponents of $\mathfrak{g}$.

If $x \in \mathfrak{g}$, let $O_x = G \cdot x$ so that $O_x = G/G^x$ where G^x is the isotropy group at x. Let $\mathfrak{g}^x = \operatorname{Lie} G^x$. Then $\dim O_x \le n - \ell$ and in current terminology (which we retain here) x is called regular if $\dim O_x = n - \ell$. (At the time paper #17 was written the word regular was reserved for semisimple regular elements. See p. 356. The latter restriction is thus abandoned now.) Let $\mathfrak{r}$ be the Zariski open set of all regular elements in $\mathfrak{g}$. If $x \in \mathfrak{r}$, then one easily has that $\mathfrak{g}^x$ is a commutative Lie subalgebra of dimension ℓ. Let $x \in \mathfrak{g}$. Then in paper #17 it is proved that

$$x \in \mathfrak{r} \iff (du_1)_x, \ldots, (du_\ell)_x \text{ are linearly independent.}$$

(See Theorem 9). Also let $x = y + z$ be the Jordan decomposition of x where y is semisimple and z is nilpotent so that in particular $\mathfrak{g}^y$ is reductive and $z \in \mathfrak{g}^y$. Then it is proved in paper #17 that

$$x \in \mathfrak{r} \iff z \text{ is principal nilpotent in } g^y.$$

See Proposition 13.

Now for any $\xi = (\xi_1, \ldots, \xi_\ell) \in \mathbb{C}^\ell$ let

$$P(\xi) = \{x \in \mathfrak{g} \mid u_i(x) = \xi_i,\ i = 1, \ldots, \ell\}.$$

In particular note that $P(\xi)$ is a closed G-stable subvariety of $\mathfrak{g}$, and if $P \subset \mathfrak{g}$ is the nilcone, note also that $P = P(0)$. Let $O^{\mathfrak{r}}(\xi) = P(\xi) \cap \mathfrak{r}$ and if $\mathfrak{s}$ is the set of semisimple elements in $\mathfrak{g}$, let $O^{\mathfrak{s}}(\xi) = P(\xi) \cap \mathfrak{s}$. In paper #17 we prove

Theorem. *There are a finite number of G-orbits in $P(\xi)$ for any $\xi \in \mathbb{C}^\ell$. In particular the nilcone P has only a finite number of G-orbits. Moreover $O^{\mathfrak{r}}(\xi)$ is the unique orbit of maximal dimension in $P(\xi)$. Furthermore $O^{\mathfrak{r}}(\xi)$ is Zariski open in $P(\xi)$ and*

$$P(\xi) = \overline{O^{\mathfrak{r}}(\xi)},$$

so that $P(\xi)$ is an irreducible variety of dimension $n - \ell$. Also $O^{\mathfrak{s}}(\xi)$ is the unique closed orbit in $P(\xi)$ and it is the unique orbit of minimal dimension in $P(\xi)$. For any $x \in \mathfrak{g}$ one has $x \in P(\xi)$ if and only if the semisimple component of x (relative to its Jordan decomposition) is in $O^{\mathfrak{s}}(\xi)$.

See Theorem 3 and (3.8.9) in paper #17.

Let (h, e, f) be a standard basis of a principal TDS. Another result in paper #17 is a (now well known) sectioning of the G-action on $\mathfrak{r}$. Let $\mathfrak{a}$ be an ad h-stable subspace complementary to $[f, \mathfrak{g}]$. Then $\dim \mathfrak{a} = \ell$. An example of a choice of $\mathfrak{a}$ is $\mathfrak{g}^e$. Let $\mathfrak{v}$ be the ℓ-dimensional affine plane defined by putting $\mathfrak{v} = f + \mathfrak{a}$. Also let $u : \mathfrak{g} \to \mathbb{C}^\ell$ be the morphism defined so that $u(x) = (u_1(x), \ldots, u_\ell(x))$. Then the following result is established in this paper.

Theorem. *One has $\mathfrak{v} \subset \mathfrak{r}$. Furthermore the restriction*

$$u : \mathfrak{v} \to \mathbb{C}^\ell$$

is an algebraic isomorphism. Moreover, $\mathfrak{v}$ meets every regular G-orbit at exactly one point. In fact if $\xi \in \mathbb{C}^\ell$, then

$$u(O^{\mathfrak{r}}(\xi) \cap \mathfrak{v}) = \xi.$$

See Theorem 7 and Remark 19′. For any $\xi \in \mathbb{C}^\ell$ let J^ξ be the ideal of codimension 1 in J generated by $u_i - \xi_i$, $i = 1, \ldots, \ell$. One notes that J^0 is the augmentation ideal J^+ in J. Also

$$J^\xi S = (u_1 - \xi_1, \ldots, u_\ell - \xi_\ell).$$

We prove (see Theorem 10)

Theorem. *For any $\xi \in \mathbb{C}^\ell$, the ideal in $J^\xi S$ defines the $n - \ell$-dimensional irreducible variety of $P(\xi)$ so that (a) $P(\xi)$ is a complete intersection and (b) $J^\xi S$ is a prime ideal. Furthermore $O^{\mathfrak{r}}(\xi)$ is the set of simple points of $P(\xi)$ and the subvariety of nonsimple points of $P(\xi)$ has codimension of at least 2 in $P(\xi)$.*

