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SOME MATHEMATICAL CONCEPTS OF THE ANALYTIC HIERARCHY PROCESS

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In this paper we give a skeletal outline of the foundations of the Analytic Hierarchy Process and some of its highlights. We also show that the principal eigenvector solution is essential for deriving the scale of priorities from the fundamental scale of judgments given in the matrix.

1. Introduction

Three principles are used in problem solving with the AHP : decomposition, comparison, and synthesis of priorities.

The decomposition principle is applied to construct a hierarchy with elements in a level independent from those in succeeding levels, working downward from the goal at the top, to criteria bearing on the goal in the second level, to subcriteria in the third level, etc., from the general (and sometimes uncertain) to the particular alternatives at the bottom level.

The principle of comparative judgments is applied to construct a pairwise comparison reciprocal matrix of the relative importance of elements in a level with respect to a criterion in the level above. Its principal eigenvector gives the priorities.

The third principle is to synthesize the priorities downward by weighting their local priorities by the priority of their corresponding criterion in the level above, and adding for each element in a level according to the criteria it affects. (The second level elements are multiplied by unity, the weight of the single top level goal). This gives the composite or global priority of that element, which is then used to weight the local priorities of the elements in the level below compared to each other with it as the criterion, and so on to the bottom level.

Four axioms govern the Analytic Hierarchy Process and utilize the notion of paired comparisons as a primitive.

Let A be a finite set of n elements called alternatives. Let C be a set of properties or attributes with respect to which elements in A are compared. We will refer to the elements of C as criteria. A criterion is a primitive.

We perform binary comparisons on the elements in $\boldsymbol{A}$ according to a criterion in $\boldsymbol{C}$. Let $>_C$ be a binary relation on $\boldsymbol{A}$ representing "more preferred than" with respect to a criterion C in $\boldsymbol{C}$. Let $\sim_E$ be the binary relation "indifferent to" with respect to a criterion C in $\boldsymbol{C}$. Hence, given two elements $A_i, A_j \epsilon A$, either $A_i >_C A_j$ or $A_j >_C A_i$ or $A_i \sim_C A_j$ for all $C \in \boldsymbol{C}$. We use $A_i \gtrsim_C A_j$ to indicate more preferred or indifferent. A given family of binary relations $>_C$ with respect to a criterion C in $\boldsymbol{C}$ is a primitive.

Let $\boldsymbol{B}$ be the set of mappings from $\boldsymbol{A}\times\boldsymbol{A}$ to $\boldsymbol{R}^+$ (the set of positive reals). Let f : $\boldsymbol{C} \to \boldsymbol{B}$. Let $P_C \in f(C)$ for $C \in \boldsymbol{C}$. P_C assigns a positive real number to every pair $(A_i, A_j) \in \boldsymbol{A}\times\boldsymbol{A}$. Let $P_C(A, A) \equiv a_{ij} \in \boldsymbol{R}^+$, $A_i, A_j \in \boldsymbol{A}$. For each $C \in \subset$, the triple $(\boldsymbol{A}\times\boldsymbol{A}, \boldsymbol{R}^+,$

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Pc) is a *fundamental* or *primitive* scale. A fundamental scale is a mapping of objects to a numerical system.

Definition: *For all* A_i, A_j **A** *and* $C \in \boldsymbol{C}$

$$A_i >_C A_j \text{ if and only if } Pc\ (A_i, A_j)>1,$$
$$A_i \sim_C A_j \text{ if and only if } Pc\ (A_i, A_j)=1.$$

If $A_i >_C A_j$ we say that A_i dominates A_j with respect to $C \in \boldsymbol{C}$. Thus Pc represents the intensity or strength of preference for one alternative over another.

Axiom 1 (the Reciprocal Axiom). *For all* $A_i, A_j \in \boldsymbol{A}$ *and* $C \in \boldsymbol{C}$

$$Pc\ (A_i, A_j)=1/Pc(A_j, A_i).$$

This axiom says that the comparison matrices we construct are formed of paired reciprocal comparisons, for if one stone is judged to be five times heavier than another, then the other must perforce be one fifth as heavy as the first. It is this simple but powerful relationship that is the basis of the AHP.

