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On the construction of metrics from fuzzy metrics and its application to the fixed point theory of multivalued mappings

Francisco Castro-Company[1], Salvador Romaguera[2] and Pedro Tirado[2*]

*Correspondence:
pedtipe@mat.upv.es
[2]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, València, 46022, Spain
Full list of author information is available at the end of the article

Abstract
We present a procedure to construct a compatible metric from a given fuzzy metric space. We use this approach to obtain a characterization of a large class of complete fuzzy metric spaces by means of a fuzzy version of Caristi's fixed point theorem, obtaining, in this way, partial solutions to a recent question posed in the literature. Some illustrative examples are also given.

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1 Introduction and preliminaries

Throughout this paper the symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all positive integers, respectively. Our basic reference for general topology is [1].

We start by recalling the notion of a continuous t-norm as well as some types of continuous t-norm which will be crucial throughout this paper.

According to [2], a binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $*$ is continuous; (iii) $a * 1 = a$ for every $a \in [0,1]$; (iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Distinguished examples of continuous t-norm are $\wedge$, Prod, and $*_L$ (the Łukasiewicz t-norm), which are defined as follows: $a \wedge b = \min\{a, b\}$, $a \operatorname{Prod} b = ab$, and $a *_L b = \max\{a + b - 1, 0\}$ for all $a, b \in [0,1]$.

The following well-known relations hold: $\wedge \geq \text{Prod} \geq *_L$. In fact, $\wedge \geq *$ for any continuous t-norm $*$.

Two important classes of continuous are the so-called Yager continuous t-norms and Hamacher continuous t-norms, which are constructed as follows.

(A) Given $p > 0$, define for each $a, b \in [0,1]$:

$$a *_{Y_p} b = 1 - \min\{1, [(1-a)^p + (1-b)^p]^{1/p}\}.$$

Then $*_{Y_p}$ is a continuous t-norm referred to in the literature as the Yager continuous t-norm (see *e.g.* [3]).

It is easy to see that $a *_{Y_{p_1}} b \geq a *_{Y_{p_2}} b$ whenever $p_1 \geq p_2$, with $a, b \in [0,1]$. In particular we see that $*_{Y_1}$ coincides with the Łukasiewicz t-norm.

(B) Hamacher constructed in [4] a family of continuous t-norms $(*_{H_\lambda})_{\lambda \in \mathbb{R}^+}$ defined as follows. Given $\lambda \in \mathbb{R}$, for each $a, b \in [0,1]$, $a *_{H_\lambda} b = 0$ if $a = b = \lambda = 0$, and

$$a *_{H_\lambda} b = \frac{ab}{\lambda + (1-\lambda)(a+b-ab)},$$

otherwise. Furthermore, we have $a *_{H_{\lambda_1}} b \leq a *_{H_{\lambda_2}} b$ whenever $\lambda_1 \geq \lambda_2$, with $a, b \in [0,1]$. In particular we see that $*_{H_1}$ coincides with the product t-norm.

In this paper we shall work with fuzzy metric spaces in the sense of Kramosil and Michalek [5] (see Definition 1 below). At this point it seems suitable to remark that George and Veeramani introduced in [6] an interesting modification of Kramosil and Michalek's notion. However, from the well-known fact that every fuzzy metric space $(X, M, *)$ in the sense of George and Veeramani can be considered as a fuzzy metric space in the sense of Kramosil and Michalek, simply putting $M(x, y, 0) = 0$ for all $x, y \in X$, we deduce that the obtained results in this paper remain valid for fuzzy metric spaces in George and Veeramani's sense.

Definition 1 (Kramosil and Michalek [5]) A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function from $X \times X \times \mathbb{R}^+$ to $[0,1]$ such that for all $x, y, z \in X$:

(FM1) $M(x, y, 0) = 0$;
(FM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(FM3) $M(x, y, t) = M(y, x, t)$;
(FM4) $M(x, z, t+s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
(FM5) $M(x, y, \_) : \mathbb{R}^+ \to [0,1]$ is left continuous.

