Theorems on Existence and Global Dynamics for the Einstein Equations

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Abstract

This article is a guide to theorems on existence and global dynamics of solutions of the Einstein equations. It draws attention to open questions in the field. The local-in-time Cauchy problem, which is relatively well understood, is surveyed. Global results for solutions with various types of symmetry are discussed. A selection of results from Newtonian theory and special relativity that offer useful comparisons is presented. Treatments of global results in the case of small data and results on constructing spacetimes with prescribed singularity structure are given. A conjectural picture of the asymptotic behaviour of general cosmological solutions of the Einstein equations is built up. Some miscellaneous topics connected with the main theme are collected in a separate section.

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1 Introduction

Systems of partial differential equations are of central importance in physics. Only the simplest of these equations can be solved by explicit formulae. Those that cannot are commonly studied by means of approximations. There is, however, another approach that is complementary. This consists in determining the qualitative behaviour of solutions, without knowing them explicitly. The first step in doing this is to establish the existence of solutions under appropriate circumstances. Unfortunately, this is often hard, and obstructs the way to obtaining more interesting information. When partial differential equations are investigated with a view to applications, existence theorems should not become a goal in themselves. It is important to remember that, from a more general point of view, they are only a starting point.

The basic partial differential equations of general relativity are Einstein's equations. In general, they are coupled to other partial differential equations describing the matter content of spacetime. The Einstein equations are essentially hyperbolic in nature. In other words, the general properties of solutions are similar to those found for the wave equation. It follows that it is reasonable to try to determine a solution by initial data on a spacelike hypersurface. Thus the Cauchy problem is the natural context for existence theorems for the Einstein equations. The Einstein equations are also nonlinear. This means that there is a big difference between the local and global Cauchy problems. A solution evolving from regular data may develop singularities.

A special feature of the Einstein equations is that they are diffeomorphism invariant. If the equations are written down in an arbitrary coordinate system then the solutions of these coordinate equations are not uniquely determined by initial data. Applying a diffeomorphism to one solution gives another solution. If this diffeomorphism is the identity on the chosen Cauchy surface up to first order then the data are left unchanged by this transformation. In order to obtain a system for which uniqueness in the Cauchy problem holds in the straightforward sense as it does for the wave equation, some coordinate or gauge fixing must be carried out.

Another special feature of the Einstein equations is that initial data cannot be prescribed freely. They must satisfy constraint equations. To prove the existence of a solution of the Einstein equations, it is first necessary to prove the existence of a solution of the constraints. The usual method of solving the constraints relies on the theory of elliptic equations.

The local existence theory of solutions of the Einstein equations is rather well understood. Section 2 points out some of the things that are not known. On the other hand, the problem of proving general global existence theorems for the Einstein equations is beyond the reach of the mathematics presently available. To make some progress, it is necessary to concentrate on simplified models. The most common simplifications are to look at solutions with various types of symmetry and solutions for small data. These two approaches are reviewed in Sections 3 and 5, respectively. A different approach is to prove the existence of solutions with a prescribed singularity structure. This is discussed in Section 6.

Section 9 collects some miscellaneous results that cannot easily be classified. Since insights about the properties of solutions of the Einstein equations can be obtained from the comparison with Newtonian theory and special relativity, relevant results from those areas are presented in Section 4.

The sections just listed are to some extent catalogues of known results, augmented with some suggestions as to how these could be extended in the future. Sections 7 and 8 complement this by looking ahead to see what the final answer to some interesting general questions might be. They are necessarily more speculative than the other sections but are rooted in the known results surveyed elsewhere in the article.

The area of research reviewed in the following relies heavily on the theory of differential equations, particularly that of hyperbolic partial differential equations. For the benefit of readers with little background in differential equations, some general references that the author has found to be useful will be listed. A thorough introduction to ordinary differential equations is given in [127]. A lot of intuition for ordinary differential equations can be obtained from [136]. The article [17] is full of information, in rather compressed form. A classic introductory text on partial differential equations, where hyperbolic equations are well represented, is [143]. Useful texts on hyperbolic equations, some of which explicitly deal with the Einstein equations, are [235, 147, 187, 171, 232, 144, 99].

An important aspect of existence theorems in general relativity that one should be aware of is their relation to the cosmic censorship hypothesis. This point of view was introduced in an influential paper by Moncrief and Eardley [178]. An extended discussion of the idea can be found in [85].

2 Local Existence

In this section basic facts about local existence theorems for the Einstein equations are recalled. Since the theory is well developed and good accounts exist elsewhere (see for instance [108]), attention is focussed on remaining open questions known to the author. In particular, the questions of the minimal regularity required to get a well-posed problem and of free boundaries for fluid bodies are discussed.

2.1 The constraints

The unknowns in the constraint equations are the initial data for the Einstein equations. These consist of a three-dimensional manifold S, a Riemannian metric h_{ab}, and a symmetric tensor k_{ab} on S, and initial data for any matter fields present. The equations are:

$$R - k_{ab}k^{ab} + (h^{ab}k_{ab})^2 = 16\pi\rho, \tag{1a}$$

$$\nabla^a k_{ab} - \nabla_b(h^{ac}k_{ac}) = 8\pi j_b. \tag{1b}$$

Here R is the scalar curvature of the metric h_{ab}, and ρ and j_a are projections of the energy-momentum tensor. Assuming that the matter fields satisfy the dominant energy condition implies that $\rho \geq (j_a j^a)^{1/2}$. This means that the trivial procedure of making an arbitrary choice of h_{ab} and k_{ab} and defining ρ and j_a by Equations (1) is of no use for producing physically interesting solutions.

The usual method for solving the Equations (1) is the conformal method [66]. In this method parts of the data (the so-called free data) are chosen, and the constraints imply four elliptic equations for the remaining parts. The case that has been studied the most is the constant mean curvature (CMC) case, where $\mathrm{tr}\, k = h^{ab}k_{ab}$ is constant. In that case there is an important simplification. Three of the elliptic equations, which form a linear system, decouple from the remaining one. This last equation, which is nonlinear, but scalar, is called the Lichnerowicz equation. The heart of the existence theory for the constraints in the CMC case is the theory of the Lichnerowicz equation.

Solving an elliptic equation is a non-local problem and so boundary conditions or asymptotic conditions are important. For the constraints, the cases most frequently considered in the literature are that where S is compact (so that no boundary conditions are needed) and that where the free data satisfy some asymptotic flatness conditions. In the CMC case the problem is well understood for both kinds of boundary conditions [52, 81, 137]. The other case that has been studied in detail is that of hyperboloidal data [4]. The kind of theorem that is obtained is that sufficiently differentiable free data, in some cases required to satisfy some global restrictions, can be completed in a unique way to a solution of the constraints. It should be noted in passing that in certain cases physically interesting free data may not be "sufficiently differentiable" in the sense it is meant here. One such case is mentioned at the end of Section 2.6. The usual kinds of differentiability conditions that are required in the study of the constraints involve the free data belonging to suitable Sobolev or Hölder spaces. Sobolev spaces have the advantage that they fit well with the theory of the evolution equations (compare the discussion in Section 2.2). In the literature nobody seems to have focussed on the question of the minimal differentiability necessary to apply the conformal method.

In the non-CMC case our understanding is much more limited although some results have been obtained in recent years (see [140, 64] and references therein). It is an important open problem to extend these so that an overview is obtained comparable to that available in the CMC case. Progress on this could also lead to a better understanding of the question of whether a spacetime that admits a compact, or asymptotically flat, Cauchy surface also admits one of constant mean curvature. Up to now there have been only isolated examples that exhibit obstructions to the existence of CMC hypersurfaces [21].

It would be interesting to know whether there is a useful concept of the most general physically reasonable solutions of the constraints representing regular initial configurations. Data of this kind should not themselves contain singularities. Thus it seems reasonable to suppose at least that the metric h_{ab} is complete and that the length of k_{ab}, as measured using h_{ab}, is bounded. Does the existence of solutions of the constraints imply a restriction on the topology of S or on the asymptotic geometry of the data? This question is largely open, and it seems that information is available only in the compact and asymptotically flat cases. In the case of compact S, where there is no asymptotic regime, there is known to be no topological restriction. In the asymptotically flat case there is also no topological restriction implied by the constraints beyond that implied by the condition of asymptotic flatness itself [241]. This shows in particular that any manifold that is obtained by deleting a point from a compact manifold admits a solution of the constraints satisfying the minimal conditions demanded above. A starting point for going beyond this could be the study of data that are asymptotically homogeneous. For instance, the Schwarzschild solution contains interesting CMC hypersurfaces that are asymptotic to the metric product of a round 2-sphere with the real line. More general data of this kind could be useful for the study of the dynamics of black hole interiors [209].

To sum up, the conformal approach to solving the constraints, which has been the standard one up to now, is well understood in the compact, asymptotically flat and hyperboloidal cases under the constant mean curvature assumption, and only in these cases. For some other approaches see [22, 23, 245]. New techniques have been applied by Corvino [90] to prove the existence of regular solutions of the vacuum constraints on $\mathbf{R}^3$ that are Schwarzschild outside a compact set.

2.2 The vacuum evolution equations

The main aspects of the local-in-time existence theory for the Einstein equations can be illustrated by restricting to smooth (*i.e.* infinitely differentiable) data for the vacuum Einstein equations. The generalizations to less smooth data and matter fields are discussed in Sections 2.3 and 2.5, respectively. In the vacuum case, the data are h_{ab} and k_{ab} on a three-dimensional manifold S, as discussed in Section 2.1. A solution corresponding to these data is given by a four-dimensional manifold M, a Lorentz metric $g_{\alpha\beta}$ on M, and an embedding of S in M. Here, $g_{\alpha\beta}$ is supposed to be a solution of the vacuum Einstein equations, while h_{ab} and k_{ab} are the induced metric and second fundamental form of the embedding, respectively.

The basic local existence theorem says that, given smooth data for the vacuum Einstein equations, there exists a smooth solution of the equations which gives rise to these data [66]. Moreover, it can be assumed that the image of S under the given embedding is a Cauchy surface for the metric $g_{\alpha\beta}$. The latter fact may be expressed loosely, identifying S with its image, by the statement that S is a Cauchy surface. A solution of the Einstein equations with given initial data having S as a Cauchy surface is called a Cauchy development of those data. The existence theorem is local because it says nothing about the size of the solution obtained. A Cauchy development of given data has many open subsets that are also Cauchy developments of that data.

It is intuitively clear what it means for one Cauchy development to be an extension of another. The extension is called proper if it is strictly larger than the other development. A Cauchy development that has no proper extension is called maximal. The standard global uniqueness theorem for the Einstein equations uses the notion of the maximal development. It is due to Choquet-Bruhat and Geroch [63]. It says that the maximal development of any Cauchy data is unique up to a diffeomorphism that fixes the initial hypersurface. It is also possible to make a statement of Cauchy stability that says that, in an appropriate sense, the solution depends continuously on the initial data. Details on this can be found in [66].

