A

Basic notations

$k(x)$ - coefficients (heterogeneous)
p - solution
v - flux (velocity)
x - space variable
t - time variable
$\mathbb{R}^d$ - d-dimensional vector space
$\mathcal{T}_h$ - coarse-scale partition
W_h - standard finite element spaces (e.g., piecewise linear functions)
$\mathcal{P}_h$ - multiscale finite element "space" for the solution[1]
$\mathcal{V}_h$ - multiscale finite element space for the flux
p_h - approximate solution obtained using MsFEM[2]
$p_{r,h}$ - fine-scale approximation of the solution (for nonlinear MsFEM only)
Ω - global domain
K - coarse grid block
h - coarse mesh size
ϵ - small physical (characteristic) scale
ϕ_j - multiscale basis functions from $\mathcal{P}_h$
ϕ_j^0 - standard (e.g., linear) basis functions from W_h
χ - solution of auxiliary periodic problem (linear case)
N - solution of auxiliary periodic problem (nonlinear case)
q_t, q_w - source terms
f - source term; also the flux function
n - outward normal
fractional flow - fraction of the displaced fluid (see (2.43))
water-cut - fraction of water in the produced fluid
PVI - pore volume injected (see (2.44))

[1] for nonlinear problems, it is not a linear space
[2] fine-scale approximation for linear problems and coarse-scale approximation for nonlinear problems

B

Review of homogenization

B.1 Linear problems

In this appendix, we use the notations commonly used in the homogenization literature and these notations can be different from those used in the main text of the book. Consider the second-order elliptic equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}\left(x/\epsilon\right)\frac{\partial}{\partial x_j}\right)u_\epsilon + a_0(x/\epsilon)u_\epsilon = f, \quad u_\epsilon|_{\partial\Omega} = 0, \tag{B.1}$$

where $a_{ij}(y)$ and $a_0(y)$ are 1-periodic in both variables of y, and satisfy $a_{ij}(y)\xi_i\xi_j \geq \alpha\xi_i\xi_i$, with $\alpha > 0$, $a_0 > \alpha_0 > 0$, and bounded. Here we have used the Einstein summation notation; that is a repeated index means summation with respect to that index.

This model equation represents a common difficulty shared by several physical problems. For porous media, it is the pressure equation described by Darcy's law with the coefficient $a_\epsilon = (a_{ij}(x/\epsilon))$ representing the permeability tensor. For composite materials, it is the steady heat conduction equation and the coefficient a_ϵ represents the thermal conductivity. For steady transport problems, it is a symmetrized form of the governing equation. In this case, the coefficient a_ϵ is a combination of transport velocity and viscosity tensor.

Homogenization theory studies the limiting behavior $u_\epsilon \to u_0$ as $\epsilon \to 0$. The main task is to find the homogenized coefficients, a^*_{ij} and a^*_0, and the homogenized equation for the limiting solution u

$$-\frac{\partial}{\partial x_i}\left(a^*_{ij}\frac{\partial}{\partial x_j}\right)u_0 + a^*_0 u_0 = f, \quad u_0|_{\partial\Omega} = 0.$$

We define the bilinear form

$$a^\epsilon(u,v) = \int_\Omega a^\epsilon_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} \quad dx + \int_\Omega a^\epsilon_0 uv\, dx.$$

The elliptic problem can also be formulated as a variational problem: find $u_\epsilon \in H_0^1$,

$$a^\epsilon(u_\epsilon, v) = (f, v), \quad \text{for all} \quad v \in H_0^1(\Omega),$$

where (f, v) is the usual L^2 inner product, $\int_\Omega fv\,dx$.

B.1.1 Special case: One-dimensional problem

Let $\Omega = (x_0, x_1)$ and take $a_0 = 0$. We have

$$-\frac{d}{dx}\left(a(x/\epsilon)\frac{du_\epsilon}{dx}\right) = f, \quad \text{in} \quad \Omega\,,$$

where $u_\epsilon(x_0) = u_\epsilon(x_1) = 0$, and $a(y) > \alpha_0 > 0$ is y-periodic with period y_0.

