Lael and Heidarpour *Fixed Point Theory and Applications* (2016) 2016:82
DOI 10.1186/s13663-016-0571-y

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for a class of generalized nonexpansive mappings

Fatemeh Lael[1]* and Zohre Heidarpour[2]

*Correspondence:
fatemehlael@yahoo.com
[1]Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce a new class of generalized nonexpansive mappings. Some new fixed point theorems for these mappings are obtained.

MSC: 47H10

Keywords: monotone mapping; nonexpansive mapping; fixed point; L_p

1 Introduction and preliminaries

A nonexpansive mapping has a Lipschitz constant equal to 1. The fixed point theory for such mappings is very rich [1–5] and has many applications in nonlinear functional analysis [6].

We first commence some basic concepts about generalization of nonexpansive mappings as formulated by Suzuki *et al.* [7, 8].

Definition 1 [8] Let C be a nonempty subset of a Banach space X. We say that a mapping $T : C \to C$ satisfies condition (C) on C if $\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$ implies $\|T(x) - T(y)\| \leq \|x - y\|$, for $x, y \in C$.

Of course, every nonexpansive mapping satisfies condition (C) but the converse is not correct and you can find some counterexamples for it in [8]. So the class of mappings which has condition (C) is broader than the class of nonexpansive mappings.

In [7], condition (C) is generalized as follows.

Definition 2 [7] Let C be a nonempty subset of a Banach space X and $\lambda \in (0,1)$. We say that a mapping $T : C \to X$ satisfies (C_λ)-condition on C if $\lambda\|x - T(x)\| \leq \|x - y\|$ implies $\|T(x) - T(y)\| \leq \|x - y\|$, for $x, y \in C$.

So if $\lambda = \frac{1}{2}$, we will have condition (C). There are examples that show the converse is false; see [7].

In [9], monotone nonexpansive mappings are defined in $L_1[0,1]$.

We next review some notions in $L_p[0,1]$. All of them can be found in [10].

Consider the Riesz Banach space $L_p[0,1]$, where $\int_0^1 |f(x)|^p\,dx < +\infty$ and $p \in (0, +\infty)$. Also, we have $f = 0$ when the set

$$\{x \in [0,1] : f(x) = 0\},$$

has Lebesgue measure zero. In this case, we say $f = 0$ almost everywhere. An element of $L_p[0,1]$ is therefore seen as a class of functions. The norm of any $f \in L_p[0,1]$ is given by $\|f\|_p = (\int_0^1 |f(x)|^p \, dx)^{\frac{1}{p}}$. Throughout this paper, we will write L_p instead of $L_p[a,b]$, $a, b \in \mathbb{R}$ and $\|\cdot\|$ instead of $\|\cdot\|_p$.

In this paper, we redefine Definition 2 on a subset of Banach space L_p and those theorems which are proved in [9] generalize to a wider class of monotone (C_λ)-condition with preserving their fixed point property.

2 Main results

Let C be a nonempty subset of L_p which is equipped with a vector order relation $\preceq$. A map $T : C \to C$ is called monotone if for all $f \preceq g$ we have $T(f) \preceq T(g)$.

We generalize the (C_λ)-condition as follows.

Definition 3 Let C be a nonempty subset of a Banach space L_p. For $\lambda \in (0,1)$, we say that a mapping T monotone (C_λ)-condition on C if T is monotone and for all $f \preceq g$, $\lambda\|f - T(f)\| \le \|g - f\|$ implies $\|T(g) - T(f)\| \le \|g - f\|$.

Note Definition 3 is a generalization of the monotone nonexpansive mapping which is defined in [9] as follows.

A map T is said to be monotone nonexpansive if T is monotone and for $f \preceq g$, we have $\|T(g) - T(f)\| \le \|g - f\|$.

The next example is a direct generalization of monotone nonexpansive mapping.

Example 1 Let $C = \{f \in L_p[0,3] : f(x) = a\}$, where $a \in [0,3]$. For $f, g \in C$, consider the partial order relation

$$f \preceq g \quad \text{iff} \quad f(x) \le g(x).$$

Let $T : C \to C$ be defined by

$$T(f) = \begin{cases} 1, & f = 3, \\ 0, & f \neq 3. \end{cases}$$

Then the mapping T satisfies the monotone $(C_{\frac{1}{2}})$-condition but it fails monotone nonexpansiveness. Indeed, whenever $f \preceq g$, if $0 \le f(x) \le g(x) < 3$, then $\|T(f) - T(g)\| \le \|f - g\|$. On the other hand, $0 \le f(x) < 3$ and $g = 3$, so if $0 \le f(x) \le 2$ and $g = 3$, then we have again $\|T(f) - T(g)\| \le \|f - g\|$, but if $2 < f(x) < 3$ and $g = 3$, then $\frac{1}{2}\|f\| \nleq \|f - 3\|$. Thus, the mapping T satisfying monotone $(C_{\frac{1}{2}})$-condition on $[0,3]$.

