Cima and Mortini *Complex Anal Synerg (2017) 3:2*
DOI 10.1186/s40627-016-0008-8

Complex Analysis and its Synergies

RESEARCH Open Access

One-component inner functions

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Dedicated to the memory of Vadim Tolokonnikov.

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Abstract

We explicitly unveil several classes of inner functions u in H^∞ with the property that there is $\eta \in]0,1[$ such that the level set $\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$ is connected. These so-called one-component inner functions play an important role in operator theory.

Keywords: Inner functions, Interpolating Blaschke products, Connected components, Level sets

Mathematics Subject Classification: Primary 30J10, Secondary 30J05

1 Background

Definition 1 An inner function u in H^∞ is said to be a one-component inner function if there is $\eta \in]0,1[$ such that the level set (also called sublevel set or filled level set) $\Omega_u(\eta) := \{z \in \mathbb{D} : |u(z)| < \eta\}$ is connected.

One-component inner functions, the collection of which we denote by $\mathfrak{I}_c$, were first studied by Cohn [10] in connection with embedding theorems and Carleson measures. It was shown that [for instance, [10], p. 355] arclength on $\{z \in \mathbb{D} : |u(z)| = \varepsilon\}$ is such a measure whenever

$$\Omega_u(\eta) = \{z \in \mathbb{D} : |u(z)| < \eta\}$$

is connected and $\eta < \varepsilon < 1$.

A thorough study of the class $\mathfrak{I}_c$ was given by Aleksandrov [1] who showed the interesting result that $u \in \mathfrak{I}_c$ if and only if there is a constant $C = C(u)$ such that for all $a \in \mathbb{D}$

$$\sup_{z\in\mathbb{D}} \left| \frac{1-\overline{u(a)}u(z)}{1-\overline{a}z} \right| \leq C\frac{1-|u(a)|^2}{1-|a|^2}.$$

Many operator-theoretic applications are given in [1–3, 7]. In our paper here, we are interested in explicit examples, which are somewhat lacking in the literature. For example, if S is the atomic inner function, which is given by

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right),$$

then all level sets $\Omega_S(\eta)$, $0 < \eta < 1$ are connected, because these sets coincide with the disks

$$D_\eta := \left\{ z \in \mathbb{D} : \left| z - \frac{L}{L+1} \right| < \frac{1}{L+1} \right\}, \quad L := \log \frac{1}{\eta}, \tag{1}$$

which are tangential to the unit circle at $p = 1$.

The scheme of our note here is as follows: in Sect. 2, we prove a general result on level sets which will be the key for our approach to the problem of unveiling classes of one-component inner functions. Then in Sect. 3, we first present several examples with elementary geometric/function theoretic methods and then we use Aleksandrov's criterion to achieve this goal. For instance, we prove that $BS, B \circ S$, and $S \circ B$ are in $\mathfrak{I}_c$ whenever B is a finite Blaschke product. Considered are also interpolating Blaschke products. It will further be shown that, under the supremum norm, $\mathfrak{I}_c$ is an open subset of the set of all inner functions and multiplicatively closed. In the final section, we give counterexamples.

2 Level sets

We first begin with a topological property of the class of general level sets. Although statement (1) is "well known" (the earliest appearance seems to be in [26, Theorem VIII, 31]), we could nowhere locate a proof. The argument that the result is a simple and direct consequence of the maximum principle is, in our viewpoint, not tenable.

Lemma 2 *Given a non-constant inner function u in H^∞ and $\eta \in]0, 1[$, let $\Omega := \Omega_u(\eta) = \{z \in \mathbb{D} : |u(z)| < \eta\}$ be a level set. Suppose that Ω_0 is a component (=maximal connected subset) of Ω. Then*

(1) *Ω_0 is a simply connected domain; that is, $\mathbb{C} \setminus \Omega_0$ has no bounded components.*[1]
(2) *$\inf_{\Omega_0} |u| = 0$.*

Proof We show that item (1) holds for every holomorphic function f in $\mathbb{D}$; that is, if Ω_0 is a component of the level set $\Omega_f(\eta)$, $\eta > 0$, then it is a simply connected domain.[2] Note that each component Ω_0 of the open set $\Omega_f(\eta)$ is an open subset of $\mathbb{D}$. We may assume that η is chosen so that $\{z \in \mathbb{D} : |f(z)| = \eta\} \neq \emptyset$.

Suppose that, on the contrary, D is a bounded component of $\mathbb{C} \setminus \Omega_0$. Note that D is closed in $\mathbb{C}$. Then, necessarily, D is contained in $\mathbb{D}$, because the unique unbounded complementary component of Ω_0 contains $\{z \in \mathbb{C} : |z| \geq 1\}$. Hence, D is a compact subset of $\mathbb{D}$. Let $G := \Omega_0^*$ be the simply connected hull of Ω_0 : the union of Ω_0 with all bounded complementary components of Ω_0. Note that G is open because it coincides with the complement of the unique unbounded complementary component of Ω_0. Then, by definition of the simply connected hull, $D \subseteq G$. Now if H is any bounded complementary component of Ω_0 then (as it was the case for D), H is a compact subset of $\mathbb{D}$ and so $\partial H \subseteq \mathbb{D}$. Moreover,

[1] A shorter proof can be given using the advanced definition that a domain G in $\mathbb{C}$ is simply connected if every curve in G is contractible in G, or equivalently, if for every Jordan curve J in G, the interior of J belongs to G. That depends, however, on the Jordan curve theorem.

