Zegeye and Tufa *Fixed Point Theory and Applications* (2018) 2018:15
https://doi.org/10.1186/s13663-018-0640-5

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH

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Halpern–Ishikawa type iterative method for approximating fixed points of non-self pseudocontractive mappings

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Abstract

In this paper, we define a Halpern–Ishikawa type iterative method for approximating a fixed point of a Lipschitz pseudocontractive non-self mapping T in a real Hilbert space settings and prove strong convergence result of the iterative method to a fixed point of T under some mild conditions. We give a numerical example to support our results. Our results improve and generalize most of the results that have been proved for this important class of nonlinear mappings.

MSC: 37C25; 47H10; 47J05

Keywords: Fixed points; Monotone mappings; Pseudocontractive mappings

1 Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and C be a nonempty subset of H. A mapping $T: C \to H$ is said to be *L-Lipschitz* if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in C. \tag{1}$$

T is said to be *contraction* if $L \in [0, 1)$ and is called *nonexpansive* mapping if $L = 1$. We observe that every contraction mapping is nonexpansive and every nonexpansive mapping is Lipschitz.

A mapping $T: C \to H$ is said to be *k-strictly pseudocontractive* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|x - y - (Tx - Ty)\big\|^2, \quad \forall x, y \in C. \tag{2}$$

We remark that every k-strictly pseudocontractive mapping is Lipschitz and hence the class of k-strictly pseudocontractive mappings includes properly the class of nonexpansive mappings.

An important class of mappings more general than the class of k-strictly pseudocontractive mappings is the class of pseudocontractive mappings. T is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \big\|x - y - (Tx - Ty)\big\|^2, \quad \forall x, y \in C. \tag{3}$$

The class of pseudocontractive mappings is related to one of the important classes of operators known as monotone mappings. A mapping $A : C \to H$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Note that a mapping $A : C \to H$ is monotone if and only if $T := I - A$ is pseudocontractive, where I is an identity mapping on C. Thus, the zeros of A are fixed points of T, that is, $N(A) := \{x \in C : Ax = 0\} = F(T) := \{x \in C : x = Tx\}$.

Several authors have studied iterative methods for approximating fixed points of nonexpansive, k-strictly pseudocontractive and pseudocontractive mappings (see, e.g., [3, 6, 15, 17, 22, 27, 28] and the references contained therein). In 1953, Mann [15] introduced the following scheme, which is refereed to as *Mann iteration method*:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{4}$$

where the initial guess $x_0 \in C$ is arbitrary and $\{\alpha_n\} \subseteq [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. The Mann iteration method has been extensively investigated for approximating fixed points of nonexpansive mappings (see, e.g., [17]). In an infinite-dimensional Hilbert space, the Mann iteration method can provide only *weak convergence* (see, e.g., [7]). To obtain strong convergence, numerous authors have modified the Mann iterative method (see, e.g., [8, 10, 11]) in many ways.

In 1967, Halpern [8] studied the following recursive formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{5}$$

where α_n is a sequence of numbers in $(0, 1)$. He proved strong convergence of $\{x_n\}$ to a fixed point of T, where $\alpha_n := n^{-a}$, for $a \in (0, 1)$, in the framework of Hilbert spaces. Halpern's scheme (5) has been studied extensively by many authors (see, e.g., [2, 12, 18, 21]). In particular, Reich [18] proved that the result of Halpern remains true in uniformly smooth Banach spaces (see also [19]).

In 1977, Lions [12] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T, where the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i)} \quad \lim_{n\to\infty} \alpha_n = 0; \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \alpha_n = \infty; \qquad \text{(iii)} \quad \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0.$$

In 2002, Xu [24] (see also [25]) improved the result of Lion in two directions. First, he weakened the condition (iii) by removing the square in the denominator so that we can choose the sequence $\alpha_n = \frac{1}{n+1}$. Second, he proved the strong convergence of Halpern's scheme (5) in the framework of real uniformly smooth Banach spaces.

For approximating fixed points of a Lipschitz pseudocontractive self-mapping T, Ishikawa [9] introduced the following process known as *Ishikawa iteration*:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions:

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum \alpha_n \beta_n = \infty$.

