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Multidirectional hybrid algorithm for the split common fixed point problem and application to the split common null point problem

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Abstract

In this article, a new multidirectional monotone hybrid iteration algorithm for finding a solution to the split common fixed point problem is presented for two countable families of quasi-nonexpansive mappings in Banach spaces. Strong convergence theorems are proved. The application of the result is to consider the split common null point problem of maximal monotone operators in Banach spaces. Strong convergence theorems for finding a solution of the split common null point problem are derived. This iteration algorithm can accelerate the convergence speed of iterative sequence. The results of this paper improve and extend the recent results of Takahashi and Yao (Fixed Point Theory Appl 2015:87, 2015) and many others .

Keywords: Split common fixed point problem, Split common null point problem, Fixed point, Metric resolvent, Multidirectional hybrid algorithm, Duality mapping

Mathematics Subject Classification: 47H05, 47H09, 47H10

Introduction and preliminaries

Let H_1 and H_2 be two real Hilbert spaces, let $D \subset H_1$ and $Q \subset H_2$ be nonempty closed, and convex subsets, let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem (Censor and Elfving 1994) is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we see that $U : H_1 \to H_1$ is an inverse strongly monotone operator (Alsulami and Takahashi 2014), where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then

$$z \in D \cap A^{-1}Q \;\Leftrightarrow\; z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem in Hilbert spaces; see, for instance, Alsulami and Takahashi (2014), Byrne et al. (2012), Censor and Segal (2009), Moudafi (2010), Takahashi et al. (2015). Recently, Takahashi (2014) and Takahashi (2015) extended an equivalent relation as in (1) in Hilbert spaces to Banach spaces and then obtained strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces. Very recently, using the hybrid method by Nakajo and Takahashi (2003) in mathematical programming, Alsulami et al. (2015) proved strong convergence theorems for finding a solution of the split feasibility problem in Banach spaces; see also Ohsawa and Takahashi (2003), Solodov and Svaiter (2000). Takahashi (2015) also obtained a result for finding a solution of the split feasibility problem in Banach space from the idea of the shrinking projection method by Takahashi et al. (2008). Takahashi and Yao (2015) presented the following hybrid iteration algorithm in a Hilbert space H: for $x_1 \in H$,

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)), \\ y_n = \alpha_n x_n + (1-\alpha_n) z_n, \\ C_n = \left\{ z \in H : \left\| y_n - z \right\| \le \| x_n - z \| \right\}, \\ D_n = \{ z \in H : \langle x_n - z, x_1 - x_n \rangle \ge 0 \} \\ x_{n+1} = P_{C_n \cap D_n} x_1. \end{cases} \tag{TY}$$

They proved the following strong convergence theorem:

Theorem TY *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^{F^*}. such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1})0 \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by (TY), where $\{\alpha_n \subset [0,1]\}$ and $\{\lambda_n\}, \{\mu_n\}$ satisfy the condition such that*

$$0 \le \alpha_n \le \alpha < 1, \ \ 0 < b \le \mu_n, \ \ 0 < c \le r_n \|T\|^2 \le d < 2,$$

for some $a, b, c \in R$. Then $\{x_n\}$ converges strongly to a point $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} x_1$.

In this article, a new multidirectional monotone hybrid iteration algorithm for finding a solution to the split common fixed point problem is presented for two countable families of quasi-nonexpansive mappings in Banach spaces. Strong convergence theorems are proved. The application of the result is to consider the split common null point problem of maximal monotone operators in Banach spaces. Strong convergence theorems for finding a solution of the split common null point problem are derived. This iteration algorithm can accelerate the convergence speed of iterative sequence.

Let E be a real Banach space with norm $\| \cdot \|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. A Banach space E is uniformly convex if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \ \ \textit{and} \ \ \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is reflexive.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \| x \| = 1\}$. The norm of E is said to be Gateaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single- valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see Takahashi (2009) and Takahashi (2000).

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\| x - z \| \leq \| x - y \|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Definition 1 Let E be a metric space, let $T : D(T) \to R(T)$ be a mapping with the domain $D(T)$ and the range $R(T)$. The mapping T is said to be quasi-nonexpansive if

$$d(Tx, p) \leq d(x, p), \ \forall\, x \in D(T), \ p \in F(T),$$

where $F(T)$ is the nonempty fixed point set of T.

