PROBLEMS FOR THE READER

Problem 1.1 Prove the complex-manifold structure of C^m, complex projective space $P_m(C)$, complex torus, orientable surfaces. What is the physical relevance of $P_2(C)$ and $P_3(C)$?

Problem 2.1 Write down the Infeld-van der Waerden symbols for Lorentzian and for real Riemannian four-manifolds.

Problem 2.2 What is the relation between $SL(2, C)$ and $SU(2)$ soldering forms (cf. Ashtekar 1988) ?

Problem 2.3 Prove that unprimed and primed spin-spaces are no longer anti-isomorphic if a Lorentzian four-dimensional space-time is replaced by a complex or real Riemannian four-manifold.

Problem 2.4 Using the Euclidean conjugation defined in section 2.1, prove that there are no Majorana spinors in real Riemannian four-manifolds.

Problem 2.5 Define a spinor conjugation in a pseudo-Riemannian four-manifold with a metric of signature (–, –, +, +). Can one find Majorana spinors in this manifold ?

Problem 2.6 Give a two-spinor description of the Riemann curvature tensor in a pseudo-Riemannian four-manifold endowed with a metric of signature (–, –, +, +). Describe in detail the differences with respect to the Lorentzian or Riemannian formalisms.

Problem 2.7 Denoting by $\Box$ the operator $\nabla_{CA'} \nabla^{CA'}$, prove that its action on solutions of the massless free-field equations is given by

$$\Box \phi_A = -6\Lambda \, \phi_A \quad , \tag{P.1}$$

$$\Box \varphi_{AB} = 2\psi_{ABCD} \, \varphi^{CD} - 8\Lambda \, \varphi_{AB} \quad , \tag{P.2}$$

$$\Box \varphi_{ABC} = 4\psi_{(AB}{}^{MN}\,\varphi_{C)MN} - 10\Lambda\,\varphi_{ABC} \quad , \qquad (P.3)$$

$$\Box \psi_{ABCD} = 6\psi_{(AB}{}^{EF}\,\psi_{CD)EF} - 12\Lambda\,\psi_{ABCD} \quad . \qquad (P.4)$$

Problem 3.1 Prove Eqs. (3.2.12)-(3.2.13).

Problem 4.1 Write the twistor equation (4.1.5) in terms of γ-matrices (Penrose 1975).

Problem 4.2 To study the relation between null twistors and null geodesics, find the equation which replaces Eq. (4.1.27) if the geodesics γ_X and γ_Y are parallel.

Problem 4.3 What is the integrability condition for β-surfaces in a complex vacuum space-time ?

Problem 4.4 Prove that, by virtue of the equations of local twistor transport, Eq. (4.3.25) holds.

Problem 5.1 One wants to evaluate explicitly the conformal structure of an anti-self-dual space-time. For this purpose, following section 5.1, prove that the right-hand side of Eq. (5.1.34) vanishes and complete the derivation of Eq. (5.1.42) therein.

Problem 5.2 Can you construct explicitly a self-dual space-time, by comparison with the work in section 5.1 on anti-self-dual space-times ?

Problem 5.3 Prove that the charge Q in Eq. (5.3.26) is such that $D^{BB'}\,Q = 0$. What is the relation between Eqs. (5.3.24) and (5.3.26) ?

Problem 5.4 Write a dissertation on (anti)-self-duality in Riemannian geometry, twistor theory and gauge theory.

Problem 6.1 Try to derive Eq. (6.4.2) by using the formulae relating curvature spinors of the Levi-Civita connection to torsion and curvature spinors of the U_4 -connection.

Problems for the Reader

Problem 6.2 Can you say how the integrability condition (6.4.2) for α-surfaces in complex space-times with non-vanishing torsion is modified if the torsion tensor is completely antisymmetric ?

Problem 7.1 If the spinor field ϕ_A solves the Weyl equation $\nabla^{AA'} \phi_A = 0$, what can we say about $\nabla_B{}^{C'} \phi_A$?

Problem 8.1 Prove that spin-raising and spin-lowering operators yield solutions of the massless free-field equations.

Problem 8.2 Check Eqs. (8.4.28)-(8.4.29).

Problem 8.3 Prove that Eq. (8.9.5) results from Eq. (8.9.4).

Problem 9.1 Following Penrose 1975 and chapter nine, write an essay on the various definitions of twistors in curved space-time.

