Guo et al. *Fixed Point Theory and Applications* (2015) 2015:223
DOI 10.1186/s13663-015-0470-7

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Strong convergence theorems for the split equality variational inclusion problem and fixed point problem in Hilbert spaces

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Abstract
In this paper, we propose and investigate two new iterative algorithms for solving the split equality variational inclusion problem in Hilbert spaces. We also prove that the sequences generated by the proposed algorithms converge strongly to a common solution of the split equality variational inclusion problem and fixed points of a family of nonexpansive mappings, which is also an unique solution of a variational inequality as an optimality condition for a minimization problem. The results presented in this paper extend and generalize a variety of existing results in this area.

Keywords: split equality problem; split equality variational inclusion problem; fixed point; variational inequality

1 Introduction

In this paper, we consider the split equality problem (SEP) proposed by Moudafi [1]. Let H_1, H_2, H_3 be real Hilbert spaces, $S_n : H_1 \to H_1$ be a family of nonexpansive mappings, $\mathrm{Fix}(S_n)$ denote the fixed points set of S_n, $n = 1, 2, \ldots$, $C = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \in H_1$, Q be the nonempty closed convex set of H_2. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The so-called SEP can mathematically be formulated as finding $x \in C$, $y \in Q$ satisfying the property:

$$x \in C, y \in Q, \quad Ax = By. \tag{1}$$

Throughout this paper, we use Γ to denote the solution set of SEP, that is,

$$\Gamma = \big\{(x, y) \in H_1 \times H_2, Ax = By, x \in C, y \in Q\big\}.$$

If $B = I$ (the identity mapping on Hilbert space H), the problem (1) is equivalent to the well-known split feasibility problem (SFP). It is easy to see that the SEP (1) includes the SFP as a special case. The split equality problems allow asymmetric and partial relations between the variables x and y. As is well known, the SEP has received much attention due to its application in various disciplines such as medical image reconstruction, game theory, decomposition methods for PDEs, and radiation therapy treatment planning [2–4].

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In 2011, Moudafi [5] introduced and studied the following split variational inclusion problem (SVIP). Let H_1, H_2 be Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, A^* be the adjoint of A, and $B_1 : H_1 \to H_1$, $B_2 : H_2 \to H_2$ be two set-valued maximal monotone mappings. SVIP is formulated as the following problem:

$$\text{find } x^* \in H_1 \text{ such that } 0 \in B_1\bigl(x^*\bigr), 0 \in B_2\bigl(Ax^*\bigr). \tag{2}$$

Recently, Byrne *et al.* [6] proposed the following iterative method to solve the problem (2): For given $x_0 \in H_1$ and $\lambda > 0$, the iterative sequence $\{x_n\}$ is generated as follows:

$$x_{n+1} = J_\lambda^{B_1}\bigl[x_n + \gamma A^*\bigl(J_\lambda^{B_2} - I\bigr)Ax_n\bigr]. \tag{3}$$

Moreover, iterative methods for nonexpansive mappings have been applied to solve minimization problem. Moudafi [7] proposed the viscosity approximation method: For every initial $x_0 \in H$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n \tag{4}$$

under some certain appropriate conditions imposed on $\{\alpha_n\}$, and it is proved that the sequence generated by (4) converges strongly to the unique solution x^* of the variational inequality

$$\bigl\langle (I - f)x^*, x - x^*\bigr\rangle \geq 0, \quad x \in C.$$

For the iterative method (4), Marino and Xu [8] introduced a new viscosity approximation method and considered the following iterative sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \tag{5}$$

and they proved that the sequence generated by (5) converges strongly to the unique solution x^* of the variational inequality

$$\bigl\langle (A - \gamma f)x^*, x - x^*\bigr\rangle \geq 0, \quad x \in C$$

which is the optimality condition for the following minimization problem:

$$\min \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf.

In 2013, Kazmi and Rizvi [9] combined the iterative method (3) and the viscosity approximation method (4) for solving a split variational inclusion and the fixed point problem of a nonexpansive mapping. Kazmi and Rizvi presented the following iteration scheme:

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n]; \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{6}$$

and they proved that the sequences $\{u_n\}$, $\{x_n\}$ converge strongly to $z \in \mathrm{Fix}(T) \cap \Gamma$, where $z = P_{\mathrm{Fix}(T)\cap\Gamma} f(z)$, Γ is the solution set of SVIP.