Let $\mathfrak{h}$ and $\mathfrak{b}$, where $\mathfrak{h} \subset \mathfrak{b}$, be a Cartan subalgebra and a Borel subalgebra. Let $D \subset \mathfrak{h}^*$ be the set of all dominant weights in the root lattice, and for any $\lambda \in D$ let $\nu^\lambda : G \to \operatorname{Aut} V^\lambda$ be an irreducible representation of G with highest weight λ. For $\lambda \in D$ let ℓ_λ be the dimension of the zero weight space in V^λ. Let Diff_+^G be the space of all G-invariant constant coefficient differental operators on $\mathfrak{g}$ with zero constant term and let

$$H = \{p \in S \mid \partial p = 0, \ \forall\ \partial \in \mathrm{Diff}_+^G\}.$$

Then H is G-stable and the polynomials in H are called harmonic. One has (see Theorem 11 and (5.3.5))

Theorem. *S is free as a J module and in fact multiplication defines a G-isomorphism*

$$J \otimes H \to S.$$

Moreover, H is the span of all powers $(e')^k$ where $e' \in \mathfrak{g}^*$ corresponds, under the Killing form isomorphism, to a nilpotent element of $\mathfrak{g}$. Furthermore H has finite multiplicities as a G-module and in fact the multiplicity of ν^λ in H is ℓ_λ.

In addition, H is complementary to the prime ideal $J^\xi S$ in S, for any $\xi \in \mathbb{C}^\ell$, so that H and S have the same restrictions to $P(\xi)$. Also the restriction of H to $P'(\xi)$ is faithful. In particular the affine rings of $P(\xi)$, over all $\xi \in \mathbb{C}^\ell$, even though not equivalent as rings, are all equivalent as G-modules.

For any $\xi \in \mathbb{C}^\ell$, let $R(O^\mathfrak{r}(\xi))$ be the ring of regular functions on the quasi-affine variety $O^\mathfrak{r}(\xi)$. Let $x \in O^\mathfrak{r}(\xi)$ so that $O_x = O^\mathfrak{r}(\xi)$ and hence as a G-module

$$\text{mult}\, \nu^\lambda \text{ in } R(O^\mathfrak{r}(\xi)) = \dim (V^\lambda)^{G^x}. \tag{17.1}$$

Using a criterion for normality due to A. Seidenberg we establish one of the main theorems in paper # 17 (see Theorem 16).

Theorem. *$P(\xi)$ is a normal variety for any $\xi \in \mathbb{C}^\ell$. In particular the nilcone is a normal variety. Furthermore $O^\mathfrak{r}(\xi)$ "sees" its closure $P(\xi)$ in the following sense: Any everywhere defined function on $O^\mathfrak{r}(\xi)$ extends to a regular function on $P(\xi)$ so that $R(O^\mathfrak{r}(\xi))$ identifies with the affine ring of $P(\xi)$ (i.e., $R(O^\mathfrak{r}(\xi))$ is an affine ring and $P(\xi)$ is its variety of maximal ideals). In particular if $x \in \mathfrak{r}$ and $\lambda \in D$, then*

$$\dim (V^\lambda)^{G^x} = \ell_\lambda. \tag{17.2}$$

Also the restriction of functions defines a G-isomorphism

$$H \to R(O^\mathfrak{r}(\xi)). \tag{17.3}$$

For $\lambda \in D$ let V_λ be the dual space to V^λ and let ν_λ be the representation of G on V_λ which is contragredient to ν^λ.

As above, let (h, e, f) be a standard basis of a principal TDS. Since the zero weight spaces of V_λ and V^λ are clearly the same dimension, and all other weights are in the root lattice, it follows from (17.2) that $V_\lambda^{G^e}$ is ℓ_λ-dimensional and admits a $\nu_\lambda(h/2)$ eigenbasis with nonnegative integral eigenvalues $m_i(\lambda)$, $i = 1, \ldots, \ell_\lambda$. The $m_i(\lambda)$ are called *generalized exponents* (the usual exponents are the special case where $\mathfrak{g}$ is simple and λ is the highest root). Let $o(\lambda) = \frac{1}{2}\lambda(h)$. Taking the generalized exponents to be in nondecreasing order, one readily has

$$m_i(\lambda) \le o(\lambda) \text{ and equality occurs } \iff \quad i = \ell_\lambda. \tag{17.4}$$

The generalized exponents "see" the degrees in which ν^λ occurs harmonically (Theorem 17).

Theorem. *Let $H = \oplus_{\lambda \in D} H^\lambda$ be the primary decomposition of H (with H^λ corresponding to ν^λ). Further decomposing we can write, as a direct sum,*

$$H^\lambda = \oplus_{i=1}^{\ell_\lambda} H_i^\lambda \tag{17.5}$$

where the H_i^λ are G-irreducible homogeneous components of H^λ. Let $n_i(\lambda)$ be the degree of H_i^λ and choose the ordering in (17.5) to be nondecreasing. Then, for $i = 1, \ldots, \ell_\lambda$,

$$n_i(\lambda) = m_i(\lambda). \tag{17.6}$$

In particular $o(\lambda)$ is the highest degree in which ν^λ occurs harmonically and this maximal occurrence has multiplicity 1.

The next result (see Theorem 18) generalizes a theorem of A.J. Coleman which determines the eigenvalues of the Coxeter element in terms of the ordinary exponents. Theorem 18 in paper #17 also gives an expression for the corresponding eigenbasis.