Definition 2 Let E be a smooth Banach space, let $S : D(T) \to R(T)$ be a mapping with the domain $D(T)$ and the range $R(T)$. The mapping S is said to be second-type quasi-nonexpansive, if

$$\langle Sx - p, J(Sx - x) \rangle \leq 0, \ \ \forall\, x \in D(T), \ \ \forall\, p \in F(S),$$

where $F(S)$ is the nonempty fixed point set of T.

Definition 3 Let E, F be two normed spaces and T be a linear operator from E into F. The adjoint operator $T^* : F^* \to E^*$ of T is defined by

$$f(T(x)) = (T^* f)(x), \ \forall\, x \in E, \, f \in F^*,$$

where E^* and F^* are the adjoint spaces of E and F, respectively.

The adjoint spaces and adjoint operators are very important in the theory of functional analysis and applications. Not only is it an important theoretical subject but it is also a very useful tool in the functional analysis and topological theory.

Definition 4 Let E be a Banach space, let C be a nonempty, closed, and convex subset of E. Let $\{T_n\}$ be sequence of mappings from C into itself with nonempty common fixed point set $F = \cap_{n=1}^{\infty} F(T_n)$. The $\{T_n\}$ is said to be uniformly closed if for any convergent sequence $\{z_n\} \subset C$ such that $\| T_n z_n - z_n \| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belong to F.

Main results

Lemma 5 *Let H be a Hilbert space, let C be a closed convex subset of H, and let $\{T_n\}$ be a uniformly closed family of countable quasi-nonexpansive mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proof Let $p_n \in F$ and $p_n \to p$ as $n \to \infty$, we have

$$\|T_n p_n - p_n\| \to 0, \quad p_n \to p$$

as $n \to \infty$. Since $\{T_n\}$ is uniformly closed, we know that $p \in F$, therefore F is closed. Next we show that F is also convex. For any $x, y \in F$, let $z = tx + (1-t)y$ for any $t \in (0,1)$, we have

$$\begin{aligned}
\|T_n z - z\|^2 &= \langle T_n z - z, T_n z - z\rangle \\
&= \|T_n z\|^2 - 2\langle T_n z, z\rangle + \|z\|^2 \\
&= \|T_n z\|^2 - 2\langle T_n z, tx + (1-t)y\rangle + \|z\|^2 \\
&= \|T_n z\|^2 - 2t\langle T_n z, x\rangle - 2(1-t)\langle T_n z, y\rangle + \|z\|^2 \\
&= t\|T_n z\|^2 + (1-t)\|T_n z\|^2 + t\|x\|^2 - t\|x\|^2 + (1-t)\|y\|^2 \\
&\quad - (1-t)\|y\|^2 - 2t\langle T_n z, x\rangle - 2(1-t)\langle T_n z, y\rangle + \|z\|^2 \\
&= t(\|T_n z\|^2 - 2t\langle T_n z, x\rangle + \|x\|^2) + (1-t)(\|T_n z\|^2 - 2t\langle T_n z, y\rangle + \|y\|^2) \\
&\quad - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= t\langle T_n z - x, T_n z - x\rangle + (1-t)\langle T_n z - y, T_n z - y\rangle \\
&\quad - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= t\|T_n z - x\|^2 + (1-t)\|T_n z - y\|^2 - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&\le t\|z - x\|^2 + (1-t)\|z - y\|^2 - t\|x\|^2 - (1-t)\|y\|^2 + \|z\|^2 \\
&= \|z\|^2 - 2\langle z, z\rangle + \|z\|^2 = 0,
\end{aligned}$$

for all n. This implies $z \in F$, therefore F is convex. This completes the proof. □

Lemma 6 *Let E be a smooth Banach space, let C be a closed convex subset of E, and let $\{S_n\}$ be a uniformly closed family of countable second-type quasi-nonexpansive mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proof Let $p_n \in F$ and $p_n \to p$ as $n \to \infty$, we have

$$\|T_n p_n - p_n\| \to 0, \quad p_n \to p$$

as $n \to \infty$. Since $\{T_n\}$ is uniformly closed, we know that $p \in F$, therefore F is closed. Next we show that F is also convex. For any $x, y \in F$, let $z = tx + (1-t)y$ for any $t \in (0,1)$, we have

$$\begin{aligned}
&\langle S_n z - x, J(S_n z - z)\rangle \le 0, \\
&\langle S_n z - y, J(S_n z - z)\rangle \le 0.
\end{aligned}$$