In 2015, Sitthithakerngkiet *et al.* [10] combined the iterative method (3) and the viscosity approximation method (5) for solving a split variational inclusion and the fixed point problem of a family of nonexpansive mappings. They proposed the following iteration algorithm:

$$\begin{cases} y_n = J_\lambda^{B_1}[x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n]; \\ x_{n+1} = \alpha_n \xi f(x_n) + (I - \alpha_n D)S_n y_n, \end{cases} \tag{7}$$

and they proved that the sequence converges strongly to a common solution of SVIP and the fixed point of a family of nonexpansive mappings.

Inspired and motivated by the corresponding convergence results of (1), (2), and (7), we consider the split equality variational inclusion problem (SEVIP):

$$\text{find } x \in U^{-1}(0) = \mathrm{Fix}\bigl(J_{u_n}^U\bigr), y \in K^{-1}(0) = \mathrm{Fix}\bigl(J_{u_n}^K\bigr) \text{ such that } Ax = By, \tag{8}$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \to 2^{H_1}$ and $K : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.

In this paper, we will introduce a more general iterative method for SEVIP (8) and a fixed point problem, which is defined in the following way:

$$\begin{cases} v_n = J_{u_n}^{(U,K)}(I - \gamma G^* G)w_n, \\ w_n = \alpha_n \sigma f(w_n) + (I - \alpha_n D)S_n v_n, \end{cases} \tag{9}$$

where $\sigma \in [0,1]$, $\alpha_n \in (0,1)$, and D is a strongly positive bounded linear operator. Note that, if $\sigma = 1$, $u_n = \lambda$, $D = I$, $B = I$, $S_n = T$, scheme (9) can be reduced to (7), that is, the iterative method (9) for solving the split equality variational inclusion problem can be reduced to the iterative method (7) for solving SVIP and SFP.

Meanwhile, we will prove that the sequences generated by (9) converge strongly to a common element of the solution set of a split equality variational inclusion problem and the common fixed point set of a family of nonexpansive mappings, which is also an unique solution of a variational inequality as an optimality condition for a minimization problem.

2 Preliminaries

We first recall that some definitions, notations, and conclusions which will be used in the proofs of our main results. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. We denote by '$\to$' strong convergence, by '$\rightharpoonup$' weak convergence. In order to establish our convergence theorems, we need the following concepts.

Definition 2.1

(1) A mapping $f : H \to H$ is k-contractive if there exists a constant $k \in (0,1)$ such that

$$\|fx - fy\| \leq k\|x - y\|, \quad \forall x, y \in H.$$

(2) A mapping T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.$$

(3) A mapping T is monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall x, y \in H.$$

(4) A mapping T is firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in H.$$

(5) A bounded linear operator D is said to be strongly positive if there exists a constant $\alpha > 0$ such that

$$\langle Dx, x\rangle \geq \alpha \|x\|^2, \quad \forall x \in H.$$

(6) A mapping P_C is called the metric projection of H onto C, if $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \min\bigl\{\|x - y\| : y \in C\bigr\}, \quad \forall x \in H.$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C.$$

Proposition 2.1 *A Banach space E is said to have the Opial property, if for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x^*$, *we have*

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|$$

$\forall y \in E$ *with* $y \neq x^*$.

Proposition 2.2 *In Hilbert spaces, the following inequalities hold*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H, \tag{10}$$

$$\langle x, y\rangle = \frac{1}{2}\bigl(\|x\|^2 + \|y\|^2 - \|x - y\|^2\bigr), \quad \forall x, y \in H. \tag{11}$$

Lemma 2.1 ([11]) *Assume D is a strongly positive linear bounded operator on a Hilbert H with coefficient* $\overline{\gamma} > 0$ *and* $0 < \alpha \leq \|D\|^{-1}$, *then* $\|I - \alpha D\| \leq 1 - \alpha\overline{\gamma}$.

Lemma 2.2 ([11]) *Let C be a nonempty, closed, and convex subset of a Hilbert space H. Assume that* $f : C \to C$ *is a contraction with a coefficient* $k \in (0,1)$ *and D is a strongly positive linear bounded operator with a coefficient* $\overline{\gamma} > 0$. *Then, for* $0 < \gamma < \frac{\overline{\gamma}}{k}$,

$$\bigl\langle x - y, (D - \gamma f)x - (D - \gamma f)y\bigr\rangle \geq (\overline{\gamma} - \gamma k)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $D - \gamma f$ *is strongly monotone with coefficient* $\overline{\gamma} - \gamma k$.

Lemma 2.3 ([12]) *Let C be a nonempty closed subset of a real Hilbert space H, and let $\{S_n\}$ be a sequence of mappings from C into itself. Suppose that $\{S_n\}$ satisfies the AKTT condition: $\sum_{n=1}^{n=\infty}\sup\{\|S_{n+1}v - S_n v\| : v \in C\} < \infty$. Then for each $x \in C$, $\{S_n x\}$ converges strongly to a point in C. Furthermore, let $S : C \to C$ be defined by*

$$Sx = S_n x, \quad \forall x \in C.$$

Then $\lim_{n\to\infty}\sup\{\|Sv - S_n v\| : v \in C\} = 0$.