Problem 9.2 Prove that, on the six-complex-dimensional space S_6 at the end of section 9.4, ω is a non-degenerate, closed two-form which defines a symplectic structure (Penrose 1975).

Problem 9.3 Following Penrose 1975, differentiate $Z^{(a)} \tilde{Z}_{(a)}$ in section 9.5 once independently with respect to $Z^{(a)}, \tilde{Z}_{(a)}$, and then set $\tilde{Z}_{(a)} = \overline{Z}_{(a)}$ Prove that this defines a Hermitian metric tensor for the asymptotic twistor space at future null infinity. Moreover, prove that this construction makes $T(\mathcal{I}^+)$ into a Kähler manifold.

Problem 9.4 Prove that the integrability condition for Eq. (9.9.12) is indeed Eq. (9.9.8). Is it necessary to make assumptions on the conformal curvature ?

Problem 9.5 Is it possible to view Eq. (8.9.4) within the framework of integrability conditions relevant for twistor theory in Einstein backgrounds ?

APPENDIX A: Clifford Algebras

In section 7.4 we have defined the total Dirac operator in Riemannian geometries as the first-order elliptic operator whose action on the sections is given by composition of Clifford multiplication with covariant differentiation. Following Ward and Wells 1990, this appendix presents a self-contained description of Clifford algebras and Clifford multiplication.

Let V be a real vector space equipped with an inner product $< , >$, defined by a non-degenerate quadratic form of signature (p,q). Let $T(V)$ be the tensor algebra of V and consider the ideal I in $T(V)$ generated by $x \otimes x + Q(x)$. By definition, I consists of sums of terms of the kind $a \otimes \left\{ x \otimes x + Q(x) \right\} \otimes b$, $x \in V$, $a, b \in T(V)$. The quotient space

$$Cl(V) \equiv Cl(V, Q) \equiv T(V)/\mathcal{I} \qquad (A.1)$$

is the Clifford algebra of the vector space V equipped with the quadratic form Q. The product induced by the tensor product in $T(V)$ is known as Clifford multiplication or the Clifford product and is denoted by $x \cdot y$, for $x, y \in Cl(V)$. The dimension of $Cl(V)$ is 2^n if $\dim(V) = n$. A basis for $Cl(V)$ is given by the scalar 1 and the products

$$e_{i_1} \cdot e_{i_2} \cdot e_{i_n} \quad i_1 < ... < i_n \quad ,$$

where $\left\{ e_1, ..., e_n \right\}$ is an orthonormal basis for V. Moreover, the products satisfy

$$e_i \cdot e_j + e_j \cdot e_i = 0 \quad i \neq j \quad , \qquad (A.2)$$

$$e_i \cdot e_i = -2 < e_i, e_i > \quad i = 1, ..., n \quad . \qquad (A.3)$$

As a vector space, $Cl(V)$ is isomorphic to $\Lambda^*(V)$, the Grassmann algebra, with

$$e_{i_1} ... e_{i_n} \longrightarrow e_{i_1} \wedge ... \wedge e_{i_n} \quad .$$

There are two natural *involutions* on $Cl(V)$. The first, denoted by $\alpha : Cl(V) \to Cl(V)$, is induced by the involution $x \to -x$ defined on V, which extends to an automorphism of $Cl(V)$. The eigenspace of α with eigenvalue +1 consists of the even elements of $Cl(V)$, and the eigenspace of α of eigenvalue –1 consists of the odd elements of $Cl(V)$.

The second involution is a mapping $x \to x^t$, induced on generators by

$$\left(e_{i_1} \dots e_{i_p}\right)^t = e_{i_p} \dots e_{i_1} \quad ,$$

where e_i are basis elements of V. Moreover, we define $x \to \bar{x}$, a third involution of $Cl(V)$, by $\bar{x} \equiv \alpha(x^t)$.