Theorem. *Let $x \in \mathfrak{r}$ and assume $a \cdot x = c\,x$ for some $a \in G$ and $c \in \mathbb{C}^\times$. Let $\lambda \in D$. Then $\nu_\lambda(a)$ stabilizes and is diagonalizable in $V_\lambda^{G^x}$. Furthermore the eigenvalues of $\nu_\lambda(a)|V_\lambda^{G^x}$ are $c^{m_i(\lambda)}$, $i = 1, \ldots, \ell_\lambda$.*

In analogy with the "separation of variables" theorem for $S(\mathfrak{g})$, the final section (18) of paper #17 gives a "separation of variables" result for the enveloping algebra U of $\mathfrak{g}$. The result (see Theorem 21) asserts that multiplication defines a G-isomorphism

$$Z \otimes E \to U \tag{17.7}$$

where $Z = \text{Cent}\, U$ and E is the span of all powers e^k where $e \in \mathfrak{g}$ is nilpotent.

Paper #17 was presented in a Bourbaki seminar given by Roger Godement.

18. Lie Algebra Cohomology and Generalized Schubert Cells, *Ann. of Math.*, **77** (1963), No. 1, 72–144.

Paper #18 is Part 2 of a paper where Part 1 is paper #13. We will freely use here some of the notation in paper #13. The problem which is dealt with in paper #18 is an understanding and proof of the "strange equality" (terminology of Raoul Bott)

$$H^q(\mathfrak{n}, M) = H^{2q}(X; \mathbb{C}) \tag{18.1}$$

empirically observed as (15.3) in Bott's paper, "Homogeneous Vector Bundles". In (18.1) $\mathfrak{n}$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$, M is an irreducible G-module and $X = G/B$ is the G-flag manifold. Since the left side of (18.1) has a basis parameterized by the Weyl group of G for any W, it suffices to consider (18.1) for the case where M is the trivial module. Paper #18 then generalizes the statement of (18.1). Going back to the notation of paper #18, first of all, W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$. Next $\mathfrak{u}$ is a standard parabolic subalgebra. That is, $\mathfrak{u}$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ and $U \subset G$ is the parabolic subgroup corresponding to $\mathfrak{u}$. Now X is the generalized flag manifold given by putting $X = G/U$. Also $\mathfrak{g}_1$ is the Levi factor of $\mathfrak{u}$ containing $\mathfrak{h}$ and $G_1 \subset G$ is the reductive subgroup of G corresponding to $\mathfrak{g}_1$. In addition $\mathfrak{n}$ is now the nilradical of $\mathfrak{u}$ so that $\mathfrak{u} = \mathfrak{g}_1 \oplus \mathfrak{n}$ is the Levi decomposition of $\mathfrak{u}$. Furthermore, $\mathfrak{n}^*$ is the unique ad g_1-stable complement of $\mathfrak{u}$ in $\mathfrak{g}$ so that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{n} \oplus \mathfrak{n}^* \tag{18.2}$$

is a triangular decomposition of $\mathfrak{g}$. As in paper #13, the Weyl group $W = W^1\, W_1$ where W_1 is the Weyl group for the pair $(\mathfrak{g}_1, \mathfrak{h})$ and if $\sigma \in W^1$, then σ is the element of minimal "length" ($\ell(\sigma)$) in the W_1 left coset $\sigma\, W_1$.

The use of the notation $\mathfrak{n}^*$ in (18.2) is suggestive in that $\mathfrak{n}^*$ identifies with the dual of $\mathfrak{n}$ using the Killing form and the adjoint action of $\mathfrak{g}_1$ on $\mathfrak{n}^*$ is contragredient to its action on $\mathfrak{n}$. On the other hand, recalling the notation of (13.3) above, we have proved in paper #13, using Lie algebra homology instead of cohomology (see Corollary 8.1 in paper # 13) that, as a $\mathfrak{g}_1$ module, $H_*(\mathfrak{n})$ is multiplicity-free and, for any $q \in \mathbb{Z}_+$, $H_q(\mathfrak{n})$ is a direct sum of the primary (and irreducible) components, $H_*(\mathfrak{n})^{\rho-\sigma\rho}$ for $\sigma \in W^1(q)$. It then follows from Schur's lemma that

$$\dim(H_q(\mathfrak{n}) \otimes H_q(\mathfrak{n}_*))^{\mathfrak{g}_1} = w^1(q) \tag{18.3}$$

where $w^1(q)$ is the cardinality of $W^1(q)$. In fact Schur's lemma implies that $(H_q(\mathfrak{n}) \otimes H_q(\mathfrak{n}_*))^{G_1}$ has, up to scalar multiplication, a natural basis $\boldsymbol{h}^\sigma$, $\sigma \in W^1(j)$. On the other hand, recalling the Kähler structure on X, one has that $H^{p,q}(X) = 0$ if $p \neq q$, and using the Schubert classes of dimension p, one also has that $\dim H^{q,q}(X, \mathbb{C}) = w^1(q)$ and that $H^{q,q}(X, \mathbb{C})$ has a natural Schubert class basis $\boldsymbol{x}^\sigma$, $\sigma \in W^1(q)$. So as a far-reaching generalization of the observation (18.1) one should establish a natural isomorphism

$$\psi : H^{q,q}(X, \mathbb{C}) \to (H_q(\mathfrak{n}) \otimes H_q(\mathfrak{n}_*))^{G_1} \tag{18.4}$$

where, for any $\sigma \in W^1(q)$, up to a scalar factor,

$$\psi(\boldsymbol{x}^\sigma) = \boldsymbol{h}^\sigma. \tag{18.5}$$

Paper #18 accomplishes (18.4) and (18.5) in two steps. To deal with the first step assume C is a finite-dimensional $\mathbb{Z}$-graded $\mathbb{C}$ vector space with a coboundary (raises degrees by 1) operator d and a boundary (lowers degrees by 1) operator ∂. One then has cohomology $H^*(C, d)$ and homology $H_*(C, \partial)$. We say that d and ∂ are disjoint in case for any $u \in C$ one has,

$$d\,\partial\,u = 0 \text{ implies } \partial\,u = 0$$
$$\partial\,d\,u = 0 \text{ implies } d\,u = 0.$$