Let $f = 2.9$ and $g = 3$. Then $f \preceq g$ while $\|T(f) - T(g)\| \nleq \|f - g\|$. Thus, T is not monotone nonexpansive.

The following lemmas will be crucial to prove the main result of this paper.

Lemma 1 *Let C be convex and T monotone. Assume that for some $f_1 \in C$, $f_1 \preceq T(f_1)$. Then the sequence f_n defined by*

$$(\star) \qquad f_{n+1} = \lambda T(f_n) + (1 - \lambda) f_n,$$

$\lambda \in (0,1)$, *satisfies*

$$f_n \preceq f_{n+1} \preceq T(f_n) \preceq T(f_{n+1}).$$

for $n \geq 1$.

Proof First, we prove that $f_n \preceq T(f_n)$. By assumption, we have $f_1 \preceq T(f_1)$. Assume that $f_n \preceq T(f_n)$, for $n \geq 1$. Then we have

$$f_n = \lambda f_n + (1-\lambda)f_n \preceq \lambda T(f_n) + (1-\lambda)f_n = f_{n+1}$$

i.e. $f_n \preceq f_{n+1}$. Since T is monotone, $T(f_n) \preceq T(f_{n+1})$. We have

$$f_{n+1} = \lambda T(f_n) + (1-\lambda)f_n \preceq \lambda T(f_n) + (1-\lambda)T(f_n) = T(f_n).$$

Thus

$$f_n \preceq f_{n+1} \preceq T(f_n) \preceq T(f_{n+1}),$$

for $n \geq 1$. The proof is closely modeled on Lemma 3.1 of [9]. □

Note that under the assumption of Lemma 1, if we assume $T(f_1) \preceq f_1$, then we have

$$T(f_{n+1}) \preceq T(f_n) \preceq f_{n+1} \preceq f_n$$

for any $n \geq 1$.

A sequence $\{f_n\}$ in C is called an almost fixed point sequence for T, if $\|f_n - T(f_n)\| \to 0$ (a.f.p.s. in short).

Lemma 2 *Let* $T : C \to L_p$ *be a monotone* (C_λ)*-condition mapping and* f_n *be a bounded a.f.p.s. for* T. *Then*

$$\liminf_n \big\| f_n - T(f) \big\| \leq \liminf_{n_k} \|f_n - f\|,$$

for $f \in C$ *which* $f_n \preceq f$ *and* $\liminf_n \|f_n - f\| > 0$, *for all* $n \geq 1$.

Proof Fix $f \in C$ such that $f_n \preceq f$. Since f_n is an a.f.p.s., for $\epsilon = \frac{1}{2} \liminf_n \|f_n - f\|$, there is n_0 such that $\|f_n - T(f_n)\| < \epsilon$, for all $n \geq n_0$. This implies that

$$\lambda \big\| f_n - T(f_n) \big\| \leq \big\| f_n - T(f_n) \big\| < \epsilon < \|f_n - f\|,$$

for all $n \geq n_0$. Since T satisfies the monotone (C_λ)-condition, we have

$$\big\| T(f_n) - T(f) \big\| \leq \|f_n - f\|, \tag{1}$$

for all $n \geq n_0$. So by the triangle inequality and (1), we have

$$\big\| f_n - T(f) \big\| \leq \big\| f_n - T(f_n) \big\| + \big\| T(f_n) - T(f) \big\| \leq \big\| f_n - T(f_n) \big\| + \|f_n - f\|.$$

Thus $\liminf_n \|f_n - T(f)\| \leq \liminf_n \|f_n - f\|$. The proof is closely modeled on Lemma 1 of [7]. □

Lemma 3 [11] *If $\{f_n\}$ is a sequence of L_p-uniformly bounded functions on a measure space, and $f_n \to f$ almost everywhere, then*

$$\liminf_n \|f_n\|^p = \liminf_n \|f_n - f\|^p + \|f\|^p,$$

for all $p \in (0, \infty)$.