One then defines $Cl^*(V)$ to be the group of invertible elements of $Cl(V)$, and the Clifford group $\Gamma(V)$ is the subgroup of $Cl^*(V)$ defined by

$$\Gamma(V) \equiv \left\{ x \in Cl^*(V) : y \in V \Rightarrow \alpha(x) y x^{-1} \in V \right\} \qquad (A.4)$$

One can show that the mapping $\rho(x) : V \to V$ given by $\rho(x)y = \alpha(x) y x^{-1}$ is an isometry of V with respect to the quadratic form Q. The mapping $x \to ||x|| \equiv x\bar{x}$ is the square-norm mapping, and enables one to define a remarkable subgroup of the Clifford group, i.e.

$$\mathrm{Pin}(V) \equiv \left\{ x \in \Gamma(V) : \|x\| = 1 \right\} \quad . \qquad (A.5)$$

APPENDIX B: Rarita-Schwinger Equations

Following Aichelburg and Urbantke 1981, one can express the Γ-potentials of (8.6.1) as (cf. (8.9.12))

$$\Gamma^{A}{}_{BB'} = \nabla_{BB'} \alpha^{A} \quad . \tag{B.1}$$

Thus, acting with $\nabla_{CC'}$ on both sides of (B.1), symmetrizing over $C'B'$ and using the spinor Ricci identity (8.7.7), one finds

$$\nabla_{C(C'} \Gamma^{AC}{}_{B')} = \tilde{\Phi}_{B'C'L}{}^{A} \alpha^{L} \quad . \tag{B.2}$$

Moreover, acting with $\nabla_{C}{}^{C'}$ on both sides of (B.1), putting $B' = C'$ (with contraction over this index), and using the spinor Ricci identity (8.7.4) leads to

$$\epsilon^{AB} \nabla_{(C}{}^{C'} \Gamma_{|A|B)C'} = -3\Lambda \alpha_{C} \quad . \tag{B.3}$$

Equations (B.1)-(B.3) rely on the conventions in Aichelburg and Urbantke 1981. However, to achieve agreement with the conventions in Penrose 1994 and in our book, the equations (8.6.3)-(8.6.6) are obtained by defining (cf. (B.1))

$$\Gamma_{B}{}^{A}{}_{B'} \equiv \nabla_{BB'} \alpha^{A} \quad , \tag{B.4}$$

and similarly for the γ-potentials of (8.6.2) (for the effect of torsion terms, see comments following equation (21) in Aichelburg and Urbantke 1981).

APPENDIX C:Fibre Bundles

The basic idea in fibre-bundle theory is to deal with topological spaces which are locally, but not necessarily globally, a product of two spaces. This appendix begins with the definition of fibre bundles and the reconstruction theorem for bundles, jointly with a number of examples, following Nash and Sen 1983. A more formal presentation of some related topics is then given, for completeness.

A fibre bundle may be defined as the collection of the following five mathematical objects:

(1) A topological space E called the total space.

(2) A topological space X, i.e. the base space, and a projection $\pi : E \rightarrow X$ of E onto X.

(3) A third topological space F, i.e. the fibre.

(4) A group G of homeomorphisms of F, called the structure group.

(5) A set $\{U_\alpha\}$ of open coordinate neighbourhoods which cover X. These reflect the *local* product structure of E. Thus, a homeomorphism ϕ_α is given

$$\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times F \quad , \tag{C.1}$$

such that the composition of the projection map π with the inverse of ϕ_α yields points of U_α, i.e.

$$\pi \, \phi_\alpha^{-1}(x, f) = x \quad x \in U_\alpha \quad , \quad f \in F \quad . \tag{C.2}$$

To see how this abstract definition works, let us focus on the Möbius strip, which can be obtained by twisting ends of a rectangular strip before joining them. In this case, the base space X is the circle S^1, while the fibre F is a line segment. For any $x \in X$, the action of π^{-1} on x yields the fibre over x. The structure group G appears on going from local coordinates $\left(U_\alpha, \phi_\alpha\right)$ to local coordinates $\left(U_\beta, \phi_\beta\right)$.

If U_α and U_β have a non-empty intersection, then $\phi_\alpha \circ \phi_\beta^{-1}$ is a continuous invertible map

$$\phi_\alpha \circ \phi_\beta^{-1} : \left(U_\alpha \cap U_\beta\right) \times F \rightarrow \left(U_\alpha \cap U_\beta\right) \times F \quad . \tag{C.3}$$