In such a case one readily shows that if $S = d\,\partial + \partial\,d$, then (a) Ker S is a space of d-cocycles and the corresponding map

$$\psi_{d,S} : \operatorname{Ker} S \to H^*(C, d), \text{ is a degree preserving isomorphism.} \tag{18.6}$$

Next (b) Ker S is a space of ∂-cycles and the corresponding map

$$\psi_{S,\partial} : \operatorname{Ker} S \to H_*(C, \partial), \text{ is a degree preserving isomorphism.} \tag{18.7}$$

In particular (18.6) and (18.7) then define a degree preserving isomorphism

$$\psi_{\partial,d} : H^*(C, d) \to H_*(C, \partial). \tag{18.8}$$

We apply (18.8) to the following case: let $\mathfrak{r}$ be the orthocomplement of $\mathfrak{g}_1$ in $\mathfrak{g}$ and let

$$C = (\wedge\mathfrak{r})^{G_1}. \tag{18.9}$$

Then, recalling the definition of relative Lie algebra cohomology, C is a cochain complex with coboundary operator d where $H^*(C, d) = H^*(\mathfrak{g}, \mathfrak{g}_1)$. In fact if K is a compact form of G, chosen so that if $K_1 \subset G_1$ where $K_1 = K \cap U$, then $X = K/K_1$ and C identifies with the space of K-invariant differential forms on X and d is induced by the exterior derivative of differential forms. Thus (Cartan–Eilenberg–de Rham theory),

$$H^*(C, d) = H^*(X, \mathbb{C}). \tag{18.10}$$

Now $\mathfrak{r} = \mathfrak{n} \oplus \mathfrak{n}^*$. Introduce a Lie algebra structure on $\mathfrak{r}$ by retaining the given Lie algebra structures on $\mathfrak{n}$ and $\mathfrak{n}^*$, but now putting $[\mathfrak{n}, \mathfrak{n}^*] = 0$. Then $\wedge\mathfrak{r}$ is a chain complex for the Lie algebra homology

$$H_*(\mathfrak{r}) = H_*(\mathfrak{n}) \otimes H_*(\mathfrak{n}_*). \tag{18.11}$$

But G_1 is a reductive group of automorphisms of the Lie algebra $\mathfrak{r}$ and hence C is stable under the boundary operator. Let ∂ be the restriction of this operator to C so that, by the reductivity of G_1, one has

$$H_*(C, \partial) = (H_*(\mathfrak{n}) \otimes H_*(\mathfrak{n}_*))^{G_1}.$$

The main problem of paper #18 is to establish the disjointness of d and ∂. This is done in Theorem 4.5 and Corollary 5.3.2 in paper #18 establishing the isomorphism (18.6). In fact $\wedge\mathfrak{r}$ is bigraded and

$$C = \oplus_{q \in \mathbb{Z}_+} C^{q,q}. \tag{18.12}$$

By Theorem 4.5 one, in fact, has the isomorphism (18.4) above. Note also that S is now a new kind of Laplacian operating on K-invariant differential forms on X and $\operatorname{Ker} S$ is the space of a new kind of harmonic forms on X. Let $q \in \mathbb{Z}_+$. Then if $s \in (\operatorname{Ker} S)^{q,q}$ (see Proposition 3.3.2) s is a K-invariant form on X of type (q, q) and, by Theorem 4.5, $\boldsymbol{s} \in H^{q,q}(X, \mathbb{C})$ where $\boldsymbol{s} = \psi_{d,S}(s)$ (see (18.6)). Let $\sigma \in W^1(q)$ so that $\boldsymbol{h}^\sigma$ is a "Schur's lemma" basal element of $(H_q(\mathfrak{n}) \otimes H_q(\mathfrak{n}^*))^{G_1}$. Let $s^\sigma \in (\operatorname{Ker} S)^{q,q}$ be defined so that

$$\psi_{\partial,d}(\boldsymbol{s}^\sigma) = \boldsymbol{h}^\sigma \tag{18.13}$$

where $\boldsymbol{s}^\sigma = \psi_{d,S}(s^\sigma)$. But now (second step) (18.5) is established as soon as one proves that

$$\boldsymbol{s}^\sigma = \boldsymbol{x}^\sigma \tag{18.14}$$

up to a scalar multiple. But this fact is proved in Theorem 6.15 of paper #18. The Schubert cells V_τ of complex dimension q are parameterized by $\tau \in W^1(q)$. The proof of Theorem 6.15 is established by showing that the integral of s^σ over V_σ is a positive number and showing that, if $\sigma \neq \tau$, then

$$\int_{V_\tau} s^\sigma = 0. \tag{18.15}$$

On the other hand (18.15) follows from a striking property of the differential form s^σ, namely that

$$s^\sigma | V_\tau \quad \text{is identically zero.} \tag{18.16}$$

(See Corollary 6.15) and this property follows from the fact $\partial s^\sigma = 0$. See Remark 6.15.