Lemma 2.4 ([13]) *Assume a_n is a sequence of nonnegative numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n$ where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that:*
(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([14]) *Let U be a set-valued maximal monotone operator on H. For $u > 0$, we define the resolvent $J_u^U = (I + uU)^{-1}$, then the following holds:*
(i) *For each $u > 0$, J_u^U is a single-valued and firmly nonexpansive mapping.*
(ii) *$D(J_u^U) = H$ and $\mathrm{Fix}(J_u^U) = U^{-1}(0) = \{x \in D(G) : 0 \in Ux\}$.*
(iii) *$\|J_\alpha^U x - J_\beta^U x\|^2 \leq \frac{\alpha-\beta}{\alpha}\langle J_\alpha^U x - J_\beta^U x, J_\alpha^U x - x\rangle$, for all $\alpha, \beta > 0$ and $x \in H$.*
In fact $\|J_\alpha^U x - J_\beta^U x\| \leq \frac{|\alpha-\beta|}{\alpha}\|J_\alpha^U x - x\|$.
(iv) *Suppose that $U^{-1}(0) \neq \emptyset$, then $\langle x - J_\alpha^U x, J_\alpha^U x - w\rangle \geq 0$ for each $x \in H$ and each $w \in U^{-1}(0)$, and each $\beta > 0$.*

Lemma 2.6 ([15]) *Let $\{S_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{S_{n_j}\}_{j\geq 0}$ of $\{S_n\}$ such that*

$$\{S_{n_j}\} < \{S_{n_j+1}\} \quad \textit{for all } j \geq 0.$$

Also consider the sequence of the integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n | S_k < S_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$, and for all $n \geq n_0$, the following two estimates hold:

$$S_{\tau(n)} \leq S_{\tau(n)+1}, \qquad S_n \leq S_{\tau(n)+1}.$$

3 Main result

In this section, the following supposed conditions always hold:
(1) Let H_1, H_2, H_3 be Hilbert spaces.
(2) Let U and K be two set-valued maximal monotone mappings.
(3) Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators and A^*, B^* be the adjoint of A and B.
(4) $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$, where f_i, $i = 1, 2$ is a contraction mapping on H_i with constant $k \in (0,1)$.
(5) Let S_n be a sequence of nonexpansive mappings on H_1, D be a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 0$.

(6) Assume the solution set of SEVIP (3) $\Gamma \neq \emptyset$,

$$J_{u_n}^{(U,K)} = \begin{bmatrix} J_{u_n}^{U} \\ J_{u_n}^{K} \end{bmatrix}, \qquad G = [A \quad -B], \qquad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

Proposition 3.1 *Let $T = I - \gamma G^*G : H_1 \times H_2 \longrightarrow H_1 \times H_2$, where $\gamma \in (0, \frac{2}{L})$, with $L = \rho(G^*G)$ being the spectral radius of the self adjoint operator G^*G on $H_1 \times H_2$, then T is a nonexpansive mapping.*

Proof In fact, for any $x, y \in H_1 \times H_2$,

$$\begin{aligned} \|Tx - Ty\|^2 &= \big\|(I - \gamma G^*G)x - (I - \gamma G^*G)y\big\|^2 \\ &= \big\|x - y - \gamma G^*G(x - y)\big\|^2 \\ &= \|x - y\|^2 + \gamma^2 \big\|G^*G(x - y)\big\|^2 - 2\gamma\big\langle x - y, G^*G(x - y)\big\rangle \\ &\leq \|x - y\|^2 + \gamma^2 L \big\|G(x - y)\big\|^2 - 2\gamma\big\langle G(x - y), G(x - y)\big\rangle \\ &= \|x - y\|^2 + \gamma^2 L \big\|G(x - y)\big\|^2 - 2\gamma \big\|G(x - y)\big\|^2 \\ &= \|x - y\|^2 - \gamma(2 - \gamma L)\big\|G(x - y)\big\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