By taking $v = u_\epsilon$ in the variational problem, we have

$$\|u_\epsilon\|_{1,\Omega} \le C.$$

Therefore one can extract a subsequence, still denoted by u_ϵ, such that

$$u_\epsilon \rightharpoonup u \ \text{ in } \ H_0^1(\Omega) \quad \text{weakly}.$$

Next, we introduce

$$\xi^\epsilon = a^\epsilon \frac{du^\epsilon}{dx}.$$

Because a^ϵ is bounded, and du^ϵ/dx is bounded in $L^2(\Omega)$, so ξ^ϵ is bounded in $L^2(\Omega)$. Moreover, because $-d\xi^\epsilon/dx = f$, we have $\xi^\epsilon \in H^1(\Omega)$. Thus we get

$$\xi^\epsilon \to \xi \quad \text{in} \quad L^2(\Omega) \quad \text{strongly},$$

so that

$$\frac{1}{a^\epsilon}\xi^\epsilon \to m(1/a)\xi \ \text{ in } L^2(\Omega) \quad \text{weakly}.$$

Furthermore, we note that $\xi^\epsilon/a^\epsilon = du^\epsilon/dx$. Therefore, we arrive at

$$\frac{du_0}{dx} = m(1/a)\xi.$$

On the other hand, $-d\xi^\epsilon/dx = f$ implies $-d\xi/dx = f$. This gives

$$-\frac{d}{dx}\left(\frac{1}{m(1/a)}\frac{du_0}{dx}\right) = f.$$

This is the correct homogenized equation for u. Note that $a^* = 1/m(1/a)$ is the harmonic average of a^ϵ. It is in general not equal to the arithmetic average $\overline{a^\epsilon} = m(a)$.

B.1.2 Multiscale asymptotic expansions.

The above analysis does not generalize to multidimensions. In this subsection, we introduce the multiscale expansion technique in deriving homogenized equations.

We look for $u_\epsilon(x)$ in the form of asymptotic expansion

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \cdots ,$$

where the functions $u_j(x, y)$ are periodic in y with period 1.

Denote by A^ϵ the second-order elliptic operator

$$A^\epsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(x/\epsilon\right)\frac{\partial}{\partial x_j}\right).$$

When differentiating a function $\phi(x, x/\epsilon)$ with respect to x, we have

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} + \frac{1}{\epsilon}\frac{\partial}{\partial y_j},$$

where y is evaluated at $y = x/\epsilon$. With this notation, we can expand A^ϵ as follows,

$$A^\epsilon = \epsilon^{-2} A_1 + \epsilon^{-1} A_2 + \epsilon^0 A_3, \tag{B.2}$$

where

$$\begin{aligned} A_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right), \\ A_2 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right), \\ A_3 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) + a_0 . \end{aligned} \tag{B.3}$$

Substituting the expansions for u_ϵ and A^ϵ into $A^\epsilon u_\epsilon = f$, and equating the terms of the same power, we get

$$A_1 u_0 = 0, \tag{B.4}$$

$$A_1 u_1 + A_2 u_0 = 0, \tag{B.5}$$

$$A_1 u_2 + A_2 u_1 + A_3 u_0 = f. \tag{B.6}$$

Equation (B.4) can be written as

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) u_0(x, y) = 0,$$

where u_0 is periodic in y. The theory of second-order elliptic PDEs [132] implies that $u_0(x, y)$ is independent of y; that is $u_0(x, y) = u_0(x)$. This simplifies (B.5) for u_1,

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)u_1 = \left(\frac{\partial}{\partial y_i}a_{ij}(y)\right)\frac{\partial u}{\partial x_j}(x).$$

Define $\chi^j = \chi^j(y)$ as the solution to the following cell problem

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\chi^j = -\frac{\partial}{\partial y_i}a_{ij}(y)\ ,$$

where χ^j is periodic in y. The general solution of (B.5) for u_1 is then given by

$$u_1(x,y) = \chi^j(y)\frac{\partial u}{\partial x_j}(x) + \tilde{u}_1(x)\ .$$

Finally, we note that the equation for u_2 is given by

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)u_2 = A_2 u_1 + A_3 u_0 - f\ . \tag{B.7}$$