The d, ∂-harmonic theory developed in this paper has been extended by Shrawan Kumar to any symmetrizable Kac–Moody setting. Thus, he has extended most of the results in this paper to any symmetrizable Kac–Moody flag varieties. He has used these results to show that any such flag variety is a formal space in the sense of rational homotopy theory.

19. Eigenvalues of a Laplacian and Commutative Lie Subalgebras, *Topology*, **13** (1965), 147–159.

Paper # 19 is a 12-page paper written in 1965 which seems to have spawned a great deal of research activity focused on abelian ideals of a Borel subalgebra of a complex semsisimple Lie algebra, paricularly in the first decade of the twenty-first century. Some of the researchers (besides myself) are D. Peterson, R. Suter, D. Panyushev, G. Rohrle, P. Cellini, P. Papi and N. Kwon. There was a conference in Italy in October 2007 devoted to this subject. Let K be a compact semisimple Lie group and let $\mathfrak{k} = \text{Lie}\, K$. Then the negative of the Killing form defines a two-sided K-invariant Riemannian structure on K. Let Lap be the Laplacian with respect to this Riemannian structure and let $\mathfrak{g}$ be the complexification of $\mathfrak{k}$. The exterior algebra $\wedge\mathfrak{g}$ identifies with the space of all complex valued left K-invariant differential forms on K. Furthermore $\wedge\mathfrak{g}$ inherits a natural Hilbert space structure $\{u, v\}$. Moreover if d is the operator on $\wedge\mathfrak{g}$ induced by exterior differentiation of forms and ∂ is the Hermitian adjoint of d, then $\wedge\mathfrak{g}$ is stable under Lap and one has

$$\text{Lap}| \wedge \mathfrak{g} = d\,\partial + \partial\, d.$$

Whereas Hodge theory focuses on $\text{Ker}\,\text{Lap}\,| \wedge \mathfrak{g}$, paper # 19 was motivated instead by the question of determining, degree-wise, the maximal eigenvalue of $\text{Lap}| \wedge \mathfrak{g}$. This quickly becomes a question in representation theory since if Cas is the Casimir operator on $\wedge\mathfrak{g}$ corresponding to the adjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}$, then one readily has that

$$\text{Cas} = 2\,\text{Lap}| \wedge\ g.$$

Let $k \in \mathbb{Z}_+$. If $\mathfrak{u} \subset \mathfrak{g}$ is a k-dimensional subspace, then $\wedge^k\mathfrak{u}$ is a 1-dimensional subspace of $\wedge^k\mathfrak{g}$. Let p be the maximal dimension of an abelian Lie subalgebra of $\mathfrak{g}$. The value of p has been determined by Malcev for all semi-simple $\mathfrak{g}$ (for example if $\mathfrak{g}$ is of type E_8, then $p = 36$.). We define the $\mathfrak{g}$-submodule $A^k \subset \wedge^k \mathfrak{g}$ as follows: If $k > p$, then put $A^k = 0$. Otherwise let A^k be the span of all the 1-dimensional subspaces $\wedge^k\mathfrak{a}$ where $\mathfrak{a}$ is any k-dimensional abelian subalgebra of $\mathfrak{g}$. Let m_k be the maximal eigenvalue of $\mathrm{Cas}|\wedge^k\mathfrak{g}$. The following theorem is proved in paper #19.

Theorem. *For any k one has*

$$m_k \leq k. \tag{19.1}$$

Furthermore one has equality in (19.1) if and only if $k \leq p$. Moreover in such a case the corresponding eigenspace for $\mathrm{Cas}\,|\wedge^k\mathfrak{g}$ *is A^k. Finally if $0 \neq u \in \wedge\mathfrak{g}$ is of the form $u = x_1 \wedge \cdots \wedge x_k$ for $x_i \in \mathfrak{g}$, then $u \in A^k$ if and only if the x_i mutually commute.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta \subset \mathfrak{h}^*$ be the set of roots for $(\mathfrak{g}, \mathfrak{h})$. For all roots φ let corresponding root vectors e_φ be chosen. Also choose a system of positive roots Δ_+ thereby defining a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. Simply order Δ_+ and if $\Phi \subset \Delta_+$ we write

$$\Phi = \{\varphi_1, \ldots, \varphi_k\} \tag{19.2}$$

in increasing order. Let $e_\Phi \subset \wedge^k\mathfrak{g}$ be defined by putting

$$e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k}. \tag{19.3}$$

Any ideal $\mathfrak{v}$ of $\mathfrak{b}$ which is contained in $\mathfrak{n}$ defines a subset $\Phi \subset \Delta_+$ of the form (19.2) and necessarily

$$\mathfrak{v} = \sum_{i=1}^{k} \mathbb{C}\, e_{\varphi_i}. \tag{19.4}$$

In such a case the G-submodule spanned by $G \cdot e_\Phi$ of $\wedge^k$ is irreducible and $\mathbb{C}\, e_\Phi$ is the highest weight space. Thus

$$\langle\Phi\rangle = \sum_{i=1}^{k} \varphi_i \tag{19.5}$$

is the highest weight. Write $\Phi = \Phi(\mathfrak{v})$. Given two such ideals $\mathfrak{v}_1$, v_2, we show in this paper that

$$\mathfrak{v}_1 = \mathfrak{v}_2 \iff \langle\Phi_1\rangle = \langle\Phi_2\rangle \tag{19.6}$$

where for $i = 1, 2$, we have put $\Phi_i = \Phi(\mathfrak{v}_i)$. In particular, distinct such ideals define inequivalent irreducible representations.

