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Application of a subordination theorem associated with certain new generalized subclasses of analytic and univalent functions

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Abstract
The prime focus of the present work is to investigate some fascinating relations of some analytic and univalent functions using a subordination theorem.

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Introduction

Let H denote the class of normalized analytic functions $f(z)$ having the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots \tag{1}$$

in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Also, let S denote the subclass of H univalent in U. Suppose that S^* denote the subclass of S consisting of the functions $f(z)$ which are starlike in U. A function $f(z) \in K$ is said to be convex in U if $f(z) \in S$ satisfies the condition that $zf'(z) \in S^*$. If $f(z) \in H$ satisfies the geometric condition:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in U$$

for some real $\beta (0 \leq \beta < 1)$, then we say that $f(z)$ belongs to the class $S^*(\beta)$ starlike of order β, and if $f(z) \in H$ satisfies the geometric condition:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad z \in U$$

for some real $\beta (0 \leq \beta < 1)$, then we say that $f(z)$ belongs to the class $K(\beta)$ convex of order β (see [1, 2]). Let the function $g(z)$ of the form:

$$g(z) = z + z^3 + z^5 + \dots \quad z \in U \tag{2}$$

be in the class S^* while the function $g(z)$ of the form:

$$g(z) = z + z^2 + z^3 + \dots \quad z \in U \tag{3}$$

be in the class K. With reference to (2) and (3), we can write that:

$$g_\alpha(z) = \frac{z}{1-z^\alpha} = z + \sum_{k=1}^{\infty} z^{1+k\alpha} \quad z \in U, \tag{4}$$

where we consider the principal value of $z^{k\alpha}$ for some real α $(0 < \alpha \leq 2)$. See Darus and Owa [3] for some properties of functions $f_\alpha(z)$ of the form (4).

Here, we present a more generalized form of (4) such that:

$$g_{\alpha,n}(z) = \frac{A^n z}{(A+Bz^\alpha)^n} = z + \sum_{k=1}^{\infty}(-1)^k \frac{B^k}{A^k} n_k z^{1+k\alpha} \quad z \in U \tag{5}$$

for some real α $(0 < \alpha \leq 2)$, $-1 \leq B < A \leq 1$, $n \geq 0$ and n_k is given by $n_k = \Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)$.

In view of (1) and (5), we introduce a class $H_{\alpha,n}$ of analytic function $f_{\alpha,n}(z)$ which is a convolution (or Hadamard product) of $f(z)$ and $g_{\alpha,n}$ $(f(z) * g_{\alpha,n}(z))$ such that:

$$f_{\alpha,n}(z) = z + \sum_{k=1}^{\infty}(-1)^k \frac{B^k}{A^k} n_k a_{k+1} z^{1+k\alpha} \quad z \in U \tag{6}$$

In addition, if $f_{\alpha,n}(z) \in H_{\alpha,n}$ satisfies the following condition:

$$\Re\left(\frac{zf'_{\alpha,n}(z)}{f_{\alpha,n}(z)}\right) > \gamma \quad z \in U \tag{7}$$

for some real α $(0 < \alpha \leq 2)$, $n > 0$, and $\gamma(0 \leq \gamma < 1)$, then $f_{\alpha,n}$ belong to the starlike class $S^*_{\alpha,n}(A,B,\gamma)$ (of order γ). Also, if $f_{\alpha,n}(z) \in H_{\alpha,n}$ satisfies the following condition:

$$\Re\left(1 + \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right) > \gamma \quad z \in U \tag{8}$$

for some real α $(0 < \alpha \leq 2)$, $n > 0$, and $\gamma(0 \leq \gamma < 1)$, then $f_{\alpha,n}$ belong to the convex class $K^*_{\alpha,n}(A,B,\gamma)$ (of order γ). Here, it is noted that $f_{\alpha,n}(z) \in H_{\alpha,n}(z)$ belong to the convex class $K_{\alpha,n}(A,B,\gamma) \Leftrightarrow zf'_{\alpha,n}(z)$ belong to the starlike class $S^*_{\alpha,n}(A,B,\gamma)$.