[2] This proof as well as two different ones, including the one mentioned in footnote 1, stem from the forthcoming book manuscript [22] of the second author together with R. Rupp.

$$\partial H \subseteq \partial \Omega_0. \tag{2}$$

In fact, given $z_0 \in \partial H$, let U be a disk centered at z_0. Then $U \cap \Omega_0 \neq \emptyset$, since otherwise $U \cup H$ would be a connected set strictly bigger than H and contained in the complement of Ω_0 : a contradiction to the maximality of H. Since $z_0 \in \partial H \subseteq H \subseteq \mathbb{C} \setminus \Omega_0$, we conclude that $z_0 \in \partial \Omega_0$.

Now $\partial H \subseteq \partial \Omega_0$ and $\Omega_0 \subseteq \Omega_f(\eta)$ imply that $|f| \leq \eta$ on ∂H, and so, by the maximum principle, $|f| \leq \eta$ on H. Consequently, $|f| \leq \eta$ on G. By the local maximum principle, $|f| < \eta$ on G. Since $\partial D \subseteq D \subseteq G$,

$$|f| < \eta \text{ on } \partial D. \tag{3}$$

On the other hand,

$$\partial D \overset{(2)}{\subseteq} \partial \Omega_0 \cap \mathbb{D} \subseteq \{z \in \mathbb{D} : |f(z)| = \eta\}. \tag{4}$$

Note that the second inclusion follows from the fact that if $|f(z_0)| < \eta$ for $z_0 \in \partial \Omega_0 \cap \mathbb{D}$, then Ω_0 would no longer be a maximal connected subset of $\Omega_f(\eta)$. Hence, $|f| = \eta$ on ∂D. This is a contradiction to (3). Thus, Ω_0 is a simply connected domain.

(2) If $\overline{\Omega}_0 \subseteq \mathbb{D}$, then, due to $\partial \Omega_0 \subseteq \{z \in \mathbb{D} : |u(z)| = \eta\}$, we obtain from the minimum principle that u must have a zero in Ω_0. Now let $E := \overline{\Omega}_0 \cap \partial \mathbb{D} \neq \emptyset$. In view of achieving a contradiction, suppose that u is bounded away from zero in Ω_0. Then $1 / |u|$ is subharmonic and bounded in Ω_0 and

$$\limsup_{\substack{\xi \to x \\ x \in \partial \Omega_0 \setminus E}} |u(\xi)|^{-1} = \eta^{-1}.$$

Since u is an inner function, E has linear measure zero (by [5, Theorem 4.2]). The maximum principle for subharmonic functions with a few exceptional points (here on the set E; see [6] or [12]) now implies that $|u|^{-1} \leq \eta^{-1}$ on Ω_0. But $|u| < \eta$ on Ω is a contradiction. We conclude that $\inf_{\Omega_0} |u| = 0$. □

Lemma 3 [10] *Let u be an inner function. Then the connectedness of $\Omega_u(\eta)$ implies the one of $\Omega_u(\eta')$ for every $\eta' > \eta$.*

Proof Because $\Omega_u(\eta)$ is connected and $\Omega_u(\eta) \subseteq \Omega_u(\eta')$, $\Omega_u(\eta)$ is contained in a unique component $U_1(\eta')$ of $\Omega_u(\eta')$. Suppose that $U_0(\eta')$ is a second component of $\Omega_u(\eta')$. Then $|u| \geq \eta$ on $U_0(\eta')$, because $U_0(\eta')$ is disjoint with $U_1(\eta')$ and hence with $\Omega_u(\eta)$. By Lemma 2 though, $\inf_{U_0(\eta')} |u| = 0$; a contradiction. Thus $\Omega_u(\eta')$ is connected. □

3 Explicit examples of one-component inner functions

Let

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z} w} \right|$$

be the pseudohyperbolic distance of z to w in $\mathbb{D}$ and

$$D_\rho(z_0, r) = \{z \in \mathbb{D} : \rho(z, z_0) < r\}$$

the associated disks, $0 < r < 1$. Here is a first class of examples of functions in $\mathfrak{I}_c$. Although the next Proposition must be known (in view of Aleksandrov's criterion [1]), see Theorem 15 below), we include a simple geometric proof for the reader's convenience.

Proposition 4 *Let B be a finite Blaschke product. Then $B \in \mathfrak{I}_c$.*

Proof Denote by $z_1, \ldots, z_N$ the zeros of B, multiplicities included. Let $\eta \in [0, 1]$ be chosen so close to 1 that $G := \bigcup_{n=1}^{N} D_\rho(z_n, \eta)$ is connected (for example by choosing η so that $z_j \in D_\rho(z_1, \eta)$ for all j). Now

$$G \subseteq \{z \in \mathbb{D} : |B(z)| < \eta\} = \Omega_B(\eta),$$

because $z \in G$ implies that for some n,

$$|B(z)| = \rho(B(z), B(z_n)) \leq \rho(z, z_n) < \sigma.$$

Since G is connected, there is a unique component Ω_1 of Ω containing G. In particular, $Z(B) \subseteq G \subseteq \Omega_1$. If, in view of achieving a contradiction, we suppose that $\Omega := \Omega_B(\eta)$ is not connected, there is a component Ω_0 of Ω which is disjoint with Ω_1, and so with G. In particular,

$$\rho(z, Z(B)) \geq \sigma \quad \text{for every} \quad z \in \Omega_0. \tag{5}$$