By a fuzzy metric space we mean a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X.

It is well known that for each $x, y \in X$, $M(x, y, \_)$ is a non-decreasing function on $\mathbb{R}^+$.

Each fuzzy metric $(M, *)$ on a set X induces a topology τ_M on X which has as a base the family of open sets $\{B_M(x, \varepsilon, t) : \varepsilon \in (0,1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $\varepsilon \in (0,1)$, $t > 0$.

A Cauchy sequence in a fuzzy metric space $(X, M, *)$ is a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that for each $\varepsilon \in (0,1)$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ satisfying $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq n_0$.

A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ converges with respect to the topology τ_M, *i.e.*, if there exists $y \in X$ such that for each $t > 0$, $\lim_n M(y, x_n, t) = 1$ (see *e.g.* [6]).

It is well known (see *e.g.* [7]) that every fuzzy metric space is metrizable, *i.e.*, given a fuzzy metric space $(X, M, *)$ there exists a metric on X whose induced topology coincides with the topology τ_M. A short and easy proof of this result consists in showing that the countable family $\{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$ for all $n \in \mathbb{N}$, is a base for a uniformity on X whose induced topology coincides with the topology τ_M, and then to apply the famous Kelley metrization lemma [1], p.185.

However, the important problem of obtaining a general procedure to construct a visual and manageable compatible metric for any fuzzy metric space, in such a way that the fixed point theory for fuzzy metric spaces could be deduced from the classical fixed point theory for metric spaces, remains unsolved.

Radu obtained some partial but interesting solutions to this problem. Thus, he proved in [8] the following theorem.

Theorem 1 [8] *Let $(X, M, *)$ be a fuzzy metric space such that $* \geq *_L$. For each $x, y \in X$ put*

$$d_R(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1 - t\}.$$

Then d_R is a metric on X such that

$$d_R(x,y) < \varepsilon \quad \Longleftrightarrow \quad M(x,y,\varepsilon) > 1 - \varepsilon,$$

for all $\varepsilon \in (0,1)$. Therefore, the uniformities, and hence the topologies, induced by $(M,*)$ and d_R coincide on X. In particular, $(X,M,*)$ is complete if and only if (X,d_R) is complete.

Theorem 1 was successfully applied (see *e.g.* [9–15]) to deduce several fixed point theorems for complete fuzzy metric spaces from the corresponding results for complete metric spaces. See also [16–18] and the references therein, for some recent contributions to the fixed point theory in fuzzy metric spaces and related structures.

Hicks [19] generalized Theorem 1, by replacing the condition $* \geq *_L$ with the following more general condition:

$$M(x,y,t) > 1 - t, \qquad M(y,z,s) > 1 - s \quad \Longrightarrow \quad M(x,z,t+s) > 1 - (t+s).$$

Later on, Radu [20], Theorem 2.1.7, obtained a substantial improvement of Hicks' result, which is established below in a slightly different form.

Theorem 2 [20] *Let $(X, M, *)$ be a fuzzy metric space. Suppose that there exists a function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions*:

(R1) *μ is continuous on $\mathbb{R}^+$*;

(R2) $\mu(t) = 0 \Longleftrightarrow t = 0$;

(R3) *$\mu(t+s) \geq \mu(t) + \mu(s)$ for all $t, s \geq 0$*;

(R4) $M(x,y,t) > 1 - \mu(t), M(y,z,s) > 1 - \mu(s) \Longrightarrow M(x,z,t+s) > 1 - \mu(t+s)$.

Then the function $d_{R\mu} : X \times X \to \mathbb{R}^+$ defined as

$$d_{R\mu}(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1 - \mu(t)\},$$

is a metric on X such that

$$d_{R\mu}(x,y) < \varepsilon \quad \Longleftrightarrow \quad M(x,y,\varepsilon) > 1 - \mu(\varepsilon),$$

for all $\varepsilon \in (0,1)$. Thus the uniformities, and hence the topologies, induced by $(M,*)$ and $d_{R\mu}$ coincide on X. Moreover, $(X,M,*)$ is complete if and only if $(X,d_{R\mu})$ is complete.

