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On Berinde's method for comparing iterative processes

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Abstract

In the literature there are several methods for comparing two convergent iterative processes for the same problem. In this note we have in view mostly the one introduced by Berinde in (Fixed Point Theory Appl. 2:97–105, 2004) because it seems to be very successful. In fact, if IP1 and IP2 are two iterative processes converging to the same element, then IP1 is faster than IP2 in the sense of Berinde. The aim of this note is to prove this almost obvious assertion and to discuss briefly several papers that cite the mentioned Berinde's paper and use his method for comparing iterative processes.

MSC: 41A99

Keywords: Faster convergence; Better convergence; Berinde's method for comparing iterative processes

1 Introduction

In the literature there are several methods for comparing two convergent iterative processes for the same problem. In this note we have in view mostly the one introduced by Berinde in [15, Definition 2.7] because it seems to be very successful. This was pointed out by Berinde himself in [18]: "This concept turned out to be a very useful and versatile tool in studying the fixed point iterative schemes and hence various authors have used it". However, it was pointed out by Popescu, using [75, Example 3.4], that Berinde's method is not consistent. The inconsistency of Berinde's method is indirectly mentioned also by Qing and Rhoades in [77, p. 2] by providing a very simple counterexample in $\mathbb{R}$ to [13, Theorem 2.1].[a] Moreover, referring to Berinde's method, Phuengrattana and Suantai say in [73, p. 218]: "It seems not to be clear if we use above definition for comparing the rate of convergence". In fact, if IP1 and IP2 are two (arbitrary) iterative processes converging to the same element, then IP1 is faster than IP2 (and vice-versa) in the sense of Berinde [15, Definition 2.7].

The aim of this note is to prove this almost obvious assertion and to discuss briefly several papers that cite [15] and refer to Berinde's method for comparing iterative processes.

2 Definitions and the main assertion

First, we quote from [15, pp. 99, 100] the text containing the definitions which we have in view; these are reproduced in many papers from our bibliography.

> Definition 2.5. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists $l = \lim_{n\to\infty} |\frac{a_n-a}{b_n-b}|$.
> (a) If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges *faster* to a than $\{b_n\}_{n=0}^{\infty}$ to b.
> (b) If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ *have the same rate of convergence.*
>
> Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to the same fixed point p, the error estimates
>
> $$\|u_n - p\| \le a_n, \quad n = 0, 1, 2, \dots \tag{2.7}$$
>
> $$\|v_n - p\| \le b_n, \quad n = 0, 1, 2, \dots \tag{2.8}$$
>
> are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero).
> Then, in view of Definition 2.5, we will adopt the following concept.
>
> Definition 2.7. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (2.7) and (2.8), respectively. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then it can be said that $\{u_n\}_{n=0}^{\infty}$ *converges faster* than $\{v_n\}_{n=0}^{\infty}$ to p.

Practically, the text above is reproduced in [18, pp. 30, 31], getting in this way Definitions 1.1 and 1.2. The only differences are: "(2.7)" and "(2.8) are available, where" are replaced by "(1.7)" and "(1.8) are available *(and these estimates are the best ones available)*, where", respectively.

Immediately after [18, Definition 1.2] it is said:

> This concept turned out to be a very useful and versatile tool in studying the fixed point iterative schemes and hence various authors have used it, see [1]–[5], [18], [22], [23], [28], [32]–[34], [37]–[41], [40], [43]–[46], [55]–[57], [66], [68]–[72], [74], [78]–[81], to cite just an incomplete list.[b]

Note that Definition 9.1 from [16] is equivalent to Definition 2.5 from [15]; replacing u_n, v_n, p, $\|u_n - p\|$, and $\|v_n - p\|$ with x_n, y_n, x^*, $d(x_n, x^*)$, and $d(y_n, x^*)$ in (2.7), (2.8), and Definition 2.7 from [15], one obtains relations (5), (6) from [16, p. 201] and an equivalent formulation of [16, Definition 9.2], respectively. Note that these definitions from Berinde's book [16] are presented in the lecture [17].