The solvability condition implies that the right-hand side of (B.7) must have mean zero in y over one periodic cell $Y = [0,1]\times[0,1]$; that is

$$\int_Y (A_2 u_1 + A_3 u_0 - f)\,dy = 0.$$

This solvability condition for second-order elliptic PDEs with periodic boundary condition [132] requires that the right-hand side of (B.7) have mean zero with respect to the fast variable y. This solvability condition gives rise to the homogenized equation for u:

$$-\frac{\partial}{\partial x_i}\left(a^*_{ij}\frac{\partial}{\partial x_j}\right)u + m(a_0)u = f\ , \tag{B.8}$$

where $m(a_0) = (1/|Y|)\int_Y a_0(y)\,dy$ and

$$a^*_{ij} = \frac{1}{|Y|}\left(\int_Y (a_{ij} - a_{ik}\frac{\partial \chi^j}{\partial y_k})\,dy\right). \tag{B.9}$$

It is often difficult to compute the homogenized coefficients when the periodic cell problem requires very fine discretization. In this case, the bounds for the homogenized coefficients can be very useful. Finding accurate bounds depending on heterogeneities is a difficult task. There have been many works in the literature where bounds are computed and the corresponding optimal microstructures are determined. In the presence of tight bounds, one can avoid solving the cell problems for the computation of the homogenized solutions. We refer to [202, 74] for descriptions of various bounds and the literature reviews.

B.1.3 Justification of formal expansions

The above multiscale expansion is based on a formal asymptotic analysis. However, we can justify its convergence rigorously.

Let $z_\epsilon = u_\epsilon - (u_0 + \epsilon u_1 + \epsilon^2 u_2)$. Applying A^ϵ to z_ϵ, we get

$$A^\epsilon z_\epsilon = -\epsilon r_\epsilon \ ,$$

where $r_\epsilon = A_2 u_2 + A_3 u_1 + \epsilon A_3 u_2$. Thus we have $\|r_\epsilon\|_{\infty,\Omega} \leq C$.

On the other hand, we have

$$z_\epsilon|_{\partial\Omega} = -(\epsilon u_1 + \epsilon^2 u_2)|_{\partial\Omega}.$$

Thus, we obtain

$$\|z_\epsilon\|_{\infty,\partial\Omega} \leq c\epsilon.$$

It follows from the maximum principle [132] that

$$\|z_\epsilon\|_{\infty,\Omega} \leq C\epsilon$$

and therefore we conclude that

$$\|u_\epsilon - u_0\|_{\infty,\Omega} \leq C\epsilon.$$

B.1.4 Boundary corrections

The above asymptotic expansion does not take into account the boundary condition of the original elliptic PDEs. If we add a boundary correction, we can obtain higher-order approximations.

Let $\theta_\epsilon \in H^1(\Omega)$ denote the solution to

$$\mathrm{div}_x(a^\epsilon \nabla_x \theta_\epsilon) = 0 \ \text{ in } \Omega, \quad \theta_\epsilon = u_1(x, x/\epsilon) \ \text{ on } \partial\Omega.$$

Then we have

$$\left(u_\epsilon - (u_0 + \epsilon u_1(x, x/\epsilon) - \epsilon\theta_\epsilon)\right)|_{\partial\Omega} = 0.$$

Moskow and Vogelius [204] have shown that

$$\begin{aligned} \|u_\epsilon - u_0 - \epsilon u_1(x, x/\epsilon) + \epsilon\theta_\epsilon\|_{0,\Omega} &\leq C_\omega \epsilon^{1+\omega} \|u_0\|_{2+\omega,\Omega}, \\ \|u_\epsilon - u_0 - \epsilon u_1(x, x/\epsilon) + \epsilon\theta_\epsilon\|_{1,\Omega} &\leq C\epsilon \|u_0\|_{2,\Omega}, \end{aligned} \tag{B.10}$$

where we assume $u \in H^{2+\omega}(\Omega)$ with $0 \leq \omega \leq 1$, and Ω is assumed to be a bounded, convex curvilinear polygon of class C^∞.

B.1.5 Nonlocal memory effect of homogenization

It is interesting to note that for certain degenerate problems, the homogenized equation may have a nonlocal memory effect.

Consider the simple 2D linear convection equation:

$$\frac{\partial u_\epsilon(x,y,t)}{\partial t} + a_\epsilon(y)\frac{\partial u_\epsilon(x,y,t)}{\partial x} = 0,$$

with initial condition $u_\epsilon(x,y,0) = u_0(x,y)$. Note that $y = x_2$ is not a fast variable here.

We assume that a_ϵ is bounded and u_0 has compact support. It is easy to write down the solution explicitly,

$$u_\epsilon(x,y,t) = u_0(x - a_\epsilon(y)t, y),$$

however, it is not an easy task to derive the homogenized equation for the weak limit of u_ϵ.