Here, we shall present a modification of Theorem 2 which can be applied to certain cases and instances where Radu's theorem does not work. From our approach we deduce a fixed point theorem of Caristi type for multivalued mappings which is valid for any complete fuzzy metric space $(X, M, *)$, as well as a characterization of those complete fuzzy metric spaces $(X, M, *)$ with $* \geq *_{Y_p}$, $p > 0$, in terms of Caristi's fixed point theorem. These results provide partial solutions to a question posed in [9]. Several illustrative examples are also given.

2 Constructing metrics from fuzzy metrics

We start this section by establishing our promised modification of Theorem 2 above (a background in this direction may be found in [21], Lemma 1.7).

Theorem 3 *Let $(X, M, *)$ be a fuzzy metric space. Suppose that there exists a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:*

(c1) *α is strictly increasing on $[0, 1]$;*

(c2) *$0 < \alpha(t) \leq t$ for all $t \in (0, 1)$ and $\alpha(t) > 1$ for all $t > 1$;*

(c3) *$(1 - \alpha(t)) * (1 - \alpha(s)) \geq 1 - \alpha(t + s)$ for all $t, s \in [0, 1]$.*

Then the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$$d_\alpha(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - \alpha(t)\},$$

is a metric on X such that $d_\alpha(x, y) \leq 1$ for all $x, y \in X$.

If, in addition, the function α is left continuous on $(0, 1]$, then

$$d_\alpha(x, y) < \varepsilon \iff M(x, y, \varepsilon) > 1 - \alpha(\varepsilon), \tag{1}$$

for all $\varepsilon \in (0, 1)$. Thus the uniformities, and hence the topologies, induced by $(M, *)$ and d_α coincide on X. Moreover, $(X, M, *)$ is complete if and only if (X, d_α) is complete.

Proof We first note that $\alpha(0) = 0$ because, by (c1) and (c2), $\alpha(0) < \alpha(t) \leq t$ for all $t \in (0, 1)$.

Now we prove that d_α is a metric on X such that $d_\alpha(x, y) \leq 1$ for all $x, y \in X$.

Let $x, y \in X$. Since $M(x, y, 0) = 0 < 1 - \alpha(0)$ we deduce that $d_\alpha(x, y) \geq 0$. Moreover, $d_\alpha(x, y) \leq 1$ because, by (c2), $\alpha(t) > 1$ for all $t > 1$.

We also have $d_\alpha(x, x) = 0$ because for each $t > 0$, $M(x, x, t) = 1 > 1 - \alpha(t)$.

Next we show that $x = y$ whenever $d_\alpha(x, y) = 0$. Indeed, suppose that $d_\alpha(x, y) = 0$. Then $M(x, y, t) > 1 - \alpha(t)$ for all $t > 0$. Choose an arbitrary $s > 0$. Then, for every $t \in (0, 1)$ with $t < s$, we obtain

$$M(x, y, s) \geq M(x, y, t) > 1 - \alpha(t) \geq 1 - t,$$

so that $M(x, y, s) = 1$. Since s is arbitrary, we deduce that $x = y$.

Furthermore, we have $d_\alpha(x, y) = d_\alpha(y, x)$ because $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

It remains to prove that d_α satisfies the triangle inequality. To this end, let $x, y, z \in X$. If $d_\alpha(x, z) + d_\alpha(z, y) \geq 1$ we immediately obtain $d_\alpha(x, y) \leq d_\alpha(x, z) + d_\alpha(z, y)$, because $d_\alpha(x, y) \leq 1$.