Since $\overline{\Omega}_0 \subseteq \overline{\Omega_B(\eta)} \subseteq \mathbb{D}$, and $|B| = \eta$ on $\partial\Omega_0$, we deduce from the minimum principle that Ω_0 contains a zero of B; a contradiction. □

We now generalize this result to a class of interpolating Blaschke products. Recall that a Blaschke product b with zero set/sequence $\{z_n : n \in \mathbb{N}\}$ is said to be an interpolating Blaschke product if $\delta(b) := \inf(1 - |z_n|^2)|b'(z_n)| > 0$. If b is an interpolating Blaschke product then, for small ε, the pseudohyperbolic disks

$$D_\rho(z_n, r) = \{z \in \mathbb{D} : \rho(z, z_n) < \varepsilon\}$$

are pairwise disjoint. Moreover, by Hoffman's Lemma (see below and also [19]), for any $\eta \in]0, 1[$, b is bounded away from zero on $\{z \in \mathbb{D} : \rho(z, Z(b)) \geq \eta\}$.

Theorem 5 (Hoffman's Lemma) ([18, p. 86, 106] and [13, p. 404, 310]) *Let δ, η and ϵ be real numbers, called Hoffman constants, satisfying* $0 < \delta < 1, 0 < \eta < (1 - \sqrt{1 - \delta^2})/\delta$, *(that is,* $0 < \eta < \rho(\delta, \eta)$*) and*

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

If B is any interpolating Blaschke product with zeros $\{z_n : n \in \mathbb{N}\}$ such that

$$\delta(B) = \inf_{n \in \mathbb{N}} (1 - |z_n|^2)|B'(z_n)| \geq \delta,$$

then

(1) *The pseudohyperbolic disks $D_\rho(a,\eta)$ for $a \in Z(B)$ are pairwise disjoint.*
(2) The following inclusions hold:

$$\{z \in \mathbb{D} : |B(z)| < \varepsilon\} \subseteq \{z \in \mathbb{D} : \rho(z, Z(B)) < \eta\} \subseteq \{z \in \mathbb{D} : |B(z)| < \eta\}.$$

A large class of interpolating Blaschke products which are one-component inner functions now is given in the following result:

Theorem 6 *Let b be an interpolating Blaschke product with zero set $\{z_n : n \in \mathbb{N}\}$. Suppose that for some $\sigma \in]0,1[$ the set*

$$G := \bigcup_n D_\rho(z_n, \sigma)$$

is connected. Then b is a one-component inner function. This holds in particular, if $\rho(z_n, z_{n+1}) < \sigma < 1$ for all n: for example if $z_n = 1 - 2^{-n}$.

Proof As in the proof of Proposition 4

$$G \subseteq \{z \in \mathbb{D} : |b(z)| < \sigma\} =: \Omega.$$

Since G is assumed to be connected, there is a unique component Ω_1 of Ω containing G. In particular, $Z(b) \subseteq G \subseteq \Omega_1$. Now, if we suppose that Ω is not connected, then there is a component Ω_0 of Ω which is disjoint with Ω_1, and so with G. In particular,

$$\rho(z, Z(b)) \geq \sigma \quad \text{for every} \quad z \in \Omega_0. \tag{6}$$

Let $\delta := \delta(b)$,

$$0 < \eta < \min\{(1 - \sqrt{1-\delta^2})/\delta, \sigma\},$$

$$0 < \varepsilon < \eta \frac{\delta - \eta}{1 - \delta\eta}.$$

By Lemma 2, $\inf_{\Omega_0} |b| = 0$. Thus, there is $z_0 \in \Omega_0$ be so that $|b(z_0)| < \varepsilon$. We deduce from Hoffman's Lemma (Theorem 5) that $\rho(z_0, Z(b)) < \eta < \sigma$. This is a contradiction to (6). We conclude that Ω must be connected. It is clear that the condition $\rho(z_n, z_{n+1}) < \sigma$ for every n implies that $\bigcup_n D_\rho(z_n, \sigma)$ is connected. For the rest, just note that

$$\rho(1 - 2^{-n}, 1 - 2^{-n-1}) = \frac{2^{-n} - 2^{-n-1}}{2^{-n} + 2^{-n-1} + 2^{-n}2^{-n-1}} = \frac{1}{3 + 2^{-n}}.$$

□

Corollary 7 *Let B be a Blaschke product with increasing real zeros x_n satisfying*

$$0 < \eta_1 \leq \rho(x_n, x_{n+1}) \leq \eta_2 < 1.$$

Then $b \in \mathfrak{I}_c$.

Proof Just use Theorem 6 and the fact that by the Vinogradov–Hayman–Newman theorem, B is interpolating if and only if

$$\sup_n \frac{1-x_{n+1}}{1-x_n} \leq s < 1$$

or equivalently

$$\inf_n \rho(x_n, x_{n+1}) \geq r > 0.$$

□

Using a result of Kam-Fook Tse [25], telling us that a sequence (z_n) of points contained in a Stolz angle (or cone) $\{z \in \mathbb{D} : |1-z| < C(1-|z|)\}$ is interpolating if and only if it is separated (meaning that $\inf_{n\neq m} \rho(z_n, z_m) > 0$), we obtain the following Corollary

Corollary 8 *Let B be a Blaschke product the zeros (z_n) of which are contained in a Stolz angle and are separated. Suppose that $\rho(z_n, z_{n+1}) \leq \eta < 1$. Then $B \in \mathfrak{I}_c$.*

Similarly, using a result by M. Weiss [27, Theorem 6] and its refinement in [4, Theorem B], we also obtain the following assertion for sequences that may be tangential at 1 (see also Wortman [28]):

Corollary 9 *Let B be a Blaschke product the zeros $z_n = r_n e^{i\theta_n}$ of which satisfy:*

$$r_n < r_{n+1},\ \theta_{n+1} < \theta_n,$$

$$r_n \nearrow 1,\ \theta_n \searrow 0,$$

$$0 < \eta_1 \leq \rho(z_n, z_{n+1}) \leq \eta_2 < 1. \tag{7}$$

Then B is an interpolating Blaschke product contained in $\mathfrak{I}_c$. This holds in particular if the zeros are located on a convex curve in $\mathbb{D}$ with endpoint 1 and satisfying (7).