For fixed $x \in U_\alpha \cap U_\beta$, such a map becomes a map $h_{\alpha\beta}$ from F to F. This is, by definition, the transition function, and yields a homeomorphism of the fibre F. The structure group G of E is then defined as the set of all these maps $h_{\alpha\beta}$ for *all* choices of local coordinates $\left(U_\alpha, \phi_\alpha\right)$. Here, it consists of just two elements $\{e, h\}$. This is best seen on considering the covering $\{U_\alpha\}$ which is given by two open arcs of S^1 denoted by U_1 and U_2. Their intersection consists of two disjoint open arcs A and B, and hence the transition functions $h_{\alpha\beta}$ are found to be

$$h_{12}(x) = e \quad \text{if} \quad x \in A \quad , \quad h \quad if \quad x \in B \quad , \tag{C.4}$$

$$h_{12}(x) = h_{21}^{-1}(x) \quad , \tag{C.5}$$

$$h_{11}(x) = h_{22}(x) = e \quad . \tag{C.6}$$

To detect the group $G = \{e, h\}$ it is enough to move the fibre once round the Möbius strip. By virtue of this operation, F is reflected in its midpoint, which implies that the group element h is responsible for such a reflection. Moreover, on squaring up the reflection one obtains the identity e, and hence G has indeed just two elements.

So far, our definition of a bundle involves the total space, the base space, the fibre, the structure group and the set of open coordinate neighbourhoods covering the base space. However, the essential information about a fibre bundle can be obtained from a smaller set of mathematical objects, i.e. the base space, the fibre, the structure group and the transition functions $h_{\alpha\beta}$. Following again Nash and Sen 1983 we now prove the reconstruction theorem for bundles, which tells us how to obtain the total space E, the projection map π and the homeomorphisms ϕ_α out of $\left(X, F, G, \{h_{\alpha\beta}\}\right)$.

First, E is obtained out of an equivalence relation, as follows. One considers the set $\widetilde{E}$ defined as the union of all products of the form $U_\alpha \times F$, i.e.

$$\widetilde{E} \equiv \bigcup_\alpha U_\alpha \times F \quad . \tag{C.7}$$

One here writes (x, f) for an element of $\widetilde{E}$, where $x \in U_\alpha$. An equivalence relation $\sim$ is then introduced by requiring that, given $(x, f) \in U_\alpha \times F$ and $(x', f') \in U_\beta \times F$, these elements are equivalent,

$$(x, f) \sim (x', f') \quad , \tag{C.8}$$

if

$$x = x' \quad \text{and} \quad h_{\alpha\beta}(x) f = f' \quad . \tag{C.9}$$

This means that the transition functions enable one to pass from f to f', while the points x and x' coincide. The desired total space E is hence given as

$$E \equiv \widetilde{E}/\sim \quad , \tag{C.10}$$

i.e. E is the set of all equivalence classes under $\sim$.

Second, denoting by $[(x,f)]$ the euivalence class containing the element (x, f) of $U_\alpha \times F$, the projection $\pi : E \to X$ is defined as the map

$$\pi : [(x, f)] \to x \quad . \tag{C.11}$$

In other words, π maps the equivalence class $[(x, f)]$ to $x \in U_\alpha$.

Third, the function ϕ_α is defined (indirectly) by giving its inverse

$$\phi_\alpha^{-1} : U_\alpha \times F \to \pi^{-1}(U_\alpha) \quad . \tag{C.12}$$

Note that, by construction, ϕ_α^{-1} satisfies the condition

$$\pi\, \phi_\alpha^{-1}(x, f) = x \in U_\alpha \quad , \tag{C.13}$$

and this is what we actually need, despite one might be tempted to think in terms of ϕ_α rather than its inverse.

Appendix C

The readers who are not familiar with fibre-bundle theory may find it helpful to see an application of this reconstruction theorem. For this purpose, we focus again on the Möbius strip. Thus, our data are the base $X = S^1$, a line segment representing the fibre, the structure group $\{e, h\}$, where h is responsible for F being reflected in its midpoint, and the transition functions $h_{\alpha\beta}$ in (C.4)-(C.6). Following the definition (C.8)-(C.9) of equivalence relation, and bearing in mind that $h_{12} = h$, one finds

$$f = f' \text{ if } x \in A \ , \tag{C.14}$$

$$hf = f' \text{ if } x \in B \ , \tag{C.15}$$

where A and B are the two open arcs whose disjoint union gives the intersection of the covering arcs U_1 and U_2. In the light of (C.14)-(C.15), if $x \in A$ then the equivalence class $[(x, f)]$ consists of (x, f) only, whereas, if $x \in B$, $[(x,f)]$ consists of two elements, i.e. (x,f) and (x, hf). Hence it should be clear how to construct the total space E by using equivalence classes, according to (C.10). What happens can be divided into three steps (Nash and Sen 1983):

(i) The base space splits into two, and one has the covering arcs U_1, U_2 and the intersection regions A and B.