Using the Laplace transform and measure theory, Luc Tartar [255] showed that the weak limit u of u_ϵ satisfies

$$\frac{\partial}{\partial t}u(x,y,t) + A_1(y)\frac{\partial}{\partial x}u(x,y,t) = \int_0^t \int \frac{\partial^2}{\partial x^2}u(x-\lambda(t-s),y,s)d\mu_y(\lambda)\,ds,$$

with $u(x,y,0) = u_0(x,y)$, where $A_1(y)$ is the weak limit of $a_\epsilon(y)$, and μ_y is a probability measure of y and has support in $[\min(a_\epsilon), \max(a_\epsilon)]$.

As we can see, the convection induces a nonlocal history-dependent diffusion term in the propagating direction (x). The homogenized equation is not amenable to coarse-scale computation in general because the measure μ_y cannot be expressed explicitly in terms of a_ϵ.

B.1.6 Convection of microstructure

It is most interesting to see if one can apply the homogenization technique to obtain an averaged equation for the large-scale quantity for incompressible Euler or Navier–Stokes equations. In 1985, McLaughlin, Papanicolaou, and Pironneau [200] attempted to obtain a homogenized equation for the 3D incompressible Euler equations with a highly oscillatory velocity field. More specifically, they considered the following initial value problem,

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\nabla p,$$

with $\nabla \cdot u = 0$ and highly oscillatory initial data

$$u(x,0) = U(x) + W(x, x/\epsilon).$$

They then constructed multiscale expansions for both the velocity field and the pressure. In doing so, they made an important assumption that the microstructure is convected by the mean flow. Under this assumption, they constructed a multiscale expansion for the velocity field as follows:

$$u^\epsilon(x,t) = u_0(x,t) + w(\tfrac{\theta(x,t)}{\epsilon}, \tfrac{t}{\epsilon}, x, t) + \epsilon u_1(\tfrac{\theta(x,t)}{\epsilon}, \tfrac{t}{\epsilon}, x, t) + O(\epsilon^2).$$

The pressure field p^ϵ is expanded similarly. From this ansatz, one can show that θ is convected by the mean velocity:

$$\frac{\partial \theta}{\partial t} + u_0 \cdot \nabla \theta = 0, \quad \theta(x,0) = x\ .$$

It is a very challenging problem to develop a systematic approach to study the large-scale solution in three-dimensional Euler and Navier–Stokes equations. The work of McLaughlin, Papanicolaou, and Pironneau provided some insightful understanding into how small scales interact with large scales and how to deal with the closure problem. However, the problem is still not completely resolved because the cell problem obtained this way does not have a unique solution. Additional constraints need to be enforced in order to derive a large-scale averaged equation. With additional assumptions, they managed to derive a variant of the $k-\epsilon$ model in turbulence modeling.

Remark B.1. One possible way to improve the work of [200] is to take into account the oscillation in the Lagrangian characteristics θ_ϵ. The oscillatory part of θ_ϵ in general could have an order-one contribution to the mean velocity of the incompressible Euler equation. In [148, 149, 150], Hou, Yang and co-workers have studied convection of the microstructure of the 2D and 3D incompressible Euler equations using a new approach. They do not assume that the oscillation is propagated by the mean flow. In fact, they found that it is crucial to include the effect of oscillations in the characteristics on the mean flow. Using this new approach, they can derive a well-posed cell problem that can be used to obtain an effective large-scale average equation.

More can be said for a passive scalar convection equation.

$$\frac{\partial v}{\partial t} + \frac{1}{\epsilon}\mathrm{div}\,(u(x/\epsilon)v) = \alpha \Delta v,$$

with $v(x,0) = v_0(x)$. Here $u(y)$ is a known incompressible periodic (or stationary random) velocity field with zero mean. Assume that the initial condition is smooth.