Let $A=(a_{ij})\equiv(Pc(A_i, A_j))$ be the set of paired comparisons of the alternatives with respect to a criterion $C \in \boldsymbol{C}$. By the definition of Pc and Axiom 1, A is a positive reciprocal matrix. The object is to obtain a scale of relative dominance (or rank order) of the alternatives from the paired comparsions given in A.

We will now show how to derive the relative dominance of a set of alternatives from a pairwise comparison matrix $\boldsymbol{A}$. Let $\boldsymbol{R}_{M(n)}$ be the set of $(n\times n)$ positive reciprocal matrices $\boldsymbol{A}=(a_{ij})\equiv(Pc(A_i, A_j))$ for all $C \in \boldsymbol{C}$. Let $[0, 1]^n$ be the n-fold cartesian product of $[0, 1]$ and let $\boldsymbol{W}: \boldsymbol{R}_{M(n)}>[0, 1]^n$ for $\boldsymbol{A}\epsilon\boldsymbol{R}_{M(n)}$, $\boldsymbol{W}(\boldsymbol{A})$ is an n-dimensional vector whose components lie in the interval $[0, 1]$. The triple $(\boldsymbol{R}_{M(n)}, [0, 1]^n, \boldsymbol{W})$ is a derived scale. A derived scale is a mapping between two numerical relational systems.

An important aspect of the AHP is the idea of consistency. If one has a scale for a property possessed by some objects and measures that property in them, then their relative weights with respect to that property are fixed. In this case there is no judgmental inconsistency (although if one has a physical scale and applies it to objects in pairs and *then* derives the relative standing of the objects on the scale from the pairwise comparison matrix, it is likely that inaccuracies will have occurred in the act of applying the physical scale and again there would be inconsistency). But when comparing with respect to a property for which there is no established scale or measure, we are trying to derive a scale though comparing the objects two at a time. Since the objects may be involved in more than one comparison and we have no standard scale, but are assigning relative values as a matter of judgment, inconsistencies may well occur. In the AHP consistency is defined in the following way, and we are able to measure inconsistency.

Definition. *The mapping Pc is said to be consistent if and only if*

$$Pc(A_i, A_j)\ Pc(A_j, A_k)=Pc(A_i, A_k) \qquad \text{for all } i, j, k$$

Similarly, the matrix A is consistent if and only if $a_{ij}\ a_{jk}=a_{ik}$ *for all i, j, and k.*

We now turn to the hierarchic axioms, 2, 3, and 4, and related definitions.

In a partially ordered set, we define $x<y$ to mean that $x<y$ and $x\neq y$, y is said to

cover (dominate) x. If $x<y$ then $x<t<y$ is possible for no t. We use the notation $x^{-}=\{y|x$ covers $y\}$ and $x^{+}=\{y|y$ covers $x\}$, for any element x in an ordered set.

Let H be a finite partially ordered set. Then H is a hierarchy if it satisfies the conditions:

a) There is a partition of H into sets L_k, $k=1, \cdots, h$, for some h where $L_k=\{b\}$, b a single element.

b) $x \in L_k$ *implies* $x^{-} \in L_{k+1}$ $\quad k=1, \cdots, h-1$.

c) $x \in L_k$ *implies* $x^{+} \in L_{k-1}$ $\quad k=2, \cdots, h$.

The notions of fundamental and derived scales can be extended to $x \in L_k$, $x \subseteq L_{k+1}$ replacing C and A respectively. The derived scale resulting form comparing the elements in x^{-} with respect to x is called a local derived scale or the local priorities for the elements in x^{-}.

Definition. Given a positive real number $p \geq 1$, a nonempty set $x^{-} \subseteq L_{k+1}$ is said to be ρ-homogeneous with respect to $x \in L_k$ if for every pair of elements $y_1, y_2 \in x^{-}$, $1/\rho \leq Pc(y_1, y_2) \leq \rho$. In particular the reciprocal axiom implies that $Pc(y_i, y_i)=1$.

Axiom 2 (the Homogeneity Axiom). *Given a hierarchy H, $x \in H$ and $x \in L$, $x^{-} \subseteq L$ is ρ-homogeneous for $k=1, \cdots, h-1$.*

Homogeneity is essential for meaningful comparisons, as the mind cannot compare widely disparate elements. For example, we cannot compare a grain of sand with an orange according to size. When the disparity is great, elements should be placed in separate clusters of comparable size, or in different levels altogether.