(ii) The space $\tilde{\boldsymbol{E}}$ defined in (C.7) splits into two. The regions $A \cap F$ are glued together without a twist, since the equivalence class $[(x,f)]$ has only the element (x, f) if $x \in A$. By contrast, a twist is necessary to glue together the regions $B \cap F$, since $[(x, f)]$ consists of two elements if $x \in B$. The identification of (x, f) and (x, hf) under the action of ~, makes it necessary to glue with twist the regions $B \cap F$.

(iii) The bundle $E \equiv \tilde{\boldsymbol{E}}/\sim$ has been obtained. Shaded regions may be drawn, which are isomorphic to $A \cap F$ and $B \cap F$ respectively.

If we now come back to the general theory of fibre bundles, we should mention some important properties of the transition functions $h_{\alpha\beta}$. They obey a set of compatibility conditions, where repeated indices are not summed over, i.e.

$$h_{\alpha\alpha}(x) = e \quad , \quad x \in U_\alpha \quad , \tag{C.16}$$

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1} \quad , \quad x \in U_\alpha \cap U_\beta \quad , \tag{C.17}$$

$$h_{\alpha\beta}(x)\, h_{\beta\gamma}(x) = h_{\alpha\gamma}(x) \quad , \quad x \in U_\alpha \cap U_\beta \cap U_\gamma \quad . \tag{C.18}$$

A simple calculation can be now made which shows that any bundle can be actually seen as an equivalence class of bundles. The underlying argument is as follows. Suppose two bundles E and E' are given, with the same base space, fibre, and group. Moreover, let $\{\phi_\alpha, U_\alpha\}$ and $\{\psi_\alpha, U_\alpha\}$ be the sets of coordinates and coverings for E and E' respectively. The map

$$\lambda_\alpha \equiv \phi_\alpha \circ \psi_\alpha^{-1} : U_\alpha \times F \to U_\alpha \times F$$

is now required to be a homeomorphism of F belonging to the structure group G. Thus, if one combines the definitions

$$\lambda_\alpha(x) \equiv \phi_\alpha \circ \psi_\alpha^{-1}(x) \quad , \tag{C.19}$$

$$h_{\alpha\beta}(x) \equiv \phi_\alpha \circ \phi_\beta^{-1}(x) \quad , \tag{C.20}$$

$$h'_{\alpha\beta}(x) \equiv \psi_\alpha \circ \psi_\beta^{-1}(x) \quad , \tag{C.21}$$

one finds

$$\lambda_\alpha^{-1}(x) h_{\alpha\beta}(x) \lambda_\beta(x) = \psi_\alpha \circ \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1} \circ \phi_\beta \circ \psi_\beta^{-1}(x) = h'_{\alpha\beta}(x) \quad . \tag{C.22}$$

Thus, since λ_α belongs to the structure group G by hypothesis, as the transition function $h_{\alpha\beta}$ varies, both $\lambda_\alpha^{-1} \, h_{\alpha\beta} \lambda_\beta$ and $h'_{\alpha\beta}$ generate all elements of G. The only difference between the bundles E and E' lies in the assignment of coordinates, and the equivalence of such bundles is expressed by (C.22). The careful reader may have noticed that in our argument the coverings of the base space for E and E' have been taken to coincide. However, this restriction is unnecessary. One may instead consider coordinates and coverings given by $\{\phi_\alpha, U_\alpha\}$ for E, and by

$\{\psi_\alpha, V_\alpha\}$ for E'. The equivalence of E and E' is then defined by requiring that the homeomorphism $\phi_\alpha \circ \psi_\beta^{-1}(x)$ should coincide with an element of the structure group G for $x \in U_\alpha \cap V_\beta$ (Nash and Sen 1983).