For the purpose of the present investigation, we shall call to mind the following definitions and lemmas.

Definition 1 (Subordination principle) *For two functions f and g analytic in U, we say that f is subordinate to g, and write $f \prec g$ in U or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$. It is known that:*

$$f(z) \prec g(z) \ \Rightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U:

$$f(z) \prec g(z) \ \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U). \tag{9}$$

Also, we say that $g(z)$ is superordinate to $f(z)$ in U (see [4–6]).

Definition 2 (Subordinating factor sequence) *A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called subordinating factor sequence if for every univalent function $f(z)$ in K, we have the subordination given by:*

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z) \quad (z \in U, \ \ a_1 = 1) \text{ (see [4–6]).} \tag{10}$$

Lemma 1 *The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if and only if:*

$$\Re\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in U). \tag{11}$$

The lemma above is due to Wilf [7]. *Interested reader can also refer to* [4–6].

Lemma 2 *Let* $s(z)$ $(s(z) \neq 0)$ *be a univalent function in* U. *Also, let* $\mu \neq 0$ *be a complex number, then we have that:*

$$\Re\left\{1+z\frac{s''(z)}{s'(z)}-z\frac{s'(z)}{s(z)}\right\} > \max\left\{0, \Re\left(\frac{\mu-1}{\mu}s(z)\right)\right\}. \tag{12}$$

Suppose that r $(r(z) \neq 0)$ *satisfies the differential equation:*

$$(1-\mu)(r(z)-1)+\mu\frac{zr'(z)}{r(z)} \prec (1-\mu)(s(z)-1)+\mu\frac{zs\prime(z)}{s(z)},\ z \in U \tag{13}$$

then $r \prec s$ *and* s *is the best dominant (see* [8] *among others).*

Lemma 3 *Let* ω *be regular in* H *with* $\omega(0) = 0$. *Also, suppose that* $|\omega(z)|$ *attains its maximum value on the circle* $|z| < 1$ *at a point* z_0, *then:*

$$z_0\omega'(z_0) = \sigma\omega(z_0), \tag{14}$$

where σ *is any real number and* $\sigma \geq 1$ *(see* [8] *among others).*

Coefficient inequality

In this section, we consider the coefficient inequalities for function $f_{\alpha,n}(z)$ given by (6) belonging to both classes $S^*_{\alpha,n}(A,B,\gamma)$ and $K_{\alpha,n}(A,B,\gamma)$ in the unit disk U.

Theorem 1 *Let the function* $f_{\alpha,n}(z)$ *of the form (6) satisfy the inequality:*

$$\sum_{k=1}^{\infty}(k\alpha-\gamma+1)\,n_k\frac{|B|^k}{A^k}|a_{k+1}| \leq 1-\gamma. \tag{15}$$

Then, $f_{\alpha,n}(z) \in S^*_{\alpha,n}(A,B,\gamma)$ *for* $0 \leq \gamma < 1$, $0 < \alpha \leq 2$, $-1 \leq B < A \leq 1$, $0 < A \leq 1$ *and* $n > 0$. *The equality holds true for* $f_{\alpha,n}(z)$ *given by:*

$$f_{\alpha,n}(z) = z + \frac{(1-\gamma)\,e^{i\pi}}{\big(k\alpha-\gamma+1\big)n_k\frac{|B|^k}{A^k}}z^{1+k\alpha} \quad (k \geq 1).$$

Proof Suppose that the function $f_{\alpha,n}(z)$ given by (6) satisfies (15), then:

$$\left|\frac{zf'_{\alpha,n}(z)}{f_{\alpha,n}(z)}-1\right| = \left|\frac{\sum_{k=1}^{\infty}(-1)^k k\alpha n_k \frac{B^k}{A^k}a_{k+1}z^{k\alpha}}{1+\sum_{k=1}^{\infty}(-1)^k n_k\frac{B^k}{A^k}a_{k+1}z^{k\alpha}}\right|$$

$$\leq \frac{\sum_{k=1}^{\infty}k\alpha n_k\frac{|B|^k}{A^k}|a_{k+1}||z|^{k\alpha}}{1-\sum_{k=1}^{\infty}n_k\frac{|B|^k}{A^k}|a_{k+1}||z|^{k\alpha}}| < \frac{\sum_{k=1}^{\infty}k\alpha n_k\frac{|B|^k}{A^k}|a_{k+1}|}{1-\sum_{k=1}^{\infty}n_k\frac{|B|^k}{A^k}|a_{k+1}|} \leq 1-\gamma.$$