Expand the solution v^ϵ in powers of ϵ

$$v^\epsilon = v(t,x) + \epsilon v_1(t,x,x/\epsilon) + \epsilon^2 v_2(t,x,x/\epsilon) + \cdots\ .$$

The coefficients of ϵ^{-1} lead to

$$\alpha \Delta_y v_1 - u \cdot \nabla_y v_1 - u \cdot \nabla_x v = 0.$$

Let $e_k,\ k = 1, 2, 3$ be the unit vectors in the coordinate directions and let $\chi^k(y)$ satisfy the cell problem:

$$\alpha \Delta_y \chi^k - u \cdot \nabla_y \chi^k - u \cdot e_k = 0.$$

Then we have

$$v_1(t, x, y) = \sum_{k=1}^{3} \chi^k(y) \frac{v(t, x)}{\partial x_k}.$$

The coefficients of ϵ^0 give

$$\alpha \Delta_y v_2 - u \cdot \nabla_y v_2 = u \cdot \nabla_x v_1 - 2\alpha \nabla_x \cdot \nabla_y v_1 - \alpha \Delta_x v + \frac{\partial v}{\partial t}.$$

The solvability condition for v_2 requires that the right-hand side have zero mean with respect to y. This gives rise to the equation for homogenized solution v,

$$\frac{\partial v}{\partial t} = \alpha \Delta_x v - \overline{u \cdot \nabla_x v_1}.$$

Using the cell problem, McLaughlin, Papanicolaou, and Pironneau obtained [200]

$$\frac{\partial v}{\partial t} = \sum_{i,j=1}^{3} (\alpha \delta_{ij} + \alpha_{T_{ij}}) \frac{\partial^2 v}{\partial x_i \partial x_j},$$

where $\alpha_{T_{ij}} = -\overline{u_i \chi^j}$.

B.2 Nonlinear problems

We briefly discuss homogenization for general nonlinear elliptic equations, $u_\epsilon \in W_0^{1,p}(\Omega)$,

$$- \operatorname{div} a_\epsilon(x, u_\epsilon, \nabla u_\epsilon) + a_{0,\epsilon}(x, u_\epsilon, \nabla u_\epsilon) = f, \tag{B.11}$$

where $a_\epsilon(x, \eta, \xi)$ and $a_{0,\epsilon}(x, \eta, \xi)$, $\eta \in \mathbb{R}$, $\xi \in \mathbb{R}^d$ satisfy assumptions given by (6.42)–(6.46), which guarantee the well-posedness of the nonlinear elliptic problem (B.11). Here $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain and ϵ denotes the small scale of the problem. The homogenization of nonlinear partial differential equations has been studied previously (see, e.g., [220]). It can be shown that a solution u_ϵ converges (up to a subsequence) to u_0 in an appropriate norm, where $u_0 \in W_0^{1,p}(\Omega)$ is a solution of a homogenized equation

$$- \operatorname{div} a^*(x, u_0, \nabla u_0) + a_0^*(x, u, \nabla u_0) = f. \tag{B.12}$$

The homogenized coefficients can be computed if we make an additional assumption on the heterogeneities, such as periodicity, almost periodicity, or when the fluxes are strictly stationary fields with respect to spatial variables.

In these cases, an auxiliary problem is formulated and used in the calculations of the homogenized fluxes a^* and a_0^*. Next, we discuss this.

We assume that a and a_0 are periodic functions with respect to the spatial variable. Then, the homogenized fluxes are defined as follows,

$$a^*(\eta, \xi) = \int_Y a(y, \eta, \xi + \nabla_y N_{\eta,\xi}(y))\, dy, \tag{B.13}$$

$$a_0^*(\eta, \xi) = \int_Y a_0(y, \eta, \xi + \nabla_y N_{\eta,\xi}(y))\, dy, \tag{B.14}$$

where a^* and a_0^* satisfy the conditions similar to (6.42)–(6.46). Here $N_{\eta,\xi} \in W^{1,p}_{per}(Y)$ is the periodic solution (with average zero) of

$$-\operatorname{div}(a(y, \eta, \xi + \nabla_y N_{\eta,\xi}(y))) = 0 \text{ in } Y, \tag{B.15}$$

where Y is a unit period. We do not present the proof of the homogenization here and refer to [220], for example.