Now let $\mathcal{C}$ be the set of all abelian ideals $\mathfrak{a}$ in $\mathfrak{b}$ and let $\mathcal{C}(k)$ be the set of abelian ideals of dimension k. One has $\mathfrak{a} \subset \mathfrak{n}$ for any $\mathfrak{a} \in \mathcal{C}$ so that the cardinality of $\mathcal{C}$ is finite by (19.4). Now for $\mathfrak{a} \in \mathcal{C}$ let $A^{\mathfrak{a}}$ be the G-module generated by $e_{\Phi(\mathfrak{a})}$. Then if $\mathfrak{a} \in \mathcal{C}(k)$,

$$A^{\mathfrak{a}} \text{ is an irreducible } G\text{-submodule of } A^k. \tag{19.7}$$

Put $A = \sum_{k=0}^{\infty} A^k$. The following result in paper #19 places the set of abelian ideals in $\mathfrak{b}$ at center stage.

Theorem A. *A is a multiplicity-free G-module. Furthermore*

$$A = \sum_{\mathfrak{a} \in \mathcal{C}} A^{\mathfrak{a}} \tag{19.8}$$

is the unique complete reduction of A as a sum of irreducible G-modules. Degree-wise, for any $k \in \mathbb{Z}_+$,

$$A^k = \sum_{\mathfrak{a} \in \mathcal{C}(k)} A^{\mathfrak{a}} \tag{19.9}$$

is the unique complete reduction of A^k as a sum of irreducible G-modules.

The abelian ideals in $\mathfrak{b}$ are characterized in paper # 19 by the following result.

Theorem B. *Let $\Phi \subset \Delta_+$. Let k be the cardinality of Φ and let the notation be as in (19.2). Then (with the usual present-day definition of ρ and the usual norm in weight-lattice) one has*

$$|\rho + \varphi_1 + \cdots + \varphi_k|^2 - |\rho|^2 \leq k \tag{19.10}$$

and equality occurs in (19.10) if and only if Φ is of the form $\Phi = \Phi(\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{C}(k)$.

Interest in the subject matter of paper # 19 was considerably stimulated by the subsequent discovery, due to Dale Peterson, of the following striking result:

$$\operatorname{Card} \mathcal{C} = 2^{\ell} \tag{19.11}$$

where $\ell = \operatorname{rank} \mathfrak{g}$. Another surprise in Peterson's proof of (19.11) was the role played by the affine Weyl group. A considerable clarification of (19.11) was provided by P. Cellini and P. Papi. If $\mathcal{V}$ is the fundamental alcove in the Weyl chamber, then $2\,\mathcal{V}$ is a union of 2^{ℓ} alcoves. Cellini and Papi established a natural bijection of $\mathcal{C}$ with these 2^{ℓ} alcoves. The dimension of an abelian ideal associated to an alcove is the number of walls separating the alcove from the fundamental alcove. Beautiful results of D. Panyushev related the maximal abelian ideals with the long roots in Δ_+. R. Suter showed that Peterson's result can be deduced from Theorem B above.

Paper # 19 became the basis of later results and conferences. In particular, it has been used by Etingof–Kac and Kumar in the solution of the Cachazo–Douglas–Seiberg–Witten conjecture on the structure of conformal algebras.

20. *Orbits, Symplectic Structures and Representation Theory*, Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965, p. 71.

In the early 60s I became interested in Hamiltonian mechanics and its symplectic manifold and Poisson bracket underlying structure. I also thought it was quite mysterious and marvelous that physicists in quantizing classical mechanics converted scalar functions (classical observables) on phase space in some fashion or other to operators on Hilbert space. Particularly striking in this process was that the classical observables were functions of position and momentum, q's and p's, whereas the elements in the Hilbert space were "functions" on half the variables (e.g., the q's or the p's). It seemed to me it would be very interesting to be able to make this process rigorous.

The ideas I developed during the early 60s to do this are now referred to as geometric quantization of Kostant–Souriau theory. It was a fortunate time to think about these matters. For one thing there was the Borel–Weil theorem, and growing out of Hirzebruch's Riemann–Roch theorem, line bundles and Chern classes were very much in the air. Bott had proved his generalization of the Borel–Weil theorem. There were also new constructions of unitary representation of Lie groups: Kirillov's complete treatment for nilpotent groups and Gelfand and Harish-Chandra's construction of such representations for semisimple groups using parabolic induction. The spark which ignited geometric quantization for me was Kirillov's observation that there is a nonsingular alternating 2-form on Lie group coadjoint orbits. Symplectic manifolds as an object of study were not in vogue at that time, but I soon realized that this 2 form is indeed symplectic and that coadjoint orbits yield a vast supply of symplectic homogeneous spaces. Much more than that I came to the realization that what the physicists were doing and the above construction of representations are in fact manifestations of the same idea. The space $C^\infty(X)$ of smooth functions on a symplectic manifold X is a Lie algebra under Poisson bracket, and as such is a central extension of the Lie algebra Ham (X) of Hamiltonian vector fields on X, thereby giving rise to a Lie algebra exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow C^\infty(X) \longrightarrow \mathrm{Ham}(X) \longrightarrow 0. \tag{20.1}$$