He showed that the sequence $\{x_n\}$ converges strongly to a fixed point of the mapping T, provided that C is a compact convex subset of a Hilbert space H. Several authors have extended the results of Ishikawa [9] to Banach spaces without compactness assumption on C (see, e.g., [13, 23]).

However, we observe that all the above results are valid only for self-mappings. For approximating fixed points of non-self mappings, several iterative schemes have been studied (see, e.g., [16, 20]) with the use of metric projection or sunny nonexpansive retraction mapping which are generally difficult to compute in practical applications.

In 2015, Colao and Marino [4] introduced a new searching strategy for the coefficient α_n which makes the Mann algorithm well-defined for non-self mappings in the setting of a real Hilbert space H. In fact, they studied the following scheme:

$$\begin{cases} x_0 \in C, \\ \alpha_0 = \max\{\frac{1}{2}, h(x_0)\}, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) T x_n, \\ \alpha_{n+1} = \max\{\alpha_n, h(x_{n+1})\}, \quad n \geq 0, \end{cases} \tag{7}$$

where $h(x) := \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}, \forall x \in C \subseteq H$ and T is a non-self mapping of C into H. Indeed, they obtained weak and strong convergence of the algorithm to a fixed point of nonexpansive non-self mappings under appropriate conditions.

Recently, Colao et al. [5] extended this result of Colao and Marino [4] to a class of k-strictly pseudocontractive mappings. We observe that these results (the results obtained in [4] and [5]) provide a way forward to avoid the use of metric projection or sunny nonexpansive mapping in constructing algorithms for approximating fixed points of a more general class of non-self mappings.

It is our purpose in this paper to construct and study a Halpern–Ishikawa type iterative scheme for non-self mappings in the setting of Hilbert spaces. As a result, we obtain strong convergence of the scheme to a fixed point of a Lipschitz pseudocontractive nonself mapping under some mild conditions. Our results extend and generalize many results in the literature.

2 Preliminaries

Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to H$ is said to be *inward* if, for any $x \in C$, we have

$$Tx \in I_C(x) := \{x + \lambda(w - x) : \text{ for some } w \in C \text{ and } \lambda \geq 1\}.$$

The set $I_C(x)$ is called *inward set* of C at x. A mapping $I - T$, where I is an identity mapping on C, is called *demiclosed* at zero if for any sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$, then $x = Tx$.

In what follows, we shall make use of the following lemmas.

Lemma 2.1 *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds*:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle.$$

Lemma 2.2 ([1]) *Let C be a convex subset of a real Hilbert space H and let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \quad \forall z \in C.$$

Lemma 2.3 ([24]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfy the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([28]) *Let C be a closed convex subset of a real Hilbert space H and $T : C \to C$ be a continuous pseudo-contractive mapping. Then*

(i) *$F(T)$ is a closed convex subset of C*;

(ii) *$I - T$ is demiclosed at zero.*

Lemma 2.5 ([14]) *Let $\{a_n\}$ be sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in N$. Then there exists a nondecreasing sequence $\{m_k\} \subset N$ such that $m_k \to \infty$ and the following properties are satisfied by all* (*sufficiently large*) *numbers $k \in N$*:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.6 ([26]) *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\alpha \in [0,1]$, the following equality holds*:

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

Lemma 2.7 ([4]) *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $T : C \to H$ be a mapping. Define $h : C \to \mathbb{R}$ by*

$$h(x) = \inf\{\lambda \geq 0 : \lambda x + (1-\lambda)Tx \in C\}.$$

Then, for any $x \in C$, the following hold:

(1) *$h(x) \in [0,1]$ and $h(x) = 0$ if and only if $Tx \in C$*;

(2) *if $\beta \in [h(x), 1]$, then $\beta x + (1-\beta)Tx \in C$*;

(3) *if T is inward, then $h(x) < 1$*;

(4) *if $Tx \notin C$, then $h(x)x + (1-h(x))Tx \in \partial C$.*

3 Results and discussion

Now, let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C \to H$ be an inward L-Lipschitz mapping. Let $\beta \in (1 - \frac{1}{1+\sqrt{L^2+1}}, 1)$ and $\{\alpha_n\} \subseteq (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. We define a Halpern–Ishikawa type iterative scheme as follows.