Given L_k, $L_{k+1} \subseteq H$, let us denote the local derived scale for $y \in x^{-}$ and $x \in L_k$ by $\Psi_{k+1}(y/x)$, $k=2, 3, \cdots, h-1$. Without loss of generality we may assume that $\Psi_{k+1}(yx)=1$. Consider the matrix $\Psi_k(L_k/L_{k-1})$ whose columns are local derived scales of elements in L_k with respect to elements in L_{k-1}.

Definition. A set A is said to be *outer dependent* on a set C if a fundamental scale can be defined on A with respect to every $C \in \boldsymbol{C}$.

The process of relating elements (e.g. alternatives) in one level of the hierarchy according to the elements of the next higher level (e.g. criteria) expresses the dependence (what is called *outer dependence*) of the lower elements on the higher so that comparisons can be made between them, the steps are repeated upward in the hierarchy through each pair of adjacent levels to the top element, the focus or goal.

The elements in a level may also depend on one another with respect to a property in another level. Input-output of industries is an example of the idea of inner dependence, formalized as follows:

Definition. Let $\boldsymbol{A}$ be outer dependent on $\boldsymbol{C}$. The elements in $\boldsymbol{A}$ are said to be *inner dependent* with respect to $C \in \boldsymbol{C}$ if for some $A \in \boldsymbol{A}$, $\boldsymbol{A}$ is outer dependent on A.

Axiom 3. *Let H be a hierarchy with levels $L_1, L_2, \cdots, L_h$. For each L_k, $k=1, 2, \cdots, h-1$,*

(1) *L_{k-1} is outer dependent on L_k,*

(2) *L_{k+1} is not inner dependent with respect to all $x \in L_k$,*
(3) *L_k^{k+1} is not outer dependent on L_{k+1}.*

Axiom 4 (the Axiom of Expectations).

$$C \subset H - L_h,\ A = L_h$$

This axiom is merely the statement that thoughtful individuals who have reasons for their beliefs should make sure that their ideas are adequately represented in the model. All alternatives, criteria and expectation (explicit and implicit) can be and should be represented in the hierarchy. This axiom does not assume. People are known at times to harbor irrational expectations and such expectations can be accommodated.

Based on the concepts in Axiom 3 we now develop a weighting function. For each $x \in H$, there is a suitable weighting function (whose nature depends on the phenomenon being hierarchically structured):

$$w_x:\ x^1 \rightarrow [0, 1] \text{ such that } \sum_{y \in x^-} w_x(y) = 1.$$

Note that $h=1$ is the last level for which x is not empty.

The sets L_i are the levels of the hierarchy, and the function w_x is the priority function of the elements in one level with respect to the objective x. We observe that even if $x \neq L$ (for some level L_k), w_x may be defined for all of L_k by setting it equal to zero for all elements in L_k not in x.

The weighting function is one of the more significant contributions towards the application of hierarchy theory.

Definition: A hierarchy is complete if, for all $x \subset L_k$, $x^+ \subset L_{k-1}$.

We can state the central question:

Basic Problem: Given any element $x \in L_\alpha$, and subset $S \subset L_\beta$, $(\alpha < \beta)$, how do we define a function $w_{x,S}:\ S \rightarrow [0, 1]$ which reflects the properties of the priority functions on the levels L_k, $k = \alpha, \cdots, \beta - 1$. Specifically, what is the function $w_{b,L_h}:\ L_h \rightarrow [0, 1]$?

In less technical terms, this can be paraphrased thus: given a social (or economic) system with a major objective b, and the set L_h of basic activities, such that the system can be modeled as a hierarchy with largest element b and lowest level L_h. What are the priorities of the elements of any level and in particular those of L_h with respect to b?

From the standpoint of optimization, to allocate a resource to the elements any interdependence may take the form of input-output relations such as, for example, the interflow of products between industries. A high priority industry may depend on flow of material from a low priority industry. In an optimization framework, the priority of the elements enables one to define the objective function to be maximized, and other hierarchies supply information regarding constraints, e.g., input-output relations.