This shows that $f_{\alpha,n}(z) \in S^*_{\alpha,n}(A,B,\gamma)$, and this ends the proof. □

Corollary 1 *Let the function $f_{\alpha,n}(z)$ of the form (6) satisfy the inequality:*

$$\sum_{k=1}^{\infty}\left(k\alpha+1\right)n_k\frac{|B|^k}{A^k}|a_{k+1}|\leq 1.$$

Then, $f_{\alpha,n}(z)\in S^*_{\alpha,n}(A,B,0)$.

Theorem 2 *Let the function $f_{\alpha,n}(z)$ of the form (6) satisfy the inequality:*

$$\sum_{k=1}^{\infty}(k\alpha+1)\,(k\alpha-\gamma+1)\,n_k\frac{|B|^k}{A^k}|a_{k+1}|\leq 1-\gamma. \tag{16}$$

Then, $f_{\alpha,n}(z)\in K_{\alpha,n}(A,B,\gamma)$ for $0\leq\gamma<1$, $0<\alpha\leq 2$, $-1\leq B<A\leq 1$, $0<A\leq 1$ and $n>0$. The equality holds true for $f_{\alpha,n}(z)$ given by:

$$f_{\alpha,n}(z)=z+\frac{(1-\gamma)\,e^{i\pi}}{(k\alpha+1)\,(k\alpha-\gamma+1)\,n_k\frac{|B|^k}{A^k}}z^{1+k\alpha}\qquad(k\geq 1).$$

Proof The proof is similar to that of Theorem 1. □

Corollary 2 *Let the function $f_{\alpha,n}(z)$ of the form (6) satisfy the inequality:*

$$\sum_{k=1}^{\infty}(k\alpha+1)^2\,n_k\frac{|B|^k}{A^k}|a_{k+1}|\leq 1.$$

Then, $f_{\alpha,n}(z)\in K_{\alpha,n}(A,B,0)$.

Remark 1 *Putting $A=n=1$ and $B=-1$ in Theorems 1 and 2, we obtain the results obtained by Darus and Owa [[3], Theorems 3 and 4].*

Next, we present some subordination results.

Some subordination results

Our prime objective here is to establish sufficient conditions for functions belonging to the analytic class $S^*_{\alpha,n}(A,B,\gamma)$.

Theorem 3 *Suppose that the function $f_{\alpha,n}(z)$ is as defined in (6). Let $0<\alpha\leq 2$, $n>0$, $\sigma\neq -1$ and μ be a non-zero complex number in U such that:*

$$\Re\left\{1+\frac{z[1-\sigma(1-2z)]}{(1-z)(1+\sigma z)}\right\}>\max\left\{0,\Re\left(\frac{\mu-1}{\mu}\right)\left(\frac{1+\sigma z}{1-z}\right)\right\}.$$

If

$$(1-\mu)\left(f'_{\alpha,n}(z)-1\right)+\mu\left(\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right)\prec(1-\mu)\left(\left(\frac{1+\sigma z}{1-z}\right)-1\right)+\mu\left(\frac{(1+\sigma)z}{(1+\sigma z)(1-z)}\right)$$

holds true, then $f_{\alpha,n}(z)\in S^*_{\alpha,n}(A,B,\gamma)$.