Besides the Möbius strip, the naturally occurring examples of bundles are the tangent and cotangent bundles and the frame bundle. The tangent bundle $T(M)$ is defined as the collection of all tangent spaces $T_p(M)$, for all points p in the manifold M, i.e.

$$T(M) \equiv \bigcup_{p \in M} T_p(M) \quad . \qquad (C.23)$$

By construction, the base space is M itself, and the fibre at $p \in M$ is the tangent space $Tp(M)$. Moreover, the projection map $\pi : T(M) \rightarrow M$ associates to any tangent vector $\in T_p(M)$ the point $p \in M$. Note that, if M is n-dimensional, the fibre at p is an n-dimensional vector space isomorphic to R^n. The *local* product structure of $T(M)$ becomes evident if one can construct a homeomorphism ϕ_α : $\pi^{-1}(U_\alpha) \rightarrow U_\alpha \times R^n$. Thus, we are expressing $T(M)$ in terms of points of M and tangent vectors at such points. This is indeed the case since, for a tangent vector V at p, its expression in local coordinates is

$$V = b^i(p) \frac{\partial}{\partial x^i} |_p \quad . \qquad (C.24)$$

Hence the desired ϕ_α has to map V to the pair $\left(p, b^i(p)\right)$. Moreover, the structure group is the general linear group $GL(n,R)$, whose action on elements of the fibre should be viewed as the action of a matrix on a vector.

The *frame* bundle of M requires taking a total space $B(M)$ as the set of all frames at all points in M. Such (linear) frames b at $x \in M$ are, of course, an ordered set $\left(b_1, b_2, ..., b_n\right)$ of basis vectors for the tangent space $T_x(M)$. The projection $\pi : B(\mathrm{M}) \rightarrow M$ acts by mapping a base b into the point of M to which b is attached. Denoting by u an element of $GL(n, R)$, the $GL(n, R)$ action on $B(M)$ is defined by

$$\left(b_1, ..., b_n\right) u \equiv \left(b_j u_{j1}, ..., b_j u_{jn}\right) \quad . \qquad (C.25)$$

The coordinates for a differentiable structure on $B(M)$ are $\left(x^1, \ldots, x^n; u_i^j\right)$, where $x^1, \ldots, x^n$ are coordinate functions in a coordinate neighbourhood $V \subset M$, while u_i^j appear in the representation of the map

$$\gamma : V \times GL(n, R) \rightarrow p^{-1}(V) \quad , \tag{C.26}$$

by means of the rule (Isham 1989)

$$(x, u) \rightarrow \left(u_1^j (\partial_j)_x, \ldots, u_n^j (\partial_j)_x\right) \quad .$$

To complete our introduction to fibre bundles, we now define cross-sections, sub-bundles, vector bundles, and connections on principal bundles, following Isham 1989.

(i) Cross-sections are very important from the point of view of physical applications, since in classical field theory the physical fields may be viewed as sections of a suitable class of bundles. The idea is to deal with functions defined on the base space and taking values in the fibre of the bundle. Thus, given a bundle (E, π, M), a *cross-section* is a map $s : M \rightarrow E$ such that the image of each point $x \in M$ lies in the fibre $\pi^{-1}(x)$ over x:

$$\pi \circ s = \mathrm{id}_M \quad . \tag{C.27}$$

In other words, one has the projection map from E to M, and the cross-section from M to E, and their composition yields the identity on the base space. In the particular case of a product bundle, a cross-section defines a unique function $\hat{s} : M \rightarrow F$ given by

$$s(x) = (x, \hat{s}(x)) \quad , \quad \forall x \in M \quad . \tag{C.28}$$

(ii) The advantage of introducing the sub-bundle E' of a given bundle E lies in the possibility to refer to a mathematical structure less complicated than the original E'. Let (E, π, M) be a fibre bundle with fibre F. A sub-bundle of (E, π, M) is a sub-space of E with the extra property that it always contains complete fibres of E, and hence is itself a fibre bundle. The formal definition demands that the following conditions on (E', π', M') should hold:

$$E' \subset E \quad , \tag{C.29}$$

$$M' \subset M \quad , \tag{C.30}$$

$$\pi' = \pi|_{E} \quad . \tag{C.31}$$

In particular, if $T \equiv (E, \pi, M)$ is a sub-bundle of the product bundle $(M \times F, \mathrm{pr}_1, M)$, then cross-sections of T have the form $s(x) = (x, \hat{s}(x))$, where $\hat{s}$: $M \rightarrow F$ is a function such that, $\forall x \in M$, $(x, \hat{s}(x)) \in E$. For example, the tangent bundle TS^n of the n-sphere S^n may be viewed as the sub-bundle of $S^n \times R^{n+1}$ (Isham 1989)

$$E(TS^n) \approx \left\{(x, y) \in S^n \times R^{n+1} : x \cdot y = 0\right\} \quad . \tag{C.32}$$