Choose $u, x_0 \in C$. Let

$$h(x_0) := \inf\{\lambda \geq 0 : \lambda x_0 + (1-\lambda)Tx_0 \in C\} \quad \text{and} \quad \lambda_0 \in \bigl[\max\{\beta, h(x_0)\}, 1\bigr).$$

Then by Lemma 2.7 it follows that $y_0 := \lambda_0 x_0 + (1-\lambda_0)Tx_0 \in C$.

Let $l(y_0) := \inf\{\theta \geq 0 : \theta x_0 + (1-\theta)Ty_0 \in C\}$ and $\theta_0 \in [\max\{\lambda_0, l(y_0)\}, 1)$. Again by Lemma 2.7, $\theta_0 x_0 + (1-\theta_0)Ty_0 \in C$, and hence it follows that

$$x_1 := \alpha_0 u + (1-\alpha_0)\bigl(\theta_0 x_0 + (1-\theta_0)Ty_0\bigr) \in C.$$

Thus, by mathematical induction, we have

$$\begin{cases} \lambda_n \in [\max\{\beta, h(x_n)\}, 1); \\ y_n = \lambda_n x_n + (1-\lambda_n)Tx_n; \\ \theta_n \in [\max\{\lambda_n, l(y_n)\}, 1); \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\theta_n x_n + (1-\theta_n)Ty_n), \end{cases} \tag{8}$$

where $h(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda)Tx_n \in C\}$ and $l(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta)Ty_n \in C\}$.

Next, we prove the following theorem.

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be an L-Lipschitz pseudocontractive inward mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by* (8). *If there exists $\epsilon > 0$ such that $\theta_n \leq 1 - \epsilon$ $\forall n \geq 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.*

Proof We make use of some ideas of the paper [27]. Let $p \in F(T)$. Then from (8) and Lemma 2.6, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \bigl\|\alpha_n u + (1-\alpha_n)\bigl(\theta_n x_n + (1-\theta_n)Ty_n\bigr) - p\bigr\|^2 \\ &\leq \alpha_n \|u-p\|^2 + (1-\alpha_n)\bigl\|\theta_n(x_n - p) + (1-\theta_n)(Ty_n - p)\bigr\|^2 \\ &\leq \alpha_n \|u-p\|^2 + (1-\alpha_n)\bigl[\theta_n\|x_n - p\|^2 + (1-\theta_n)\|Ty_n - p\|^2\bigr] \\ &\quad - (1-\alpha_n)\theta_n(1-\theta_n)\|Ty_n - x_n\|^2, \end{aligned}$$

and hence from (3) we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|u-p\|^2 + (1-\alpha_n)\theta_n\|x_n - p\|^2 + (1-\alpha_n)(1-\theta_n) \\ &\quad \times \bigl[\|y_n - p\|^2 + \|y_n - Ty_n\|^2\bigr] - (1-\alpha_n)\theta_n(1-\theta_n)\|Ty_n - x_n\|^2 \\ &\leq \alpha_n\|u-p\|^2 + (1-\alpha_n)(1-\theta_n)\|y_n - p\|^2 \end{aligned}$$

$$+ (1-\alpha_n)(1-\theta_n)\|y_n - Ty_n\|^2$$
$$+ (1-\alpha_n)\theta_n\big(\|x_n - p\|^2 - (1-\theta_n)\|Ty_n - x_n\|^2\big). \quad (9)$$

Moreover, from (8), Lemma 2.6, and (3), we have

$$\begin{aligned}\|y_n - p\|^2 &= \big\|\lambda_n(x_n - p) + (1-\lambda_n)(Tx_n - p)\big\|^2\\ &= \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|Tx_n - p\|^2\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - Tx_n\|^2\\ &\le \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\big[\|x_n - p\|^2 + \|x_n - Tx_n\|^2\big]\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - Tx_n\|^2\\ &= \|x_n - p\|^2 + (1-\lambda_n)^2\|x_n - Tx_n\|^2. \end{aligned} \quad (10)$$