Because of the parentheses in "(converging to zero)" in the preamble of [15, Definition 2.7] (and [16, Definition 9.2], [18, Definition 1.2]), the convergence to 0 of (a_n) and (b_n) seems to be optional. This is probably the reason for the absence of this condition in [29, p. 3]; note that (a_n) is a constant sequence in [90].

In the next result we use the version for metric spaces of [15, Definition 2.7] (see [16, Definition 9.2]).

Proposition 1 *Let (X, d) be a metric space and $(x_n)_{n\ge1}$, $(y_n)_{n\ge1}$ be two sequences from X converging to $x^* \in X$. Then (x_n) converges faster than (y_n) to x^*.*

Proof For each $n \geq 1$, let us consider

$$0 < a_n := d(x_n, x^*) + d(y_n, x^*) + \frac{1}{n}, \qquad 0 < b_n := \begin{cases} \sqrt{a_n} & \text{if } a_n \leq 1, \\ d(y_n, x^*) & \text{otherwise.} \end{cases}$$

It follows that $a_n \to 0$, $b_n \to 0$,

$$d(x_n, x^*) \leq a_n, \qquad d(y_n, x^*) \leq b_n, \quad \forall n \geq 1,$$

and $a_n/b_n = \sqrt{a_n}$ for sufficiently large n; it follows that $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \sqrt{a_n} = 0$. Therefore, (x_n) converges faster to x^* than (y_n) does. □

From our point of view, the preceding "result" shows that Berinde's notion of rapidity for fixed point iterative schemes, recalled above, is not useful, even if Berinde in [18, p. 35] claims that "Of all concepts of rapidity of convergence presented above for numerical sequences, the one introduced by us in Definition 1.2 [14] appears to be the most suitable in the study of fixed point iterative methods". Berinde (see [18, p. 36]) mentions that he "tacitly admitted in Definition 1.2 that *the estimates (1.7) and (1.8)* taken into consideration *are the best possible*". Clearly, "the estimates are the best ones available" and "the estimates … are the best possible" are very different in meaning.[c]

Of course, *the best possible estimates in relations (1.7) and (1.8)* from [18] (that is, in relations (2.7) and (2.8) from [15] recalled above) *are*

$$a_n := \|u_n - p\|, \qquad b_n := \|v_n - p\| \quad (n \geq 0). \tag{1}$$

Assuming that $d(x_n, x^*) \to 0$, getting (better) upper estimates for $d(x_n, x^*)$ depends on the proof, including the author's ability to majorize certain expressions. Surely, *the best available estimates are* exactly *those obtained by the authors in their proofs*.

The use of Berinde's method for comparing the speeds of convergence is very subjective. It is analogue to deciding that $a/b \leq c/d$ knowing only that $0 < a \leq c$ and $0 < b \leq d$!

Taking a_n and b_n defined by (1) in [15, Definition 2.7], one obtains Definition 3.5 of Popescu from [75].[d] Popescu's definition is used explicitly by Rhoades and Xue (see [81, p. 3]), but they wrongly attribute it to [15]; this attribution is wrong because [75, Definition 3.5] reduces to [15, Definition 2.5] only in the case in which the involved normed vector space is $\mathbb{R}$. Note that Rhoades knew about Popescu's definition because [75] is cited in [77, p. 2].

Notice that Popescu's definition is extended to metric spaces by Berinde, Khan, and Păcurar in [20, p. 32], as well as by Fukhar-ud-din and Berinde in [31, p. 228]; also observe that Popescu's paper [75] is not cited in [20] and [31].

Even if in [15] it is not defined when two iteration schemes have the same rate of convergence, Dogan and Karakaya obtained that "the iteration schemes $\{k_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ have the same rate of convergence to p of $\wp$" in [27, Theorem 2.4]; the proof of [27, Theorem 2.4] is based on the fact that they found the same upper estimates for $\|k_{n+1} - p\|$ and $\|l_{n+1} - p\|$ when $l_0 = k_0$.