A somewhat stronger form of the local existence theorem is to say that the solution exists on a uniform time interval in all of space. The meaning of this is not *a priori* clear, due to the lack of a preferred time coordinate in general relativity. The following is a formulation that is independent of coordinates. Let p be a point of S. The temporal extent $T(p)$ of a development of data on S is the supremum of the length of all causal curves in the development passing through p. In this way, a development defines a function T on S. The development can be regarded as a solution that exists on a uniform time interval if T is bounded below by a strictly positive constant. For compact S this is a straightforward consequence of Cauchy stability. In the case of asymptotically flat data it is less trivial. In the case of the vacuum Einstein equations it is true, and in fact the function T grows at least linearly as a function of spatial distance at infinity [81]. It should follow from the results of [156] that the constant of proportionality in the linear lower bound for T can be chosen to be unity, but this does not seem to have been worked out explicitly.

When proving the above local existence and global uniqueness theorems it is necessary to use some coordinate or gauge conditions. At least no explicitly diffeomorphism-invariant proofs have been found up to now. Introducing these extra elements leads to a system of reduced equations, whose solutions are determined uniquely by initial data in the strict sense, and not just uniquely up to diffeomorphisms. When a solution of the reduced equations has been obtained, it must be checked that it is a solution of the original equations. This means checking that the constraints and gauge conditions propagate. There are many methods for reducing the equations. An overview of the possibilities may be found in [104]. See also [108].

2.3 Questions of differentiability

Solving the Cauchy problem for a system of partial differential equations involves specifying a set of initial data to be considered, and determining the differentiability properties of solutions. Thus, two regularity properties are involved – the differentiability of the allowed data, and that of the corresponding solutions. Normally, it is stated that for all data with a given regularity, solutions with a certain type of regularity are obtained. For instance, in Section 2.2 we chose both types of regularity to be "infinitely differentiable". The correspondence between the regularity of data and that of solutions is not a matter of free choice. It is determined by the equations themselves, and in general the possibilities are severely limited. A similar issue arises in the context of the Einstein constraints, where there is a correspondence between the regularity of free data and full data.

The kinds of regularity properties that can be dealt with in the Cauchy problem depend, of course, on the mathematical techniques available. When solving the Cauchy problem for the Einstein equations, it is necessary to deal at least with nonlinear systems of hyperbolic equations. (There may be other types of equations involved, but they will be ignored here.) For general nonlinear systems of hyperbolic equations the standard technique is the method of energy estimates. This method is closely connected with Sobolev spaces, which will now be discussed briefly.

Let u be a real-valued function on $\mathbf{R}^n$. Let

$$\|u\|_s = \left(\sum_{i=0}^{s} \int |D^i u|^2(x)\, dx \right)^{1/2}. \quad (2)$$

The space of functions for which this quantity is finite is the Sobolev space $H^s(\mathbf{R}^n)$. Here, $|D^i u|^2$ denotes the sum of the squares of all partial derivatives of u of order i. Thus, the Sobolev space H^s is the space of functions, all of whose partial derivatives up to order s are square integrable. Similar spaces can be defined for vector valued functions by taking a sum of contributions from the separate components in the integral. It is also possible to define Sobolev spaces on any Riemannian manifold, using covariant derivatives. General information on this can be found in [18]. Consider now a solution u of the wave equation in Minkowski space. Let $u(t)$ be the restriction of this function to a time slice. Then it is easy to compute that, provided u is smooth and $u(t)$ has compact support for each t, the quantity $\|Du(t)\|_s^2 + \|\partial_t u(t)\|_s^2$ is time independent for each s. For $s = 0$ this is just the energy of a solution of the wave equation. For a general nonlinear hyperbolic system, the Sobolev norms are no longer time-independent. The constancy in time is replaced by certain inequalities. Due to the similarity to the energy for the wave equation, these are called energy estimates. They constitute the foundation of the theory of hyperbolic equations. It is because of these estimates that Sobolev spaces are natural spaces of initial data in the Cauchy problem for hyperbolic equations. The energy estimates ensure that a solution evolving from data belonging to a given Sobolev space on one spacelike hypersurface will induce data belonging to the same Sobolev space on later spacelike hypersurfaces. In other words, the property of belonging to a Sobolev space is propagated by the equations. Due to the locality properties of hyperbolic equations (existence of a finite domain of dependence), it is useful to introduce the spaces H^s_{loc}, which are defined by the condition that whenever the domain of integration is restricted to a compact set, the integral defining the space H^s is finite.

In the end, the solution of the Cauchy problem should be a function that is differentiable enough so that all derivatives that occur in the equation exist in the usual (pointwise) sense. A square integrable function is in general defined only almost everywhere and the derivatives in the above formula must be interpreted as distributional derivatives. For this reason, a connection between Sobolev spaces and functions whose derivatives exist pointwise is required. This is provided by the Sobolev embedding theorem. This says that if a function u on $\mathbf{R}^n$ belongs to the Sobolev space H^s_{loc} and if $k < s - n/2$, then there is a k times continuously differentiable function that agrees with u except on a set of measure zero.

In the existence and uniqueness theorems stated in Section 2.2, the assumptions on the initial data for the vacuum Einstein equations can be weakened to say that h_{ab} should belong to H^s_{loc} and k_{ab} to H^{s-1}_{loc}. Then, provided s is large enough, a solution is obtained that belongs to H^s_{loc}. In fact, its restriction to any spacelike hypersurface also belongs to H^s_{loc}, a property that is *a priori* stronger. The details of how large s must be would be out of place here, since they involve examining the detailed structure of the energy estimates. However, there is a simple rule for computing the required value of s. The value of s needed to obtain an existence theorem for the Einstein equations using energy estimates is that for which the Sobolev embedding theorem, applied to spatial slices, just ensures that the metric is continuously differentiable. Thus the requirement is that $s > n/2+1 = 5/2$, since $n = 3$. It follows that the smallest possible integer s is three. Strangely enough, uniqueness up to diffeomorphisms is only known to hold for $s \geq 4$. The reason is that in proving the uniqueness theorem a diffeomorphism must be carried out, which need not be smooth. This apparently leads to a loss of one derivative. It would be desirable to show that uniqueness holds for $s = 3$ and to close this gap, which has existed for many years. There exists a definition of Sobolev spaces for an arbitrary real number s, and hyperbolic equations can also be solved in the spaces with s not an integer [234]. Presumably these techniques could be applied to prove local existence for the Einstein equations with s any real number greater than $5/2$. However, this has apparently not been done explicitly in the literature.

Consider now C^∞ initial data. Corresponding to these data there is a development of class H^s for each s. It could conceivably be the case that the size of these developments shrinks with increasing s. In that case, their intersection might contain no open neighbourhood of the initial hypersurface, and no smooth development would be obtained. Fortunately, it is known that the H^s developments cannot shrink with increasing s, and so the existence of a C^∞ solution is obtained for C^∞ data. It appears that the H^s spaces with s sufficiently large are the only spaces containing the space of smooth functions for which it has been proved that the Einstein equations are locally solvable.

What is the motivation for considering regularity conditions other than the apparently very natural C^∞ condition? One motivation concerns matter fields and will be discussed in Section 2.5. Another is the idea that assuming the existence of many derivatives that have no direct physical significance seems like an admission that the problem has not been fully understood. A further reason for considering low regularity solutions is connected to the possibility of extending a local existence result to a global one. If the proof of a local existence theorem is examined closely it is generally possible to give a continuation criterion. This is a statement that if a solution on a finite time interval is such that a certain quantity constructed from the solution is bounded on that interval, then the solution can be extended to a longer time interval. (In applying this to the Einstein equations we need to worry about introducing an appropriate time coordinate.) If it can be shown that the relevant quantity is bounded on any finite time interval where a solution exists, then global existence follows. It suffices to consider the maximal interval on which a solution is defined, and obtain a contradiction if that interval is finite. This description is a little vague, but contains the essence of a type of argument that is often used in global existence proofs. The problem in putting it into practice is that often the quantity whose boundedness has to be checked contains many derivatives, and is therefore difficult to control. If the continuation criterion can be improved by reducing the number of derivatives required, then this can be a significant step toward a global result. Reducing the number of derivatives in the continuation criterion is closely related to reducing the number of derivatives of the data required for a local existence proof.

A striking example is provided by the work of Klainerman and Machedon [155] on the Yang–Mills equations in Minkowski space. Global existence in this case was first proved by Eardley and Moncrief [97], assuming initial data of sufficiently high differentiability. Klainerman and Machedon gave a new proof of this, which, though technically complicated, is based on a conceptually simple idea. They prove a local existence theorem for data of finite energy. Since energy is conserved this immediately proves global existence. In this case finite energy corresponds to the Sobolev space H^1 for the gauge potential. Of course, a result of this kind cannot be expected for the Einstein equations, since spacetime singularities do sometimes develop from regular initial data. However, some weaker analogue of the result could exist.

2.4 New techniques for rough solutions

Recently, new mathematical techniques have been developed to lower the threshold of differentiability required to obtain local existence for quasilinear wave equations in general and the Einstein equations in particular. Some aspects of this development will now be discussed following [154, 157]. A central aspect is that of Strichartz inequalities. These allow one to go beyond the theory based on L^2 spaces and use Sobolev spaces based on the Lebesgue L^p spaces for $p \neq 2$. The classical approach to deriving Strichartz estimates is based on the Fourier transform and applies to flat space. The new ideas allow the use of the Fourier transform to be limited to that of Littlewood–Paley theory and facilitate generalizations to curved space.

The idea of Littlewood–Paley theory is as follows (see [1] for a good exposition of this). Suppose that we want to describe the regularity of a function (or, more generally, a tempered distribution) u on $\mathbf{R}^n$. Differentiability properties of u correspond, roughly speaking, to fall-off properties of its Fourier transform $\hat{u}$. This is because the Fourier transform converts differentiation into multiplication. The Fourier transform is decomposed as $\hat{u} = \sum \phi_i u$, where ϕ_i is a dyadic partition of unity. The statement that it is dyadic means that all the ϕ_i except one are obtained from each other by scaling the argument by a factor which is a power of two. Transforming back we get the decomposition $u = \sum u_i$, where u_i is the inverse Fourier transform of $\phi_i u$. The component u_i of u contains only frequencies of the order 2^i. In studying rough solutions of the Einstein equations, the Littlewood–Paley decomposition is applied to the metric itself. The high frequencies are discarded to obtain a smoothed metric which plays an important role in the arguments.

Another important element of the proofs is to rescale the solution by a factor depending on the cut-off λ applied in the Littlewood–Paley decomposition. Proving the desired estimates then comes down to proving the existence of the rescaled solutions on a time interval depending on λ in a particular way. The rescaled data are small in some sense and so a connection is established to the question of long-time existence of solutions of the Einstein equation for small initial data. In this way, techniques from the work of Christodoulou and Klainerman on the stability of Minkowski space (see Section 5.2) are brought in.

What is finally proved? In general, there is a close connection between proving local existence for data in a certain space and showing that the time of existence of smooth solutions depends only on the norm of the data in the given space. Klainerman and Rodnianski [157] demonstrate that the time of existence of solutions of the reduced Einstein equations in harmonic coordinates depends only on the $H^{2+\epsilon}$ norm of the initial data for any $\epsilon > 0$. The reason that this does not allow them to assert an existence result in the same space is that the constraints are needed in their proof and that an understanding of solving the constraints at this low level of differentiability is lacking.