This completes the proof of Proposition 3.1. □

Lemma 3.1 ([13]) *Let H_1, H_2, H_3, A, B, A^*, B^*, U, K, $J_{u_n}^{(U,K)}$, G, G^*, f, S_n, D, S be the same as above. If $\Gamma \neq \emptyset$ (the solution set of SEVIP (8)), then $w^* = (x^*, y^*) \in H_1 \times H_2$ is a solution of SEVIP (8) if and only if for any given $\gamma > 0$ and $u > 0$*

$$w^* = J_u^{(U,K)}\big(I - \gamma G^*G\big)w^*.$$

Theorem 3.1 *Let H_1, H_2, H_3, A, B, A^*, B^*, U, K, $J_{u_n}^{(U,K)}$, G, G^*, f, S_n, D, S be the same as above. Let w_n be generated by*

$$\begin{cases} v_n = J_{u_n}^{(U,K)}(I - \gamma G^*G)w_n; \\ w_n = \alpha_n \sigma f(w_n) + (I - \alpha_n D)S_n v_n. \end{cases} \tag{12}$$

Suppose S_n satisfies the AKTT condition, $\mathrm{Fix}(S) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_n)$.

If the solution set $\Omega = \mathrm{Fix}(S) \cap \Gamma$ is nonempty and the following conditions are satisfied:

(i) $\alpha_n \in (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $0 < \overline{\gamma} < \frac{1}{\alpha_n}$, $0 < \sigma < \frac{\overline{\gamma}}{k}$.

Then the sequence w_n converges strongly to a point w^*, where $w^* = P_\Omega(I - D - \sigma f)(w^*)$ is a unique solution of the variational inequalities

$$\big\langle (D - \sigma f)w^*, w^* - z\big\rangle \leq 0, \quad z \in \Omega. \tag{13}$$

Proof First, we show that w_n defined by (12) is well defined.

We define a mapping

$$W_n = \alpha_n \sigma f(w_n) + (I - \alpha_n D) S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr) w_n, \quad n \geq 0. \tag{14}$$

By Lemma 2.1, Proposition 3.1, and (14), for any $x, y \subseteq H_1$, we have

$$\begin{aligned}
&\bigl\| W_n(x) - W_n(y) \bigr\| \\
&\quad = \bigl\| \alpha_n \sigma f(x) - \alpha_n \sigma f(y) + (I - \alpha_n D)\bigl(S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr)x - S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr)y\bigr) \bigr\| \\
&\quad \leq \alpha_n \sigma \bigl\| f(x) - f(y) \bigr\| + \| I - \alpha_n D \| \bigl\| S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr)x - S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr)y \bigr\| \\
&\quad \leq \alpha_n \sigma k \| x - y \| + (1 - \alpha_n \overline{\gamma}) \| x - y \| \\
&\quad = \bigl(1 - \alpha_n(\overline{\gamma} - \sigma k)\bigr) \| x - y \|.
\end{aligned}$$

Since $0 < 1 - \alpha_n(\overline{\gamma} - \sigma k) < 1$, it follows that W_n is a contraction. Therefore, by the Banach contraction principle, W_n has a unique fixed point in H_1, denoted by w_n, that is,

$$w_n = \alpha_n \sigma f(w_n) + (I - \alpha_n D) S_n J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr) w_n, \tag{15}$$

which is exactly (12).

Second, we claim that w_n is bounded.

Indeed, take any $z \in \Omega$, we have $z = J_{u_n}^{(U,K)}(I - \gamma G^* G)z$ and $z \in \operatorname{Fix}(S) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n)$,

$$\begin{aligned}
\| v_n - z \| &= \bigl\| J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr) w_n - z \bigr\| \\
&= \bigl\| J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr) w_n - J_{u_n}^{(U,K)}\bigl(I - \gamma G^* G\bigr) z \bigr\| \\
&\leq \| w_n - z \|.
\end{aligned}$$

Thus, we derive that

$$\begin{aligned}
\| w_n - z \| &= \bigl\| \alpha_n \sigma f(w_n) + (I - \alpha_n D) S_n v_n - z \bigr\| \\
&= \bigl\| \alpha_n \sigma f(w_n) - \alpha_n D z + (I - \alpha_n D) S_n v_n - (I - \alpha_n D) S_n z \bigr\| \\
&\leq \alpha_n \bigl\| \sigma f(w_n) - D z \bigr\| + \| I - \alpha_n D \| \| v_n - z \| \\
&\leq \alpha_n \bigl\| \sigma f(w_n) - D z \bigr\| + (1 - \alpha_n \overline{\gamma}) \| w_n - z \| \\
&\leq \alpha_n \bigl\| \sigma \bigl(f(w_n) - f(z)\bigr) + \bigl(\sigma f(z) - D z\bigr) \bigr\| + (1 - \alpha_n \overline{\gamma}) \| w_n - z \| \\
&\leq \alpha_n \sigma k \| w_n - z \| + \alpha_n \bigl\| \sigma f(z) - D z \bigr\| + (1 - \alpha_n \overline{\gamma}) \| w_n - z \| \\
&= \bigl(1 - \alpha_n(\overline{\gamma} - \sigma k)\bigr) \| w_n - z \| + \alpha_n(\overline{\gamma} - \sigma k) \frac{\| \sigma f(z) - D z \|}{\overline{\gamma} - \sigma k}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\alpha_n(\overline{\gamma} - \sigma k) \| w_n - z \| \leq \alpha_n(\overline{\gamma} - \sigma k) \frac{\| \sigma f(z) - D z \|}{\overline{\gamma} - \sigma k}, \\
&\| w_n - z \| \leq \frac{\| \sigma f(z) - D z \|}{\overline{\gamma} - \sigma k}.
\end{aligned} \tag{16}$$