Other classes of this type can be deduced from [14]. Here are two explicit examples of interpolating Blaschke products in $\mathfrak{I}_c$ the zeros of which are given by iteration of the zero of a hyperbolic, respectively, parabolic automorphism of $\mathbb{D}$. These functions appear, for instance, in the context of isometries on the Hardy space H^p (see [8]).

Example 10 • Let $\varphi(z) = \dfrac{z-1/2}{1-(1/2)z}$. Then φ is an hyperbolic automorphism with fixed points ± 1. If $\varphi_j := \underbrace{\varphi \circ \cdots \circ \varphi}_{j\text{-times}}$, then $\varphi_j \in \mathrm{Aut}(\mathbb{D})$ and vanishes exactly at the point

$$x_j := \frac{3^j - 1}{3^j + 1} = 1 - \frac{2}{3^j + 1}.$$

This can readily be seen using that $x_{j+1} = \varphi^{-1}(x_j)$ and

$$\varphi_{j+1}(z) = (\varphi_j \circ \varphi)(z) = \frac{z - \frac{\frac{1}{2}+x_j}{1+\frac{1}{2}x_j}}{1 - z\frac{\frac{1}{2}+x_j}{1+\frac{1}{2}x_j}}.$$

Since

$$\rho(x_j, x_{j+1}) = \frac{3^{j+1} - 3^j}{3^{j+1} + 3^j} = \frac{1}{2},$$

we deduce from Corollary 7 that the Blaschke product

$$B(z) := \prod_{j=1}^{\infty} \frac{x_j - z}{1 - x_j z}$$

associated with these zeros is in $\mathfrak{I}_c$.

• Let $\sigma \in \mathrm{Aut}(\mathbb{D})$ and $\tau = \sigma \circ \varphi \circ \sigma^{-1}$. Then τ is also a hyperbolic automorphism fixing the points $\sigma(\pm 1)$, and where $\xi := \sigma(1)$ is the Denjoy–Wolff point with $\tau'(\xi) < 1$. The zeros of the n-th iterate τ_n of τ are given by

$$z_n = \tau_n^{-1}(0) = (\sigma \circ \varphi_n^{-1} \circ \sigma^{-1})(0).$$

By the grand iteration theorem [23, p.78], since 1 is an attracting fixpoint with $(\varphi^{-1})'(1) = 1/3 < 1$, the sequence $(\varphi_n^{-1}(\sigma^{-1}(0)))$ converges nontangentially to 1. Hence, the points z_n are located in a cone with cusp at ξ. Moreover, if $n > k$ and $a = \sigma^{-1}(0)$,

$$\begin{aligned}\rho(z_n, z_k) &= \rho\big((\varphi_n^{-1} \circ \sigma^{-1})(0), (\varphi_k^{-1} \circ \sigma^{-1})(0)\big)\\ &= \rho\big(\varphi_{n-k}^{-1}(a), a\big)\end{aligned}$$

Thus, $\rho(z_n, z_{n+1}) = \rho(\varphi(a), a)$ for all n and $\inf_{n \neq k} \rho(z_n, z_k) > 0$. Now (z_n) is a Blaschke sequence[3] ([23, Ex. 6, p. 85]); in fact, use d'Alembert's quotient criterion and observe that by the Denjoy–Wolff theorem,

$$\frac{1 - |z_{n+1}|}{1 - |z_n|} = \frac{1 - |\tau^{-1}(z_n)|}{1 - |z_n|} \to (\tau^{-1})'(\xi) < 1.$$

Hence, by Corollary 8, (z_n) is an interpolating sequence (see also [11, p.80]) and the associated Blaschke product $b = \prod_{n=1}^{\infty} e^{i\theta_n} \tau_n$ belongs to $\mathfrak{I}_c$ (here θ_n is chosen so that the nth Blaschke factor is positive at the origin).

• Let $\psi(z) = i\,\dfrac{z - \frac{1+i}{2}}{1 - \frac{1-i}{2}\,z}$. Then ψ is a parabolic automorphism with attracting fixed point 1. The automorphism ψ is conjugated to the translation $w \mapsto w + 2$ on the upper half-plane (see Fig. 1) via the map $M(z) = i(1+z)/(1-z)$ and $\psi_n = M^{-1} \circ T_n \circ M$. The zeros of the n-th iterate ψ_n of ψ are given by

$$z_n = \frac{n}{n - i};$$

just use that $z_n = (M^{-1} \circ T_n^{-1} \circ M)(0)$. These zeros satisfy $\left|z_n - \frac{1}{2}\right| = \frac{1}{2}$ and of course also the Blaschke condition $\sum_{n=1}^{\infty} 1 - |z_n|^2 < \infty$. Moreover,

$$\rho(z_n, z_{n+1}) = \frac{1}{\sqrt{2}}.$$

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[3] This also follows form the inequalities $1 - |\sigma(\xi_n)|^2 = \frac{(1-|a|^2)(1-|\xi_n|^2)}{|1-\overline{a}\xi_n|^2} \leq \frac{1+|a|}{1-|a|}(1 - |\xi_n|^2)$ and $1 - |\psi_n(a)|^2 \leq \frac{1+|a|}{1-|a|}(1 - |w_n|^2)$, whenever (w_n) is a Blaschke sequence and $\psi_n(w_n) = \sigma(a) = 0$.