From the two inequalities given above, we have that

$$\langle t(S_n z - x) + (1-t)(S_n z - y), J(S_n z - z)\rangle \le 0$$

which implies

$$\langle S_n z - z, J(S_n z - z)\rangle \le 0.$$

Therefore $\|S_n z - z\|^2 \le 0$, that is $\|S_n z - z\|^2 = 0$, so that $z \in F$. Therefore F is convex. This completes the proof. □

Lemma 7 (Alber 1996) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $x \in E$. Then*

$$\|z - P_C x\|^2 + \|P_C x - x\|^2 \le \|z - x\|^2, \quad \forall\, z \in C.$$

Next, we present a new hybrid algorithm so-called the multidirectional hybrid algorithm for finding the common fixed point of a uniformly closed family of countable quasi-nonexpansive mappings and a uniformly closed family of countable second-type quasi-nonexpansive mappings.

Theorem 8 *Let H be a Hilbert space and let E be a uniformly convex and smooth Banach space. Let J be the duality mapping on E. Let $\{T_n\} : H \to H$ be a uniformly closed family of countable quasi-nonexpansive mappings with the nonempty common fixed point set $\cap_{n=1}^{\infty} F(T_n)$ and $\{S_n\} : E \to E$ be a uniformly closed family of countable second-type quasi-nonexpansive mappings with the nonempty common fixed point sets $\cap_{n=1}^{\infty} F(S_n)$. Suppose that $F = (\cap_{n=1}^{\infty} F(T_n)) \cap (T^{-1}(\cap_{n=1}^{\infty} F(S_n))) \ne \emptyset$. Let $T : H \to E$ be a bounded linear operator such that $T \ne 0$ and let T^* be the adjoint operator of T. Let $x_{1,i} \in H, i = 1, 2, 3, \ldots, N$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences generated by*

$$\begin{cases} z_n = T_n(x_n - r_n T^* J(Tx_n - S_n Tx_n)), \\ C_n = \{z \in C_{n-1} : \|z_n - z\| \le \|x_n - z\|\}, \\ C_0 = H,\ n = 1, 2, 3, \ldots, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i P_{C_n} x_{1,i},\ \sum_{i=1}^{N} \lambda_i = 1, \end{cases}$$

where $\{r_n\}$ satisfy the condition such that

$$0 < a \le r_n \|T\|^2 \le b < 2,$$

for some constants a, b and $\lambda \in [0, 1]$ is a constant. Then the following conclusions hold:

(1) $\{x_n\}$ and $\{z_n\}$ *converge strongly to a point $w \in F$;*
(2) *the limits* $\lim_{n\to\infty} P_{C_n} x_{1,i} = P_F x_{1,i},\ i = 1, 2, 3, \ldots, N$;
(3) $w = \sum_{i=1}^{N} \lambda_i \lim_{n\to\infty} P_{C_n} x_{1,i}$.

Proof It is not hard to see that, C_n is closed and convex for all $n \ge 0$. Let us show that, $F \subset C_n$ for all $n \ge 0$. For any $z \in F$, we have

$$
\begin{aligned}
\|z_n - z\|^2 &= \|T_n(x_n - r_n T^* J(Tx_n - S_n Tx_n)) - z\|^2 \\
&\le \|x_n - r_n T^* J(Tx_n - S_n Tx_n)) - z\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, r_n T^* J(Tx_n - S_n Tx_n)\rangle \\
&\quad + \|r_n T^* J(Tx_n - S_n Tx_n))\|^2 \\
&\le \|x_n - z\|^2 - 2r_n\langle Tx_n - Tz, J(Tx_n - S_n Tx_n)\rangle \\
&\quad + r_n^2\|T\|^2\|J(Tx_n - S_n Tx_n))\|^2 \\
&= \|x_n - z\|^2 + r_n^2\|T\|^2\|Tx_n - S_n Tx_n)\|^2 \\
&\quad - 2r_n\langle Tx_n - S_n Tx_n + S_n Tx_n - Tz, J(Tx_n - S_n Tx_n)\rangle \\
&= \|x_n - z\|^2 + r_n^2\|T\|^2\|Tx_n - S_n Tx_n)\|^2 \\
&\quad - 2r_n\langle S_n Tx_n - Tz, J(Tx_n - S_n Tx_n)\rangle - 2r_n\|Tx_n - S_n Tx_n)\|^2 \\
&= \|x_n - z\|^2 + r_n^2\|T\|^2\|Tx_n - S_n Tx_n)\|^2 - 2r_n\|Tx_n - S_n Tx_n)\|^2 \\
&= \|x_n - z\|^2 + r_n(r_n\|T\|^2 - 2)\|Tx_n - S_n Tx_n)\|^2 \\
&\le \|x_n - z\|^2. \qquad (2)
\end{aligned}
$$

So, $z \in C_n$, which implies that $F \subset C_n$ for all $n \ge 0$.