Hence the sequence $\{w_n\}$ of (15) is bounded, so are $\{v_n\}$, $\{f(w_n)\}$, and $\{S_n v_n\}$.
Third, we show that $\|w_n - v_n\| \to 0$.
Indeed, for any $z \in \Omega$, we have

$$\begin{aligned}\|v_n - z\|^2 &= \big\|J_{u_n}^{(U,K)}\big(I - \gamma G^*G\big)w_n - z\big\|^2 \\ &\leq \big\|w_n - z - \gamma G^*Gw_n\big\|^2 \\ &= \|w_n - z\|^2 + \gamma^2\big\|G^*Gw_n\big\|^2 - 2\gamma\langle w_n - z, G^*Gw_n\rangle \\ &\leq \|w_n - z\|^2 + \gamma^2 L\|Gw_n\|^2 - 2\gamma\|Gw_n\|^2 \\ &= \|w_n - z\|^2 - \gamma(2 - \gamma L)\|Gw_n\|^2. \end{aligned} \tag{17}$$

It follows from (10) and (17) that

$$\begin{aligned}\|w_n - z\|^2 &= \big\|\alpha_n\sigma f(w_n) + (I - \alpha_n D)S_n v_n - z\big\|^2 \\ &= \big\|(I - \alpha_n D)(S_n v_n - z) + \alpha_n\big(\sigma f(w_n) - Dz\big)\big\|^2 \\ &\leq \big\|(I - \alpha_n D)(S_n v_n - z)\big\|^2 + 2\alpha_n\langle\sigma f(w_n) - Dz, w_n - z\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|v_n - z\|^2 + 2\alpha_n\langle\sigma f(w_n) - Dz, w_n - z\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})\|w_n - z\|^2 - (1 - \alpha_n\overline{\gamma})\gamma(2 - \gamma L)\|Gw_n\|^2 \\ &\quad + 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\|. \end{aligned} \tag{18}$$

This implies that

$$\begin{aligned}(1 - \alpha_n\overline{\gamma})\gamma(2 - \gamma L)\|Gw_n\|^2 &\leq -\alpha_n\overline{\gamma}\|w_n - z\|^2 + 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\| \\ &\leq 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\|.\end{aligned}$$

Since both w_n and $f(w_n)$ are bounded and $\alpha_n \to 0$, we have $\|Gw_n\| \to 0$.
Then from (11) and Lemma 2.5, we derive that

$$\begin{aligned}\|v_n - z\|^2 &= \big\|J_{u_i}^{(U_i,K_i)}\big(I - \gamma G^*G\big)w_n - z\big\|^2 \\ &\leq \langle v_n - z, w_n - \gamma G^*Gw_n - z\rangle \\ &= \frac{1}{2}\big\{\|v_n - z\|^2 + \|w_n - z\|^2 - \gamma(2 - \gamma L)\|Gw_n\|^2 - \|w_n - v_n\|^2 \\ &\quad - \big\|\gamma G^*Gw_n\big\|^2 + 2\langle w_n - v_n, \gamma G^*Gw_n\rangle\big\}.\end{aligned}$$

This implies that

$$\|v_n - z\|^2 \leq \|w_n - z\|^2 - \|w_n - v_n\|^2 + 2\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\|. \tag{19}$$

By (18) and (19), we have

$$\begin{aligned}\|w_n - z\|^2 &\leq (1 - \alpha_n\overline{\gamma})^2\|v_n - z\|^2 + 2\alpha_n\langle\sigma f(w_n) - Dz, w_n - z\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})\|v_n - z\|^2 + 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\|\end{aligned}$$

$$\le (1-\alpha_n\overline{\gamma})\|w_n - z\|^2 - (1-\alpha_n\overline{\gamma})\|w_n - v_n\|^2$$
$$+ 2(1-\alpha_n\overline{\gamma})\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\| + 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\|.$$

Hence, we obtain

$$(1-\alpha_n\overline{\gamma})\|w_n - v_n\|^2 \le 2(1-\alpha_n\overline{\gamma})\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\|$$
$$+ 2\alpha_n\big\|\sigma f(w_n) - Dz\big\|\|w_n - z\|.$$

Since $\alpha_n \to 0$, $\|Gw_n\| \to 0$, it follows that $\|w_n - v_n\| \to 0$.