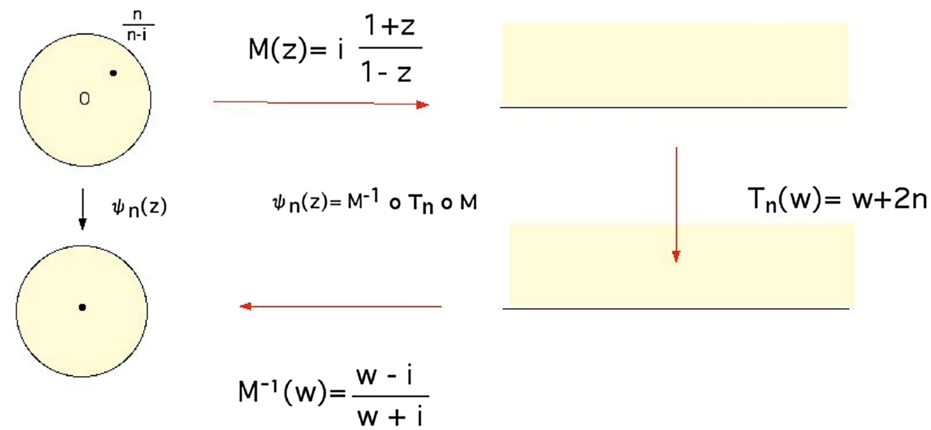

Fig. 1 The parabolic automorphism

Thus, by, Corollary 9, the Blaschke product associated with these zeros is interpolating and belongs to $\mathfrak{I}_c$.

Proposition 11 *Let B be a finite Blaschke product or an interpolating Blaschke product with real zeros clustering at* $p = 1$. *Then* $f := BS \in \mathfrak{I}_c$.

Proof (i) Let B be a finite Blaschke product. Chose $\eta \in [0,1]$ so close to 1 that the disk D_η in (1), which coincides with the level set $\Omega_S(\eta)$, contains all zeros of B. Note that $D_\eta = \Omega_S(\eta) \subseteq \Omega_f(\eta)$. Now $\Omega_f(\eta)$ must be connected, since otherwise there would be a component Ω_0 of $\Omega_f(\eta)$ disjoint from the component Ω_1 containing D_η. But f is bounded away from zero outside D_η; hence, $f = BS$ is bounded away from zero on Ω_0. This is a contradiction to Lemma 2 (2).

(ii) If B is an interpolating Blaschke product with zeros (z_n), then, by Hoffman's Lemma (Theorem 5), B is bounded away from zero outside $R := \bigcup D_\rho(z_n, \varepsilon)$ for every $\varepsilon \in [0,1]$. Now, if the zeros of B are real, and bigger than $-\sigma$ for some $\sigma \in [0,1]$, this set R is contained in a cone with cusp at 1 and aperture-angle strictly less than π (for instance, [21]). Hence, R is contained in D_η for all η close to 1. Thus, as above, we can deduce that $\Omega_{BS}(\eta)$ is connected. □

The previous result shows, in particular, that certain non one-component inner functions (for example a thin Blaschke product with positive zeros, see Corollary 21), can be multiplied by a one-component inner function into $\mathfrak{I}_c$. In particular, $uv \in \mathfrak{I}_c$ does not imply that u and v belong to $\mathfrak{I}_c$. The reciprocal, though, is true: that is, $\mathfrak{I}_c$ itself is stable under multiplication, as we proceed to show below.

Proposition 12 *Let* u, v *be two inner functions in* $\mathfrak{I}_c$. *Then* $uv \in \mathfrak{I}_c$.

Proof Let $\Omega_u(\eta)$ and $\Omega_v(\eta')$ be two connected level sets. Due to monotonicity (Lemma 3), and the fact that $\bigcup_{\lambda \in [\lambda_0, 1[} \Omega_f(\lambda) = \mathbb{D}$, we may assume that σ satisfies

$$\max\{\eta, \eta'\} \le \sigma < 1$$

and is so close to 1 that $0 \in \Omega_u(\sigma) \cap \Omega_v(\sigma) \neq \emptyset$. Hence, $U := \Omega_u(\sigma) \cup \Omega_v(\sigma)$ is connected. Now

$$\Omega_u(\sigma) \cup \Omega_v(\sigma) \subseteq \Omega_{uv}(\sigma).$$

If we suppose that $\Omega_{uv}(\sigma)$ is not connected, then there is a component Ω_0 of $\Omega_{uv}(\sigma)$ which is disjoint from U. In particular, u and v are bounded away from zero on Ω_0. This contradicts Lemma 2 (2). Hence, $\Omega_{uv}(\sigma)$ is connected and so $uv \in \mathfrak{I}_c$. □

Theorem 13 *The set of one-component inner functions is open inside the set of all inner functions (with respect to the uniform norm topoplogy).*