Hence, we assume, without loss of generality, that $d_\alpha(x, z) + d_\alpha(z, y) < 1$. In this case, we shall use the following relation for any $a \in (0, 1)$:

$$M(x, y, a) \geq 1 - \alpha(a) \implies d_\alpha(x, y) \leq a. \tag{2}$$

(To show (2) suppose that $d_\alpha(x,y) > a$. Then there exists $t \in (a,1]$ such that $M(x,y,t) \leq 1-\alpha(t)$. Since $t > a$, we deduce that $\alpha(t) > \alpha(a)$, and thus, $1-\alpha(a) \leq M(x,y,a) \leq M(x,y,t) \leq 1-\alpha(t)$, so $\alpha(t) \leq \alpha(a)$, a contradiction.)

Choose an arbitrary $\varepsilon > 0$ such that $d_\alpha(x,z) + d_\alpha(z,y) + 2\varepsilon < 1$. Then, from the definition of d_α and condition (c3), we deduce

$$\begin{aligned} M\bigl(x,y,d_\alpha(x,z)+d_\alpha(z,y)+2\varepsilon\bigr) &\geq M\bigl(x,z,d_\alpha(x,z)+\varepsilon\bigr) * M\bigl(z,y,d_\alpha(z,y)+\varepsilon\bigr) \\ &\geq \bigl(1-\alpha\bigl(d_\alpha(x,z)+\varepsilon\bigr)\bigr) * \bigl(1-\alpha\bigl(d_\alpha(z,y)+\varepsilon\bigr)\bigr) \\ &\geq 1-\alpha\bigl(d_\alpha(x,z)+d_\alpha(z,y)+2\varepsilon\bigr). \end{aligned}$$

It follows from (2) that $d_\alpha(x,y) \leq d_\alpha(x,z) + d_\alpha(z,y) + 2\varepsilon$. Since ε is arbitrary, we conclude that $d_\alpha(x,y) \leq d_\alpha(x,z) + d_\alpha(z,y)$.

We have proved that d_α is a metric on X.

Now suppose that α is left continuous on $(0,1]$. If $d_\alpha(x,y) < \varepsilon$, with $\varepsilon \in (0,1)$, then $M(x,y,t) > 1-\alpha(\varepsilon)$ by the definition of d_α. Conversely, if $M(x,y,\varepsilon) > 1-\alpha(\varepsilon)$, then $d_\alpha(x,y) \leq \varepsilon$ by the definition of d_α. In that case, if $d_\alpha(x,y) = \varepsilon$, left continuity of $M(x,y,\_)$ and of α at ε, provide a contradiction. So $d_\alpha(x,y) < \varepsilon$, and thus we have shown the equivalence (1).

From this equivalence it immediately follows that the uniformities, and hence, the topologies induced by $(M,*)$ and d_α coincide. In particular, a sequence in X is a Cauchy sequence in $(X,M,*)$ if and only if it is a Cauchy sequence in (X,d_α). Consequently $(X,M,*)$ is complete if and only if (X,d_α) is complete. This concludes the proof. □

Example 1 Let $(X,M,*)$ be a fuzzy metric space such that $* \geq *_{Y_p}$ for some $p \geq 1$. Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\alpha(t) = t^p$ for all $t \in \mathbb{R}^+$. Clearly, α is continuous and strictly increasing on $\mathbb{R}^+$, and satisfies conditions (c2) and (c3) of Theorem 3. Hence, the function $d : X \times X \to \mathbb{R}^+$ defined as

$$d(x,y) = \sup\bigl\{t \geq 0 : M(x,y,t) \leq 1-t^p\bigr\},$$

is a metric on X such that

$$d(x,y) < \varepsilon \quad \Longleftrightarrow \quad M(x,y,\varepsilon) > 1-\varepsilon^p,$$

for all $\varepsilon \in (0,1)$. Thus the uniformities, and hence the topologies, induced by $(M,*)$ and d coincide on X. Furthermore $(X,M,*)$ is complete if and only if (X,d) is complete.