Nextly, we show $\|Sv_n - v_n\| \to 0$.

$$\begin{aligned}
\|w_n - S_n w_n\| &= \|w_n - S_n v_n + S_n v_n - S_n w_n\| \\
&\le \|w_n - S_n v_n\| + \|v_n - w_n\| \\
&= \big\|\alpha_n \sigma f(w_n) + (I - \alpha_n D)S_n v_n - S_n v_n\big\| + \|v_n - w_n\| \\
&= \alpha_n \big\|f(w_n) - DS_n v_n\big\| + \|v_n - w_n\|.
\end{aligned}$$

Since $\{f(w_n)\}$ and $\{S_n v_n\}$ are bounded, $\alpha_n \to 0$, $\|w_n - v_n\| \to 0$, then $\|w_n - S_n w_n\| \to 0$. Thus,

$$\begin{aligned}
\|v_n - S_n v_n\| &= \|v_n - w_n + w_n - S_n w_n + S_n w_n - S_n v_n\| \\
&\le \|v_n - w_n\| + \|w_n - S_n w_n\| + \|w_n - v_n\|.
\end{aligned}$$

Since $\|w_n - v_n\| \to 0$, $\|w_n - S_n w_n\| \to 0$, we get $\|v_n - S_n v_n\| \to 0$.

Moreover, we note that

$$\begin{aligned}
\|Sv_n - v_n\| &\le \|Sv_n - S_n v_n\| + \|S_n v_n - v_n\| \\
&\le \sup\big\{\|Sw - S_n w\| : w \in \{v_n\}\big\} + \|S_n v_n - v_n\|.
\end{aligned}$$

By Lemma 2.3, we have $\|Sv_n - v_n\| \to 0$.

Now, we prove that $\widetilde{w} \in \Omega$.

Since $\{v_n\}$ is bounded, we may assume that there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to a point $\widetilde{w}$, *i.e.* $v_{n_i} \rightharpoonup \widetilde{w}$ as $i \to \infty$. Suppose that $\widetilde{w} \notin \mathrm{Fix}(S)$, since $v_{n_i} \rightharpoonup \widetilde{w}$ and $S\widetilde{w} \neq \widetilde{w}$. Applying Opial's property, we obtain

$$\begin{aligned}
\liminf_{i\to\infty}\|v_{n_i} - \widetilde{w}\| &< \liminf_{i\to\infty}\|v_{n_i} - S\widetilde{w}\| \\
&\le \liminf_{i\to\infty}\big\{\|v_{n_i} - Sv_{n_i}\| + \|Sv_{n_i} - S\widetilde{w}\|\big\} \\
&\le \liminf_{i\to\infty}\|v_{n_i} - \widetilde{w}\|.
\end{aligned}$$

This is a contraction, then $\widetilde{w} \in \mathrm{Fix}(S) = \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_n)$.

Since $\{w_n\}$ and $\{v_n\}$ are bounded, $\|w_n - v_n\| \to 0$, $\{w_n\}$ and $\{v_n\}$ have the same asymptotical behavior, we may assume that there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which also converges weakly to the point $\widetilde{w}$, *i.e.* $w_{n_i} \rightharpoonup \widetilde{w}$ as $n_i \to \infty$. Suppose that $\widetilde{w} \neq J_{u_n}^{(U,K)}(I - \gamma G^*G)\widetilde{w}$, Applying Opial's property, we have

$$\begin{aligned}
&\liminf_{i\to\infty}\|w_{n_i}-\widetilde{w}\| \\
&\quad<\liminf_{i\to\infty}\big\|w_{n_i}-J_{u_n}^{(U,K)}\big(I-\gamma G^*G\big)\widetilde{w}\big\| \\
&\quad\le\liminf_{i\to\infty}\big\{\big\|w_{n_i}-J_{u_n}^{(U,K)}\big(I-\gamma G^*G\big)w_{n_i}\big\| \\
&\qquad+\big\|J_{u_n}^{(U,K)}\big(I-\gamma G^*G\big)w_{n_i}-J_{u_n}^{(U,K)}\big(I-\gamma G^*G\big)\widetilde{w}\big\|\big\} \\
&\quad\le\liminf_{i\to\infty}\big\{\|w_{n_i}-v_{n_i}\|+\|w_{n_i}-\widetilde{w}\|\big\} \\
&\quad\le\liminf_{i\to\infty}\|w_{n_i}-\widetilde{w}\|.
\end{aligned}$$

This is a contraction, then $\widetilde{w}=J_{u_n}^{(U,K)}(I-\gamma G^*G)\widetilde{w}$, by Lemma 3.1 we have $\widetilde{w}\in\Gamma$. Thus, $\widetilde{w}$ is a solution of SEVIP, *i.e.* $\widetilde{w}\in\Omega=\mathrm{Fix}(S)\cap\Gamma$.