Next, we also present the homogenization results for the random homogeneous case. Homogenization in random homogeneous media for linear problems ([43, 164]) has been a pioneering work in this direction. We start with a description of random homogeneous fields on $\mathbb{R}^d$ which is shown to be useful in homogenization problems (e.g., [164]). Let (U, Σ, μ) be a probability space. A random homogeneous field is a measurable function on U and $f(T(x)\omega)$ are realizations of the random field. The realizations are well-defined measurable functions on $\mathbb{R}^d$ for almost all $\omega \in U$. Consider a d-dimensional dynamical system on U, $T(x) : U \to U$, $x \in \mathbb{R}^d$, that satisfies the following conditions: (1) $T(0) = I$, and $T(x + y) = T(x)T(y)$; (2) $T(x) : U \to U$ preserve the measure μ on U; and (3) for any measurable function $f(\omega)$ on U, the function $f(T(x)\omega)$ defined on $\mathbb{R}^d \times U$ is also measurable (see [164]). Let $L^p(U)$ denote the space of all p-integrable functions on U. Then $U(x)f(\omega) = f(T(x)\omega)$ defines a d-parameter group of isometries in the space $L^p(U)$, and $U(x)$ is strongly continuous [164, 220]. We further assume that the dynamical system T is ergodic; that is, any measurable T-invariant function on U is constant. Denote by $\langle \cdot \rangle_\mu$ the mean value over U,

$$\langle f \rangle_\mu = \int_U f(\omega) d\mu(\omega) = E(f).$$

Denote by D^i_ω the generator of $U(x)$ along the ith coordinate direction; that is,

$$D^i_\omega = \lim_{\delta \to 0} \frac{f(T(x_i)\omega) - f(\omega)}{\delta}.$$

The domains ∂_i of D^i_ω are dense in $L^2(U)$, and the intersection of all D^i_ω is also dense.

Next following [220] we define potential and solenoidal fields. A vector field $f \in L^p(U)$ is said to be potential (or solenoidal, respectively) if its generic realization $f(T_x\omega)$ is a potential (or solenoidal respectively) vector field in $\mathbb{R}^d$. Denote by $L^p_{pot}(U)$ (respectively, $L^p_{sol}(U)$) the subspace of $L^p(U)$ that consists of all potential (respectively, solenoidal) vector fields. Introduce the following notations,

$$V^p_{\rm pot} = \{f \in L^p_{\rm pot}(U), \langle f\rangle_\mu = 0\}, \quad V^p_{\rm sol} = \{f \in L^p_{\rm sol}(U), \langle f\rangle_\mu = 0\}.$$

The following properties are known (see [220], page 138)

$$L^p_{\rm pot}(U) = V^p_{\rm pot} \oplus \mathbb{R}^d, \quad L^p_{\rm sol}(U) = V^p_{\rm sol} \oplus \mathbb{R}^d,$$

$$L^q_{\rm sol}(U) = (V^p_{\rm pot})^\perp, \quad L^q_{\rm pot}(U) = (V^p_{\rm sol})^\perp.$$

Next, we consider (B.11) with the assumptions given by (6.42)–(6.46), which guarantee the well-posedness of the nonlinear elliptic problem.

It is known (e.g., [220]) that as $\epsilon \to 0$ ∇u_ϵ converges to ∇u_0 weakly in $L^p(\Omega)$ for a.e. ω, and u_0 is the solution of

$$-\operatorname{div}(a^*(u_0, \nabla u_0)) + a_0^*(u_0, \nabla u_0) = f, \; u_0 \in W_0^{1,p}(\Omega). \tag{B.16}$$

Furthermore, a^* and a_0^* can be constructed using the solution of the following auxiliary problem. Given $\eta \in R$ and $\xi \in \mathbb{R}^d$ define $w_{\eta,\xi} \in V^p_{\rm pot}$ such that

$$a(\omega, \eta, \xi + w_{\eta,\xi}(\omega)) \in L^q_{\rm sol}(U)^d. \tag{B.17}$$

Then $a^*(\eta, \xi)$ and $a_0(\eta, \xi)$ are defined as

$$\begin{aligned} a^*(\eta, \xi) &= \langle a(\omega, \eta, \xi + w_{\eta,\xi}(\omega))\rangle_\mu, \\ a_0^*(\eta, \xi) &= \langle a_0(\omega, \eta, \xi + w_{\eta,\xi}(\omega))\rangle_\mu. \end{aligned} \tag{B.18}$$

Moreover, $a^*(\eta, \xi)$ and $a_0^*(\eta, \xi)$ satisfy similar estimates as a and a_0 with different constants [220].