Let $u_{n+1,i} = P_{C_n} x_{1,i}$ for all $n \ge 1$, $i = 1, 2, 3, \ldots, N$. Since F is nonempty, closed, and convex, there exist $p_{1,i} = P_F x_{1,i}$ such that

$$\|u_{n+1,i} - p_{1,i}\| \le \|x_{1,i} - p_{1,i}\|, \quad i = 1, 2, 3, \ldots, N.$$

This means that $\{u_{n,i}\}$ is bounded for all $i = 1, 2, 3, \ldots, N$.

From $u_{n+1,i} = P_{C_n} x_{1,i}$ and $C_n \subset C_{n-1}$, we have that

$$\|u_{n,i} - x_{1,i}\| \le \|u_{n+1,i} - x_{1,i}\|, \quad i = 1, 2, 3, \ldots, N,$$

for all $n \in N$. This implies that $\{\|u_{n,i} - x_1\|\}$ is bounded and nondecreasing for all $i = 1, 2, 3, \ldots, N$. Then there exist the limits of $\{\|u_{n,i} - x_{1,i}\| : i = 1, 2, 3, \ldots, N\}$. Put

$$\lim_{n\to\infty} \|u_{n,i} - x_{1,i}\| = c_i, \quad i = 1, 2, 3, \ldots, N.$$

On the other hand, $u_{n+m,i} \in C_{n-1}$, $i = 1, 2, 3, \ldots, N$, by using Lemma 7, we have, for any positive integer m, that

$$\|u_{n+m,i} - u_{n,i}\|^2 \le \|u_{n+m,i} - x_{1,i}\|^2 - \|u_{n,i} - x_{1,i}\|^2.$$

So that $\{u_{n,i}\}$ is Cauchy sequences in C for all $i = 1, 2, 3, \ldots, N$, therefore there exit two points $p_i \in C$ such that

$$\lim_{n\to\infty} u_{n,i} = p_i, \quad i = 1, 2, 3, \ldots, N.$$

That is

$$\lim_{n\to\infty} P_{C_n} x_{1,i} = p_i, \quad i = 1, 2, 3, \ldots, N.$$

Therefore

$$\lim_{n\to\infty} x_n = \sum_{i=1}^{N} \lambda_i p_i.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$

which implies

$$\lim_{n\to\infty} z_n = \sum_{i=1}^{N} \lambda_i p_i.$$

From (2), we have, for any $z \in F$, that

$$r_n(2 - r_n\|T\|^2)\|Tx_n - Q_{t_n}Tx_n)\|^2 \leq \|x_n - z\|^2 - \|z_n - z\|^2 \to 0$$

as $n \to \infty$. This implies

$$\lim_{n\to\infty} \|Tx_n - S_nTx_n)\| = 0. \tag{3}$$

Since

$$\lim_{n\to\infty} Tx_n = T\left(\sum_{i=1}^{N} \lambda_i p_i\right)$$

and the sequence $\{S_n\}$ is uniformly closed, so that

$$T\left(\sum_{i=1}^{N} \lambda_i p_i\right) \in \cap_{n=1}^{\infty} F(S_n).$$

That is

$$\sum_{i=1}^{N} \lambda_i p_i \in T^{-1}(\cap_{n=1}^{\infty} F(S_n)).$$

On the other hand, from

$$z_n = T_n(x_n - r_nT^*J(Tx_n - S_nTx_n)),$$

we have

$$\begin{aligned}\|z_n - T_nz_n\| &= \|T_n(x_n - r_nT^*J(Tx_n - S_nTx_n)) - T_nz_n\| \\ &\leq \|(x_n - r_nT^*J(Tx_n - S_nTx_n)) - z_n\|.\end{aligned}$$

This together with (3) implies that

$$\lim_{n\to\infty} \|z_n - T_nz_n\| = 0.$$

Since

$$\lim_{n\to\infty} z_n = \sum_{i=1}^{N} \lambda_i p_i,$$

and the sequence $\{T_n\}$ is uniformly closed, so that

$$\sum_{i=1}^{N} \lambda_i p_i \in \cap_{n=1}^{\infty} F(T_n).$$

From above two hands, we have $\sum_{i=1}^{N} \lambda_i p_i \in F$.