The techniques discussed in this section, which have been stimulated by the desire to understand the Einstein equations, are also helpful in understanding other nonlinear wave equations. Thus, this is an example where information can flow from general relativity to the theory of partial differential equations.

2.5 Matter fields

Analogues of the results for the vacuum Einstein equations given in Section 2.2 are known for the Einstein equations coupled to many types of matter model. These include perfect fluids, elasticity theory, kinetic theory, scalar fields, Maxwell fields, Yang–Mills fields, and combinations of these. An important restriction is that the general results for perfect fluids and elasticity apply only to situations where the energy density is uniformly bounded away from zero on the region of interest. In particular, they do not apply to cases representing material bodies surrounded by vacuum. In cases where the energy density, while everywhere positive, tends to zero at infinity, a local solution is known to exist, but it is not clear whether a local existence theorem can be obtained that is uniform in time. In cases where the fluid has a sharp boundary, ignoring the boundary leads to solutions of the Einstein–Euler equations with low differentiability (cf. Section 2.3), while taking it into account explicitly leads to a free boundary problem. This will be discussed in more detail in Section 2.6. In the case of kinetic or field theoretic matter models it makes no difference whether the energy density vanishes somewhere or not.

2.6 Free boundary problems

In applying general relativity one would like to have solutions of the Einstein–matter equations modelling material bodies. As will be discussed in Section 3.1 there are solutions available for describing equilibrium situations. However, dynamical situations require solving a free boundary problem if the body is to be made of fluid or an elastic solid. We will now discuss the few results which are known on this subject. For a spherically symmetric self-gravitating fluid body in general relativity, a local-in-time existence theorem was proved in [151]. This concerned the case in which the density of the fluid at the boundary is non-zero. In [202] a local existence theorem was proved for certain equations of state with vanishing boundary density. These solutions need not have any symmetry but they are very special in other ways. In particular, they do not include small perturbations of the stationary solutions discussed in Section 3.1. There is no general result on this problem up to now.

Remarkably, the free boundary problem for a fluid body is also poorly understood in classical physics. There is a result for a viscous fluid [226], but in the case of a perfect fluid the problem was wide open until very recently. Now, a major step forward has been taken by Wu [244], who obtained a result for a fluid that is incompressible and irrotational. There is a good physical reason why local existence for a fluid with a free boundary might fail. This is the Rayleigh–Taylor instability which involves perturbations of fluid interfaces that grow with unbounded exponential rates (cf. the discussion in [26]). It turns out that in the case considered by Wu this instability does not cause problems, and there is no reason to expect that a self-gravitating compressible fluid with rotation in general relativity with a free boundary cannot also be described by a well-posed free boundary value problem. For the generalization of the problem considered by Wu to the case of a fluid with rotation, Christodoulou and Lindblad [80] have obtained estimates that look as if they should be enough to obtain an existence theorem. It has, however, not yet been possible to complete the argument. This point deserves some further comment. In many problems the heart of an existence proof is obtaining suitable estimates. Then more or less standard approximation techniques can be used to obtain the desired conclusion (for a discussion of this see [108], Section 3.1). In the problem studied in [80] it is an appropriate approximation method that is missing.

One of the problems in tackling the initial value problem for a dynamical fluid body is that the boundary is moving. It would be very convenient to use Lagrangian coordinates, since in those coordinates the boundary is fixed. Unfortunately, it is not at all obvious that the Euler equations in Lagrangian coordinates have a well-posed initial value problem, even in the absence of a boundary. It was, however, recently shown by Friedrich [105] that it is possible to treat the Cauchy problem for fluids in general relativity in Lagrangian coordinates.

In the case of a fluid with non-vanishing boundary density it is not only the evolution equations that cause problems. It is already difficult to construct suitable solutions of the constraints. A theorem on this has recently been obtained by Dain and Nagy [91]. There remains an undesirable technical restriction, but the theorem nevertheless provides a very general class of physically interesting initial data for a self-gravitating fluid body in general relativity.

3 Global Symmetric Solutions

An obvious procedure to obtain special cases of the general global existence problem for the Einstein equations that are amenable to attack is to make symmetry assumptions. In this section, we discuss the results obtained for various symmetry classes defined by different choices of number and character of Killing vectors.

3.1 Stationary solutions

Many of the results on global solutions of the Einstein equations involve considering classes of spacetimes with Killing vectors. A particularly simple case is that of a timelike Killing vector, *i.e.* the case of stationary spacetimes. In the vacuum case there are very few solutions satisfying physically reasonable boundary conditions. This is related to no hair theorems for black holes and lies outside the scope of this review. More information on the topic can be found in the book of Heusler [134] and in his Living Review [133] (see also [37] where the stability of the Kerr metric is discussed). The case of phenomenological matter models has been reviewed in [215]. The account given there will be updated in the following.

The area of stationary solutions of the Einstein equations coupled to field theoretic matter models has been active in recent years as a consequence of the discovery by Bartnik and McKinnon [24] of a discrete family of regular, static, spherically symmetric solutions of the Einstein–Yang–Mills equations with gauge group $SU(2)$. The equations to be solved are ordinary differential equations, and in [24] they were solved numerically by a shooting method. The first existence proof for a solution of this kind is due to Smoller, Wasserman, Yau and McLeod [230] and involves an arduous qualitative analysis of the differential equations. The work on the Bartnik–McKinnon solutions, including the existence theorems, has been extended in many directions. Recently, static solutions of the Einstein–Yang–Mills equations that are not spherically symmetric were discovered numerically [158]. It is a challenge to prove the existence of solutions of this kind. Now the ordinary differential equations of the previously known case are replaced by elliptic equations. Moreover, the solutions appear to still be discrete, so that a simple perturbation argument starting from the spherical case does not seem feasible. In another development, it was shown that a linearized analysis indicates the existence of stationary non-static solutions [50]. It would be desirable to study the question of linearization stability in this case, which, if the answer were favourable, would give an existence proof for solutions of this kind.

Now we return to phenomenological matter models, starting with the case of spherically symmetric static solutions. Basic existence theorems for this case have been proved for perfect fluids [218], collisionless matter [195, 189], and elastic bodies [185]. The last of these is the solution to an open problem posed in [215]. All these theorems demonstrate the existence of solutions that are everywhere smooth and exist globally as functions of area radius for a general class of constitutive relations. The physically significant question of the finiteness of the mass of these configurations was only answered in these papers under restricted circumstances. For instance, in the case of perfect fluids and collisionless matter, solutions were constructed by perturbing about the Newtonian case. Solutions for an elastic body were obtained by perturbing about the case of isotropic pressure, which is equivalent to a fluid. Further progress on the question of the finiteness of the mass of the solutions was made in the case of a fluid by Makino [172], who gave a rather general criterion on the equation of state ensuring the finiteness of the radius. Makino's criterion was generalized to kinetic theory in [197]. This resulted in existence proofs for various models that have been considered in galactic dynamics and which had previously been constructed numerically (cf. [38, 227] for an account of these models in the non-relativistic and relativistic cases, respectively). Most of the work uoted up to now refers to solutions where the support of the density is a ball. For matter with anisotropic pressure the support may also be a shell, *i.e.* the region bounded by two concentric spheres. The existence of static shells in the case of the Einstein–Vlasov equations was proved in [193].

In the case of self-gravitating Newtonian spherically symmetric configurations of collisionless matter, it can be proved that the phase space density of particles depends only on the energy of the particle and the modulus of its angular momentum [25]. This is known as Jeans' theorem. It was already shown in [189] that the naive generalization of this to the general relativistic case does not hold if a black hole is present. Recently, counterexamples to the generalization of Jeans' theorem to the relativistic case, which are not dependent on a black hole, were constructed by Schaeffer [225]. It remains to be seen whether there might be a natural modification of the formulation that would lead to a true statement.

For a perfect fluid there are results stating that a static solution is necessarily spherically symmetric [167]. They still require a restriction on the equation of state, which it would be desirable to remove. A similar result is not to be expected in the case of other matter models, although as yet no examples of non-spherical static solutions are available. In the Newtonian case examples have been constructed by Rein [193]. (In that case static solutions are defined to be those in which the particle current vanishes.) For a fluid there is an existence theorem for solutions that are stationary but not static (models for rotating stars) [129]. At present there are no corresponding theorems for collisionless matter or elastic bodies. In [193], stationary, non-static configurations of collisionless matter were constructed in the Newtonian case.

Two obvious characteristics of a spherically symmetric static solution of the Einstein–Euler equations that has a non-zero density only in a bounded spatial region are its radius R and its total mass M. For a given equation of state there is a one-parameter family of solutions. These trace out a curve in the (M, R) plane. In the physics literature, pictures of this curve indicate that it spirals in on a certain point in the limit of large density. The occurrence of such a spiral and its precise asymptotic form have been proved rigorously by Makino [173].

For some remarks on the question of stability see Section 4.1.

3.2 Spatially homogeneous solutions

A solution of the Einstein equations is called spatially homogeneous if there exists a group of symmetries with three-dimensional spacelike orbits. In this case there are at least three linearly independent spacelike Killing vector fields. For most matter models the field equations reduce to ordinary differential equations. (Kinetic matter leads to an integro-differential equation.) The most important results in this area have been reviewed in a recent book edited by Wainwright and Ellis [237]. See, in particular, Part Two of the book. There remain a host of interesting and accessible open questions. The spatially homogeneous solutions have the advantage that it is not necessary to stop at just existence theorems; information on the global qualitative behaviour of solutions can also be obtained.

An important question that has been open for a long time concerns the mixmaster model, as discussed in [213]. This is a class of spatially homogeneous solutions of the vacuum Einstein equations, which are invariant under the group $SU(2)$. A special subclass of these $SU(2)$-invariant solutions, the (parameter-dependent) Taub–NUT solution, is known explicitly in terms of elementary functions. The Taub–NUT solution has a simple initial singularity which is in fact a Cauchy horizon. All other vacuum solutions admitting a transitive action of $SU(2)$ on spacelike hypersurfaces (Bianchi type IX solutions) will be called generic in the present discussion. These generic Bianchi IX solutions (which might be said to constitute the mixmaster solution proper) have been believed for a long time to have singularities that are oscillatory in nature where some curvature invariant blows up. This belief was based on a combination of heuristic considerations and numerical calculations. Although these together do make a persuasive case for the accepted picture, until very recently there were no mathematical proofs of the these features of the mixmaster model available. This has now changed. First, a proof of curvature blow-up and oscillatory behaviour for a simpler model (a solution of the Einstein–Maxwell equations) which shares many qualitative features with the mixmaster model, was obtained by Weaver [240]. In the much more difficult case of the mixmaster model itself, corresponding results were obtained by Ringström [223]. Later he extended this in several directions in [222]. In that paper more detailed information was obtained concerning the asymptotics and an attractor for the evolution was identified. It was shown that generic solutions of Bianchi type IX with a perfect fluid whose equation of state is $p = (\gamma - 1)\rho$ with $1 \leq \gamma < 2$ are approximated near the singularity by vacuum solutions. The case of a stiff fluid ($\gamma = 2$) which has a different asymptotic behaviour was analysed completely for all models of Bianchi class A, a class which includes Bianchi type IX.