Example 2 Let $(X,M,*)$ be a fuzzy metric space such that $* \geq *_{Y_p}$ for some $p \in (0,1)$. Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $\alpha(t) = t^{1/p}$ for all $t \in \mathbb{R}^+$. Clearly, α is continuous and strictly increasing on $\mathbb{R}^+$, and satisfies conditions (c2) and (c3) of Theorem 3. Hence, the function $d : X \times X \to \mathbb{R}^+$, defined as

$$d(x,y) = \sup\bigl\{t \geq 0 : M(x,y,t) \leq 1-t^{1/p}\bigr\},$$

is a metric on X such that

$$d(x,y) < \varepsilon \quad \Longleftrightarrow \quad M(x,y,\varepsilon) > 1-\varepsilon^{1/p},$$

for all $\varepsilon \in (0,1)$. Thus the uniformities, and hence the topologies, induced by $(M,*)$ and d coincide on X. Furthermore $(X,M,*)$ is complete if and only if (X,d) is complete.

Although Theorem 2 can we also applied to Examples 1 and 2 above, the following provides an instance where Theorem 3 works but not Theorem 2.

Example 3 Let $(X,M,*)$ be a fuzzy metric space such that $* \geq *_{H_\lambda}$ for some $\lambda \in (0,2]$. Let $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ defined as $\alpha(t) = \lambda t/2$ if $t \in [0,1]$, and $\alpha(t) = (2t+1)/3$ if $t > 1$. It is clear that α is left continuous and strictly increasing on $\mathbb{R}^+$, and it satisfies condition (c2) of Theorem 3.

Next we show that it also satisfies condition (c3). Indeed, for $\lambda \in (0,2]$ fixed, and $t,s \in [0,1]$, we have

$$\begin{aligned}\alpha(t+s) + \bigl(\bigl(1-\alpha(t)\bigr) *_{H_\lambda} \bigl(1-\alpha(s)\bigr)\bigr) &= \alpha(t+s) + \frac{(1-\alpha(t))(1-\alpha(s))}{\lambda + (1-\lambda)(1-\alpha(t)\alpha(s))} \\ &= \alpha(t+s) + \frac{4+\lambda^2 ts - 2\lambda(t+s)}{4+\lambda^2(\lambda-1)ts}.\end{aligned}$$

If $t+s > 1$, we obtain $\alpha(t+s) = (2(t+s)+1)/3 > 1$, so, in this case, the inequality

$$\alpha(t+s) + \bigl(\bigl(1-\alpha(t)\bigr) *_{H_\lambda} \bigl(1-\alpha(s)\bigr)\bigr) \geq 1,$$

is trivially satisfied.

If $t+s \leq 1$, we have $\alpha(t+s) = \lambda t/2$, and an easy computation shows that condition

$$\alpha(t+s) + \bigl(\bigl(1-\alpha(t)\bigr) *_{H_\lambda} \bigl(1-\alpha(s)\bigr)\bigr) \geq 1$$

is equivalent to condition $\lambda(\lambda-1)(t+s) \geq 2\lambda - 4$, which is clearly satisfied because $\lambda \in (0,2]$.

Therefore, we can apply Theorem 3. However, the function α is not continuous at $t = 1$, so it does not satisfy condition (R1) of Theorem 2. Moreover, it does not satisfy condition (R3) of Theorem 2 because for any $\lambda \in (5/3,2]$ we have

$$\alpha(1+1) = \alpha(2) = \frac{5}{3} < \lambda = \alpha(1) + \alpha(1).$$

3 Application to the fixed point theory of multivalued mappings on fuzzy metric spaces

Given a non-empty set X we shall denote by $\mathcal{P}_0(X)$ the collection of all non-empty subsets of X.

Let (X,d) be a metric space. A multivalued mapping $T : X \to \mathcal{P}_0(X)$ is said to be a Caristi multivalued mapping (on (X,d)) if there is a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$ there is $y_x \in Tx$ satisfying $d(x,y_x) \leq \varphi(x) - \varphi(y_x)$.

In particular, a self-mapping T of a metric space (X,d) is said to be a Caristi mapping if there is a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that $d(x,Tx) \leq \varphi(x) - \varphi(Tx)$, for all $x \in X$.

