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BOOK REVIEW

Vincent Guedj, Ahmed Zeriahi: "Degenerate Complex Monge-Ampére Equations"

EMS, 2017, 496 pp.

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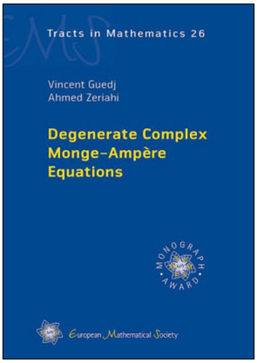

The Monge-Ampère equation in real variables is a standard example of a nonlinear elliptic equation related to geometric problems involving various curvatures. Its solutions are convex functions. The plurisubharmonic functions solve the complex counterpart of the equation. The Dirichlet problem is well posed in strictly pseudoconvex domains. For smooth nondegenerate data it was solved by Caffarelli, Kohn, Nirenberg, Spruck by a method similar to the one applied in the real case. Even earlier Bedford and Taylor showed the existence of weak solutions for the continuous data. Building on this they developed what was later named pluripotential theory. It allowed them to discover new properties of plurisubharmonic functions such as the fact that negligible sets (for this class of functions) are pluripolar—a long standing problem posed by Lelong.

The Monge-Ampère equation is also one of the central topics in complex geometry since its solution provides canonical Kähler-Einstein metrics on a given Kähler manifold with definite first Chern class (the cohomology class of the Ricci form of any Kähler form). A metric is Kähler-Einstein if the metric tensor is proportional to the Ricci tensor. The Calabi-Yau theorem settled the problem for compact manifolds having metrics of Ricci curvature zero, and the Aubin-Yau result for $c_1(X) < 0$. In the Fano case, $c_1(X) > 0$, the classical canonical metrics may not exist. Then one can hope for the existence of canonical metrics away from some analytic singularity set. Furthermore, the metrics can be studied on singular spaces. In those situations one is

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forced to study the degenerate forms of the Monge-Ampère equation. In the last two decades we have been seeing the intensive studies of the Kähler-Ricci flow, where the evolution of the Kähler metric is governed by its Ricci tensor, the studies of the conical metrics and Calabi-Yau cohomology classes degenerations. Some form of the Monge-Ampère equation does appear in those situations as well. This is the subject of this book.

Part I of the book under review "The local theory" is the introduction to pluripotential theory developed mainly in fundamental papers of E. Bedford and B.A. Taylor [1, 2]. The first section reviews the classical results on harmonic, subharmonic and plurisubharmonic functions. In the second section the basics on positive currents are given and the singularities of plurisubharmonic functions are studied by means of Lelong numbers and integrability theorems of Skoda. After those preparations one can present the Bedford-Taylor theory. The Monge-Ampère (MA for brevity) operator is defined for locally bounded plurisubharmonic functions. It is symmetric and continuous with respect to monotone sequences, which is proven with a use of Chern-Levine-Nirenberg inequalities. The theory has its limitations since by examples of Shiffman, Kiselman and Cegrell the domain of the MA operator cannot be extended to all plurisubharmonic functions, and the operator is not continuous on sequences of bounded functions converging only in L^1_{loc}. Some of the central notions are the extremal functions and the Monge-Ampère capacity (often called "relative capacity" or "Bedford-Taylor capacity" in literature). The capacity is applied to define and prove the quasi-continuity of plurisubharmonic functions. The next section is devoted to the Dirichlet problem for the Monge-Ampère equation in a ball, and in strictly pseudoconvex domains. The main results of Bedford-Taylor [1] are proven, in particular the existence of continuous solutions when the density of the right hand side and the boundary data are continuous. This is supplemented by some generalizations of the existence theorem and some stability statements. Let us remark that the elliptic PDEs approach of Caffarelli-Kohn-Nirenberg-Spruck which gives smooth solutions for smooth, nondegenerate data is only mentioned. The last section of the first part presents the viscosity solutions method adopted to the complex Monge-Ampère equations by Eyssidieux and the Authors.

Part II is titled "Pluripotential theory on compact manifolds". One of the problems which arise for the authors of this kind of book is that many results and proofs are parallel to those from Part I. So, perhaps to avoid at least some repetitions certain topics, like Cegrell's theory of finite energy classes, were skipped in Part I and are discussed only on compact manifolds. The second part begins with a short introduction to Kähler geometry which covers notions and results (without proofs) used in the sequel. Quasi-plurisubharmonic functions are treated in the next section with the focus on possibility of approximation by smooth functions within this class. Next, pluripotential theory is implemented in the context of Kähler manifolds. The final section deals with finite energy classes. Cegrell [5] defined and studied those classes in strictly pseudoconvex domains showing in particular that the MA operator can be reasonably extended to those classes which include unbounded plurisubharmonic functions. He also characterized the measures on the right hand side of the Monge-Ampère equation for which solutions of the Dirichlet problem in such classes exist. The authors of this book carried over this theory to compact Kähler manifolds and applied it also in more general settings. In this context important functionals are studied—the primitive of the MA operator and the energy bifunctional. By means of the latter the convergence in energy is defined. The MA operator is continuous on sequences converging in this manner. The domain of definition of the MA operator acting on plurisubharmonic function was described by Błocki and Cegrell. The corresponding result on compact Kähler manifolds finishes the second part of the book.