Ringström's analysis of the mixmaster model is potentially of great significance for the mathematical understanding of singularities of the Einstein equations in general. Thus, its significance goes far beyond the spatially homogeneous case. According to extensive investigations of Belinskii, Khalatnikov and Lifshitz (see [164, 30, 31] and references therein), the mixmaster model should provide an approximate description for the general behaviour of solutions of the

Einstein equations near singularities. This should apply to many matter models as well as to the vacuum equations. The work of Belinskii, Khalatnikov, and Lifshitz (BKL) is hard to understand and it is particularly difficult to find a precise mathematical formulation of their conclusions. This has caused many people to remain sceptical about the validity of the BKL picture. Nevertheless, it seems that nothing has ever been found to indicate any significant flaws in the final version. As long as the mixmaster model itself was not understood this represented a fundamental obstacle to progress on understanding the BKL picture mathematically. The removal of this barrier opens up an avenue to progress on this issue. The BKL picture is discussed in more detail in Section 8.

Some recent and qualitatively new results concerning the asymptotic behaviour of spatially homogeneous solutions of the Einstein–matter equations, both close to the initial singularity and in a phase of unlimited expansion, (and with various matter models) can be found in [219, 220, 200, 238, 183, 135]. These show in particular that the dynamics can depend sensitively on the form of matter chosen. (Note that these results are consistent with the BKL picture.) The dynamics of indefinitely expanding cosmological models is discussed further in Section 7.

3.3 Spherically symmetric solutions

The most extensive results on global inhomogeneous solutions of the Einstein equations obtained up to now concern spherically symmetric solutions of the Einstein equations coupled to a massless scalar field with asymptotically flat initial data. In a series of papers, Christodoulou [68, 67, 70, 69, 71, 72, 73, 77] has proved a variety of deep results on the global structure of these solutions. Particularly notable are his proofs that naked singularities can develop from regular initial data [73] and that this phenomenon is unstable with respect to perturbations of the data [77]. In related work, Christodoulou [74, 75, 76] has studied global spherically symmetric solutions of the Einstein equations coupled to a fluid with a special equation of state (the so-called two-phase model). A generalization of the results of [68] to the case of a nonlinear scalar field has been given by Chae [57].

The rigorous investigation of the spherically symmetric collapse of collisionless matter in general relativity was initiated by Rein and the author [194], who showed that the evolution of small initial data leads to geodesically complete spacetimes where the density and curvature fall off at large times. Later, it was shown [198] that independent of the size of the initial data the first singularity, if there is one at all, must occur at the centre of symmetry. This result uses a time coordinate of Schwarzschild type; an analogous result for a maximal time coordinate was proved in [214]. The question of what happens for general large initial data could not yet be answered by analytical techniques. In [199], numerical methods were applied to try to make some progress in this direction. The results are discussed in the next paragraph.

Despite the range and diversity of the results obtained by Christodoulou on the spherical collapse of a scalar field, they do not encompass some of the most interesting phenomena that have been observed numerically. These are related to the issue of critical collapse. For sufficiently small data the field disperses. For sufficiently large data a black hole is formed. The question is what happens in between. This can be investigated by examining a one-parameter family of initial data interpolating between the two cases. It was found by Choptuik [61] that there is a critical value of the parameter below which dispersion takes place and above which a black hole is formed, and that the mass of the black hole approaches zero as the critical parameter value is approached. This gave rise to a large literature in which the spherical collapse of different kinds of matter was computed numerically and various qualitative features were determined. For reviews of this see [120, 119]. In the calculations of [199] for collisionless matter, it was found that in the situations considered the black hole mass tended to a strictly positive limit as the critical parameter was approached from above. These results were confirmed and extended by Olabarrieta and Choptuik [184]. There are no rigorous mathematical results available on the issue of a mass gap for either a scalar field or collisionless matter and it is an outstanding challenge for mathematical relativists to change this situation.

Another aspect of Choptuik's results is the occurrence of a discretely self-similar solution. It would seem hard to prove the existence of a solution of this kind analytically. For other types of matter, such as a perfect fluid with linear equation of state, the critical solution is continuously self-similar and this looks more tractable. The problem reduces to solving a system of singular ordinary differential equations subject to certain boundary conditions. This problem was solved in [73] for the case where the matter model is given by a massless scalar field, but the solutions produced there, which are continuously self-similar, cannot include the Choptuik critical solution. Bizoń and Wasserman [42] studied the corresponding problem for the Einstein equations coupled to a wave map with target $SU(2)$. They proved the existence of continuously self-similar solutions including one which, according the results of numerical calculations, appears to play the role of critical solution in collapse. Another case where the question of the existence of the critical solution seems to be a problem that could possibly be solved in the near future is that of a perfect fluid. A good starting point for this is the work of Goliath, Nilsson, and Uggla [115, 116]. These authors gave a formulation of the problem in terms of dynamical systems and were able to determine certain qualitative features of the solutions. See also [53, 54].

A possible strategy for learning more about critical collapse, pursued by Bizoń and collaborators, is to study model problems in flat space that exhibit features similar to those observed numerically in the case of the Einstein equations. Until now, only models showing continuous self-similarity have been found. These include wave maps in various dimensions and the Yang–Mills equations in spacetimes of dimension greater than four. As mentioned in Section 2.3, it is known that in four dimensions there exist global smooth solutions of the Yang–Mills equations corresponding to rather general initial data [97, 155]. In dimensions greater than five it is known that there exist solutions that develop singularities in finite time. This follows from the existence of continuously self-similar solutions [41]. Numerical evidence indicates that this type of blow-up is stable, *i.e.* occurs for an open set of initial data. The numerical work also indicates that there is a critical self-similar solution separating this kind of blow-up from dispersion. The spacetime dimension five is critical for Yang–Mills theory. Apparently singularities form, but in a different way from what happens in dimension six. There is as yet no rigorous proof of blow-up in five dimensions.

The various features of Yang–Mills theory just mentioned are mirrored in two dimensions less by wave maps with values in spheres [40]. In four dimensions, blow-up is known while in three dimensions there appears (numerically) to be a kind of blow-up similar to that found for Yang–Mills in dimension five. There is no rigorous proof of blow-up. What is seen numerically is that the collapse takes place by scaling within a one-parameter family of static solutions. The case of wave maps is the most favourable known model problem for proving theorems about critical phenomena associated to singularity formation. The existence of a solution having the properties expected of the critical solution for wave maps in four dimensions has been proved in [39]. Some rigorous support for the numerical findings in three dimensions has been given by work of Struwe (see the preprints available from [233]). He showed, among other things, that if there is blow-up in finite time it must take place in a way resembling that observed in the numerical calculations.

Self-similar solutions are characteristic of what is called Type II critical collapse. In Type I collapse an analogous role is played by static solutions and quite a bit is known about the existence of these. For instance, in the case of the Einstein–Yang–Mills equations, it is one of the Bartnik–McKinnon solutions mentioned in Section 3.1 which does this. In the case of collisionless matter the results of [184] show that at least in some cases critical collapse is mediated by a static solution in the form of a shell. There are existence results for shells of this kind [192] although no connection has yet been made between those shells whose existence has been proved and those which have been observed numerically in critical collapse calculations. Note that Martín-García and Gundlach [174] have presented a (partially numerical) construction of self-similar solutions of the Einstein–Vlasov system.

3.4 Cylindrically symmetric solutions

Solutions of the Einstein equations with cylindrical symmetry that are asymptotically flat in all directions allowed by the symmetry represent an interesting variation on asymptotic flatness. Since black holes are apparently incompatible with this symmetry, one may hope to prove geodesic completeness of solutions under appropriate assumptions. (It would be interesting to have a theorem making the statement about black holes precise.) A proof of geodesic completeness has been achieved for the Einstein vacuum equations and for the source-free Einstein–Maxwell equations in [34], building on global existence theorems for wave maps [83, 82]. For a quite different point of view on this question involving integrable systems see [243]. A recent paper of Hauser and Ernst [128] also appears to be related to this question. However, due to the great length of this text and its reliance on many concepts unfamiliar to this author, no further useful comments on the subject can be made here.

3.5 Spatially compact solutions

In the context of spatially compact spacetimes it is first necessary to ask what kind of global statements are to be expected. In a situation where the model expands indefinitely it is natural to pose the question whether the spacetime is causally geodesically complete towards the future. In a situation where the model develops a singularity either in the past or in the future one can ask what the qualitative nature of the singularity is. It is very difficult to prove results of this kind. As a first step one may prove a global existence theorem in a well-chosen time coordinate. In other words, a time coordinate is chosen that is geometrically defined and that, under ideal circumstances, will take all values in a certain interval (t_-, t_+). The aim is then to show that, in the maximal Cauchy development of data belonging to a certain class, a time coordinate of the given type exists and exhausts the expected interval. The first result of this kind for inhomogeneous spacetimes was proved by Moncrief in [176]. This result concerned Gowdy spacetimes. These are vacuum spacetimes with a two-dimensional Abelian group of isometries acting on compact orbits. The area of the orbits defines a natural time coordinate (areal time coordinate). Moncrief showed that in the maximal Cauchy development of data given on a hypersurface of constant time, this time coordinate takes on the maximal possible range, namely $(0, \infty)$. This result was extended to more general vacuum spacetimes with two Killing vectors in [33]. Andréasson [8] extended it in another direction to the case of collisionless matter in a spacetime with Gowdy symmetry.

Another attractive time coordinate is constant mean curvature (CMC) time. For a general discussion of this see [209]. A global existence theorem in this time for spacetimes with two Killing vectors and certain matter models (collisionless matter, wave maps) was proved in [212]. That the choice of matter model is important for this result was demonstrated by a global non-existence result for dust in [211]. As shown in [141], this leads to examples of spacetimes that are not covered by a CMC slicing. Results on global existence of CMC foliations have also been obtained for spherical and hyperbolic symmetry [206, 51].

A drawback of the results on the existence of CMC foliations just cited is that they require as a hypothesis the existence of one CMC Cauchy surface in the given spacetime. More recently, this restriction has been removed in certain cases by Henkel using a generalization of CMC foliations called prescribed mean curvature (PMC) foliations. A PMC foliation can be built that includes any given Cauchy surface [130] and global existence of PMC foliations can be proved in a way analogous to that previously done for CMC foliations [131, 132]. These global foliations provide barriers that imply the existence of a CMC hypersurface. Thus, in the end it turns out that the unwanted condition in the previous theorems on CMC foliations is in fact automatically satisfied. Connections between areal, CMC, and PMC time coordinates were further explored in [9]. One important observation there is that hypersurfaces of constant areal time in spacetimes with symmetry often have mean curvature of a definite sign.

Once global existence has been proved for a preferred time coordinate, the next step is to investigate the asymptotic behaviour of the solution as $t \to t_{\pm}$. There are few cases in which this has been done successfully. Notable examples are Gowdy spacetimes [84, 139, 87] and solutions of the Einstein–Vlasov system with spherical and plane symmetry [190]. Progress in constructing spacetimes with prescribed singularities will be described in Section 6. In the future this could lead in some cases to the determination of the asymptotic behaviour of large classes of spacetimes as the singularity is approached.