Finally, we prove $p_i = P_F x_{1,i},\ i = 1, 2, 3, \ldots, N$. From Lemma 7, we have

$$\|p_i - P_F x_{1,i}\|^2 + \|P_F x_{1,i} - x_{1,i}\|^2 \leq \|p_i - x_{1,i}\|^2. \tag{4}$$

On the other hand, since $x_{n+1,i} = P_{C_n} x_{1,i}$ and $F \subset C_n$, for all n. Also from Lemma 7, we have

$$\|P_F x_{1,i} - x_{n+1,i}\|^2 + \|x_{n+1,i} - x_{1,i}\|^2 \leq \|P_F x_{1,i} - x_{1,i}\|^2. \tag{5}$$

Since

$$\lim_{n\to\infty} \|x_{n+1,i} - x_{1,i}\| = \|p_i - x_{1,i}\|. \tag{6}$$

Combining (4), (5) and (6), we know that $\|p_i - x_{1,i}\| = \|P_F x_{1,i} - x_{1,i}\|$. Therefore, it follows from the uniqueness of $P_F x_{1,i}$ that $p_i = P_F x_{1,i}$. This completes the proof. □

By using Theorem 8 and setting $N = 1$, we can get the following result.

Theorem 9 *Let H be a Hilbert space and let E be a uniformly convex and smooth Banach space. Let J be the duality mapping on E. Let $\{T_n\} : H \to H$ be a uniformly closed family of countable quasi-nonexpansive mappings with the nonempty common fixed point set $\cap_{n=1}^{\infty} F(T_n)$ and $\{S_n\} : E \to E$ be a uniformly closed family of countable second-type quasi-nonexpansive mappings with the nonempty common fixed point sets $\cap_{n=1}^{\infty} F(S_n)$. Suppose that $F = (\cap_{n=1}^{\infty} F(T_n)) \cap (T^{-1}(\cap_{n=1}^{\infty} F(S_n))) \neq \emptyset$. Let $T : H \to E$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = T_n(x_n - r_n T^* J(Tx_n - S_n Tx_n)), \\ C_n = \{z \in C_{n-1} : \|z_n - z\| \leq \|x_n - z\|\}, \\ C_0 = H,\ n = 1, 2, 3, \ldots, \\ x_{n+1} = P_{C_n} x_1, \end{cases}$$

where $\{r_n\}$ satisfy the condition such that

$$0 < a \leq r_n \|T\|^2 \leq b < 2,$$

for some constants a, b. Then $\{x_n\}$ converges strongly to a point $z_0 = P_F x_1$.

Application for common null point problem

Let E be a Banach space, let A be a multi-valued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if it's graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well-known.

Theorem 10 (Rockafellar 1970). *Let E be a reflexive, strictly convex and smooth Banach space and let A be a monotone operator from E to E^*. Then A is maximal if and only if $R(J + rA) = E^*$. for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 10 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r.$$

Then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A. We know that J_r is a nonexpansive mapping and $A^{-1}0 = F(J_r)$ for all $r > 0$, see Takahashi (2000, 2009), Alber (1996).

Lemma 11 (Aoyama et al. 2009) *Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Then*

$$\langle J_r x - p, J(x - J_r x)\rangle \geq 0, \quad \forall\, x \in E, \quad \forall \quad p \in A^{-1}0, \quad \forall\, r > 0,$$

where J_r is the resolvent of A.

From Lemma 11, we know that, J_r is a second-type quasi-nonexpansive mapping, where J_r is the resolvent of A with $r > 0$.

Definition 12 Let E be a Banach space, let C be a nonempty, closed, and convex subset of E. Let $\{T_n\}$ be sequence of mappings from C into itself with nonempty common fixed point set $F = \cap_{n=1}^{\infty} F(T_n)$. The $\{T_n\}$ is said to be uniformly weak closed if for any weak convergent sequence $\{z_n\} \subset C$ such that $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the weak limit of $\{z_n\}$ belong to F.