In the following, let C be a nonempty, convex, and bounded set and $T : C \to C$ be a monotone (C_λ)-condition, for some $\lambda \in (0,1)$.

Theorem 1 *Let $f_1 \in C$ such that $f_1 \preceq T(f_1)$. Then f_n defined in $(\star)$ is an a.f.p.s.*

Proof Since $f_{n+1} = \lambda T(f_n) + (1-\lambda)f_n$, for $n \geq 1$, we have

$$\lambda \|f_n - T(f_n)\| = \|f_n - f_{n+1}\|.$$

By Lemma 1, we have $f_n \preceq f_{n+1}$. Therefore, monotone (C_λ)-condition implies that $\|T(f_n) - T(f_{n+1})\| \leq \|f_n - f_{n+1}\|$. Now, we can apply Lemma 3 of [1] to conclude that $\lim_n \|f_n - T(f_n)\| = 0$. □

Example 2 We show that T, which is defined in Example 1, has an a.f.p.s. It is easy to see that C is a nonempty, convex, and bounded subset of L_p. Also, we proved T obeys the monotone $(C_{\frac{1}{2}})$-condition. Moreover, $0 \preceq T(0)$. Thus, by Theorem 1, T has an a.f.p.s.

Now, we construct an a.f.p.s. according $(\star)$. Let $f_1 = 0$. So $f_n = 0$. Therefore

$$\|f_n - T(f_n)\| = 0.$$

Thus f_n is an a.f.p.s.

Theorem 2 *Let C be compact. Assume there exists $f_1 \in C$ such that f_1 and $T(f_1)$ are comparable. Then T has a fixed point.*

Proof Let f_n be a sequence which is defined in $(\star)$. By Theorem 1, f_n is an a.f.p.s. Since C is compact, f_n has a convergent subsequence f_{n_k} to f. By triangle inequality, we get

$$\liminf_{n_k} \|T(f_{n_k}) - T(f)\| \leq \lim_{n_k} \|T(f_{n_k}) - f_{n_k}\| + \liminf_{n_k} \|f_{n_k} - T(f)\|.$$

Since f_n is an a.f.p.s., we have

$$\liminf_{n_k} \|T(f_{n_k}) - T(f)\| \leq \liminf_{n_k} \|f_{n_k} - T(f)\|. \tag{2}$$

Again, by triangle inequality, we have

$$\liminf_{n_k} \|f_{n_k} - T(f)\| \leq \lim_{n_k} \|f_{n_k} - T(f_{n_k})\| + \liminf_{n_k} \|T(f) - T(f_{n_k})\|.$$

Therefore,

$$\liminf_{n_k} \|f_{n_k} - T(f)\| \leq \liminf_{n_k} \|T(f_{n_k}) - T(f)\|. \tag{3}$$

From equations (2) and (3), we have

$$\liminf_{n_k} \|f_{n_k} - T(f)\| = \liminf_{n_k} \|T(f_{n_k}) - T(f)\|. \tag{4}$$

By using the partially order and convergent properties $f_{n_k} \preceq f$. Lemma 1 implies $f_{n_k} \preceq f_{n_k+1} \preceq f$. So $\|f_{n_k+1} - f_{n_k}\| \leq \|f - f_{n_k}\|$. Since $f_{n_k+1} - f_{n_k} = \lambda(f_{n_k} - T(f_{n_k}))$, we get

$$\lambda \|f_{n_k} - T(f_{n_k})\| = \|f_{n_k+1} - f_{n_k}\|.$$

Therefore

$$\lambda \|(f_{n_k} - T(f_{n_k}))\| \leq \|f - f_{n_k}\|.$$

Thus the monotone (C_λ)-condition implies

$$\|T(f_{n_k}) - T(f)\| \leq \|f_{n_k} - f\|. \tag{5}$$

Since f_{n_k} is bounded, Lemma 3 implies

$$\liminf_{n_k} \|f_{n_k} - T(f)\| = \liminf_{n_k} \|f_{n_k} - f\| + \|f - T(f)\|.$$

From equation (4), we get

$$\liminf_{n_k} \|f_{n_k} - f\| + \|f - T(f)\| = \liminf_{n_k} \|T(f_{n_k}) - T(f)\|.$$

From equation (5), we get

$$\liminf_{n_k} \|f_{n_k} - f\| + \|f - T(f)\| \leq \liminf_{n_k} \|f_{n_k} - f\|.$$

This implies that $T(f) = f$. □

By Theorem 2, we can see that T in Example 1, has a fixed point.