Proof Let $u \in \mathfrak{I}_c$. Then, by Lemma 3, $\Omega_u(\eta)$ is connected for all $\eta \in [\eta_0, 1[$. Choose $0 < \varepsilon < \min\{\eta, 1-\eta\}$ and let Θ be an inner function with $||u - \Theta|| < \varepsilon$. We claim that $\Theta \in \mathfrak{I}_c$, too. Toward this end, we note that

$$\Omega_\Theta(\eta - \varepsilon) \subseteq \Omega_u(\eta) \subseteq \Omega_\Theta(\eta + \varepsilon),$$

where $0 < \eta - \varepsilon < \eta + \varepsilon < 1$. As usual, if we suppose that $\Omega_\Theta(\eta + \varepsilon)$ is not connected, then there is a component Ω_0 of $\Omega_\Theta(\eta + \varepsilon)$ which is disjoint from the connected set $\Omega_u(\eta)$, and hence disjoint with $\Omega_\Theta(\eta - \varepsilon)$. In other words, $|\Theta| \geq \eta - \varepsilon > 0$ on Ω_0. This contradicts Lemma 2 (2). H $\Omega_\Theta(\eta + \varepsilon)$ is connected and so $\Theta \in \mathfrak{I}_c$. □

Next we look at right-compositions of S with finite Blaschke products. We first deal with the case where $B(z) = z^2$.

Example 14 The function $S(z^2)$ is a one-component inner function.

Proof Let $\Omega_S(\eta)$ be the η-level set of S. Then, as we have already seen, this is a disk tangent to the unit circle at the point 1 (Fig. 2). We may choose $0 < \eta < 1$ so close to 1 that 0 belongs to $\Omega_S(\eta)$. Let $U = \Omega_S(\eta) \setminus]-\infty, 0]$. Then U is a simply connected slitted disk on which exists a holomorphic square root q of z. The image of U under q is a simply connected domain V in the semi-disk $\{z : |z| < 1, \text{Re } z > 0\}$. Let V^* be its reflection along the imaginary axis. Then $E := \overline{V^* \cup V}$ is mapped by z^2 onto the closed disk $\overline{\Omega_S(\eta)}$. Then $E \setminus \partial E$ coincides with $\Omega_{S(z^2)}(\eta)$. □

Using Aleksandrov's criterion (see below), we can extend this by replacing z^2 with any finite Blaschke product. Recall that the spectrum $\rho(\Theta)$ of an inner function Θ is the set of all boundary points ζ for which Θ does not admit a holomorphic extension—or equivalently, for which $Cl(\Theta, \zeta) = \overline{\mathbb{D}}$, where

$$Cl(\Theta, \zeta) = \{w \in \mathbb{C} : \exists (z_n) \in \mathbb{D}^{\mathbb{N}}, \ \lim z_n = \zeta \ \text{ and } \ \lim \Theta(z_n) = w\}$$

is the cluster set of Θ at ζ (see [13, p. 80]).

Theorem 15 (Aleksandrov) [1, Theorem 11 and Remark 2, p. 2915] *Let Θ be an inner function. The following assertions are equivalent:*

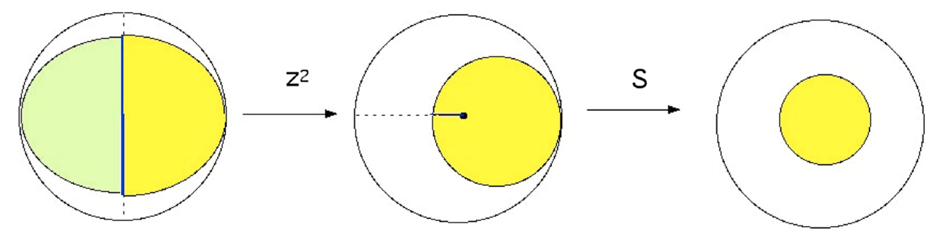

Fig. 2 The level sets of $S(z^2)$

(1) $\Theta \in \mathfrak{I}_c$.
(2) *There is a constant* $C > 0$ *such that for every* $\zeta \in \mathbb{T} \setminus \rho(\Theta)$ *we have*

i) $|\Theta''(\zeta)| \leq C\,|\Theta'(\zeta)|^2$,

and

ii) $\liminf_{r\to 1} |\Theta(r\zeta)| < 1$ for all $\zeta \in \rho(\Theta)$.

Note that, due to the above theorem, $\Theta \in \mathfrak{I}_c$ necessarily implies that $\rho(\Theta)$ has measure zero.

Proposition 16 *Let* B *be a finite Blaschke product. Then* $S \circ B \in \mathfrak{I}_c$.

Proof Let us note first that $\rho(S \circ B) = B^{-1}(\{1\})$. Since the derivative of B on the boundary never vanishes (due to

$$z\frac{B'(z)}{B(z)} = \sum_{n=1}^{N} \frac{1-|a_n|^2}{|a_n - z|^2}, \quad |z| = 1, B(a_n) = 0,) \tag{8}$$

B is schlicht in a neighborhood of 1. The angle conservation law now implies that for $\zeta \in B^{-1}(1)$ the curve $r \mapsto B(r\zeta)$ stays in a Stolz angle at 1 in the image space of B. Hence $\liminf_{r\to 1} S(B(r\zeta)) = 0$ for $\zeta \in \rho(S \circ B)$. Now let us calculate the following derivatives:

$$S'(z) = -S(z)\frac{2}{(1-z)^2},$$

$$S''(z) = S(z)\left[\frac{4}{(1-z)^4} - \frac{4}{(1-z)^3}\right],$$

$$(S \circ B)' = (S' \circ B)B'$$

$$(S \circ B)'' = (S'' \circ B)B'^2 + (S' \circ B)B''$$

$$\begin{aligned} A := \frac{(S \circ B)''}{[(S \circ B)']^2} &= \frac{S'' \circ B}{(S' \circ B)^2} + \frac{(S' \circ B)}{(S' \circ B)^2}\frac{B''}{B'^2} \\ &= \frac{S'' \circ B}{(S' \circ B)^2} + \frac{1}{S' \circ B}\frac{B''}{B'^2}. \end{aligned} \tag{9}$$