Proof Suppose that we let:

$$r(z)=f'_{\alpha,n}(z)\quad\text{and}\quad s(z)=\frac{1+\sigma z}{1-z}. \tag{17}$$

Then,

$$\Re\left\{1+\frac{zs''(z)}{s'(z)}-\frac{zs'(z)}{s(z)}\right\}>\max\left\{0,\Re\left(\frac{\mu-1}{\mu}\right)\left(\frac{1+\sigma z}{1-z}\right)\right\}=\max\left\{0,\Re\left(\frac{\mu-1}{\mu}s(z)\right)\right\}$$

and

$$(1-\mu)(r(z)-1)+\mu\frac{zr'(z)}{r(z)}=(1-\mu)\left(f'_{\alpha,n}(z)-1\right)+\mu\left(\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right)$$

$$\prec(1-\mu)\left(\left(\frac{1+\sigma z}{1-z}\right)-1\right)+\mu\left(\frac{(1+\sigma)z}{(1+\sigma z)(1-z)}\right)=(1-\mu)(s(z)-1)+\mu\frac{zs'(z)}{s(z)}. \tag{18}$$

Using Lemma 2 in (18), then we obtain the desired result. □

Theorem 4 *Let the analytic function $f_{\alpha,n}(z)$ be defined as in (6). Suppose that $f_{\alpha,n}(z)$ satisfies the condition that:*

$$\Re\left\{\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right\}<-\frac{1+\sigma}{2(1-\sigma)}. \tag{19}$$

Then, for $0<\alpha\le 2$, $n>0$ and $\sigma>1$, $f_{\alpha,n}(z)\in S^*_{\alpha,n}(A,B,\gamma)$.

Proof Setting:

$$f'_{\alpha,n}(z)=\left(\frac{1+\sigma\omega(z)}{1-\omega(z)}\right),\quad \omega(z)\neq 1.$$

Then, ω is regular in U, and since $\sigma\neq -1$, then $\omega(0)=0$. Also, it follows that:

$$\Re\left\{\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right\}=\Re\left\{\frac{(1+\sigma)z\omega'(z)}{(1-\omega(z))(1+\sigma\omega(z))}\right\}<\frac{\sigma+1}{2(\sigma-1)},\quad \sigma\neq 1.$$

Next, we show that $|\omega(z)|<1$. So, let there exists a point $z_0\in U$ such that for $|z|\le|z_0|$:

$$\max|\omega(z)|=|\omega(z)|=1.$$

Then, appealing to Lemma 3 and setting $\omega(z_0)=e^{i\theta}$, $z_0\omega'(z_0)=\delta e^{i\theta}$ and for $\delta\ge 1$, $\sigma>1$, we have that:

$$\Re\left\{\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right\}=\Re\left\{\frac{(1+\sigma)z_0\omega'(z_0)}{(1-\omega(z_0))(1+\sigma\omega(z_0))}\right\}=\Re\left\{\frac{\delta e^{i\theta}(1+\sigma)}{(1-e^{i\theta})(1+\sigma e^{i\theta})}\right\}$$
$$=\frac{\delta(\sigma+1)}{2(\sigma-1)}\ge\frac{(\sigma+1)}{2(\sigma-1)}.$$

Therefore,

$$\Re\left\{\frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right\}\ge-\frac{1+\sigma}{2(1-\sigma)}\quad z\in U$$

which negates the hypothesis (19).

Hence, we conclude that $|\omega(z)|<1$ for all $z\in U$ and:

$$f'_{\alpha,n}(z)\prec\left(\frac{1+\sigma z}{1-z}\right),\quad \sigma\neq 1,\ z\in U$$

and this obviously ends the proof. □

Application of a subordination theorem

Let $\overline{S}^*_{\alpha,n}(A,B,\gamma)$ and $\overline{K}_{\alpha,n}(A,B,\gamma)$ denote the classes of functions $f_{a,n}\in H_{a,n}$ whose coefficients satisfy conditions (15) and (16), respectively. We note that $\overline{S}^*_{\alpha,n}(A,B,\gamma)\subseteq S^*_{\alpha,n}(A,B,\gamma)$ and $\overline{K}_{\alpha,n}(A,B,\gamma)\subseteq K_{\alpha,n}(A,B,\gamma)$. Here, we consider an application of the subordination result given in Lemma 1 to both classes $\overline{S}^*_{\alpha,n}(A,B,\gamma)$ and $\overline{K}_{\alpha,n}(A,B,\gamma)$.