Caristi proved in [22] his celebrated theorem that every Caristi mapping on a complete metric space has a fixed point.

Kirk proved in [23] that Caristi's fixed point theorem actually characterizes the metric completeness.

On the other hand, it is well known that Caristi's fixed point theorem admits an easy a natural multivalued generalization (see *e.g.* [24]).

These results are usually combined as follows.

Theorem 4 [22–24] *For a metric space (X,d) the following are equivalent.*

(1) (X,d) *is complete.*

(2) *Every Caristi multivalued mapping $T : X \to \mathcal{P}_0(X)$ has a fixed point, i.e., there is $z \in X$ such that $z \in Tz$.*

(3) *Every Caristi mapping $T : X \to X$ has a fixed point.*

In a recent paper [9] it was obtained a fuzzy version of Theorem 4. To this end, the authors of [9] introduced the following notion.

Let $(X,M,*)$ be a fuzzy metric space. A multivalued mapping $\dot{T} : X \to \mathcal{P}_0(X)$ is called a fuzzy Caristi multivalued mapping if there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that for each $x \in X$ there exists $y_x \in Tx$ satisfying the following condition:

$$\varphi(x) - \varphi(y_x) < t \quad \Longrightarrow \quad M(x, y_x, t) > 1 - t, \quad t > 0.$$

The notion of a fuzzy Caristi mapping for a self-mapping $T : X \to X$ is defined in the obvious manner (see [9], Definition 2).

Then in Theorem 3 of [9] the following was proved.

Theorem 5 [9] *Let $(X,M,*)$ be a fuzzy metric space such that $* \geq *_L$. The following are equivalent.*

(1) $(X,M,*)$ *is complete.*

(2) *Every fuzzy Caristi multivalued mapping $T : X \to \mathcal{P}_0(X)$ has a fixed point.*

(3) *Every fuzzy Caristi's mapping $T : X \to X$ has a fixed point.*

Remark 1 Actually Theorem 5 was proved in [9] for multivalued mappings from X to the set $\mathcal{C}_0(X)$ of all non-empty closed subsets of $(X,M,*)$. However, the proof remains valid, without changes, for the case that T take values in $\mathcal{P}_0(X)$.

We are going to improve Theorem 5 in two directions, which will provide partial solutions to a question posed in [9], p.1220.

Definition 2 Let $(X,M,*)$ be a fuzzy metric space and $\dot{T} : X \to \mathcal{P}_0(X)$ a multivalued mapping. We say that T is an α-fuzzy Caristi multivalued mapping if there exist a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$, and a function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ which is left continuous on $(0,1]$ and satisfies conditions (c1)-(c3) of Theorem 3, and such that for each $x \in X$ there exists $y_x \in Tx$ for which the following holds:

$$\varphi(x) - \varphi(y_x) < t \quad \Longrightarrow \quad M(x, y_x, t) > 1 - \alpha(t), \quad t > 0. \tag{3}$$

The notion of an α-fuzzy Caristi mapping for a self-map $T : X \to X$ is defined in the obvious manner.

Then we obtain the following.

Theorem 6 *Let $(X, M, *)$ be a complete fuzzy metric space. Then every α-fuzzy Caristi multivalued mapping has a fixed point.*

Proof Let $T : X \to \mathcal{P}_0(X)$ be an α-fuzzy Caristi multivalued mapping. Then there exist a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$, and a left continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions of Definition 2. By Theorem 3, the function $d_\alpha : X \times X \to \mathbb{R}^+$ defined as

$$d_\alpha(x,y) = \sup\bigl\{t \geq 0 : M(x,y,t) \leq 1 - \alpha(t)\bigr\},$$

is a metric on X such that

$$d_\alpha(x,y) < \varepsilon \quad \Longleftrightarrow \quad M(x,y,\varepsilon) > 1 - \alpha(\varepsilon),$$

for all $\varepsilon \in (0,1)$. Thus the topologies, induced by $(M, *)$ and d_α coincide on X, and $(X, M, *)$ is complete if and only if (X, d_α) is complete.