Hence, for $\zeta \in \mathbb{T} \setminus \rho(S \circ B)$, $|B(\zeta)| = 1$, but $\xi := B(\zeta) \neq 1$, and so, by (8),

$$|A(\zeta)| \leq \sup_{\xi \neq 1} \frac{|S''(\xi)|}{|S'(\xi)|^2} + 2 \sup_{\xi \neq 1} \frac{|1-\xi|^2}{|S(\xi)|} C$$
$$\leq C' \sup_{\xi \neq 1} \frac{|1-\xi|^4}{|1-\xi|^4} + 8C \leq \infty.$$

□

Corollary 17 *Let S_μ be a singular inner function with finite spectrum $\rho(S_\mu)$. Then $S_\mu \in \mathfrak{I}_c$.*

Proof Since S is the universal covering map of $\mathbb{D} \setminus \{0\}$, each singular inner function S_μ is written as $S_\mu = S \circ \nu$ for some inner function ν. Since $\rho(S_\mu)$ is finite, ν necessarily is a finite Blaschke product. (This can also be seen from [15, Proof of Theorem 2]). The assertion now follows from Proposition 16. □

Note that the above result also follows in an elementary way from Proposition 12 and the fact that every such S_μ is a finite product of powers of the atomic inner function S. We now consider left-compositions with finite Blaschke products.

Proposition 18 *Let Θ be a one-component inner function. Then each Frostman shift $(a - \Theta)/(1 - \overline{a}\Theta) \in \mathfrak{I}_c$, is too. Here $a \in \mathbb{D}$.*

Proof Let $\tau(z) = (a - z)/(1 - \overline{a}z)$. Then $\rho(\tau \circ \Theta) = \rho(\Theta)$. As stated above,

$$\liminf_{r \to 1} |\tau \circ \Theta(r\zeta)| < 1$$

for every $\zeta \in \rho(\tau \circ \Theta)$. Now

$$\tau(z) = \frac{1}{\overline{a}} + \frac{|a|^2 - 1}{\overline{a}} \frac{1}{1 - \overline{a}z},$$

from which we easily deduce the first and second derivatives. Using the formulas 9, we obtain

$$A := \left| \frac{(\tau \circ \Theta)''}{[(\tau \circ \Theta)']^2} \right| \leq C \frac{|1 - \overline{a}\Theta|^4}{|1 - \overline{a}\Theta|^3} + C' |1 - \overline{a}\Theta|^2 \frac{|\Theta''|}{|\Theta'|^2}.$$

Hence, the assumption $\Theta \in \mathfrak{I}_c$ now yields (via Aleksandrov's criterion, Theorem 15) that $\sup_{\zeta \in \rho(\tau \circ \Theta)} A(\zeta) < \infty$. Thus, $\tau \circ \Theta \in \mathfrak{I}_c$. □

Corollary 19 *Given $a \in \mathbb{D} \setminus \{0\}$, the interpolating Blaschke products $(S - a)/(1 - \overline{a}S)$ belong to $\mathfrak{I}_c$.*

This also follows from Corollary 9 by noticing that the a-points of S are located on a disk tangent at 1 and that the pseudohyperbolic distance between two consecutive ones is constant (see [20]). In the cited reference, it is also shown that the Frostman shift $(S - a)/(1 - \overline{a}S)$ is an interpolating Blaschke product.

Corollary 20 *Let B be a finite Blaschke product and $\Theta \in \mathfrak{I}_c$. Then $B \circ \Theta \in \mathfrak{I}_c$.*

Proof This is a combination of Propositions 12 and 18. □

4 Inner functions not belonging to $\mathfrak{I}_c$

Here we present a class of Blaschke products that are not one-component inner functions. Recall that a Blaschke product b with zero-sequence (z_n) is *thin* if

$$\lim_n \prod_{k \neq n} \rho(z_k, z_n) = \lim_{n \to 1} (1 - |z_n|^2)|b'(z_n)| = 1.$$

It was shown by Tolokonnikov [24, Theorem 3] that b is thin if and only if

$$\lim_{|z| \to 1} (|b(z)|^2 + (1 - |z|^2)|b'(z)|) = 1.$$

Corollary 21 *Thin Blaschke products are never one-component inner functions.*

Proof Let $\varepsilon \in]0, 1[$ be arbitrary close to 1. Choose $\eta > 0$ and $\delta > 0$ so close to 1 so that

$$\varepsilon < \eta^2 \text{ and } \eta < (1 - \sqrt{1 - \delta^2})/\delta.$$

By deleting finitely many zeros, say $z_1, \ldots, z_N$ of b, we obtain a tail b_N such that $(1 - |z_n|^2)|b_N'(z_n)| \geq \delta$ for every $n > N$. Hence, by Theorem 5,

$$\{z \in \mathbb{D} : |b_N(z)| < \varepsilon\} \subseteq \{z \in \mathbb{D} : \rho(z, Z(b_N)) < \eta\} \tag{10}$$

and the disks $D_\rho(z_n, \eta)$ are pairwise disjoint. This implies that the level set $\{z \in \mathbb{D} : |b_N(z)| < \varepsilon\}$ is not connected. Now choose r so close to 1 that

$$p(z) := \prod_{n=1}^{N} \rho(z, z_n) \geq \varepsilon$$

for every z with $r \leq |z| < 1$. We show that the level set $\{|b| < \varepsilon^2\}$ is not connected. In fact, for some $r \leq |z| < 1$ we have $|b(z)| < \varepsilon^2$, then

$$|b_N(z)| = \frac{|b(z)|}{|p(z)|} < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