4 Newtonian Theory and Special Relativity

To put the global results discussed in this article into context it is helpful to compare with Newtonian theory and special relativity. Some of the theorems that have been proved in those contexts and that can offer insight into questions in general relativity will now be reviewed. It should be noted that even in these simpler contexts open questions abound.

4.1 Hydrodynamics

Solutions of the classical (compressible) Euler equations typically develop singularities, *i.e.* discontinuities of the basic fluid variables, in finite time [228]. Some of the results of [228] were recently generalized to the case of a relativistic fluid [124]. The proofs of the development of singularities are by contradiction and so do not give information about what happens when the smooth solution breaks down. One of the things that can happen is the formation of shock waves and it is known that, at least in certain cases, solutions can be extended in a physically meaningful way beyond the time of shock formation. The extended solutions only satisfy the equations in the weak sense. For the classical Euler equations there is a well-known theorem on global existence of weak solutions in one space dimension which goes back to [114]. This has been generalized to the relativistic case. Smoller and Temple treated the case of an isentropic fluid with linear equation of state [229] while Chen analysed the cases of polytropic equations of state [59] and flows with variable entropy [60]. This means that there is now an understanding of this question in the relativistic case similar to that available in the classical case.

In space dimensions higher than one there are no general global existence theorems. For a long time there were also no uniqueness theorems for weak solutions even in one dimension. It should be emphasized that weak solutions can easily be shown to be non-unique unless they are required to satisfy additional restrictions such as entropy conditions. A reasonable aim is to find a class of weak solutions in which both existence and uniqueness hold. In the one-dimensional case this has recently been achieved by Bressan and collaborators (see [47, 49, 48] and references therein).

It would be desirable to know more about which quantities must blow up when a singularity forms in higher dimensions. A partial answer was obtained for classical hydrodynamics by Chemin [58]. The possibility of generalizing this to relativistic and self-gravitating fluids was studied by Brauer [45]. There is one situation in which a smooth solution of the classical Euler equations is known to exist for all time. This is when the initial data are small and the fluid initially is flowing uniformly outwards. A theorem of this type has been proved by Grassin [118]. There is also a global existence result due to Guo [121] for an irrotational charged fluid in Newtonian physics, where the repulsive effect of the charge can suppress the formation of singularities.

A question of great practical interest for physics is that of the stability of equilibrium stellar models. Since, as has already been pointed out, we know so little about the global time evolution for a self-gravitating fluid ball, even in the Newtonian case, it is not possible to say anything rigorous about nonlinear stability at the present time. We can, however, make some statements about linear stability. The linear stability of a large class of static spherically symmetric solutions of the Einstein–Euler equations within the class of spherically symmetric perturbations has been proved by Makino [172] (cf. also [165] for the Newtonian problem). The spectral properties of the linearized operator for general (*i.e.* non-spherically symmetric) perturbations in the Newtonian problem have been studied by Beyer [36]. This could perhaps provide a basis for a stability analysis, but this has not been done.

4.2 Kinetic theory

Collisionless matter is known to admit a global singularity-free evolution in many cases. For self-gravitating collisionless matter, which is described by the Vlasov–Poisson system, there is a general global existence theorem [186, 169]. There is also a version of this which applies to Newtonian cosmology [196]. A more difficult case is that of the Vlasov–Maxwell system, which describes charged collisionless matter. Global existence is not known for general data in three space dimensions but has been shown in two space dimensions [111, 112] and in three dimensions with one symmetry [110] or with almost spherically symmetric data [188].

The nonlinear stability of static solutions of the Vlasov–Poisson system describing Newtonian self-gravitating collisionless matter has been investigated using the energy–Casimir method. For information on this see [122] and its references. The energy–Casimir method has been applied to the Einstein equations in [242].

For the classical Boltzmann equation, global existence and uniqueness of smooth solutions has been proved for homogeneous initial data and for data that are small or close to equilibrium. For general data with finite energy and entropy, global existence of weak solutions (without uniqueness) was proved by DiPerna and Lions [94]. For information on these results and on the classical Boltzmann equation in general see [56, 55]. Despite the non-uniqueness it is possible to show that all solutions tend to equilibrium at late times. This was first proved by Arkeryd [16] by non-standard analysis and then by Lions [168] without those techniques. It should be noted that since the usual conservation laws for classical solutions are not known to hold for the DiPerna–Lions solutions, it is not possible to predict which equilibrium solution a given solution will converge to. In the meantime, analogues of several of these results for the classical Boltzmann equation have been proved in the relativistic case. Global existence of weak solutions was proved in [96]. Global existence and convergence to equilibrium for classical solutions starting close to equilibrium was proved in [113]. On the other hand, global existence of classical solutions for small initial data is not known. Convergence to equilibrium for weak solutions with general data was proved by Andréasson [7]. There is still no existence and uniqueness theorem in the literature for general spatially homogeneous solutions of the relativistic Boltzmann equation. (A paper claiming to prove existence and uniqueness for solutions of the Einstein–Boltzmann system which are homogeneous and isotropic [180] contains fundamental errors.)

4.3 Elasticity theory

There is an extensive literature on mathematical elasticity theory but the mathematics of self-gravitating elastic bodies seems to have been largely neglected. An existence theorem for spherically symmetric elastic bodies in general relativity was mentioned in Section 3.1. More recently, Beig and Schmidt [27] proved an existence theorem for static elastic bodies subject to Newtonian gravity, which need not be spherically symmetric.

5 Global Existence for Small Data

An alternative to symmetry assumptions is provided by "small data" results, where solutions are studied that develop from data close to those for known solutions. This leads to some simplification in comparison to the general problem, but with present techniques it is still very hard to obtain results of this kind.

5.1 Stability of de Sitter space

In [101], Friedrich proved a result on the stability of de Sitter space. He gives data at infinity but the same type of argument can be applied starting from a Cauchy surface in spacetime to give an analogous result. This concerns the Einstein vacuum equations with positive cosmological constant and is as follows. Consider initial data induced by de Sitter space on a regular Cauchy hypersurface. Then all initial data (vacuum with positive cosmological constant) near enough to these data in a suitable (Sobolev) topology have maximal Cauchy developments that are geodesically complete. The result gives much more detail on the asymptotic behaviour than just this and may be thought of as proving a form of the cosmic no hair conjecture in the vacuum case. (This conjecture says roughly that the de Sitter solution is an attractor for expanding cosmological models with positive cosmological constant.) This result is proved using conformal techniques and, in particular, the regular conformal field equations developed by Friedrich.

There are results obtained using the regular conformal field equations for negative or vanishing cosmological constant [103, 106], but a detailed discussion of their nature would be out of place here (cf. however Section 9.1).

5.2 Stability of Minkowski space

Another result on global existence for small data is that of Christodoulou and Klainerman on the stability of Minkowski space [79]. The formulation of the result is close to that given in Section 5.1, but now de Sitter space is replaced by Minkowski space. Suppose then that initial data for the vacuum Einstein equations are prescribed that are asymptotically flat and sufficiently close to those induced by Minkowski space on a hyperplane. Then Christodoulou and Klainerman prove that the maximal Cauchy development of these data is geodesically complete. They also provide a wealth of detail on the asymptotic behaviour of the solutions. The proof is very long and technical. The central tool is the Bel–Robinson tensor, which plays an analogous role for the gravitational field to that played by the energy-momentum tensor for matter fields. Apart from the book of Christodoulou and Klainerman itself, some introductory material on geometric and analytic aspects of the proof can be found in [44, 78], respectively. More recently, the result for the vacuum Einstein equations has been generalized to the case of the Einstein–Maxwell system by Zipser [246].

In the original version of the theorem, initial data had to be prescribed on all of $\mathbf{R}^3$. A generalization described in [156] concerns the case where data need only be prescribed on the complement of a compact set in $\mathbf{R}^3$. This means that statements can be obtained for any asymptotically flat spacetime where the initial matter distribution has compact support, provided attention is confined to a suitable neighbourhood of infinity. The proof of the new version uses a double null foliation instead of the foliation by spacelike hypersurfaces previously used and leads to certain conceptual simplifications.

5.3 Stability of the (compactified) Milne model

The interior of the light cone in Minkowski space foliated by the spacelike hypersurfaces of constant Lorentzian distance from the origin can be thought of as a vacuum cosmological model, sometimes known as the Milne model. By means of a suitable discrete subgroup of the Lorentz group it can be compactified to give a spatially compact cosmological model. With a slight abuse of terminology the latter spacetime will also be referred to here as the Milne model. A proof of the stability of the latter model by Andersson and Moncrief has been announced in [3]. The result is that, given data for the Milne model on a manifold obtained by compactifying a hyperboloid in Minkowski space, the maximal Cauchy developments of nearby data are geodesically complete in the future. Moreover, the Milne model is asymptotically stable in the sense that any other solution in this class converges towards the Milne model in terms of suitable dimensionless variables.

The techniques used by Andersson and Moncrief are similar to those used by Christodoulou and Klainerman. In particular, the Bel–Robinson tensor is crucial. However, their situation is much simpler than that of Christodoulou and Klainerman, so that the complexity of the proof is not so great. This has to do with the fact that the fall-off of the fields towards infinity in the Minkowksi case is different in different directions, while it is uniform in the Milne case. Thus it is enough in the latter case to always contract the Bel–Robinson tensor with the same timelike vector when deriving energy estimates. The fact that the proof is simpler opens up a real possibility of generalizations, for instance by adding different matter models.

5.4 Stability of the Bianchi type III form of flat spacetime

Another vacuum cosmological model whose nonlinear stability has been investigated is the Bianchi III form of flat spacetime. To obtain this model, first do the construction described in the last section with the difference that the starting solution is three-dimensional Minkowski space. Then, take the metric product of the resulting three-dimensional Lorentz manifold with a circle. This defines a flat spacetime that has one Killing vector, which is the generator of rotations of the circle. It has been shown by Choquet-Bruhat and Moncrief [65] that this solution is stable under small vacuum perturbations preserving the one-dimensional symmetry. More precisely, the result is proved only for the polarized case, but the authors suggest that this restriction can be lifted at the expense of doing some more work. As in the case of the Milne model, a natural task is to generalize this result to spacetimes with suitable matter content. The reasons it is necessary to restrict to symmetric perturbations in this analysis, in contrast to what happens with the Milne model, are discussed in detail in [65].

One of the main techniques used is a method of modified energy estimates that is likely to be of more general applicability. The Bel–Robinson tensor plays no role. The other main technique is based on the fact that the problem under study is equivalent to the study of the 2+1-dimensional Einstein equations coupled to a wave map (a scalar field in the polarized case). This helps to explain why the use of the Dirichlet energy could be imported into this problem from the work of [5] on 2+1 vacuum gravity.

6 Prescribed Singularities

If it is too hard to get information on the qualitative nature of solutions by evolving from a regular initial hypersurface toward a possible singularity, an alternative approach is to construct spacetimes with given singularities. Recently, the latter method has made significant progress and the new results are presented in this section.