A uniformly weak closed family of countable quasi-nonexpansive mappings must be a uniformly closed family of countable quasi-nonexpansive mappings.

Theorem 13 *Let $r_n \geq c > 0$, for some constant c, then $\{J_{r_n}^A\}_{n=0}^{\infty}$ is a uniformly weak closed family of countable quasi-nonexpansive mappings with the nonempty common fixed point sets $\bigcap_{n=0}^{\infty} F(J_{r_n}^A) = A^{-1}0$.*

Proof It is well-known that, $\bigcap_{n=0}^{\infty} F(J_{r_n}^A) = A^{-1}0 \neq \emptyset$ and $\{J_{r_n}^A\}_{n=0}^{\infty}$ is a family of countable nonexpansive mappings. Let $\{z_n\} \subset E$ be a sequence such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty} \|z_n - J_{r_n}^A z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}^A z_n) \to 0.$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ^A_{r_n}z_n) \in AJ^A_{r_n}z_n$$

and the monotonicity of A that

$$\left\langle w - J^A_{r_n}z_n, w^* - \frac{1}{r_n}(Jz_n - JJ^A_{r_n}z_n)\right\rangle \geq 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have $\langle w - p, w^*\rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore from the maximality of A, we obtain $p \in A^{-1}0$. That is $p \in \bigcap_{n=0}^{\infty} F(J^A_{r_n})$. This completes the proof. □

Theorem 14 *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^F such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_r be the resolvent of A for $r > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $W = A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{r_n}(x_n - r_nT^*J_F(Tx_n - Q_{t_n}Tx_n)), \\ C_n = \{z \in C_{n-1} : \|z_n - z\| \leq \|x_n - z\|\}, \\ C_0 = H, \ n = 1, 2, 3, \ldots, \\ x_{n+1} = P_{C_n}x_1, \end{cases}$$

where $\{r_n\}$ satisfy the condition such that

$$0 < a \leq r_n\|T\|^2 \leq b < 2, \quad 0 < c \leq t_n,$$

for some constants a, b, c. Then $\{x_n\}$ converges strongly to a point $z_0 = P_W x_1$.

Proof Let $T_n = J_{r_n}$, $S_n = Q_{\mu_n}$ for all $n \geq 1$, then $\{T_n\}$, $\{S_n\}$ satisfy the all conditions of Theorem 8, and

$$F = \cap_{n=1}^{\infty}F(T_n) \cap T^{-1}(\cap_{n=1}^{\infty}F(S_n)) = A^{-1}0 \cap T^{-1}(B^{-1}0) = W.$$

By using Theorem 9, we obtain the conclusion of Theorem 14. This completes the proof. □

Theorem 15 *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let A and B be maximal monotone operators of H into 2^H and F into 2^F such that $A^{-1}0 \neq \emptyset$ and $B^{-1}0 \neq \emptyset$, respectively. Let J_r be the resolvent of A for $r > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \to F$ be a bounded linear operator such that $T \neq 0$ and let T^* be the adjoint operator of T. Suppose that $W = A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $x_{1,i} \in H$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences generated by*

$$\begin{cases} z_n = J_{r_n}(x_n - r_n T^* J_F(Tx_n - Q_{t_n} Tx_n)), \\ C_n = \{z \in C_{n-1} : \|z_n - z\| \le \|x_n - z\|\}, \\ C_0 = H, \ n = 1, 2, 3, \ldots, \\ x_{n+1} = \sum_{i=1}^{N} \lambda_i P_{C_n} x_{1,i}, \ \sum_{i=1}^{N} \lambda_i = 1, \end{cases}$$

where $\{r_n\}$ *satisfy the condition such that*

$$0 < a \le r_n \|T\|^2 \le b < 2,$$

for some constants a, b *and* $\lambda \in [0, 1]$ *is a constant. Then the following conclusions hold:*

(1) $\{x_n\}$ and $\{z_n\}$ converge strongly to a point $w \in W$;

(2) the limits $\lim_{n\to\infty} P_{C_n} x_{1,i} = P_W x_{1,i}, \ i = 1, 2, 3, \ldots, N$;

(3) $w = \sum_{i=1}^{N} \lambda_i \lim_{n\to\infty} P_{C_n} x_{1,i}$.