Furthermore, (8) and Lemma 2.6 imply that

$$\begin{aligned}\|y_n - Ty_n\|^2 &= \big\|\lambda_n(x_n - Ty_n) + (1-\lambda_n)(Tx_n - Ty_n)\big\|^2\\ &= \lambda_n\|x_n - Ty_n\|^2 + (1-\lambda_n)\|Tx_n - Ty_n\|^2\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - Tx_n\|^2\\ &\le \lambda_n\|x_n - Ty_n\|^2 + (1-\lambda_n)L^2\|x_n - y_n\|^2\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - Tx_n\|^2\\ &= \lambda_n\|x_n - Ty_n\|^2 + (1-\lambda_n)^3L^2\|x_n - Tx_n\|^2\\ &\quad - \lambda_n(1-\lambda_n)\|x_n - Tx_n\|^2\\ &= \lambda_n\|x_n - Ty_n\|^2\\ &\quad - (1-\lambda_n)\big(\lambda_n - L^2(1-\lambda_n)^2\big)\|x_n - Tx_n\|^2. \end{aligned} \quad (11)$$

Substituting (10) and (11) into (9), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|u - p\|^2 + (1-\alpha_n)(1-\theta_n)\big(\|x_n - p\|^2\\ &\quad + (1-\lambda_n)^2\|x_n - Tx_n\|^2\big) + (1-\alpha_n)(1-\theta_n)\big(\lambda_n\|x_n - Ty_n\|^2\\ &\quad - (1-\lambda_n)\big(\lambda_n - L^2(1-\lambda_n)^2\big)\|x_n - Tx_n\|^2\big)\\ &\quad + (1-\alpha_n)\theta_n\|x_n - p\|^2 - (1-\alpha_n)\theta_n(1-\theta_n)\|Ty_n - x_n\|^2\\ &= \alpha_n\|u - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)(1-\theta_n)(1-\lambda_n)\\ &\quad \times \big(1 - \big(L^2(1-\lambda_n)^2 + 2(1-\lambda_n)\big)\big)\|x_n - Tx_n\|^2\\ &\quad + (1-\alpha_n)(1-\theta_n)(\lambda_n - \theta_n)\|Ty_n - x_n\|^2. \end{aligned} \quad (12)$$

Then since, from the hypothesis, we have

$$1 - 2(1-\lambda_n) - L^2(1-\lambda_n)^2 \ge 1 - 2(1-\beta) - L^2(1-\beta)^2 > 0, \quad (13)$$

and

$$\theta_n \geq \lambda_n, \quad \text{for all } n \geq 0, \tag{14}$$

inequality (12) implies that

$$\|x_{n+1} - p\|^2 \leq \alpha_n \|u - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2. \tag{15}$$

Thus, by induction,

$$\|x_{n+1} - p\|^2 \leq \max\{\|u - p\|^2, \|x_0 - p\|^2\}, \quad \forall n \geq 0,$$

which provides that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Now, let $x^* = P_{F(T)}(u)$. Then, using (8), Lemma 2.1, and following the methods used to get (12), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n u + (1 - \alpha_n)(\theta_n x_n + (1 - \theta_n)Ty_n) - x^*\|^2 \\
&= \|\alpha_n(u - x^*) + (1 - \alpha_n)[\theta_n x_n + (1 - \theta_n)Ty_n - x^*]\|^2 \\
&\leq (1 - \alpha_n)\|\theta_n x_n + (1 - \theta_n)Ty_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n)\theta_n\|x_n - x^*\|^2 + (1 - \alpha_n)(1 - \theta_n)\|Ty_n - x^*\|^2 \\
&\quad - (1 - \alpha_n)\theta_n(1 - \theta_n)\|Ty_n - x_n\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)\theta_n\|x_n - x^*\|^2 \\
&\quad + (1 - \alpha_n)(1 - \theta_n)[\|y_n - x^*\|^2 + \|y_n - Ty_n\|^2] \\
&\quad - (1 - \alpha_n)\theta_n(1 - \theta_n)\|Ty_n - x_n\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&\leq (1 - \alpha_n)\theta_n\|x_n - x^*\|^2 + (1 - \alpha_n)(1 - \theta_n) \\
&\quad \times [\|x_n - x^*\|^2 + (1 - \lambda_n)^2\|x_n - Tx_n\|^2] + (1 - \alpha_n)(1 - \theta_n) \\
&\quad \times [\lambda_n\|x_n - Ty_n\|^2 - (1 - \lambda_n)(\lambda_n - L^2(1 - \lambda_n)^2)\|x_n - Tx_n\|^2] \\
&\quad - (1 - \alpha_n)\theta_n(1 - \theta_n)\|Ty_n - x_n\|^2 + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n)\|x_n - x^*\|^2 - (1 - \alpha_n)(1 - \theta_n)(1 - \lambda_n) \\
&\quad \times [1 - L^2(1 - \lambda_n)^2 - 2(1 - \lambda_n)]\|x_n - Tx_n\|^2 \\
&\quad + (1 - \alpha_n)(1 - \theta_n)(\lambda_n - \theta_n)\|x_n - Ty_n\|^2 \\
&\quad + 2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \qquad (16) \\
&\leq (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, x_n - x^*\rangle \\
&\quad + 2\alpha_n\|u - x^*\|\|x_{n+1} - x_n\|. \qquad (17)
\end{aligned}$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq n_0$. Then it follows that $\{\|x_n - x^*\|\}$ is convergent. Thus, from (16), (13), and (14), we have