We now present the method to solve the Basic Problem. Assume that $Y = \{y_1, \cdots, y_{m_k}\} \subset L_k$ and that $X = \{x_1, \cdots, x_{m_{k+1}}\} \subset L_{k+1}$. Without loss of generality we may assume that $X = L_{k+1}$, and that there is an element $z \in L_k$ such that $y_i \in z$. Then consider the priority functions

$$w_z:\ Y \rightarrow [0, 1] \text{ and } w_{y_j}:\ X \rightarrow [0, 1] \quad j = 1, \cdots, m_k.$$

Construct the priority function of the elements in X with respect to z, denoted w, $w: X \to [0, 1]$, by

$$w(x_i) = \sum_{j=1}^{m_k} w_{y_j}(x_i) w_z(y_j), \quad i = 1, \cdots, m_{k+1}$$

It is obvious that is no more than the process of weighting the influence of the element y_j on the priority of x_i by multiplying it with the importance of x_i with respect to z.

The algorithms involved will be simplified it one combines the $w_{y_j}(x_i)$ into a matrix $\boldsymbol{B}$ by setting $b_{ij} = w_{y_j}(x_i)$. If one further sets $w_i = w(x_i)$ and $w'_j = w_z(y_j)$, then the above formula becomes

$$w_i = \sum_{j=1}^{m_k} b_{ij} w_j' \qquad i = 1, \cdots, n_{k+1}$$

Thus, one may speak of the priority vector $\boldsymbol{w}$ and, indeed, of the priority matrix $\boldsymbol{B}$ of the $(k+1)$st level; this gives the final formulation

$$w = \boldsymbol{B}\boldsymbol{w}'$$

The following is easy to prove:

Theorem. *Let H be a complete hierarchy with largest element b and h levels. Let $\boldsymbol{B}_k$ be the priority matrix of the kth level, $k = 1, \cdots, h$. If $\boldsymbol{w}'$ is the priority vector of the pth level with respect to some element z in the $(p-1)$st level, then the priority vector $\boldsymbol{w}$ of the qth level $(p < q)$ with respect to z is given by.*

$$\boldsymbol{w} = \boldsymbol{B}_q \ \boldsymbol{B}_{q-1} \cdots \boldsymbol{B}_{p+1} \ \boldsymbol{w}'$$

Thus, the priority vector of the lowest level with respect to the element b *is* given by:

$$\boldsymbol{w} = \boldsymbol{B}_h \boldsymbol{B}_{h-1} \cdots \boldsymbol{B}_2 \boldsymbol{b}_1$$

if L_1 has a single element, $\boldsymbol{b}_1 = 1$. Otherwise, $\boldsymbol{b}_1$ is a prescribed vector. We note that the pairwise comparison process takes into consideration nonlinearities. Such nonlinearities are captured by the composition weighting process.

2. Network systems

Often alternatives depend on criteria and criteria on alternatives and there should be a cycle connecting the two which is more accurately studied with the network feedback approach. The AHP has been generalized to deal with feedback as shown below, although people generally prefer to simplify and arrange their thinking in terms of a linear hierarchy even if the answers are only approximate.

A network is a set of nodes (each of which consists of a set of elements) and a set of arcs which indicate the order of interaction among the components. The priorities of the elements in each node are components of the principal eigenvector of the matrix of pairwise of the relative impact of these elements with respect to an element or node with which they interact. The interaction is indicated by an arc of the network. All such eigenvectors define what is known as a supermatrix of impact priorities. By weighting the eigenvectors corresponding to each component by the priority of that component in the system, the supermatrix is transformed into a stochastic matrix. The limiting impact priorities are obtained by computing large powers of this matrix. Formally we have:

Definition A partially ordered set S is a network system if

a) There is a partition of S into sets C_k, $k=1, \cdots, s$

b) There is an ordering on C_k, $k=1, \cdots, s$ such that $x \subseteq C_k$ implies either $\boldsymbol{x}^-$ or $\boldsymbol{x}^+$ is in C_{k_j} for some k_j or both $x^- \subseteq C_{k_j}$, $x^+ \subseteq C_{k_j}$ for one or more k_j.

c) For each $x \in S$, there is a suitable weighting function

$w_x: x^- \rightarrow [0, 1]$ such that $\sum_{y \in x^-} w_x(y) = 1$

and for $C_k \subseteq S$, $k=1, \cdots, s$ there is a weighting function

$w_{(k)}: C_k^- \rightarrow [0, 1]$

where $C_k^- = \{C_h \mid C_k \text{ covers } C_h\}$.