Proof Let $T_n = J_{r_n}$, $S_n = Q_{\mu_n}$ for all $n \ge 1$, then $\{T_n\}$, $\{S_n\}$ satisfy the all conditions of Theorem 9, and

$$F = \cap_{n=1}^{\infty} F(T_n) \cap T^{-1}(\cap_{n=1}^{\infty} F(S_n)) = A^{-1}0 \cap T^{-1}(B^{-1}0) = W.$$

By using Theorem 8, we obtain the conclusion of Theorem 15. This completes the proof. □

Examples

It is easy to see that, a uniformly weak closed family $\{T_n\}$ of countable quasi-nonexpansive mappings must be a uniformly closed family $\{T_n\}$ of countable quasi-nonexpansive mappings. Next we will give an example which is a uniformly closed family of countable quasi-nonexpansive mappings, but not a uniformly weak closed family of countable quasi-nonexpansive mappings.

Conclusion 16 *Let* H *be a Hilbert space,* $\{x_n\}_{n=1}^{\infty} \subset H$ *be a sequence such that it converges weakly to a non-zero element* x_0 *and* $\|x_i - x_j\| \ge 1$ *for any* $i \ne j$. *Define a sequence of mappings* $T_n : H \to H$ *as follows*

$$T_n(x) = \begin{cases} L_n x_n & \text{if} \quad x = x_n (\exists\, n \ge 1), \\ -x & \text{if} \quad x \ne x_n (\forall\, n \ge 1), \end{cases}$$

where $L_n \le 1$ *and* $\lim_{n\to\infty} L_n = 1$. *Then* $\{T_n\}$ *is a uniformly closed family of countable quasi-nonexpansive mappings with the common fixed point set* $F = \{0\}$, *but not a uniformly weak closed family of countable quasi-nonexpansive mappings.*

Proof It is obvious that, $\{T_n\}$ has a unique common fixed point 0. Next, we prove that, $\{T_n\}$ is uniformly closed. In fact that, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, there exists sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then $T_n z_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_0 \in F$. From the definition of $\{T_n\}$, we have

$$\|T_n x - 0\| = \|T_n x\| \leq \|L_n x\| = \|x - 0\|, \quad \forall x \in H.$$

so that $\{T_n\}$ is a uniformly closed family of countable quasi-nonexpansive mappings. Next, we prove the $\{T_n\}$ is not weak closed. Since $\{x_n\}$ converges weakly to x_0 and

$$\|T_n x_n - x_n\| = \|L_n x_n - x_n\| = (L_n - 1)\|x_n\| \to 0$$

as $n \to \infty$, but x_0 is not a fixed point. □

Conclusion

In the multidirectional iteration algorithm, the C_n is a closed convex set, and $F \subset C_n$ for any $n \geq 1$. If we use one initial $x_{1,1}$, the projection point $x_n = P_{C_n} x_{1,1}$ belongs to the boundary of the C_n. If we use N initials $x_{1,1}, x_{1,2}, x_{1,3}, \ldots, x_{1,N}$, the element $x_n = \sum_{i=1}^{N} \lambda_i P_{C_n} x_{1,i}$ belongs to the interior of the C_n. In general, the distance $d(\sum_{i=1}^{N} \lambda_i P_{C_n} x_{1,i}, F)$ is less than the distance $d(P_{C_n} x_{1,1}, F)$, so the multidirectional iteration algorithm can accelerate the convergence speed of iterative sequence $\{x_n\}$. We give a simple experimental example in the following.

Example Let $X = R^2$, $C_n = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}$, $x_{1,1} = (1, 1)$, $x_{1,2} = (-1, 1)$, $F = \{0\}$. Case 1, take only one initial $x_{1,1}$, $x_n = P_{C_n} x_{1,1} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, then $d(x_n, F) = 1$. Case 2, take two initials $x_{1,1}, x_{1.2}$,

$$x_n = \frac{1}{2} P_{C_n} x_{1,1} + \frac{1}{2} P_{C_n} x_{1,2} = \left(0, \frac{\sqrt{2}}{2}\right),$$

then $d(x_n, F) = \frac{\sqrt{2}}{2}$. From the inequality "$\frac{\sqrt{2}}{2} < 1$", we can see that, the multidirectional iteration algorithm can accelerate the convergence speed of iterative sequence $\{x_n\}$.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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