Accepting such an argument and taking $a_n := b_n := d(x_n, x^*) + d(y_n, x^*) + \frac{1}{n}$ in the proof of Proposition 1, one should obtain that any pair of sequences $(x_n)_{n\geq1}$, $(y_n)_{n\geq1} \subset (X, d)$ with the same limit $x^* \in X$ have the same rate of convergence.

Recall that Rhoades in [80, pp. 742, 743] says that having "$\{x_n\}$, $\{z_n\}$ two iteration schemes which converge to the same fixed point p, we shall say that $\{x_n\}$ is better than $\{z_n\}$ if $|x_n - p| \le |z_n - p|$ for all n"; having in view the previous definition and [15, Example 2.8], Berinde claims that "The previous example shows that Definition 2.7 introduces a sharper concept of rate of convergence than the one considered by Rhoades [11]". In this context we propose the following definition.

Definition 2 Let (X,d) be a metric space, and let $(x_n)_{n\ge1}$, $(y_n)_{n\ge1} \subset (X,d)$, and $x, y \in X$ be such that $x_n \to x$, $y_n \to y$. One says that (x_n) converges better to x than (y_n) to y if there exists some $\alpha > 0$ such that $d(x_n,x) \le \alpha d(y_n,y)$ for sufficiently large n; one says that (x_n) and (y_n) have the same rate of convergence if (x_n) converges better to x than (y_n) to y, and (y_n) converges better to y than (x_n) to x.

Using the conventions $\frac{0}{0} := 1$ and $\frac{\alpha}{0} := \infty$ for $\alpha > 0$, [(x_n) converges better to x than (y_n) to y] if and only if $\limsup_{n\to\infty} \frac{d(x_n,x)}{d(y_n,y)} < \infty$; consequently, [$(x_n)$ and (y_n) have the same rate of convergence] (in the sense of Definition 2) if and only if $0 < \liminf_{n\to\infty} \frac{d(x_n,x)}{d(y_n,y)} \le \limsup_{n\to\infty} \frac{d(x_n,x)}{d(y_n,y)} < \infty$.[e]

Example 3 Consider the sequences $(x_n)_{n\ge1}$, $(y_n)_{n\ge1} \subset \mathbb{R}$ defined by

$$x_n := \begin{cases} n^{-1} & \text{if } n \text{ is odd,} \\ (2n)^{-1} & \text{if } n \text{ is even,} \end{cases} \qquad y_n := \begin{cases} (2n)^{-1} & \text{if } n \text{ is odd,} \\ n^{-1} & \text{if } n \text{ is even.} \end{cases}$$

Clearly $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$, and it is very natural to consider that they have the same rate of convergence; this is confirmed using Definition 2. It is obvious that neither (x_n) is better (faster) than (y_n), nor (y_n) is better (faster) than (x_n) in the senses of Rhoades [80, pp. 742, 743], or Berinde [15, Definition 2.5], or Popescu [75, Definition 3.5], or Berinde, Khan, and Păcurar [20, p. 32], or Fukhar-ud-din and Berinde [31, p. 228].

3 Remarks on the use of Berinde and Popescu's notions in papers citing [15]

Practically, all the papers mentioned in the sequel were found on the internet when searching, with Google Scholar, for the works citing Berinde's article [15].