6.1 Isotropic singularities

The existence and uniqueness results discussed in this section are motivated by Penrose's Weyl curvature hypothesis. Penrose suggests that the initial singularity in a cosmological model should be such that the Weyl tensor tends to zero or at least remains bounded. There is some difficulty in capturing this by a geometric condition, and it was suggested in [117] that a clearly formulated geometric condition (which, on an intuitive level, is closely related to the original condition) is that the conformal structure should remain regular at the singularity. Singularities of this type are known as conformal or isotropic singularities.

Consider now the Einstein equations coupled to a perfect fluid with the radiation equation of state $p = \rho/3$. Then, it has been shown [181, 182, 88] that solutions with an isotropic singularity are determined uniquely by certain free data given at the singularity. The data that can be given are, roughly speaking, half as much as in the case of a regular Cauchy hypersurface. The method of proof is to derive an existence and uniqueness theorem for a suitable class of singular hyperbolic equations. In [13] this was extended to the equation of state $p = (\gamma - 1)\rho$ for any γ satisfying $1 < \gamma \leq 2$.

What happens to this theory when the fluid is replaced by a different matter model? The study of the case of a collisionless gas of massless particles was initiated in [14]. The equations were put into a form similar to that which was so useful in the fluid case and therefore likely to be conducive to proving existence theorems. Then theorems of this kind were proved in the homogeneous special case. These were extended to the general (*i.e.* inhomogeneous) case in [12]. The picture obtained for collisionless matter is very different from that for a perfect fluid. Much more data can be given freely at the singularity in the collisionless case.

These results mean that the problem of isotropic singularities has largely been solved. There do, however, remain a couple of open questions. What happens if the massless particles are replaced by massive ones? What happens if the matter is described by the Boltzmann equation with non-trivial collision term? Does the result in that case look more like the Vlasov case or more like the Euler case?

6.2 Fuchsian equations

The singular equations that arise in the study of isotropic singularities are closely related to what Kichenassamy [147] calls Fuchsian equations. He has developed a rather general theory of these equations (see [147, 146, 145], and also the earlier papers [19, 148, 149]). In [150] this was applied to analytic Gowdy spacetimes on T^3 to construct a family of vacuum spacetimes depending on the maximum number of free functions (for the given symmetry class) whose singularities can be described in detail. The symmetry assumed in that paper requires the two-surfaces orthogonal to the group orbits to be surface-forming (vanishing twist constants). In [138] a corresponding result was obtained for the class of vacuum spacetimes with polarized $U(1) \times U(1)$ symmetry and non-vanishing twist. The analyticity requirement on the free functions in the case of Gowdy spacetimes on T^3 was reduced to smoothness in [217]. There are also Gowdy spacetimes on S^3 and $S^2 \times S^1$, which have been less studied than those on T^3. The Killing vectors have zeros, defining axes, and these lead to technical difficulties. In [231] Fuchsian techniques were applied to Gowdy spacetimes on S^3 and $S^2 \times S^1$. The maximum number of free functions was not obtained due to difficulties on the axes.

Anguige [11] has obtained results on solutions with perfect fluid that are general under the condition of plane symmetry, which is stronger than Gowdy symmetry. He also extended this to polarized Gowdy symmetry in [10].

Work related to these Fuchsian methods was done earlier in a somewhat simpler context by Moncrief [177], who showed the existence of a large class of analytic vacuum spacetimes with Cauchy horizons.

As a result of the BKL picture, it cannot be expected that the singularities in general solutions of the Einstein equations in vacuum or with a non-stiff fluid can be handled using Fuchsian techniques (cf. Section 8.1). However, things look better in the presence of a massless scalar field or a stiff fluid. For these types of matter it has been possible [6] to prove a theorem analogous to that of [150] without requiring symmetry assumptions. The same conclusion can be obtained for a scalar field with mass or with a potential of moderate growth [216].

The results included in this review concern the Einstein equations in four spacetime dimensions. Of course, many of the questions discussed have analogues in other dimensions and these may be of interest for string theory and related topics. In [92] Fuchsian techniques were applied to the Einstein equations coupled to a variety of field theoretic matter models in arbitrary dimensions. One of the highlights is the result that it is possible to apply Fuchsian techniques without requiring symmetry assumptions to the vacuum Einstein equations in spacetime dimension at least eleven. Many new results are also obtained in four dimensions. For instance, the Einstein–Maxwell–dilaton and Einstein–Yang–Mills equations are treated. The general nature of the results is that, provided certain inequalities are satisfied by coupling constants, solutions with prescribed singularities can be constructed that depend on the same number of free functions as the general solution of the given Einstein–matter system.

7 Asymptotics of Expanding Cosmological Models

The aim of this section is to present a picture of the dynamics of forever-expanding cosmological models, by which we mean spacetimes that are maximal globally hyperbolic developments and which can be covered by a foliation by Cauchy surfaces whose mean curvature $\mathrm{tr}\, k$ is strictly negative. In contrast to the approach to the big bang considered in Section 8, the spatial topology can be expected to play an important role in the present considerations. Intuitively, it may well happen that gravitational waves have time to propagate all the way around the universe. It will be assumed, as the simplest case, that the spacetimes considered admit a compact Cauchy surface. Then the hypersurfaces of negative mean curvature introduced above have finite volume and this volume is a strictly increasing function of time.

7.1 Lessons from homogeneous solutions

Which features should we focus on when thinking about the dynamics of forever expanding cosmological models? Consider for a moment the Kasner solution

$$-dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3}$$

where $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. These are the first and second Kasner relations. They imply that not all p_i can be strictly positive. Taking the coordinates x, y and z to be periodic, gives a vacuum cosmological model whose spatial topology is that of a three-torus. The volume of the hypersurfaces $t = \mathrm{const.}$ grows monotonically. However, the geometry does not expand in all directions, since not all p_i are positive. This can be reformulated in a way which is more helpful when generalizing to inhomogeneous models. In fact the quantities $-p_i$ are the eigenvalues of the second fundamental form. The statement then is that the second fundamental form is not negative definite. Looking at other homogeneous models indicates that this behaviour of the Kasner solution is not typical of what happens more generally. On the contrary, it seems reasonable to conjecture that in general the second fundamental form eventually becomes negative definite, at least in the presence of matter.

Some examples will now be presented. The following discussion makes use of the Bianchi classification of homogenous cosmological models (see *e.g.* [237]). If we take the Kasner solution and add a perfect fluid with equation of state $p = (\gamma - 1)\rho$, $1 \le \gamma < 2$, maintaining the symmetry (Bianchi type I), then the eigenvalues λ_i of the second fundamental satisfy $\lambda_i/\mathrm{tr}\, k \to 1/3$ in the limit of infinite expansion. The solution isotropizes. More generally this does not happen. If we look at models of Bianchi type II with non-tilted perfect fluid, *i.e.* where the fluid velocity is orthogonal to the homogeneous hypersurfaces, then the quantities $p_i = \lambda_i/\mathrm{tr}\, k$ converge to limits that are positive but differ from 1/3 (see [237], p. 138.) There is partial but not complete isotropization. The quantities p_i just introduced are called generalized Kasner exponents, since in the case of the Kasner solution they reduce to the p_i in the metric form (3). This kind of partial isotropization, ensuring the definiteness of the second fundamental form at late times, seems to be typical.

Intuitively, a sufficiently general vacuum spacetime should resemble gravitational waves propagating on some metric describing the large-scale geometry. This could even apply to spatially homogeneous solutions, provided they are sufficiently general. Hence, in that case also there should be partial isotropization. This expectation is confirmed in the case of vacuum spacetimes of Bianchi type VIII [224]. In that case the generalized Kasner exponents converge to non-negative limits different from 1/3. For a vacuum model this can only happen if the quantity $\hat{R} = R/(\mathrm{tr}\, k)^2$, where R is the spatial scalar curvature, does not tend to zero in the limit of large time.

The Bianchi models of type VIII are the most general indefinitely expanding models of class A. Note, however, that models of class VI_h for all h together are just as general. The latter models with perfect fluid and equation of state $p = (\gamma - 1)\rho$ sometimes tend to the Collins model for an open set of values of h for each fixed γ (cf. [237], p. 160). These models do not in general exhibit partial isotropization. It is interesting to ask whether this is connected to the issue of spatial boundary conditions. General models of class B cannot be spatially compactified in such a way as to be locally spatially homogeneous while models of Bianchi type VIII can. See also the discussion in [20].

Another issue is what assumptions on matter are required in order that it have the effect of (partial) isotropization. Consider the case of Bianchi I. The case of a perfect fluid has already been mentioned. Collisionless matter described by kinetic theory also leads to isotropization (at least under the assumption of reflection symmetry), as do fluids with almost any physically reasonable equation of state [210]. There is, however, one exception. This is the stiff fluid, which has a linear equation of state with $\gamma = 2$. In that case the generalized Kasner exponents are time-independent, and may take on negative values. In a model with two non-interacting fluids with linear equation of state the one with the smaller value of γ dominates the dynamics at late times [89], and so the isotropization is restored. Consider now the case of a magnetic field and a perfect fluid with linear equation of state. A variety of cases of Bianchi types I, II and VI_0 have been studied in [161, 162, 163], with a mixture of rigorous results and conjectures being obtained. The general picture seems to be that, apart from very special cases, there is at least partial isotropization. The asymptotic behaviour varies with the parameter γ in the equation of state and with the Bianchi type (only the case $\gamma \geq 1$ will be considered here). At one extreme, Bianchi type I models with $\gamma \leq 4/3$ isotropize. At the other extreme, the long time behaviour resembles that of a magnetovacuum model. This occurs for $\gamma > 5/3$ in type I, for $\gamma > 10/7$ in type II and for all $\gamma > 1$ in type VI_0. In all these cases there is partial isotropization.

Under what circumstances can a spatially homogeneous spacetime have the property that the generalized Kasner exponents are independent of time? The strong energy condition says that $R_{\alpha\beta}n^\alpha n^\beta \geq 0$ for any causal vector n^α. It follows from the Hamiltonian constraint and the evolution equation for $\mathrm{tr}\, k$

that if the generalized Kasner exponents are constant in time in a spacetime of Bianchi type I, then the normal vector n^α to the homogeneous hypersurfaces gives equality in the inequality of the strong energy condition. Hence the matter model is in a sense on the verge of violating the strong energy condition and this is a major restriction on the matter model.

A further question that can be posed concerning the dynamics of expanding cosmological models is whether $\hat{\rho} = \rho/(\mathrm{tr}\, k)^2$ tends to zero. This is of cosmological interest since $\hat{\rho}$ is (up to a constant factor) the density parameter Ω used in the cosmology literature. Note that it is not hard to show that $\mathrm{tr}\, k$ and ρ each tend to zero in the limit for any model with $\Lambda = 0$ which exists globally in the future and where the matter satisfies the dominant and strong energy conditions. First, it can be seen from the evolution equation for $\mathrm{tr}\, k$ that this quantity is monotone increasing and tends to zero as $t \to \infty$. Then it follows from the Hamiltonian constraint that ρ tends to zero.

A reasonable condition to be demanded of an expanding cosmological model is that it be future geodesically complete. This has been proved for many homogeneous models in [207].