For parabolic problems, the homogenization also yields the macroscopic equations of the same class. If we consider

$$\frac{\partial u_\epsilon}{\partial t} - \operatorname{div}(a_\epsilon(x, t, u_\epsilon, \nabla u_\epsilon)) + a_{0,\epsilon}(x, t, u_\epsilon, \nabla u_\epsilon) = f, \tag{B.19}$$

where $a_\epsilon(x, t, \eta, \xi) = a(x/\epsilon^\beta, t/\epsilon^\alpha, \eta, \xi)$ $a_{0,\epsilon}(x, t, \eta, \xi) = a_0(x/\epsilon^\beta, t/\epsilon^\alpha, \eta, \xi)$.

The homogenization of nonlinear parabolic equations depends on the ratio between α and β and is presented in [111]. The following cases are distinguished: (1) self-similar case ($\alpha = 2\beta$); (2) nonself-similar case ($\alpha < 2\beta$); (3) nonself-similar case ($\alpha > 2\beta$); (4) spatial case ($\alpha = 0$); and (5) temporal case ($\beta = 0$). To introduce the homogenized operator, we introduce fast variables $y = x/\epsilon^\beta$ and $\tau = t/\epsilon^\alpha$. Moreover, denote by $\langle\cdot\rangle_{y,\tau}$ the average over y and τ. If a single variable y or τ is used as a subscript, then the average is taken with respect to that variables. Similarly, we denote by $\Pi_{y,\tau}$ the periodic box in space and time, and correspondingly Π_y and Π_τ are periods in space and temporal variable. The homogenized operator is given by

$$\frac{\partial u_0}{\partial t} - \mathrm{div}(a^*(x,t,u,\nabla u_0)) + a_0^*(x,t,u,\nabla u_0) = f,$$

where the homogenized coefficients are defined below.

- For self-similar case ($\alpha = 2\beta$),

$$a^*(\eta,\xi) = \langle a(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$
$$a_0{}^*(\eta,\xi) = \langle a_0(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$

  where $N_{\eta,\xi}$ is the unique solution of

$$\frac{\partial N_{\eta,\xi}}{\partial \tau} - \mathrm{div}_y\ a(\omega,\eta,\xi+\nabla_y N_{\eta,\xi}) = 0 \tag{B.20}$$

  in $\Pi_{y,\tau}$.
- For nonself-similar case ($\alpha < 2\beta$),

$$a^*(\eta,\xi) = \langle a(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$
$$a_0{}^*(\eta,\xi) = \langle a_0(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$

  where $N_{\eta,\xi}$ is the unique solution of

$$-\,\mathrm{div}_y\ a(y,\tau,\eta,\xi+\nabla_y N_{\eta,\xi}) = 0 \tag{B.21}$$

  in $\Pi_{y,\tau}$.
- For nonself-similar case ($\alpha > 2\beta$),

$$a^*(\eta,\xi) = \langle a(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$
$$a_0{}^*(\eta,\xi) = \langle a_0(y,\tau,\eta,\xi+\nabla N_{\eta,\xi})\rangle_{y,\tau},$$

  where $N_{\eta,\xi}$ is the unique solution of

$$-\,\mathrm{div}_y\ \overline{a}(y,\eta,\xi+\nabla_y N_{\eta,\xi}) = 0. \tag{B.22}$$

  $\overline{a}(y,\eta,\xi) = \langle a(y,\tau,\eta,\xi)\rangle_\tau$.
- For spatial case ($\alpha = 0$),

$$a^*(t,\eta,\xi) = \langle a(y,t,\eta,\xi+\nabla N_{\eta,\xi})\rangle_y$$
$$a_0^*(t,\eta,\xi) = \langle a_0(y,t,\eta,\xi+\nabla N_{\eta,\xi})\rangle_y,$$

  where $N_{\eta,\xi}$ satisfies

$$-\,\mathrm{div}_y\ a(y,t,\eta,\xi+\nabla_y N_{\eta,\xi}) = 0 \tag{B.23}$$

  in Π_y for each t (assuming the coefficients smoothly depend on t).
- For temporal case ($\beta = 0$), the homogenized fluxes are defined by

$$\begin{aligned} a^*(x,\eta,\xi) &= \langle a(x,\tau,\eta,\xi)\rangle_\tau, \\ a_0^*(x,\eta,\xi) &= \langle a_0(\omega,\eta,\xi)\rangle_\tau. \end{aligned} \tag{B.24}$$

For the results concerning the random homogenization of nonlinear parabolic equations we refer to [111].

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