$$x_n - Tx_n \to 0 \quad \text{as } n \to \infty. \tag{18}$$

Moreover, from (8) and (18), we obtain

$$\|y_n - x_n\| = (1 - \lambda_n)\|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty, \tag{19}$$

and hence the Lipschitz continuity of T, (19), and (18) imply that

$$\begin{aligned} \|Ty_n - x_n\| &\leq \|Ty_n - Tx_n\| + \|Tx_n - x_n\| \\ &\leq L\|y_n - x_n\| + \|Tx_n - x_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{20}$$

In addition, from (3.1) and (18), we obtain

$$\|x_{n+1} - x_n\| \leq \alpha_n\|u - x_n\| + (1 - \alpha_n)(1 - \theta_n)\|Ty_n - x_n\| \to 0. \tag{21}$$

Furthermore, since $\{x_n\}$ is a bounded subset of H which is reflexive, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \rightharpoonup w \quad \text{and} \quad \limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle.$$

Then from (18) and Lemma 2.4, we have $w \in F(T)$. Therefore, by Lemma 2.2, we immediately obtain

$$\begin{aligned} \limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle &= \lim_{i\to\infty}\langle u - x^*, x_{n_i} - x^*\rangle \\ &= \langle u - x^*, w - x^*\rangle \leq 0. \end{aligned} \tag{22}$$

Then it follows from (17), (22), and Lemma 2.3 that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Consequently, $x_n \to x^* = P_{F(T)}(u)$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|, \quad \forall i \in \mathbb{N}.$$

Then, by Lemma 2.5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \tag{23}$$

for all $k \in N$. Now, from (16), (13), and (14), it follows that $x_{m_k} - Tx_{m_k} \to 0$ as $k \to \infty$. Thus, like in Case 1, we obtain

$$\limsup_{k\to\infty}\langle u - x^*, x_{m_k} - x^*\rangle \leq 0. \tag{24}$$

Now, from (17), we have

$$\begin{aligned}\|x_{m_k+1}-x^*\|^2 &\le (1-\alpha_{m_k})\|x_{m_k}-x^*\|^2+2\alpha_{m_k}\langle u-x^*,x_{m_k}-x^*\rangle\\&\quad+2\alpha_{m_k}\|u-x^*\|\|x_{m_k+1}-x_{m_k}\|,\end{aligned}\tag{25}$$

and hence (23) and (25) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k}-x^*\|^2 &\le \|x_{m_k}-x^*\|^2-\|x_{m_k+1}-x^*\|^2+2\alpha_{m_k}\langle u-x^*,x_{m_k}-x^*\rangle\\&\quad+2\alpha_{m_k}\|u-x^*\|\|x_{m_k+1}-x_{m_k}\|\\&\le 2\alpha_{m_k}\langle u-x^*,x_{m_k}-x^*\rangle+2\alpha_{m_k}\|u-x^*\|\|x_{m_k+1}-x_{m_k}\|.\end{aligned}$$

Thus, using (21), (24), and the fact that $\alpha_{m_k}>0$, we obtain

$$\|x_{m_k}-x^*\|^2\le 0 \quad \text{and hence} \quad \|x_{m_k}-x^*\|\to 0 \quad \text{as } k\to\infty.$$