7.2 Inhomogeneous solutions with $\Lambda = 0$

For inhomogeneous models with vanishing cosmological constant there is little information available about what happens in general. Fischer and Moncrief [100] have made an interesting proposal that attempts to establish connections between the evolution of a suitably conformally rescaled version of the spatial metric in an expanding cosmological model and themes in Riemannian geometry such as the Thurston geometrization conjecture [236], degeneration of families of metrics with bounded curvature [2], and the Ricci flow [126]. A key element of this picture is the theorem on the stability of the Milne model discussed in Section 5.3. More generally, the rescaled metric is supposed to converge to a hyperbolic metric (metric of constant negative curvature) on a region that is large in the sense that the volume of its complement tends to zero. If the topology of the Cauchy surface is such that it is consistent with a metric of some Bianchi type, then the hyperbolic region will be missing and the volume of the entire rescaled metric will tend to zero. In this situation it might be expected that the metric converges to a (locally) homogeneous metric in some sense. Evidently the study of the nonlinear stability of Bianchi models is very relevant to developing this picture further.

Independently of the Fischer–Moncrief picture the study of small (but finite) perturbations of Bianchi models is an avenue for making progress in understanding expanding cosmological models. There is a large literature on linear perturbations of cosmological models and it would be desirable to determine what insights the results of this work might suggest for the full nonlinear dynamics. Just as it is interesting to know under what circumstances homogeneous cosmological models become isotropic in the course of expansion, it is interesting to know when more general models become homogeneous. This does happen in the case of small perturbations of the Milne model. On the other hand, there is an apparent obstruction in other cases. This is the Jeans instability [170, 43]. A linear analysis indicates that under certain circumstances (*e.g.* perturbations of a flat Friedmann model) inhomogeneities grow with time. As yet there are no results on this available for the fully nonlinear case. A comparison that should be useful is that with Landau damping in plasma physics, where rigorous results are available [123].

The most popular matter model for spatially homogeneous cosmological models is the perfect fluid. Generalizing this to inhomogeneous models is problematic since formation of shocks or (in the case of dust) shell-crossing must be expected to occur. These signal an end to the interval of evolution of the cosmological model, which can be treated mathematically with known techniques. Criteria for the development of shocks (or their absence) should be developed, based on the techniques of classical hydrodynamics.

In the case of polarized Gowdy spacetimes there is a description of the late-time asymptotics in the literature [87], although the proofs have unfortunately never been published. The central object in the analysis of these spacetimes is a function P that satisfies the equation $P_{tt} + t^{-1}P_t = P_{\theta\theta}$. The picture that emerges is that the leading asymptotics are given by $P = A \log t + B$ for constants A and B, this being the form taken by this function in a general Kasner model, while the next order correction consists of waves whose amplitude decays like $t^{-1/2}$, where t is the usual Gowdy time coordinate. The entire spacetime can be reconstructed from P by integration. It turns out that the generalized Kasner exponents converge to $(1, 0, 0)$ for inhomogeneous models. This shows that if it is stated that these models are approximated by Kasner models at late times it is necessary to be careful in what sense the approximation is supposed to hold. Information on the asymptotics is also available in the case of small but finite perturbations of the Milne model and the Bianchi type III form of flat spacetime, as discussed in Sections 5.3 and 5.4, respectively.

There are not too many results on future geodesic completeness for inhomogeneous cosmological models. A general criterion for geodesic completeness is given in [62]. It does not apply to cases like the Kasner solution but is well-suited to the case where the second fundamental form is eventually negative definite.

7.3 Inflationary models

One important aspect of the fragmentary picture of the dynamics of expanding cosmological models presented in the last two sections is that it seems to be complicated. A situation where we can hope for a simpler, more unified picture is that where a positive cosmological constant is present. Recall first that when the cosmological constant vanishes and the matter satisfies the usual energy conditions, spacetimes of Bianchi type IX recollapse [166] and so never belong to the indefinitely expanding models. When $\Lambda > 0$ this is no longer true. Then Bianchi IX spacetimes show complicated features, which will not be considered here (cf. [93]). In discussing homogeneous models we restrict to the other Bianchi types. Then a general theorem of Wald [239] states that any model whose matter content satisfies the strong and dominant energy conditions and which expands for an infinite proper time t is such that all generalized Kasner exponents tend to $1/3$ as $t \to \infty$. A positive cosmological constant leads to isotropization. The mean curvature tends to the constant value $-\sqrt{3\Lambda}$ as $t \to \infty$, while the scale factors increase exponentially.

Wald's result is only dependent on energy conditions and uses no details of the matter field equations. The question remains whether solutions corresponding to initial data for the Einstein equations with positive cosmological constant, coupled to reasonable matter, exist globally in time under the sole condition that the model is originally expanding. It can be shown that this is true for various matter models using the techniques of [207]. Suppose we have a solution on an interval (t_1, t_2). It follows from [239] that the mean curvature is increasing and no greater than $-\sqrt{3\Lambda}$. Hence, in particular, $\operatorname{tr} k$ is bounded as t approaches t_2. Now we wish to verify condition (7) of [207]. This says that if the mean curvature is bounded as an endpoint of the interval of definition of a solution is approached then the solution can be extended to a longer interval. As in [207] it can be shown that if $\operatorname{tr} k$ is bounded, then g_{ij}, k_{ij}, and $(\det g)^{-1}$ are bounded. Thus, in the terminology of [207], it is enough to check $(7)'$ for a given matter model in order to get the desired global existence theorem. This condition involves the behaviour of a fluid in a given spacetime. Since the Euler equation does not contain Λ, the result of [207] applies directly. It follows that global existence holds for perfect fluids and mixtures of non-interacting perfect fluids. A similar result holds when the matter is described by collisionless matter satisfying the Vlasov equation. Here it suffices to note that the proof of Lemma 2.2 of [204] generalizes without difficulty to the case where a cosmological constant is present.

The effect of a cosmological constant can be mimicked by a suitable exotic matter field that violates the strong energy condition: for example, a nonlinear scalar field with exponential potential. In the latter case, an analogue of Wald's theorem has been proved by Kitada and Maeda [152]. For a potential of the form $e^{-\lambda\phi}$ with λ smaller than a certain limiting value, the qualitative picture is similar to that in the case of a positive cosmological constant. The difference is that the asymptotic rate of decay of certain quantities is not the same as in the case with positive Λ. In [153] it is discussed how the limiting value of λ can be increased. The behaviour of homogeneous and isotropic models with general λ has been investigated in [125].

Both models with a positive cosmological constant and models with a scalar field with exponential potential are called inflationary because the rate of (volume) expansion is increasing with time. There is also another kind of inflationary behaviour that arises in the presence of a scalar field with power law potential like ϕ^4 or ϕ^2. In that case the inflationary property concerns the behaviour of the model at intermediate times rather than at late times. The picture is that at late times the universe resembles a dust model without cosmological constant. This is known as reheating. The dynamics have been analysed heuristically by Belinskii *et al.* [29]. Part of their conclusions have been proved rigorously in [200]. Calculations analogous to those leading to a proof of isotropization in the case of a positive cosmological constant or an exponential potential have been done for a power law potential in [179]. In that case, the conclusion cannot apply to late time behaviour. Instead, some estimates are obtained for the expansion rate at intermediate times.

Consider what happens to Wald's proof in an inhomogeneous spacetime with positive cosmological constant. His arguments only use the Hamiltonian constraint and the evolution equation for the mean curvature. In Gauss coordinates spatial derivatives of the metric only enter these equations via the spatial scalar curvature in the Hamiltonian constraint. Hence, as noticed in [142], Wald's argument applies to the inhomogeneous case, provided we have a spacetime that exists globally in the future in Gauss coordinates and which has everywhere non-positive spatial scalar curvature. Unfortunately, it is hard to see how the latter condition can be verified starting from initial data. It is not clear whether there is a non-empty set of inhomogeneous initial data to which this argument can be applied.

In the vacuum case with positive cosmological constant, the result of Friedrich discussed in Section 5.1 proves local homogenization of inhomogeneous spacetimes, *i.e.* that all generalized Kasner exponents corresponding to a suitable spacelike foliation tend to 1/3 in the limit. To see this, consider (part of) the de Sitter metric in the form $-dt^2 + e^{2t}(dx^2 + dy^2 + dz^2)$. This choice, which is different from that discussed in [101], simplifies the algebra as much as possible. Letting $\tau = e^{-t}$ shows that the above metric can be written in the form $\tau^{-2}(-d\tau^2 + dx^2 + dy^2 + dz^2)$. This exhibits the de Sitter metric as being conformal to a flat metric. In the construction of Friedrich the conformal class and conformal factor are perturbed. The corrections to the metric in terms of coordinate components are of relative order $\tau = e^{-t}$. Thus, the trace-free part of the second fundamental forms decays exponentially, as desired.

There have been several numerical studies of inflation in inhomogeneous spacetimes. These are surveyed in Section 3 of [15].

8 Structure of General Singularities

The aim of this section is to present a picture of the nature of singularities in general solutions of the Einstein equations. It is inspired by the ideas of Belinskii, Khalatnikov, and Lifshitz (BKL). To fix ideas, consider the case of a solution of the Einstein equations representing a cosmological model with a big bang singularity. A central idea of the BKL picture is that near the singularity the evolution at different spatial points decouples. This means that the global spatial topology of the model plays no role. The decoupled equations are ordinary differential equations. They coincide with the equations for spatially homogeneous cosmological models, so that the study of the latter is of particular significance.

8.1 Lessons from homogeneous solutions

In the BKL picture a Gaussian coordinate system (t, x^a) is introduced such that the big bang singularity lies at $t = 0$. It is not *a priori* clear whether this should be possible for very general spacetimes. A positive indication is given by the results of [6], where coordinates of this type are introduced in one very general class of spacetimes. Once these coordinates have been introduced, the BKL picture says that the solution of the Einstein equations should be approximated near the singularity by a family of spatially homogeneous solutions depending on the coordinates x^a as parameters. The spatially homogeneous solutions satisfy ordinary differential equations in t.

Spatially homogeneous solutions can be classified into Bianchi and Kantowski–Sachs solutions. The Bianchi solutions in turn can be subdivided into types I to IX according to the Lie algebra of the isometry group of the spacetime. Two of the types, VI_h and VII_h are in fact one-parameter families of non-isomorphic Lie algebras labelled by h. The generality of the different symmetry types can be judged by counting the number of parameters in the initial data for each type. The result of this is that the most general types are Bianchi VIII, Bianchi IX, and Bianchi $\mathrm{VI}_{-1/9}$. The usual picture is that Bianchi VIII and Bianchi IX have more complicated dynamics than all other types and that the dynamics is similar in both these cases. This leads one to concentrate on Bianchi type IX and the mixmaster solution (see Section 3.2). Bianchi type $\mathrm{VI}_{-1/9}$ was apparently never mentioned in the work of BKL and has been largely ignored in the literature. This is a gap in understanding that should be filled. Here we follow the majority and focus on Bianchi type IX.