We shall now turn to the calculation procedures for the weights and for the inconsistency index.

3. The eigenvector solution for weights and consistency

There is an infinite number of ways to derive the vector of priorities form the matrix (a_{ij}). But emphasis on consistency leads to an eigenvalue formulation.

If a_{ij} represents the importance of alternative i over alternative j and a_{jk} represents the importance of alternative j over alternative k then a_{ik}, the importance of alternative i over alternative k, must equal $a_{ij}a_{jk}$ for the judgments to be consistent. If we do not have a scale at all, or do not have it conveniently as in the case of some measuring devices, we cannot give the precise values of w_i/w_j but only an estimate. Our problem becomes $\boldsymbol{A}' \boldsymbol{w}' = \lambda_{\max} \boldsymbol{w}'$ where $\lambda_{\max}$ is the largest or principal eigenvalue of $\boldsymbol{A}' = (a'_{ij})$ the perturbed value of $\boldsymbol{A} = (a_{ij})$ with $a'_{ji} = 1/a'_{ij}$ forced. To simplify the notation we shall continue to write $\boldsymbol{A}\boldsymbol{w} = \lambda_{\max} \boldsymbol{w}$ where $\boldsymbol{A}$ is the matrix of pairwise comparisons.

The solution is obtained by raising the matrix to a sufficiently large power then summing over the rows and normalizing to obtain the priority vector $\boldsymbol{w} = (w_1, \cdots, w_n)$. The process is stopped when the difference between components of the priority vector obtained at the kth power and at the $(k+1)$th power is less than some pre-determined small value.

An easy way to get an approximation to the priorities is to normalize the geometric means of the rows. This result coincides with the eigenvector for $n \leq 3$. A second way to obtain an aproximation is by normalizing the elements in each column of the judgment matrix and the averaging over each row.

We would like to caution the reader that for important applications one should only use the eigenvector derivation procedure because approximations can lead to rank reversal in spite of the closeness of the result to the eigenvector. It is easy to prove that for an arbitrary estimate x of the priority vector

$$\lim_{k \to \infty} \frac{1}{\lambda_{\max}^k} \boldsymbol{A}^k \boldsymbol{x} = c \boldsymbol{w}$$

where c is a positive constant and w is the principal eigenvector of $\boldsymbol{A}$. This may be interpreted roughly to say that if we begin with an estimate and operate on it successively by $A/\lambda_{\max}$ to get new estimates, the result converges to a constant multiple of the principal eigenvector.

A simple way to obtain the exact value (or an estimate) of $\lambda_{\max}$ when the exact value (or an estimate) of $\boldsymbol{w}$ is available in normalized form is to add the columns of $\boldsymbol{A}$ and multiply the resulting vector with the vector $\boldsymbol{w}$. The resulting number is $\lambda_{\max}$ (or an estimate). This follows from

$$\sum_{j=1}^{n} a_{ij}w_j = \lambda_{\max} w_i$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}w_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}\right)w_j = \sum_{i=1}^{n} \lambda_{\max} w_i = \lambda_{\max}$$

The problem is now, how good is the principal eigenvector estimate $\boldsymbol{w}$? Note that if we obtain $\boldsymbol{w}=(w_1, \cdots, w_n)$ by solving this problem, the matrix whose entries are w_i/w_j is a consistent matrix which is our consistent estimate of the matrix $\boldsymbol{A}$. The original matrix $\boldsymbol{A}$ itself need not be consistent. In fact, the entries of $\boldsymbol{A}$ need not even be transitive; i.e, A_1 may be preferred to A_2 and A_2 to A_3 but A_3 may be preferred to A_1. What we would like is a measure of the error due to inconsistency. It turns out that $\boldsymbol{A}$ is consistent if and only if $\lambda_{\max}=n$ and that we always have $\lambda_{\max}\geq n$. this suggests using $\lambda_{\max}-n$ as an index of departure from consistency. But

$$\lambda_{\max} - n = -\sum_{i=2}^{n} \lambda_i\,; \qquad \lambda_{\max} = \lambda_1,$$

where λ_i, $i=1, \cdots, n$ are the eigenvalues of $\boldsymbol{A}$. We adopt the average value $(\lambda_{\max}-n)/(n-1)$, which is the (negative) average of λ_i, $i=2, \cdots, n$ (some of which may be complex conjugates).