This together with (25) implies that $\|x_{m_k+1}-x^*\|\to 0$ as $k\to\infty$. But, since $\|x_k-x^*\|\le\|x_{m_k+1}-x^*\|$, for all $k\in\mathbb{N}$, it follows that $x_k\to x^*=P_{F(T)}(u)$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to the fixed point of T nearest to u. □

If, in Theorem 3.1, we assume that T is k-strictly pseudocontractive, then T is Lipschitz pseudocontractive with $L=\frac{1+k}{k}$, and hence we get the following corollary.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T:C\to H$ be a k-strictly pseudocontractive inward mapping with $F(T)\neq\emptyset$. Let $\beta\in(1-\frac{k}{k+\sqrt{(k+1)^2+k^2}},1)$ and $\{\alpha_n\}\subseteq(0,1)$ such that $\lim_{n\to\infty}\alpha_n=0$ and $\sum\alpha_n=\infty$. Let a sequence $\{x_n\}$ be generated from arbitrary $x_0,u\in C$ by*

$$\begin{cases}\lambda_n\in[\max\{\beta,h(x_n)\},1);\\ y_n=\lambda_n x_n+(1-\lambda_n)Tx_n;\\ \theta_n\in[\max\{\lambda_n,l(y_n)\},1);\\ x_{n+1}=\alpha_n u+(1-\alpha_n)(\theta_n x_n+(1-\theta_n)Ty_n),\end{cases}\tag{26}$$

where $h(x_n):=\inf\{\lambda\ge 0:\lambda x_n+(1-\lambda)Tx_n\in C\}$ and $l(y_n):=\inf\{\theta\ge 0:\theta x_n+(1-\theta)Ty_n\in C\}$.

If there exists $\epsilon>0$ such that $\theta_n\le 1-\epsilon$ $\forall n\ge 0$, then $\{x_n\}$ converges strongly to a fixed point of T nearest to u.

If, in Theorem 3.1, we assume that T is nonexpansive, then we have that T is Lipschitz pseudocontractive with $L=1$, and hence we get the following corollary.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T:C\to H$ be a nonexpansive inward mapping with $F(T)\neq\emptyset$. Let $\beta\in(2-\sqrt{2},1)$ and $\{\alpha_n\}\subseteq(0,1)$ such that $\lim_{n\to\infty}\alpha_n=0$ and $\sum\alpha_n=\infty$. Let a sequence $\{x_n\}$ be generated* *from arbitrary* $x_0, u \in C$ *by*

$$\begin{cases} \lambda_n \in [\max\{\beta, h(x_n)\}, 1); \\ y_n = \lambda_n x_n + (1-\lambda_n)Tx_n; \\ \theta_n \in [\max\{\lambda_n, l(y_n)\}, 1); \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\theta_n x_n + (1-\theta_n)Ty_n), \end{cases} \tag{27}$$

where $h(x_n) := \inf\{\lambda \geq 0 : \lambda x_n + (1-\lambda)Tx_n \in C\}$ *and* $l(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta)Ty_n \in C\}$.

If there exists $\epsilon > 0$ *such that* $\theta_n \leq 1-\epsilon$ $\forall n \geq 0$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T *nearest to* u.

We now state and prove a convergence result for a monotone mapping.

Corollary 3.4 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be an L-Lipschitz monotone inward mapping with* $N(A) \neq \emptyset$. *Let* $\beta \in (1 - \frac{1}{1+\sqrt{1+(1+L)^2}}, 1)$ *and* $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Let a sequence* $\{x_n\}$ *be generated from arbitrary* $x_0, u \in C$ *by*

$$\begin{cases} \lambda_n \in [\max\{\beta, h(x_n)\}, 1); \\ y_n = x_n - (1-\lambda_n)Ax_n; \\ \theta_n \in [\max\{\lambda_n, l(y_n)\}, 1); \\ x_{n+1} = \alpha_n u + (1-\alpha_n)(\theta_n x_n + (1-\theta_n)(I-A)y_n), \end{cases} \tag{28}$$

where $h(x_n) := \inf\{\lambda \geq 0 : x_n - (1-\lambda)Ax_n \in C\}$ *and* $l(y_n) := \inf\{\theta \geq 0 : \theta x_n + (1-\theta)(I-A)y_n \in C\}$.