The following example shows that monotone (C_λ)-condition is a direct generalization of (C_λ)-condition.

Example 3 Let $C = co\{x, \sin(x)\}$, where $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Define a partial order on C as follows:

$$f \preceq g \quad \text{iff} \quad f(x) \leq g(x).$$

Let $T : C \to C$ be

$$T(f) = \begin{cases} \sin(x) & f \neq x, \\ x & f = x. \end{cases}$$

Since C is convex hull of a compact set $\{x, \sin(x)\}$, so it is a nonempty, convex and compact subset of L_p. Put $f = x$. Then f and $T(f)$ are comparable. Also, T obeys the monotone (C_λ)-condition. Thus, by Theorem 2, T has a fixed point.

Note, for $\lambda \in (0,1)$, T does not obey the (C_λ)-condition. Because, for $f = x$ and $g = \frac{x}{2} + \frac{1}{2}\sin(x)$, we have $\lambda\|f - T(f)\| \leq \|f - g\|$, but $\|T(g) - T(f)\| \nleq \|f - g\|$.

Theorem 3 *Let C be a weakly compact subset of L_2. Assume, there is $f_1 \in C$ such that $f_1 \preceq T(f_1)$. Then T has a fixed point.*

Proof By Theorem 1, T has an a.f.p.s. f_n. Since C is weakly compact, there is a weakly convergent subsequence f_{n_k} to some $f \in C$. If $\liminf_{n_k} \|f_{n_k} - f\| = 0$, then f_{n_k} is convergent and we will have the same proof of Theorem 2. On the other hand, if $\liminf_{n_k} \|f_{n_k} - f\| > 0$, then by Lemma 2,

$$\liminf_{n_k} \|f_{n_k} - T(f)\| \leq \liminf_{n_k} \|f_{n_k} - f\|. \tag{6}$$

We claim that $f = T(f)$. Because if $f \neq T(f)$, since L_2 satisfies Opial condition, we have

$$\liminf_{n_k} \|f_{n_k} - f\| < \liminf_{n_k} \|f_{n_k} - T(f)\|,$$

which is a contradiction with inequality (6). □

This result is a generalization of the original existence theorem in [7, 9] form monotone nonexpansive to monotone (C_λ)-condition. Therefore this class is bigger and is used to answer the question asked by T Benavides [12]: Does X also satisfy the fixed point property for Suzuki-type mappings?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran. [2]Department of Mathematics, Payame Noor University, Tehran, 19395-3697, Iran.

Acknowledgements
The first author acknowledges Buein Zahra Technical University for supporting this research.

Received: 13 February 2016 Accepted: 20 July 2016 Published online: 01 August 2016

References

1. Goebel, K, Kirk, WA: Iteration processes for nonexpansive mappings. Contemp. Math. **21**, 115-123 (1983)
2. Ishikawa, S: Fixed points and iteration of a nonexpansive mapping in a Banach space. Proc. Am. Math. Soc. **59**, 65-71 (1976)
3. Khan, AR, Hussain, N: Iterative approximation of fixed points of nonexpansive maps. Sci. Math. Jpn. **54**, 503-511 (2001)
4. Kirk, WA: Fixed point theory for nonexpansive mappings. In: Fixed Point Theory. Lecture Notes in Mathematics, vol. 886, pp. 485-505 (1981)
5. Opial, Z: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 595-597 (1967)
6. Browder, FE: Nonexpansive nonlinear operators in a Banach space. Proc. Natl. Acad. Sci. USA **54**, 1041-1044 (1965)
7. Falset, JG, Fuster, EL, Suzuki, T: Fixed point theory for a class of generalized nonexpansive mappings. J. Math. Anal. Appl. **375**, 185-195 (2011)
8. Suzuki, T: Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. **340**, 1088-1095 (2008)

9. Khamsi, MA, Khan, AR: On monotone nonexpansive mappings in $L_1([0,1])$. Fixed Point Theory Appl. **2015**, Article ID 94 (2015). doi:10.1186/s13663-015-0346-x
10. Beauzamy, B: Introduction to Banach Spaces and Their Geometry. North-Holland, Amsterdam (1985)
11. Brezis, H, Lieb, E: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486-490 (1983)
12. Benavides, TD, Medina Perez, B: The fixed point property for some generalized nonexpansive mappings and renormings. J. Math. Anal. Appl. **429**, 800-813 (2015)