We now show that $\limsup_{n\to\infty}\langle\sigma f(w^*)-Dw^*,w_n-w^*\rangle\le 0$, where $w^*=P_\Omega(I-D+\sigma f)(w^*)$ is the unique solution of VI (13).

Indeed, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\big\langle\sigma f\big(w^*\big)-Dw^*,w_n-w^*\big\rangle=\lim_{n\to\infty}\big\langle\sigma f\big(w^*\big)-Dw^*,w_{n_i}-w^*\big\rangle.$$

We also assume that $w_{n_i}\rightharpoonup\widetilde{w}$. Therefore

$$\begin{aligned}
&\limsup_{n\to\infty}\big\langle\sigma f\big(w^*\big)-Dw^*,w_n-w^*\big\rangle \\
&\quad=\lim_{n_i\to\infty}\big\langle\sigma f\big(w^*\big)-Dw^*,w_{n_i}-w^*\big\rangle \\
&\quad=\big\langle\sigma f\big(w^*\big)-Dw^*,\widetilde{w}-w^*\big\rangle \\
&\quad=\big\langle(I-D+\sigma f)w^*-w^*,\widetilde{w}-w^*\big\rangle \\
&\quad=\big\langle(I-D+\sigma f)w^*-P_\Omega(I-D+\sigma f)w^*,\widetilde{w}-P_\Omega(I-D+\sigma f)w^*\big\rangle \\
&\quad\le 0.
\end{aligned}$$

Then

$$\limsup_{n\to\infty}\big\langle\sigma f\big(w^*\big)-Dw^*,w_n-w^*\big\rangle\le 0. \tag{20}$$

On the other hand, we will prove that $w^*=P_\Omega(I-D+\sigma f)(w^*)$ is the unique solution of VI (13).

Suppose $w^*\in\Omega$ and $w^{**}\in\Omega$ both are solutions to VI (13), then

$$\big\langle\sigma f\big(w^*\big)-Dw^*,w^{**}-w^*\big\rangle\le 0$$

and

$$\big\langle\sigma f\big(w^{**}\big)-Dw^{**},w^*-w^{**}\big\rangle\le 0.$$

From the above inequalities we have

$$\langle (D-\sigma f)w^{**}-(D-\sigma f)w^{*}, w^{**}-w^{*}\rangle \leq 0.$$

By Lemma 2.2, we have $D-\sigma f$ is strongly monotone, then $w^{**}=w^{*}$, the uniqueness is proved.

Finally, we show that w_n converges strongly to w^* as $n\to\infty$.

$$\begin{aligned}
&\|w_n-w^*\|^2\\
&\quad=\|\alpha_n\sigma f(w_n)+(I-\alpha_nD)S_nv_n-w^*\|^2\\
&\quad=\|(I-\alpha_nD)(S_nv_n-w^*)+\alpha_n(\sigma f(w_n)-Dw^*)\|^2\\
&\quad\leq\|(I-\alpha_nD)(S_nv_n-w^*)\|^2+2\alpha_n\langle\sigma f(w_n)-Dw^*,w_n-w^*\rangle\\
&\quad\leq(1-\alpha_n\overline{\gamma})^2\|v_n-w^*\|^2+2\alpha_n\sigma\langle f(w_n)-f(w^*),w_n-w^*\rangle\\
&\qquad+2\alpha_n\langle\sigma f(w^*)-Dw^*,w_n-w^*\rangle\\
&\quad\leq(1-\alpha_n\overline{\gamma})^2\|w_n-w^*\|^2+2\alpha_n\sigma k\|w_n-w^*\|^2+2\alpha_n\langle\sigma f(w^*)-Dw^*,w_n-w^*\rangle.
\end{aligned}$$

This implies that

$$2(\overline{\gamma}-\sigma k)\|w_n-w^*\|^2\leq\alpha_n\overline{\gamma}^2\|w_n-w^*\|^2+2\langle\sigma f(w^*)-Dw^*,w_n-w^*\rangle.$$