If there exists $\epsilon > 0$ *such that* $\theta_n \leq 1-\epsilon$ $\forall n \geq 0$, *then* $\{x_n\}$ *converges strongly to the zero point of* A *nearest to* u.

Proof Let $Tx := (I-A)x$. Then T is a Lipschitz pseudocontractive mapping with Lipschitz constant $L' := (1+L)$ and $F(T) = N(A) \neq \emptyset$. Moreover, if A is replaced with $(I-T)$, then scheme (28) reduces to scheme (8), and hence the conclusion follows from Theorem 3.1. □

We observe that the method of proof of Theorem 3.1 provides the following result for approximating the minimum-norm point of fixed points of Lipschitz pseudocontractive non-self mappings.

Theorem 3.5 *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H *containing* 0, *and let* $T : C \to H$ *be an L-Lipschitz pseudocontractive inward mapping with* $F(T) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence defined by* (8) *with* $u = 0$. *If there exists* $\epsilon > 0$ *such that* $\theta_n \leq 1-\epsilon$ $\forall n \geq 0$, *then* $\{x_n\}$ *converges strongly to the minimum-norm point* x^* *of* $F(T)$.

Remark 3.6 Note that, in the above results, the coefficients λ_n and θ_n can be chosen simply as follows: $\lambda_n = \max\{\beta, h(x_n)\}$ and $\theta_n = \max\{\lambda_n, l(y_n)\}$.

Remark 3.7 If, in all the above theorems and corollaries, the set $F(T)$ is a subset of interior of C, then the assumption that there exists $\epsilon > 0$ such that $\theta_n \leq 1-\epsilon$ $\forall n \geq 0$ may not be required.

4 Numerical example

Now, we give an example of a nonlinear mapping which satisfies the conditions of Theorem 3.1.

Example 4.1 Let $H = \mathbb{R}$ with Euclidean norm. Let $C = [-1,1]$ and $T : C \to \mathbb{R}$ be defined by

$$Tx = \begin{cases} -3x, & x \in [-1,0], \\ x, & x \in (0,1]. \end{cases} \tag{29}$$

Then we observe that T satisfies the inward condition and $F(T) = [0,1]$. One can also easily verify that

$$\langle x - Tx - (y - Ty), x - y \rangle \geq 0, \quad \forall x, y \in C.$$

Thus, $I - T$ is monotone and hence T is a pseudocontractive mapping. To show that T is a Lipschitz mapping, we consider the following cases.

Case 1: Let $x, y \in [-1,0]$. Then we have

$$|Tx - Ty| = |-3x + 3y| = 3|x - y|.$$

Case 2: Let $x, y \in (0,1]$. Then we have

$$|Tx - Ty| = |x - y|.$$

Case 3: Let $x \in [-1,0]$ and $y \in (0,1]$. Then we have

$$\begin{aligned} |Tx - Ty| &= |-3x - y| \\ &= |3x + y| \\ &= |x - y + 2x + 2y| \\ &\leq |x - y| + 2|x + y| \\ &\leq |x - y| + 2|x - y| \\ &= 3|x - y|. \end{aligned}$$

From the above cases, it follows that T is L-Lipschitz with $L = 3$.

Now, let $\beta = \frac{5}{6}, u = \frac{1}{2}, x_0 = -1$, and $\alpha_n = \frac{2}{n+5}$. Then $Tx_0 = 3$ and

$$\begin{aligned} h(x_0) &= \inf\{\lambda \geq 0 : \lambda x_0 + (1 - \lambda)Tx_0 \in C\} \\ &= \inf\{\lambda \geq 0 : -\lambda + 3(1 - \lambda) \in C\} \\ &= \frac{1}{2}. \end{aligned}$$