The point of departure, in quantization, was the critical recognition that the symplectic 2-form, ω — constrained only by an integrality condition for the corresponding de Rham class $[\omega]$ — should be regarded as the curvature of a line bundle L, with connection, over X. I then found that the Lie algebra $C^\infty(X)$ operates, via what I called prequantization, on the space $\Gamma(L)$ of smooth sections of L. So functions become operators. Moreover, in the spirit of the Heisenberg uncertainty principle, under prequantization, the constant function operates as a nonzero scalar operator so that, unlike in classical mechanics, the action does not descend (see (20.1)) to $\mathrm{Ham}(X)$.

If a Lie group G with Lie algebra $\mathfrak{g}$ operates symplectically on X in such a fashion that the action induces a homomorphism $\sigma : \mathfrak{g} \to \mathrm{Ham}(X)$ (this is always the case if X is simply connected) then one says that X is a *Hamiltonian G-space* if σ lifts to a homomorphism $\sigma' : X \to C^{\infty}(X)$. I introduced this terminology but restricted my considerations to the case where G operated transitively on X. It has since become standardized terminology but without the assumption of homogeneity. If X is a Hamiltonian G-space, then the points of X define linear functionals on $\mathfrak{g}$ giving rise to a map

$$\mu : X \to \mathfrak{g}^* \tag{20.2}$$

now well known as the moment or momentum map. In the homogeneous case (20.2) is a covering of a coadjoint orbit and using this, one of early results was a classification of all symplectic homogeneous spaces for G. For example, if $\mathfrak{g}$ is semisimple, then the most general symplectic homogeneous space is a covering of a coadjoint orbit. In case G is also compact, then the coajoint orbits are themelves simply-connected so that one obtains a generalization of a theorem of H. C. Wang on the classification of all compact Kähler homogeneous spaces for G.

To carry out geometric quantization one requires some additional structures, the main one involving a choice of what I called a polarization F of (X, ω). This is a choice of a complex involutory distribution of half the dimension of X whose "leaves" (in a complex sense) are Lagrangian (e.g., a Kähler structure). This "explains" the choice of half the variables in constructing the Hilbert space of states for physicists and parabolic induction in representation theory. The term polarization has been widely accepted and is now in common usage. Another ingredient required for geometric quantization (in order to obtain a Hilbert space structure) was the introduction of what I called half-forms.

Given a polarization F, and inspired by the Bott–Borel–Weil theorem, one is led to introduce the sheaf $\mathcal{S}$ of germs of local sections of L which are constant along the leaves of F and then to consider the sheaf cohomology $H(X, \mathcal{S})$. If X is an integral coadjoint orbit of G and F is invariant under the action of G, then G operates on $H(X, \mathcal{S})$. Although there are many unresolved questions there are still a large number of examples where irreducible unitary representations of G can be extracted from this action.

Except for half-forms, I gave a course at MIT in 1965 on the above subject. Notes of these lectures by N. Iwahori were widely distributed. See J. Wolf, *Bull. AMS* Vol 75 (1969) and *Représentations des groupes de Lie résolubles*, P. Bernat et al, Dunod, vol. 4 (1972) for a reference to these notes. I spoke about this subject at a 1965 conference in Differential Geometry in Kyoto, Japan. An all too brief outline, paper # 20, appears in the 1965 proceedings of this conference published by Nippon Hyoronsha Co. I also presented the material above as Phillips lecturer at Haverford college in 1965. I finally published some of the material in Vol. 170 of the Lecture Notes in Mathematics, Springer, 1970.

21. Groups Over $\mathbb{Z}$, *Proc. Symposia in Pure Math.*, **9** (1966), 90–98.

I became interested in the theory of Hopf algebras in the early 60s. I was mainly inspired by a paper of Milnor and Moore. They proved a theorem which asserted that a "connected cocommutative Hopf algebra H over a field of characteristic zero is the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra of primitive elements in H." If one discards connectedness, then H may contain elements g with augmentation value 1 such that $\delta(g) = g \otimes g$ where δ is the diagonal homomorphism. I called such elements group-like since the set of elements form a group. This terminology has been adopted and become standardized terminology in Hopf algebra theory. I then went on to prove that the most general cocommutative Hopf algebra H over, say $\mathbb{C}$, is the smash product

$$H = \mathbb{C}[G] \# U(\mathfrak{g}) \tag{21.1}$$

where $\mathbb{C}[G]$ is the group algebra over the group G of group-like elements in H and $\mathfrak{g}$ is the Lie algebra of primitive elements in H. I did not publish the theorem but it appears in a well-known (and by now classic) book, *Hopf Algebras*, written by one of my students at that time, Moss Sweedler. See the introduction in *Hopf Algebras* for the proper citation of this theorem. If H is a Hopf algebra, let H' be the space of those linear functionals on H which vanish on an ideal of finite codimension in H. We will say that H is dualizable if H' is nonsingularly paired to H. In such a case H' is a dualizable Hopf algebra and