First we give the list of articles, mentioning their authors and results, in which Berinde's Definition 2.7 from [15] is used (even if not said explicitly):

Berinde and Berinde—[19, Theorem 3.3]; Babu and Prasad—[13, Theorem 2.1] and [14, Theorems 3.1, 3.3]; Olaleru—[69, Theorem 1] and [70, Theorems 1, 2];[f] Sahu—[82, Theorem 3.6]; Akbulut and Özdemir—[5, Theorem 2.3]; Hussain et al.—[43, Theorems 18, 19]; Karahan and Ozdemir—[49, Theorem 1]; Abbas and Nazir—[2, Theorem 3]; Gürsoy and Karakaya—[39, Theorem 3]; Kadioglu and Yildirim—[47, Theorem 5]; Karakaya et al.—[52, Theorem 3][g] and [53, Theorem 2.2]; Kumar—[59, Theorem 3.1]; Öztürk Çeliker—[71, Theorem 8]; Thakur et al.—[88, Theorem 2.3][h] and [87, Theorem 3.1]; Chugh et al.—[23, Theorem 3.1] and [24, Theorem 13]; Fathollahi et al.—[29, Propositions 3.1, 3.2, Theorems 3.1, 4.1–4.4, Lemmas 3.1–3.4]; Gursoy—[35, Theorem 3]; Jami and Abed—[46, Theorems 3.1–3.4]; Yadav—[93, Example 2]; Abed and Abbas, [3, Theorem (3.8)]; Asaduzzaman et al.—[11, Theorem 3.3]; Mogbademu—[66, Theorem 2.1]; Rani and Jyoti—[79, Theorem 13]; Sahu et al.—[83, Theorem 4.1]; Sintunavarat and Pitea—[86, Theorem 2.1]; Verma et al.—[90];[i] Alecsa—[8, Theorems 3.1, 3.3–3.12]; Karakaya et al.—[50, Theorem 2.4]; Okeke and Abbas—[68, Proposition 2.1]; Sharma and Imdad—[84, Proposition 4.9]; Yildirim and Abbas—[95, Theorem 2]; Abass et al.—[1, Remark 2]; Akhtar and Khan—[6, Theorem 3.1–3.3]; Alagoz et al.—[7, Theorem 2.1]; Dogan—[25, Theorem 3.3.1]; Fathollahi and Rezapour—[30, Propositions 2.1–2.3, 3.1, Theorem 3.2]; Garodia and Uddin—[32, Theorem 3.1]; Hussain et al.—[45, Theorem 3.4]; Kumar and Chauhan—[60, Theorems 1, 2]; Piri et al. [74, Lemmas 3.1, 3.2, Theorem 3.3]; Yildirim—[94, Theorem 2], Asaduzzaman and Ali—[10, Theorem 3.3], Ertürk and Gürsoy—[28, Theorem 2.3]; Gürsoy et al.—[37, Theorem 6]; Kumar and Chugh—[61, Theorem 2.2]; Malik and Choudhary—[64, Theorem 6]; Mebawondu and Mewomo—[65, Theorem 3.2]; Okeke—[67, Theorem 3.3]; Aibinu and Kim—[4, Theorem 3.2]; Garodia and Uddin—[33, Theorem 3.1] and [34, Theorem 3.1]; Gürsoy et al.—[40, Theorem 2.3].

As mentioned in Sect. 2, Dogan and Karakaya obtained that "the iteration schemes $\{k_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$ have the same rate of convergence to p of $\wp$" in [27, Theorem 2.4] because they found the same upper estimates for $\|k_{n+1} - p\|$ and $\|l_{n+1} - p\|$ when $l_0 = k_0$ (see [27, p. 156]).

It is worth repeating that Popescu (in [75]) recalls [15, Definition 2.7], mentions its inconsistency, introduces his direct comparison of iterative processes in [75, Definition 3.5], and uses this definition in [75, Theorem 3.7].