Hence

$$\{z : r < |z| < 1, |b(z)| < \varepsilon^2\} \subseteq \{|b_N(z)| < \varepsilon\} \overset{(3.1)}{\subseteq} \bigcup_{n > N} D_\rho(z_n, \eta).$$

Since the disks $D_\rho(z_n, \eta)$ are pairwise disjoint if $n > N$, we are done with the tasks. □

Corollary 22 *No finite product B of thin interpolating Blaschke products belongs to $\mathfrak{I}_c$.*

Proof Let $\varepsilon \in [0, 1]$ be arbitrary close to 1. By Corollary 21, if b_j, $(j = 1, 2)$, are two thin Blaschke products with zero-sequence $(z_n^{(j)})_n$,

$$\Omega_{b_j}(\varepsilon) \subseteq \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta)$$

for suitable η, the disks $D_\rho(z_n^{(j)}, \eta)$, being pairwise disjoint for n large. Since $\lim_n \rho(z_n^{(j)}, z_{n+1}^{(j)}) = 1$, we see that a disk $D_\rho(z_n^{(1)}, \eta)$ can meet at the most at one disk $D_\rho(z_m^{(2)}, \eta)$ for n large. Hence

$$\Omega_{b_1 b_2}(\varepsilon^2) \subseteq \bigcup_{j=1}^{2} \bigcup_{n=1}^{\infty} D_\rho(z_n^{(j)}, \eta),$$

where the set on the right-hand side obviously is disconnected. The general case works via induction. □

Remark 23 The conditions

$$\eta^* := \sup_{n\in\mathbb{N}} \rho(z_n, Z(b) \setminus \{z_n\}) < 1, \tag{11}$$

or equivalently

$$D(z_n, \eta) \cap \bigcup_{m\neq n} D(z_m, \eta) \neq \emptyset \text{ for some } \eta \in]0, 1[, \tag{12}$$

are not sufficient to guarantee that the interpolating Blaschke product b is a one-component inner function.

Proof Take $z_{2n} = 1 - n^{-n}$ and $z_{2n+1} = 1 - (n^{-n} + n^{-n})$. Then (z_{2n}) and (z_{2n+1}) are (thin) interpolating sequences by [16, Corollary 2.4]. Using $a = n^{-n}$ and $b = 2a$, and the identity,

$$\rho(1 - a, 1 - b) = \frac{|a - b|}{a + b - ab},$$

we conclude that

$$\rho(z_{2n}, z_{2n+1}) = \frac{n^{-n}}{1 - z_{2n} z_{2n+1}} \to 1/3,$$

and so the union (z_n) is an interpolating sequence satisfying (12). By Corollary 22, the Blaschke product formed with the zero-sequence (z_n) is not in $\mathfrak{I}_c$. □

Using the following theorem in [5], we can exclude a much larger class of Blaschke products from being one-component inner functions:

Theorem 24 (Berman) *Let u be an inner function. Then, for every $\varepsilon \in]0, 1[$, all the components of the level sets $\{z \in \mathbb{C} : |u(z)| < \varepsilon\}$ have compact closures in $\mathbb{D}$ if and only if u is a Blaschke product and*

$$\limsup_{r\to 1} |u(r\xi)| = 1 \textit{ for every } \xi \in \mathbb{T}.$$

In particular this condition is satisfied by finite products of thin Blaschke products (see [17, Proposition 2]) as well as by the class of uniform Frostman Blaschke products whose zero sequence (z_n) satisfies

$$\sup_{\xi \in \mathbb{T}} \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|\xi - z_n|} < \infty.$$

Note that this Frostman condition implies that the associated Blaschke product has radial limits of modulus one everywhere [9, p. 33]. As a byproduct of Theorem 6, we therefore obtain

Corollary 25 *If b is a uniform Frostman Blaschke product with zeros (z_n) clustering at a single point, then* $\limsup_\rho (z_n, z_{n+1}) = 1$.

Questions 25 To conclude, we would like to ask two questions and present three problems:

(1) Can every inner function u whose boundary spectrum $\rho(u)$ has measure zero, be multiplied by a one-component inner function into $\mathfrak{I}_c$?
(2) Let S_μ be a singular inner function with countable spectrum. Give a characterization of those measures μ such that $S_\mu \in \mathfrak{I}_c$. Do the same for singular continuous measures.
(3) In terms of the zeros, give a characterization of those interpolating Blaschke products that belong to $\mathfrak{I}_c$.
(4) Does the Blaschke product B with zeros $z_n = 1 - n^{-2}$ belong to $\mathfrak{I}_c$?

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Acknowledgements
We thank Rudolf Rupp and Robert Burckel for their valuable comments concerning Lemma 2(1), the proof of which was originally developed for the upcoming monograph [22].

Received: 7 September 2016 Accepted: 16 December 2016
Published online: 25 January 2017

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