Cross-sections of TS^n are vector fields on the n-sphere. It is also instructive to introduce the normal bundle $v(S^n)$ of S^n, i.e. the set of all vectors in R^{n+1} which are normal to points on S^n (Isham 1989):

$$E(\nu(S^n)) \equiv \left\{(x, y) \in S^n \times R^{n+1} : \exists k \in R : y = kx\right\} \quad . \tag{C.33}$$

(iii) In the case of vector bundles, the fibres are isomorphic to a vector space, and the space of cross-sections has the structure of a vector space. Vector bundles are relevant for theoretical physics, since gauge theory may be formulated in terms of vector bundles (Ward and Wells 1990), and the space of cross-sections can replace the space of functions on a manifold (although, in this respect, the opposite point of view may be taken). By definition, a n-dimensional real (resp. complex) vector bundle (E, π, M) is a fibre bundle in which each fibre is isomorphic to a n-dimensional real (resp. complex) vector space. Moreover, $\forall x \in M$, a neighbourhood $U \subset M$ of x exists, jointly with a *local* trivialization $\rho : U \times R^n \rightarrow \pi^{-1}(U)$ such that, $\forall y \in U$, $\rho : \{y\} \times R^n \rightarrow \pi^{-1}(y)$ is a *linear map.*

The simplest examples are the product space $M \times R^n$, and the tangent and cotangent bundles of a manifold *M.* A less trivial example is given by the normal bundle (cf. (C.33)). If M is a m-dimensional sub-manifold of R^n, its *normal bundle* is a $(n - m)$-dimensional vector bundle $v(M)$ over *M,* with total space (Isham 1989)

$$E(\nu(M)) \equiv \{(x,v) \in M \times R^n : v \cdot w = 0, \forall w \in T_x(M)\} \quad , \tag{C.34}$$

and projection map $\pi : E(v(M)) \to M$ defined by $\pi(x,v) \equiv x$. Last, but not least, we should mention the *canonical real line bundle* γ_n over the real projective space RP^n, with total space

$$E(\gamma_n) \equiv \{([x],v) \in RP^n \times R^{n+1} : v = \lambda\, x, \lambda \in R\} \quad , \tag{C.35}$$

where $[x]$ denotes the line passing through $x \in R^{n+1}$. The projection map π : $E(\gamma_n) \to RP^n$ is defined by the condition

$$\pi([x], v) \equiv [x] \quad . \tag{C.36}$$

Its inverse is therefore the line in R^{n+1} passing through x. Note that γ_n is a one-dimensional vector bundle.

(iv) In Nash and Sen 1983, principal bundles are defined by requiring that the fibre F should be (isomorphic to) the structure group. However, a more precise definition, such as the one given in Isham 1989, relies on the theory of Lie groups. Since it is impossible to describe such a theory in a short appendix, we refer the reader to Isham 1989 and references therein for the theory of Lie groups, and we limit ourselves to the following definitions.

A bundle (E, π, M) is a G-bundle if E is a right G-space and if (E, π, M) is isomorphic to the bundle $(E, \sigma, E/G)$, where E/G is the orbit space of the G-action on E, and σ is the usual projection map. Moreover, if G acts freely on E, then (E, π, M) is said to be a *principal* G-bundle, and G is the structure group of the bundle. Since G acts freely on E by hypothesis, each orbit is homeomorphic to G, and hence one has a fibre bundle with fibre G (see earlier remarks).

Appendix C

To define connections in a principal bundle, with the associated covariant differentiation, one has to look for vector fields on the bundle space P that point from one fibre to another. The first basic remark is that the tangent space $T_p(P)$ at a point $p \in P$ admits a natural direct-sum decomposition into two sub-spaces $V_p(P)$ and $H_p(P)$, and the connection enables one to obtain such a split of $T_p(P)$. Hence the elements of $T_p(P)$ are uniquely decomposed into a sum of components lying in $V_p(P)$ and $H_p(P)$ by virtue of the connection. The first sub-space, $V_p(P)$, is defined as

$$V_p(P) \equiv \{t \in T_p(P) : \pi_* t = 0\} \quad , \tag{C.37}$$

where $\pi : P \rightarrow M$ is the projection map from the total space to the base space. The elements of $V_p(P)$ are, by construction, *vertical* vectors in that they point along the fibre. The desired vectors, which point away from the fibres, lie instead in the horizontal sub-space $H_p(P)$. By definition, a *connection* in a principal bundle $P \rightarrow M$ with group G is a smooth assignment, to each $p \in P$, of a *horizontal* sub-space $H_p(P)$ of $T_p(P)$ such that

$$T_p(P) \approx V_p(P) \oplus H_p(P) \quad . \tag{C.38}$$

By virtue of (C.38), a connection is also called, within this framework, a *distribution.* Moreover, the decomposition (C.38) is required to be compatible with the right action of G on P.