Theorem 5 *Let* $f_{\alpha,n}(z) \in \overline{S}^*_{\alpha,n}(A, B, \gamma)$. *If* $0 \le \gamma < 1$, $0 < \alpha \le 2$, $-1 \le B < A \le 1$, $0 < A \le 1$ *and* $n > 0$, *then:*

$$\frac{n(\alpha-\gamma+1)|B|}{2\big[n\alpha|B|+(1-\gamma)(A+n|B|)\big]}\left(f_{\alpha,n} * g_\alpha\right)(z) \prec g_\alpha(z) \tag{20}$$

for every function g_α *in* K_α *and:*

$$\Re\left(f_{\alpha,n}(z)\right) > -\frac{[n\alpha|B|+(1-\gamma)(A+n|B|)]}{n(\alpha-\gamma+1)|B|}. \tag{21}$$

The constant factor:

$$\frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B|+(1-\gamma)(A+n|B|)\right]}$$

in the subordination result (20) is sharp.

Proof Let $f_{\alpha,n} \in \overline{S}^*_{\alpha,n}(A, B, \gamma)$ and let g_α be any function in K_α. Then:

$$\frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B|+(1-\gamma)(A+n|B|)\right]}\left(f_{\alpha,n} * g_\alpha\right)(z) \prec g_\alpha(z)$$
$$= \frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B|+(1-\gamma)(A+n|B|)\right]}\left(z+\sum_{k=1}^{\infty} a_{k+1}b_{k+1}z^{k\alpha+1}\right).$$

Thus, by Definition 2, the subordination result (20) will hold true if:

$$\left\{\frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B|+(1-\gamma)(A+n|B|)\right]}a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$, appealing to Lemma 1, this is equivalent to:

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{n(\alpha-\gamma+1)|B|}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}a_k z^{(k-1)\alpha+1}\right\} > 0 \quad (z \in U). \tag{22}$$

Since $n_k\,(k\alpha-\gamma+1)\,\frac{|B|^k}{A^k}$ is an increasing function of k $(k \ge 1)$, we have that:

$$\Re\left\{1+\sum_{k=1}^{\infty}\frac{n(\alpha-\gamma+1)|B|}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}a_k z^{(k-1)\alpha+1}\right\}$$
$$= \Re\left\{1+\frac{n(\alpha-\gamma+1)|B|}{M}z+\frac{A}{M}\sum_{k=2}^{\infty}n(\alpha-\gamma+1)\frac{|B|}{A}a_k z^{(k-1)\alpha+1}\right\}$$
$$\ge 1-\frac{n(\alpha-\gamma+1)|B|}{M}r-\frac{A}{M}\sum_{k=2}^{\infty}n_{k-1}\Big((k-1)\alpha-\gamma+1\Big)\frac{|B|^{k-1}}{A^{k-1}}a_k r^{(k-1)\alpha+1}$$
$$> 1-\frac{n(\alpha-\gamma+1)|B|}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}r-\frac{A\,(1-\gamma)}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}r^{\alpha+1}$$
$$> 1-\frac{n(\alpha-\gamma+1)|B|}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}r-\frac{A\,(1-\gamma)}{[n\alpha|B|+(1-\gamma)(A+n|B|)]}r = 1-r > 0 \tag{23}$$

$$(|z| = r < 1)\,,$$

where $M = [n\alpha|B| + (1-\gamma)(A + n|B|)]$.