Now, let $\lambda_0 = \frac{5}{6}$. Then $y_0 = \lambda_0 x_0 + (1 - \lambda_0)Tx_0 = -\frac{1}{3}$ and $Ty_0 = 1$, which gives

$$l(x_0) = \inf\{\theta \geq 0 : \theta x_0 + (1 - \theta)Ty_0 \in C\} = 0.$$

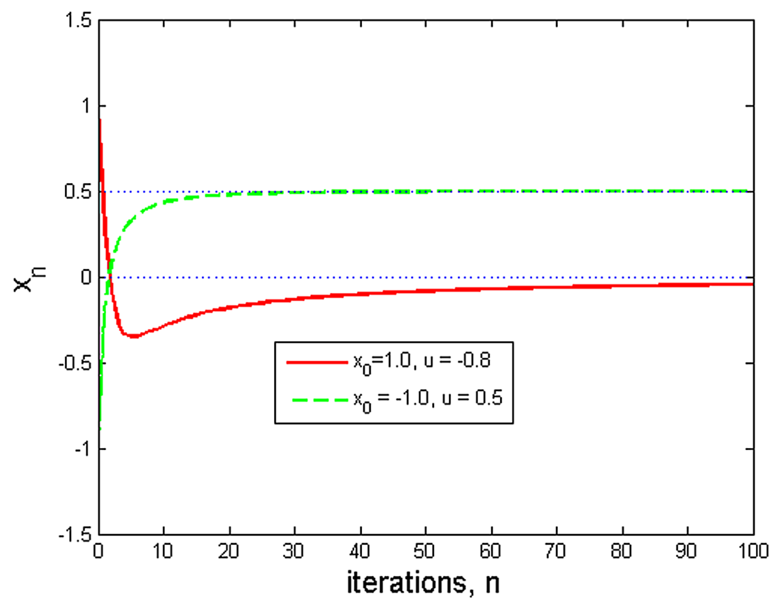

Figure 1 Convergence of x_n with different values of x_0 and u

If we choose $\theta_0 = \frac{5}{6}$, then we have

$$x_1 = \alpha_0 u + (1-\alpha_0)\big[\theta_0 x_0 + (1-\theta_0)Ty_0\big] = -\frac{1}{5}.$$

Thus, $Tx_1 = \frac{3}{5}$, which implies that $h(x_1) = 0$. Now, if we choose $\lambda_1 = \frac{5}{6}$, then we obtain

$$y_1 = \lambda_1 x_1 + (1-\lambda_1)Tx_1 = -\frac{1}{15}, \qquad Ty_1 = \frac{1}{5} \quad \text{and} \quad l(y_1) = 0.$$

Again, we can choose $\theta_1 = \frac{5}{6}$, which yields $x_2 = 0.0778$. In general, we observe that for $u = 0.5, x_0 = -1$ and $\alpha_n = \frac{2}{n+5}$, we can choose $\lambda_n = \theta_n = \frac{5}{6}$. Thus, all the conditions of Theorem 3.1 are satisfied and x_n converges to $0.5 = P_{F(T)}u$ (see Fig. 1).

On the other hand, for $u = -0.8, x_0 = 1$, and $\alpha_n = \frac{2}{n+5}$, we obtain that x_n converges to $0.0 = P_{F(T)}u$. Figure 1 is obtained using MATLAB version 7.5.0.342(R2007b).

5 Conclusion

In this paper, we have constructed and studied a Halpern–Ishikawa type iterative scheme for non-self mappings in the setting of Hilbert spaces. As a result, we obtained strong convergence of the scheme to a fixed point of a Lipschitz pseudocontractive non-self mapping under some mild conditions. In addition, we provided a numerical example to support our results. Our study can open the door for further research activity in the field for a more general class of mappings in Hilbert and/or Banach spaces more general than Hilbert spaces. Our results extend and generalize many results in the literature. More particularly, Theorem 3.1 extends Theorem 8 of Colao et al. [5] in the sense that it provides a convergent scheme for approximating fixed points of Lipschitz pseudocontractive non-self mappings more general than that of k-strictly pseudocontractive non-self mappings.

Acknowledgements
The authors appreciate the support of their institutes.

Funding
The second author is supported by the International Mathematical Union (IMU) Breakout Graduate Fellowship Program through The World Academy of Sciences (TWAS).

Abbreviations
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing the article. Both authors read and approved the final manuscript.

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Received: 26 October 2017 Accepted: 12 April 2018 Published online: 04 June 2018

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