The constructions outlined in this appendix are the first step towards a geometrical and intrinsic formulation of gauge theories, and they are frequently applied also in twistor theory (sections 5.1-5.3, 9.6-9.7). Thus, despite the incompleteness, we hope that the reader will find them useful.

APPENDIX D: Sheaf Theory

In chapter four we have given an elementary introduction to sheaf cohomology. However, to understand the language of section 9.6, it may be helpful to supplement our early treatment by some more precise definitions. This is here achieved by relying on Chern 1979.

The definition of a sheaf of Abelian groups involves two topological spaces S and M, jointly with a map $\pi : S \rightarrow M$. The sheaf of Abelian groups is then the pair (S, π) such that:

(i) π is a local homeomorphism;

(ii) $\forall x \in M$, the set $\pi^{-1}(x)$, i.e. the *stalk* over x, is an Abelian group;

(iii) the group operations are continuous in the topology of S.

Denoting by U an open set of M, a *section* of the sheaf S over U is a continuous map $f: U \rightarrow S$ such that its composition with π yields the identity (cf. appendix C). The set $\Gamma(U, S)$ of all (smooth) sections over U is an Abelian group, since if $f, g \in \Gamma(U, S)$, one can define f-g by the condition $(f-g)(x) \equiv f(x)-g(x)$, $x \in U$. The *zero* of $\Gamma(U, S)$ is the zero section assigning the zero of the stalk $\pi^{-1}(x)$ to every $x \in U$.

The following step is the definition of *presheaf* of Abelian groups over M. This is obtained on considering the homomorphism between sections over U and sections over V, for V an open subset of U. More precisely, by a presheaf of Abelian groups over M we mean (Chern 1979):

(i) a basis for the open sets of M;

(ii) an Abelian group S_U assigned to each open set U of the basis;

(iii) a homomorphism $\rho_{VU} : S_U \rightarrow S_V$ associated to each inclusion $V \subset U$, such that

$$\rho_{WV} \, \rho_{VU} = \rho_{WU} \quad \text{whenever} \quad W \subset V \subset U \, .$$

The sheaf is then obtained from the presheaf by a limiting procedure (cf. chapter four).

For a given complex manifold M, the following sheaves play a very important role (cf. section 9.6):

(i) The sheaf $A_{p\ q}$ of germs of complex-valued C^∞ forms of type $(p,\ q)$. In particular, the sheaf of germs of complex-valued C^∞ functions is denoted by A_{00}.

(ii) The sheaf $C_{p\ q}$ of germs of complex-valued C^∞ forms of type $(p,\ q)$, closed under the operator $\overline{\partial}$. The sheaf of germs of holomorphic functions (i.e. zero-forms) is denoted by $\mathcal{O} = C_{00}$. This is the most important sheaf in twistor theory (as well as in the theory of complex manifolds, cf. Chern 1979).

(iii) The sheaf $\mathcal{O}^*$ of germs of nowhere-vanishing holomorphic functions. The group operation is the multiplication of germs of holomorphic functions.

Following again Chern 1979, we complete this brief review by introducing *fine sheaves.* They are fine in that they admit a partition of unity subordinate to *any* locally finite open covering, and play a fundamental role in cohomology, since the corresponding cohomology groups $H^q(M,\ S)$ vanish $\forall q \geq 1$. Partitions of unity of a sheaf of Abelian groups, subordinate to the locally finite open covering U of M, are a collection of sheaf homomorphisms $\eta_i : S \rightarrow S$ such that:

(i) η_i is the zero map in an open neighbourhood of $M - U_i$;

(ii) $\Sigma_i \eta_i$ equals the identity map of the sheaf $(S,\ \pi)$.

The sheaf of germs of complex-valued C^∞ forms is indeed fine, while C_{pq} and the constant sheaf are not fine.

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