Therefore, (22) holds true in U and this obviously proves the inequality (20) while (21) follows by taking:

$$g_\alpha(z) = \frac{z}{1-z^\alpha} \in K_\alpha$$

in (20). Now, suppose that we consider the function $q_{\alpha,n}(z)$ of the form:

$$q_{\alpha,n}(z) = z - \frac{1-\gamma}{n(\alpha-\gamma+1)\frac{|B|}{A}} z^{\alpha+1}$$

which belongs to the class $\overline{S}^*_{\alpha,n}(A,B,\gamma)$. Then, using (20), we have that:

$$\frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B| + (1-\gamma)(A+n|B|)\right]}.q_{\alpha,n}(z) \prec \frac{z}{1-z^{\alpha}} \quad (z \in U)$$

which can easily be verified that for $0 \le \gamma < 1$, $0 < \alpha \le 2$, $-1 \le B < A \le 1$, $0 < A \le 1$, $n \ge 0$ and $|z| \le r$:

$$\min\left\{\Re\left(\frac{n(\alpha-\gamma+1)|B|}{2\left[n\alpha|B| + (1-\gamma)(A+n|B|)\right]}.q_{\alpha,n}(z)\right)\right\} = -\frac{1}{2} \quad (z \in U)$$

and this evidently completes the proof of Theorem 5. For various choices of the parameters involved, several interesting results are obtained. Given below are few instances. □

Corollary 3 *Let* $f_{\alpha,n}(z) \in \overline{S}^*_{\alpha,n}(1,-1,\gamma)$. *Then:*

$$\frac{n(\alpha-\gamma+1)}{2\left[n\alpha + (1-\gamma)(1+n)\right]}\left(f_{\alpha,n} * g_{\alpha}\right)(z) \prec g_{\alpha}(z)$$

for every function g_{α} *in* K_{α} *and:*

$$\Re\left(f_{\alpha,n}(z)\right) > -\frac{\left[n\alpha + (1-\gamma)(1+n)\right]}{n(\alpha-\gamma+1)}.$$

The constant factor:

$$\frac{n(\alpha-\gamma+1)}{2\left[n\alpha + (1-\gamma)(1+n)\right]}$$

is sharp.

Corollary 4 *Let* $f_{\alpha,1}(z) \in \overline{S}^*_{\alpha,1}(1,-1,\gamma)$. *Then:*

$$\frac{(\alpha-\gamma+1)}{2(\alpha-2\gamma+2)}\left(f_{\alpha,1} * g_{\alpha}\right)(z) \prec g_{\alpha}(z)$$

for every function g_{α} *in* K_{α} *and:*

$$\Re\left(f_{\alpha,1}(z)\right) > -\frac{(\alpha-2\gamma+2)}{(\alpha-\gamma+1)}.$$

The constant factor:

$$\frac{(\alpha-\gamma+1)}{2\,(\alpha-2\gamma+2)}$$

is sharp.

Corollary 5 [9, 10] *Let* $f_{1,1}(z) \in \overline{S}^*_{1,1}(1,-1,\gamma)$. *Then:*

$$\frac{(2-\gamma)}{2\,(3-2\gamma)}\left(f_{1,1} * g_{1}\right)(z) \prec g_{1}(z)$$

for every function g_1 *in* K_1 *and:*

$$\Re\left(f_{1,1}(z)\right) > -\frac{(3-2\gamma)}{(2-\gamma)}.$$

The constant factor:

$$\frac{(2-\gamma)}{2\,(3-2\gamma)}$$

is sharp.

Corollary 6 [9–11] *Let* $f_{1,1}(z) \in \overline{S}^*_{1,1}(1,-1,0)$. *Then:*

$$\frac{1}{3}\left(f_{1,1} * g_1\right)(z) \prec g_1(z)$$

for every function g_1 *in* K_1 *and:*

$$\Re\left(f_{1,1}(z)\right) > -\frac{3}{2}.$$

Theorem 6 *Let* $f_{\alpha,n}(z) \in \overline{K}_{\alpha,n}(A,B,\gamma)$. *If* $0 \le \gamma < 1$, $0 < \alpha \le 2$, $-1 \le B < A \le 1$ *and* $n > 0$, *then:*

$$\frac{n(\alpha+1)(\alpha-\gamma+1)|B|}{2\left[n\alpha(\alpha+1)|B| + (1-\gamma)(A+n(\alpha+1)|B|)\right]}\left(f_{\alpha,n} * g_\alpha\right)(z) \prec g_\alpha(z) \tag{24}$$

for every function g_α *in* K_α *and:*

$$\Re\left(f_{\alpha,n}(z)\right) > -\frac{\left[n\alpha(\alpha+1)|B| + (1-\gamma)(A+n(\alpha+1)|B|)\right]}{n(\alpha+1)(\alpha-\gamma+1)|B|}. \tag{25}$$