$$H \subset H''. \tag{21.2}$$

But now (21.1) and (21.2) provide a possible algebraic device for constructing a group G associated to a Lie algebra $\mathfrak{g}$ without appealing to the usual Lie theoretic machinery, i.e., $G \subset H''$ for the case, where under suitable conditions, $H = U(\mathfrak{g})$. This was part of the motivation which led to paper #21. In more detail Chevalley in his famous Tohoku paper introduced a group $G(F)$, where F is any field, "modeled" after a complex simple Lie group. If $\mathfrak{g}$ is a complex simple Lie algebra, Chevalley found a lattice $\mathfrak{g}_{\mathbb{Z}}$ in $\mathfrak{g}$ and root vectors $e_\varphi \in \mathfrak{g}_{\mathbb{Z}}$ with the property that $\mathfrak{g}_{\mathbb{Z}}$ was stable under $\frac{1}{n!}(\operatorname{ad} e_\varphi)^n$ for any $n \in \mathbb{Z}_+$ and any root $\varphi \in \Delta$ where Δ is the set of roots with respect to a Cartan subalgebra $\mathfrak{h}$. Tensor product by F replaces $\mathfrak{g}_{\mathbb{Z}}$ by $\mathfrak{g}_F$ and introduces F-parameter groups $e_\varphi(t)$, where $t \in F$, with an automorphism action on $\mathfrak{g}_F$. $G(F)$ is the group generated by these F-parameter groups.

The key objective of paper # 21 was to do the above in a Hopf algebra context so that hopefully we would get the affine ring of the desired algebraic group as a Hopf dual, construct the hyperalgebra at the identity, and find $G(F)$ in the double dual. The first problem was to replace $\mathbb{C}$ by $\mathbb{Z}$ and construct a $\mathbb{Z}$-form $U_{\mathbb{Z}}(\mathfrak{g})$. We defined $U_{\mathbb{Z}}(\mathfrak{g})$ as the algebra over $\mathbb{Z}$ in $U(\mathfrak{g})$ generated by all elements of the form $\frac{1}{n!}\, e_\varphi^n$ for $n \in \mathbb{Z}_+$ and $\varphi \in \Delta$. Let Δ_+ be a choice of positive roots and order, $\Delta_+ = \{\varphi_1, \ldots, \varphi_r\}$ so that if $\varphi_j - \varphi_i$ is a sum of positive roots, then $j > i$. Let $\ell = \operatorname{rank} \mathfrak{g}$ and if the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, let $h_i = [e_{\alpha_i}, e_{-\alpha_i}]$. For $N, M \in \mathbb{Z}_+^r$ and $K \in \mathbb{Z}_+^\ell$ put

$$b(N, K, M) = \frac{e_{-\varphi_1}^{n_1}}{n_1!} \cdots \frac{e_{-\varphi_r}^{n_r}}{n_r!} \binom{h_1}{k_1} \cdots \binom{h_\ell}{k_\ell} \frac{e_{\varphi_1}^{m_1}}{m_1!} \cdots \frac{e_{\varphi_r}^{m_r}}{m_r!} \tag{21.3}$$

where $N = \{n_1, \ldots, n_r\}$, $K = \{k_1, \ldots, k_\ell\}$ and $M = \{m_1, \ldots, m_r\}$. Let $d = \dim \mathfrak{g}$. The main theorem of Theorem in paper # 21 asserts the following.

Theorem 1. *The elements $b(N, K, M)$ for $(N, K, M) \in \mathbb{Z}_+^d$ are a $\mathbb{Z}$-basis of $U_{\mathbb{Z}}(\mathfrak{g})$ and also a (PBW) $\mathbb{C}$-basis of $U(\mathfrak{g})$ so that*

$$U(\mathfrak{g}) = \mathbb{C} \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}). \tag{21.4}$$

Furthermore the Hopf structure on $U(\mathfrak{g})$ induces a $\mathbb{Z}$-Hopf structure on $U_{\mathbb{Z}}(\mathfrak{g})$. In fact the $b(N, K, M)$ are a d-multisequence of divided powers. In addition if V is a finite-dimensional $U(\mathfrak{g})$-module, then $U_{\mathbb{Z}}$ stabilizes a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ in V. Moreover $V_{\mathbb{Z}}$ is the sum of its intersections with the weight spaces in V.

The $\mathbb{Z}$-algebra $U_{\mathbb{Z}}(\mathfrak{g})$ has been referred to as the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ and is well known in Lie theory. The last statement in Theorem 1 above implies that $U_{\mathbb{Z}}(\mathfrak{g})$ has a Hopf dual H', where the definition of the latter is modified so that $\mathbb{Z}$ replaces $\mathbb{C}$. If A is any commutative ring, then $H'_A = H' \otimes_{\mathbb{Z}} A$ has the structure of a Hopf algebra and the group-like elements $G(A)$ in its dual define a functor, $A \to G(A)$, from commutative rings to groups. In case A is an algebraically closed field, I had hoped at some later point to show that $G(A)$ was the Chevalley group, modeled on G, and associated to A, and that H'_A is the affine ring of $G(A)$. However I did not succeed in doing this. An unsolved problem for me was even to show that H'_A is Noetherian. (Theorem 3, attributed to Chevalley, in paper # 21 should be ignored since it is a misunderstanding on my part of a statement of Chevalley.). However the result is true and was proved by George Lusztig. See his paper entitled "Study of a $\mathbb{Z}$-form of the coordinate ring of a reductive group", *Jour. AMS*, March 31, 2008, posted online. Lusztig also establishes that this Hopf algebra approach to Chevalley theory generalizes to the quantum case.