In Part III "Solving the Monge-Ampère equations" the equation is studied in a general form

$$MA(\varphi) = e^{-\lambda\varphi}\mu,$$

where λ is a real constant (the sign of which influences the solvability of the equation), μ is a probability measure on a compact Kähler manifold X, φ is the searched function quasi-plurisubharmonic with respect to a closed, semipositive and big form ω on X. The chronological presentation would start with Calabi-Yau and Aubin-Yau theorems where smooth nondegenerate data is considered and then gradually weaker assumptions would be used to obtain less regular solutions. The authors choose a different path starting with a recent variational approach of Berman, Boucksom, Guedj, Zeriahi [3]. The Monge-Ampère equation is the Euler-Lagrange equation for a maximizer of the functional $\mathcal{F}_\lambda$. The functional is well defined on finite energy class $\mathcal{E}^1$. It is shown that if the functional has certain properness property then there exists a solution of the Monge-Ampère equation in $\mathcal{E}^1$ which maximizes $\mathcal{F}_\lambda$. In particular such solutions exist for $\lambda < 0$ and μ which vanishes on pluripolar sets. If $\lambda = 0$ then we have solvability for μ such that $\mathcal{E}^1 \subset L^1(\mu)$. For $\lambda > 0$ solutions do not always exist even for smooth data. One of the sufficient conditions—Tian's invariant criterion is generalized to weak solutions in $\mathcal{E}^1$. Next the uniqueness of solutions is discussed, with proofs by Calabi—for smooth solutions, by Błocki—for bounded ones, and finally, by Dinew—in the class $\mathcal{E}$. The following section is devoted to L^∞ estimates and the continuity and Hölder continuity of solutions when X is Kähler and the right hand side $d\mu = f\omega^n$ has the density f in L^p , $p > 1$ (due to the reviewer). There are also some related stability results. Then the viscosity method is employed to establish existence of solutions when the background form is closed, smooth and semipositive, and the measure on the right hand side is continuous with respect to the volume form. Here one of the nontrivial things to prove is the identity of viscosity and pluripotential solutions. In the last section the classical solutions are treated. Thus, there is a review of the continuity method used in the proofs of Aubin and Yau with later simplifications by Siu, including the adoption, to the complex case, of the Evans-Krylov method of getting $C^{2,\alpha}$ a priori estimates. More recent results of Eyssidieux and the Authors with a generalization due to Di Nezza and Lu give regularity, off a divisor, of solutions for smooth semipositive and big background form θ such that the cohomology class $\{\theta\} - c_1(E)$ is Kähler for an effective divisor E with simple normal crossings.

Part IV "Singular Kähler-Einstein metrics" discusses applications of the theory of the MA operator to proving the existence and some regularity properties of Kähler-Einstein metrics with singularities. Calabi reduced the problem of existence of a Kähler-Einstein metric on a given Kähler metric with definite first Chern class to solving the appropriate Monge-Ampère equation. For Fano manifolds such metrics may not exist, but then there still may exist singular Kähler-Einstein metrics on $X \setminus D$

for a divisor D. They appear as weak/degenerate solutions of the Monge-Ampère equation. Another context where the equation occurs in geometry is the study of the Riemannian space of Kähler metrics introduced by Mabuchi and studied by Semmes, Donaldson, Chen and others. The geodesics joining given metrics in this space are found by solving the homogeneous Monge-Ampère equation on the product of the given manifold and the annulus. The delicate problem of finding a smooth geodesic translates into a question about regularity of those solutions. Canonical metrics, like constant scalar curvature metric or Kähler-Einstein metric are critical points of some functionals (named after Calabi, Mabuchi, Yau, Aubin) on the (Mabuchi) space of Kähler metrics or its completion. The authors give some background on those functionals and some deeper results without proofs. They also briefly discuss the existence of Kähler-Einstein metrics on Fano manifolds—the Futaki invariant, K-stability, and the recent breakthrough on the Yau-Tian-Donaldson conjecture. The last section is devoted to singular Kähler-Einstein metrics on normal Kähler spaces, obtained as solutions of the degenerate Monge-Ampère equation. Those are main results of Eyssidieux, Guedj, Zeriahi [6], Boucksom, Eyssidieux, Guedj, Zeriahi [4] and a few later papers.

Writing such a book is a very ambitious task since the subject is too large for a complete coverage and it is still growing. Furthermore the proofs of many important theorems are long and sometimes tedious. The authors had to make a selection of results and proofs. I guess that the guiding principles were: to give a detailed background at the expense of complete, long proofs of some deeper results, and to focus on developments related to the research of the authors. Thus, for instance, the Kähler-Ricci flow theory is omitted, though it is closely related to the main subject—degenerate Monge-Ampère equations. I have no doubt, however, that it is a very useful publication, especially for many young researchers starting in this rapidly growing area of research at the juncture of complex geometry, elliptic PDEs, complex analysis and pluripotential theory. A beginner needs to learn basics of all those fields and the introductory sections of the book contain many items which are difficult to find elsewhere. An ample selection of exercises is also very helpful. It should also be observed that the exposition is really good and clear.

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