The constant factor:

$$\frac{n(\alpha+1)(\alpha-\gamma+1)|B|}{2\left[n\alpha(\alpha+1)|B| + (1-\gamma)(A+n(\alpha+1)|B|)\right]}$$

in the subordination result (24) cannot be replaced by a larger one, and the proof of which is similar to that of Theorem 3.

Corollary 7 *Let* $f_{\alpha,n}(z) \in \overline{K}_{\alpha,n}(1,-1,\gamma)$. *Then:*

$$\frac{n(\alpha+1)(\alpha-\gamma+1)}{2\left[n\alpha(\alpha+1) + (1-\gamma)(1+n(\alpha+1))\right]}\left(f_{\alpha,n} * g_\alpha\right)(z) \prec g_\alpha(z) \tag{26}$$

for every function g_α *in* K_α *and:*

$$\Re\left(f_{\alpha,n}(z)\right) > -\frac{\left[n\alpha(\alpha+1) + (1-\gamma)(1+n(\alpha+1))\right]}{n(\alpha+1)(\alpha-\gamma+1)}. \tag{27}$$

The constant factor:

$$\frac{n(\alpha+1)(\alpha-\gamma+1)}{2\left[n\alpha(\alpha+1) + (1-\gamma)(1+n(\alpha+1))\right]}$$

cannot be replaced by a larger one.

Corollary 8 *Let* $f_{\alpha,1}(z) \in \overline{K}_{\alpha,1}(1,-1,\gamma)$. *Then:*

$$\frac{(\alpha+1)(\alpha-\gamma+1)}{2\left[\alpha(\alpha+1) + (1-\gamma)(\alpha+2)\right]}\left(f_{\alpha,1} * g_\alpha\right)(z) \prec g_\alpha(z) \tag{28}$$

for every function g_α *in* K_α *and:*

$$\Re\left(f_{\alpha,1}(z)\right) > -\frac{\left[\alpha(\alpha+1) + (1-\gamma)(\alpha+2)\right]}{(\alpha+1)(\alpha-\gamma+1)}. \tag{29}$$

The constant factor:

$$\frac{(\alpha+1)(\alpha-\gamma+1)}{2\left[\alpha(\alpha+1) + (1-\gamma)(\alpha+2)\right]}$$

cannot be replaced by a larger one.

Corollary 9 [9, 10] *Let* $f_{1,1}(z) \in \overline{K}_{1,1}(1, -1, \gamma)$. *Then:*

$$\frac{2-\gamma}{5-3\gamma}\left(f_{1,1} * g_1\right)(z) \prec g_1(z) \tag{30}$$

for every function g_1 *in* K_1 *and:*

$$\Re\left(f_{1,1}(z)\right) > -\frac{5-3\gamma}{2(2-\gamma)}. \tag{31}$$

The constant factor:

$$\frac{2-\gamma}{5-3\gamma}$$

cannot be replaced by a larger one.

Corollary 10 [9, 10] *Let* $f_{1,1}(z) \in \overline{K}_{1,1}(1, -1, 0)$. *Then:*

$$\frac{2}{5}\left(f_{1,1} * g_1\right)(z) \prec g_1(z) \tag{32}$$

for every function g_1 *in* K_1 *and:*

$$\Re\left(f_{1,1}(z)\right) > -\frac{5}{4}. \tag{33}$$

The constant factor:

$$\frac{2}{5}$$

cannot be replaced by a larger one.

For further illustrations on the applications of the subordination result stated in Lemma 1, interested reader can see [4, 6, 8–11].

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