We now show that T is a Caristi multivalued mapping on (X, d_α). Indeed, by Definition 2, for each $x \in X$ there exists $y_x \in Tx$ for which condition (3) is satisfied. Hence, for each $t > 0$ such that $M(x,y,t) \leq 1 - \alpha(t)$, we deduce that $\varphi(x) - \varphi(y_x) \geq t$. Consequently, $d_\alpha(x,y) \leq \varphi(x) - \varphi(y_x)$.

Finally, T has a fixed point by Theorem 4, (1) $\Longrightarrow$ (2). □

Theorem 7 *Let $(X, M, *)$ be a fuzzy metric space such that $* \geq *_{Y_p}$ for some $p > 0$. The following are equivalent.*

(1) *$(X, M, *)$ is complete.*
(2) *Every α-fuzzy Caristi multivalued mapping has a fixed point.*
(3) *Every α-fuzzy Caristi mapping has a fixed point.*

Proof (1) $\Longrightarrow$ (2) follows from Theorem 6, and (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1) Construct the metric d of Example 1 if $p \geq 1$, or the metric d of Example 2 if $p \in (0,1)$. In any case, we shall prove that every Caristi mapping on (X, d) has a fixed point. Indeed, let $T : X \to X$ such that there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

for all $x \in X$. We claim that T is an α-fuzzy Caristi mapping for α taken as in Example 1 or Example 2, respectively. In fact, if for some $x \in X$ and $t > 0$, we have $\varphi(x) - \varphi(Tx) < t$ but $M(x, Tx, t) \leq 1 - \alpha(t)$, it follows that $d(x, Tx) \geq t$, and thus $\varphi(x) - \varphi(Tx) \geq t$, a contradiction. We conclude that T is an α-fuzzy Caristi mapping, and by hypothesis, it has a fixed point. Therefore, by Theorem 4, (X, d) is complete, and thus $(X, M, *)$ is complete by Theorem 3 (or by Examples 1 and 2). □

We conclude the paper with an example that illustrates our results in this section.

Example 4 Let $X = [0,1]$ and let $*$ be a continuous t-norm such that $* \geq *_{Y_p}$, $p \in (0,1]$. Define $M : X \times X \times \mathbb{R}^+ \to [0,1]$ as $M(x,y,0) = 0$, $M(x,y,t) = x * y$ if $x \neq y$ and $t > 0$, and

$M(x,x,t) = 1$ for all $t > 0$. It is well known, and easy to check, that $(X, M, *)$ is a complete fuzzy metric space. For each $x \in X\backslash\{1\}$, fix a subset A_x of X such that $x \notin A_x$ and $1 \in A_x$. Define $T : X \to \mathcal{P}_0(X)$ as $T1 = \{1\}$ and $Tx = A_x$ for all $x \in X\backslash\{1\}$. We show that T is an α-fuzzy Caristi multivalued mapping on X for $\varphi : X \to \mathbb{R}^+$ given by $\varphi(1) = 0$ and $\varphi(x) = 1$ for all $x \in X\backslash\{1\}$ and $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\alpha(t) = t^{1/p}$ for all $t \geq 0$. Indeed, let $x \in X\backslash\{1\}$. Take $y_x = 1 \in T_x$. If $\varphi(x) - \varphi(y_x) < t$, then $1 < t$, and thus, $1 < t^{1/p}$. Hence $M(x, y_x, t) \geq 0 > 1 - t^{1/p} = 1 - \alpha(t)$. Therefore T is an α-fuzzy Caristi multivalued mapping, and all conditions of Theorem 6 are satisfied (note that we can also apply Theorem 7, (1) $\Longrightarrow$ (2)).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final version.

Author details
[1]Gilmation S.L., Calle 232, 66 La Cañada, Paterna, 46182, Spain. [2]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, València, 46022, Spain.

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