It is interesting to note that $2(\lambda_{\max}-n)/(n-1)$ is the variance of the error incurred in estimating a_{ij}. This can be shown by writing

$$a_{ij}=(w_i/w_j)\epsilon_{ij},\ \epsilon_{ij}>0 \text{ and } \epsilon_{ij}=1+\delta_{ij},\ \delta_{ij}>-1.$$

and substituting in the expression for $\lambda_{\max}$. It is δ_{ij} that concerns us as the error component and its value $|\delta_{ij}|<1$ for an unbiased estimator. Normalizing the principal eigenvector yields a uniques estimate of a ratio scale underlying the judgments.

The consistency index of a matrix of comparisons is given by C.I. $=\lambda_{\max}-n/n-1$. The consistency ratio (C.R.) is obtained by comparing the C.I. with the appropriate one of the following set of numbers each of which is an average random consistency index derived from a sample of size 500 of randomly generated reciprocal matrix using the scale 1/9, 1/8, $\cdots$, 1, $\cdots$, 8, 9 to see if it is about 0.10 or less (0.20 may be tolerated but not more). If it is not less than 0.10 study the problem and revise the judgments.

n	1	2	3	4	5	6	7	8	9	10
Random Consistency Index (R.I.)	0	0	.58	.90	1.12	1.24	1.32	1.41	1.45	1.49

Why Tolerate 10% Inconsistency? The priority of consistency to obtain a coherent explanation of a set of facts must differ by an order of magnitude from the priority or inconsistency which is an error in the measurement of consistency. Thus on a scale from zero to one, inconsistency should not exceed. 10 by very much. Note that the requirement of 10% should not be made much smaller such as 1% or .1%, the reason is that inconsistency itself if important, for without it new knowledge which changes preference order cannot be admitted. Assuming all knowledge to be consistent contradicts experience which requires continued adjustment in understanding. Thus the objective of developing a wide ranging consistent framework depends on admitting some inconsistency.

This also accounts for why the number of elements compared should be small. If the number of elements is large, their relative priorities would be small and error can distort these priorities considerably. If the number of items is small and the priorities are comparable a small error does not affect the order of magnitude of the answers and hence the relative priorities would be about the same. For this to happen, the items must be less than ten so their values on the whole would be over 10% each and hence remain relatively unaffected by 1% error for example.

The consistency index for an entire hierarchy is defined by

$$C_H=\sum_{j=1}^{h}\sum_{i=1}^{n_{ij}+1} w_{ij}\mu_{i,j+1}$$

where $w_{ij}=1$ for $j=1$, and n_{ij+1} is the number of elements of the $(j+1)$st level with respect to the ith criterion of the jth level.

Let $|C_{\bar{k}}|$ be the number of elements of $C_{\bar{k}}$, and let $w_{(k)(h)}$ be the priority of the impact of the hth component on the kth component i.e., $w_{(k)(h)}=w_{(k)}(C_h)$or $w_{(k)}:\ C_h\to w_{(k)(h)}$.

If we label the components of a system along lines similar to those we followed for a hierarchy, and denote by w_{jk} the limiting priority of the jth element in the kth component, w_e have

$$C_S=\sum_{k=1}^{s}\sum_{j=1}^{n_k} w_{ij}\sum_{h=1}^{|C_{\bar{k}}|} w_{(k)(h)}\mu_k(j,\ h)$$

where $\mu_k(j,\ h)$ is the consistency index of the pairwise comparison matrix of the elements in the kth component with respect to the jth element in the hth component.

4. Conclusion

The interested reader will find several hundred papers and several books written on the subject. These include both extensive theoretical developments of the subject such as how to synthesize group judgments and how to generalize the mathematics of the subject to the continuous case, and diverse practical applications in science, technology, government and business. A useful reference is F. Zahedi's article.

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