Other papers in which [75, Definition 3.5] is used, without citing it (but possibly recalling [15, Definition 2.5 or/and Definition 2.7]), are: Xue—[92, Theorems 2.1, 2.2]; Rhoades and Xue—[81, Theorems 2.1, 2.2, 3.1, 3.2]; Thong—[89, Theorems 2.1, 2.3, 2.5]; Alotaibi et al.—[9, Theorem 3.1]; Hussain et al.—[43, Theorems 14–17];[j] Phuengrattana and Suantai—[73, Theorems 2.4, 2.6]; Khan et al.—[55, Theorem 3.1]; Fukhar-ud-din and Berinde—[31, Theorems 2.5, 2.7]; Gürsoy—[36, Theorem 2.4]; Khan et al.—[54, Theorem 3]; Gürsoy et al.—[41, Theorem 2.3]; Kosol—[57, Theorem 2.2]; Pansuwan and Sintunavarat—[72, Theorem 3.7]; Atalan and Karakaya—[12, Theorem 3.3]; Ertürk and Gürsoy—[28, Theorem 2.3]; Kumam et al.—[58, Theorems 3.4, 3.5]; Gürsoy et al.—[38, Theorem 4].

It is also worth noticing that by taking simple examples in $\mathbb{R}$, Rafiq et al.—[78, Example 11]; Hussain et al.—[44, Example 9]; Chugh et al.—[22, Example 4.1]; Hussain et al.—[42, Example 3.1, 3.2]; Kang et al.—[48, Example 11]; Karakaya et al.—[51, Example 4]; Kumar et al.—[63, Example 9]; Doğan and Karakaya—[26, Example 10]; Prasad and Goyal—[76, Example 2.1]; Chauhan et al.—[21, Example 3.1]; Sintunavarat—[85, Example 13]; Wahab and Rauf—[91, Example 11, Remarks 12–17] "prove" that certain iteration processes are faster than others.

Final remark I wish to point out that this paper is not about the correctness of the results in the cited papers; I did not check the proofs of the results. My aim is to emphasize again, as Popescu [75] and Phuengrattana and Suantai [73] did, that Berinde's method is inconsistent, and so what is obtained using it is useless from my point of view. The other remarks mainly concern wrong attributions of notions as well as the fact that one cannot claim the validity of general assertions using some examples.

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Endnotes

[a] Note that Berinde's paper [15] is not cited in [77]; see also [18, Remark 3.2].

[b] Throughout this paper the references mentioned in the quoted texts are those in the works from where the texts are taken.

[c] Among the 35 papers from our bibliography published in the period 2017–2020, our reference [18] is mentioned only in [28, 37, 38], and [40]. However, [15, Definition 2.7] is used in [28, 37], and [40] without any mention that the obtained estimates are the best possible.

[d] Of course, when $(X, \|\cdot\|)$ is $(\mathbb{R}, |\cdot|)$, Definition 3.5 of Popescu [75] reduces to Definition 2.5(a) of Berinde [15] when $a = b$.

[e] This manner of comparing the rate of convergence for sequences of real numbers is attributed to Knopp [56] in [41, Definition 1.2]; a more precise presentation of this topic is done in [18, p. 34].

[f] Kumar (see [62, p. 1320]) shows that [70, Theorem 2] "is not consistent" by using [75, Definition 3.5] and a simple example in $\mathbb{R}$.

[g] Note that [52, Definition 1] is [15, Definition 2.5 (a)] for "$\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ two sequences of real numbers with limits a and b respectively", but in the proof of [52, Theorem 3] one uses [15, Definition 2.7]. Similar remarks are valid for [53, Definition 1.1, Theorem 2.2], [50, Definition 1.4, Theorem 2.4], [84, Definition 1.11, Proposition 4.9.].

[h] In [88, p. 3] one appreciates that "In recent years, Definition 2.2 has been used as a standard tool to compare the fastness of two fixed point iterations", Definition 2.2 being [15, Definition 2.7].

[i] See estimates (23) and (24), as well as the very strange arguments to get the conclusion on page SMC_ 2016 001606.

[j] Note the strange quantity $\|\frac{\mathrm{JN}_{n+1-p}}{\mathrm{JI}_{n+1-p}}\|$, the numerator and denominator being in $(X, \|\cdot\|)$ "an arbitrary Banach space".

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