Another aspect of the BKL picture is that most types of matter should become negligible near the singularity for suitably general solutions. In the case of perfect fluid solutions of Bianchi type IX with a linear equation of state, this has been proved by Ringström [222]. In the case of collisionless matter it remains an open issue, since rigorous results are confined to Bianchi types I, II and III and Kantowski–Sachs, and have nothing to say about Bianchi type IX. If it is accepted that matter is usually asymptotically negligible then vacuum solutions become crucial. The vacuum solutions of Bianchi type IX (mixmaster solutions) play a central role. They exhibit complicated oscillatory behaviour, and essential aspects of this have been captured rigorously in the work of Ringström [223, 222] (compare Section 3.2).

Some matter fields can have an important effect on the dynamics near the singularity. A scalar field or stiff fluid leads to the oscillatory behaviour being replaced by monotone behaviour of the basic quantities near the singularity, and thus to a great simplification of the dynamics. An electromagnetic field can cause oscillatory behaviour that is not present in vacuum models or models with perfect fluid of the same symmetry type. For instance, models of Bianchi type I with an electromagnetic field show oscillatory, mixmaster-like behaviour [161]. However, it seems that this does not lead to anything essentially new. It is simply that the effects of spatial curvature in the more complicated Bianchi types can be replaced by electromagnetic fields in simpler Bianchi types.

A useful heuristic picture that systematizes much of what is known about the qualitative dynamical behaviour of spatially homogeneous solutions of the Einstein equations is the idea developed by Misner [175] of representing the dynamics as the motion of a particle in a time-dependent potential. In the approach to the singularity the potential develops steep walls where the particle is reflected. The mixmaster evolution consists of an infinite sequence of bounces of this kind.

8.2 Inhomogeneous solutions

Consider now inhomogeneous solutions of the Einstein equations where, according to the BKL picture, oscillations of mixmaster type are to be expected. This is for instance the case for general solutions of the vacuum Einstein equations. There is only one rigorous result to confirm the presence of these oscillations in an inhomogeneous spacetime of any type, and that concerns a family of spacetimes depending on only finitely many parameters [35]. They are obtained by applying a solution-generating technique to the mixmaster solution. Perhaps a reason for the dearth of results is that oscillations usually only occur in combination with the formation of local spatial structure discussed in Section 8.3. On the other hand, there is a rich variety of numerical and heuristic work supporting the BKL picture in the inhomogeneous case [32].

A situation where there is more hope of obtaining rigorous results is where the BKL picture suggests that there should be monotone behaviour near the singularity. This is the situation for which Fuchsian techniques can often be applied to prove the existence of large classes of spacetimes having the expected behaviour near the initial singularity (see Section 6.2). It would be desirable to have a stronger statement than these techniques have provided up to now. Ideally, it should be shown that a non-empty open set of solutions of the given class (by which is meant all solutions corresponding to an open set of initial data on a regular Cauchy surface) lead to a singularity of the given type. The only results of this type in the literature concern polarized Gowdy spacetimes [139], plane symmetric spacetimes with a massless scalar field [208], spacetimes with collisionless matter and spherical, plane or hyperbolic symmetry [190], and a subset of general Gowdy spacetimes [85]. The work of Christodoulou [69] on spherically symmetric solutions of the Einstein equations with a massless scalar field should also be mentioned in this context, although it concerns the singularity inside a black hole rather than singularities in cosmological models. Note that all these spacetimes have at least two Killing vectors so that the PDE problem to be solved reduces to an effective problem in one space dimension.

8.3 Formation of localized structure

Numerical calculations and heuristic methods such as those used by BKL lead to the conclusion that, as the singularity is approached, localized spatial structure will be formed. At any given spatial point the dynamics is approximated by that of a spatially homogeneous model near the singularity, and there will in general be bounces (cf. Section 8.1). However, there will be exceptional spatial points where the bounce fails to happen. This leads to a situation in which the spatial derivatives of the quantities describing the geometry blow up faster than these quantities themselves as the singularity is approached. In general spacetimes there will be infinitely many bounces before the singularity is reached, and so the points where the spatial derivatives are large will get more and more closely separated as the singularity is approached.

In Gowdy spacetimes only a finite number of bounces are to be expected and the behaviour is eventually monotone (no more bounces). There is only one essential spatial dimension due to the symmetry and so large derivatives in general occur at isolated values of the one interesting spatial coordinate. Of course, these correspond to surfaces in space when the symmetry directions are restored. The existence of Gowdy solutions showing features of this kind has been proved in [221]. This was done by means of an explicit transformation that makes use of the symmetry. Techniques should be developed which can handle this type of phenomenon more directly and more generally.

The formation of spatial structure calls the BKL picture into question (cf. the remarks in [28]). The basic assumption underlying the BKL analysis is that spatial derivatives do not become too large near the singularity. Following the argument to its logical conclusion then indicates that spatial derivatives do become large near a dense set of points on the initial singularity. Given that the BKL picture has given so many correct insights, the hope that it may be generally applicable should not be abandoned too quickly. However, the problem represented by the formation of spatial structure shows that at the very least it is necessary to think carefully about the sense in which the BKL picture could provide a good approximation to the structure of general spacetime singularities.

9 Further Results

This section collects miscellaneous results that do not fit into the main line of the exposition.

9.1 Evolution of hyperboloidal data

In Section 2.1, hyperboloidal initial data were mentioned. They can be thought of as generalizations of the data induced by Minkowski space on a hyperboloid. In the case of Minkowski space the solution admits a conformal compactification where a conformal boundary, null infinity, can be added to the spacetime. It can be shown that in the case of the maximal development of hyperboloidal data a piece of null infinity can be attached to the spacetime. For small data, *i.e.* data close to that of a hyperboloid in Minkowski space, this conformal boundary also has completeness properties in the future allowing an additional point i_+ to be attached there (see [102] and references therein for more details). Making contact between hyperboloidal data and asymptotically flat initial data is much more difficult and there is as yet no complete picture. (An account of the results obtained up to now is given in [106].) If the relation between hyperboloidal and asymptotically flat initial data could be understood it would give a very different approach to the problem treated by Christodoulou and Klainerman (Section 5.2). It might well also give more detailed information on the asymptotic behaviour of the solutions.

9.2 The Newtonian limit

Most textbooks on general relativity discuss the fact that Newtonian gravitational theory is the limit of general relativity as the speed of light tends to infinity. It is a non-trivial task to give a precise mathematical formulation of this statement. Ehlers systematized extensive earlier work on this problem and gave a precise definition of the Newtonian limit of general relativity that encodes those properties that are desirable on physical grounds (see [98].) Once a definition has been given, the question remains whether this definition is compatible with the Einstein equations in the sense that there are general families of solutions of the Einstein equations that have a Newtonian limit in the sense of the chosen definition. A theorem of this kind was proved in [205], where the matter content of spacetime was assumed to be a collisionless gas described by the Vlasov equation. (For another suggestion as to how this problem could be approached, see [109].) The essential mathematical problem is that of a family of equations, depending continuously on a parameter λ, which are hyperbolic for $\lambda \neq 0$ and degenerate for $\lambda = 0$. Because of the singular nature of the limit it is by no means clear *a priori* that there are families of solutions that depend continuously on λ. That there is an abundant supply of families of this kind is the result of [205]. Asking whether there are families which are k times continuously differentiable in their dependence on λ is related to the issue of giving a mathematical justification of post-Newtonian approximations. The approach of [205] has not even been extended to the case $k = 1$, and it would be desirable to do this. Note however that when k is too large, serious restrictions arise [203]. The latter fact corresponds to the well-known divergent behaviour of higher order post-Newtonian approximations.

It may be useful for practical projects, for instance those based on numerical calculations, to use hybrid models in which the equations for self-gravitating Newtonian matter are modified by terms representing radiation damping. If we expand in terms of the parameter λ as above then at some stage radiation damping terms should play a role. The hybrid models are obtained by truncating these expansions in a certain way. The kind of expansion that has just been mentioned can also be done, at least formally, in the case of the Maxwell equations. In that case a theorem on global existence and asymptotic behaviour for one of the hybrid models has been proved in [160]. These results have been put into context and related to the Newtonian limit of the Einstein equations in [159].

9.3 Newtonian cosmology

Apart from the interest of the Newtonian limit, Newtonian gravitational theory itself may provide interesting lessons for general relativity. This is no less true for existence theorems than for other issues. In this context, it is also interesting to consider a slight generalization of Newtonian theory, the Newton–Cartan theory. This allows a nice treatment of cosmological models, which are in conflict with the (sometimes implicit) assumption in Newtonian gravitational theory that only isolated systems are considered. It is also unproblematic to introduce a cosmological constant into the Newton–Cartan theory.

Three global existence theorems have been proved in Newtonian cosmology. The first [46] is an analogue of the cosmic no hair theorem (cf. Section 5.1) and concerns models with a positive cosmological constant. It asserts that homogeneous and isotropic models are nonlinearly stable if the matter is described by dust or a polytropic fluid with pressure. Thus, it gives information about global existence and asymptotic behaviour for models arising from small (but finite) perturbations of homogeneous and isotropic data. The second and third results concern collisionless matter and the case of vanishing cosmological constant. The second [196] says that data that constitute a periodic (but not necessarily small) perturbation of a homogeneous and isotropic model that expands indefinitely give rise to solutions that exist globally in the future. The third [191] says that the homogeneous and isotropic models in Newtonian cosmology that correspond to a $k = -1$ Friedmann–Robertson–Walker model in general relativity are non-linearly stable.

9.4 The characteristic initial value problem

In the standard Cauchy problem, which has been the basic set-up for all the previous sections, initial data are given on a spacelike hypersurface. However, there is also another possibility, where data are given on one or more null hypersurfaces. This is the characteristic initial value problem. It has the advantage over the Cauchy problem that the constraints reduce to ordinary differential equations. One variant is to give initial data on two smooth null hypersurfaces that intersect transversely in a spacelike surface. A local existence theorem for the Einstein equations with an initial configuration of this type was proved in [201]. Another variant is to give data on a light cone. In that case local existence for the Einstein equations has not been proved, although it has been proved for a class of quasilinear hyperbolic equations that includes the reduced Einstein equations in harmonic coordinates [95].

Another existence theorem that does not use the standard Cauchy problem, and which is closely connected to the use of null hypersurfaces, concerns the Robinson–Trautman solutions of the vacuum Einstein equations. In that case the Einstein equations reduce to a parabolic equation. Global existence for this equation has been proved by Chruściel [86].

9.5 The initial boundary value problem

In most applications of evolution equations in physics (and in other sciences), initial conditions need to be supplemented by boundary conditions. This leads to the consideration of initial boundary value problems. It is not so natural to consider such problems in the case of the Einstein equations since in that case there are no physically motivated boundary conditions. (For instance, we do not know how to build a mirror for gravitational waves.) An exception is the case of a fluid boundary discussed in Section 2.6.

For the vacuum Einstein equations it is not *a priori* clear that it is even possible to find a well-posed initial boundary value problem. Thus, it is particularly interesting that Friedrich and Nagy [107] have been able to prove the well-posedness of certain initial boundary value problems for the vacuum Einstein equations. Since boundary conditions come up quite naturally when the Einstein equations are solved numerically, due to the need to use a finite grid, the results of [107] are potentially important for numerical relativity. The techniques developed there could also play a key role in the study of the initial value problem for fluid bodies (cf. Section 2.6).

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