From condition (i) and (20), we can obtain the desired conclusion

$$\lim_{n\to\infty}\|w_n-w^*\|=0.$$

This completes the proof. □

Theorem 3.2 *Let H_1, H_2, H_3, A, B, A^*, B^*, U, K, $J_{u_n}^{(U,K)}$, G, G^*, f, S_n, D, S be the same as them of Theorem* 3.1. *Let w_n be generated by*

$$\begin{cases} v_n=J_{u_n}^{(U,K)}(I-\gamma G^*G)w_n;\\ w_{n+1}=\alpha_n\sigma f(w_n)+(I-\alpha_nD)S_nv_n,\end{cases}\tag{21}$$

suppose S_n satisfies the AKTT condition, $\mathrm{Fix}(S)=\bigcap_{i=1}^{\infty}\mathrm{Fix}(S_n)$. *If the solution set* $\Omega=\mathrm{Fix}(S)\cap\Gamma$ *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n\in(0,1)$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;
(ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;
(iii) $\sum_{n=0}^{\infty}|u_{n+1}-u_n|<\infty$;
(iv) $0<\overline{\gamma}<\frac{1}{\alpha_n}$, $0<\sigma<\frac{\overline{\gamma}}{k}$,

then the sequence w_n converges strongly to a point w^*, where $w^*=P_\Omega(I-D-\sigma f)(w^*)$ is a unique solution of the variational inequalities

$$\langle(D-\sigma f)w^*,w^*-z\rangle\leq 0,\quad z\in\Omega.\tag{22}$$

Proof We first prove the w_n is bounded.

For any given $z \in \Omega$, we have $z = J_{u_i}^{(U_i,K_i)}(I - \gamma G^*G)z$ and $z \in F(S) = \bigcap_{n=1}^{\infty} F(S_n)$. By Lemma 2.1 and Proposition 3.1, we have

$$\begin{aligned}
\|w_{n+1} - z\| &= \left\|\alpha_n \sigma f(w_n) + (I - \alpha_n D)S_n v_n - z\right\| \\
&= \left\|\alpha_n \sigma f(w_n) - \alpha_n Dz + (I - \alpha_n D)S_n v_n - (I - \alpha_n D)S_n z\right\| \\
&\le \alpha_n \left\|\sigma f(w_n) - Dz\right\| + \|I - \alpha_n D\| \|v_n - z\| \\
&\le \alpha_n \left\|\sigma f(w_n) - Dz\right\| + (1 - \alpha_n \overline{\gamma}) \left\|J_{u_i}^{(U_i,K_i)}\big(I - \gamma G^*G\big)w_n - z\right\| \\
&\le \alpha_n \left\|\sigma f(w_n) - Dz\right\| + (1 - \alpha_n \overline{\gamma}) \|w_n - z\| \\
&\le \alpha_n \left\|\sigma \big(f(w_n) - f(z)\big) + \big(\sigma f(z) - Dz\big)\right\| + (1 - \alpha_n \overline{\gamma}) \|w_n - z\| \\
&\le \alpha_n \sigma k \|w_n - z\| + \alpha_n \left\|\sigma f(z) - Dz\right\| + (1 - \alpha_n \overline{\gamma}) \|w_n - z\| \\
&= \big(I - \alpha_n (\overline{\gamma} - \sigma k)\big) \|w_n - z\| + \alpha_n (\overline{\gamma} - \sigma k) \frac{\|\sigma f(z) - Dz\|}{\overline{\gamma} - \sigma k} \\
&\le \max\left\{\|w_n - z\|, \frac{\|\sigma f(z) - Dz\|}{\overline{\gamma} - \sigma k}\right\}.
\end{aligned}$$

By a simple induction, we have

$$\|w_n - z\| \le \max\left\{\|w_0 - z\|, \frac{\|\sigma f(z) - Dz\|}{\overline{\gamma} - \sigma k}\right\}.$$

Therefore, $\{w_n\}$ is bounded, and so are $\{v_n\}$, $\{f(w_n)\}$, $\{S_n v_n\}$.

From (17), by a similar argument to the proof of Theorem 3.1, we derive that

$$\begin{aligned}
&\|w_{n+1} - z\|^2 \\
&= \left\|\alpha_n \sigma f(w_n) + (I - \alpha_n D)S_n v_n - z\right\|^2 \\
&= \left\|(I - \alpha_n D)(S_n v_n - z) + \alpha_n \big(\sigma f(w_n) - Dz\big)\right\|^2 \\
&\le \left\|(I - \alpha_n D)(S_n v_n - z)\right\|^2 + 2\alpha_n \big\langle \sigma f(w_n) - Dz, w_{n+1} - z\big\rangle \\
&\le (1 - \alpha_n \overline{\gamma})^2 \|v_n - z\|^2 + 2\alpha_n \big\langle \sigma f(w_n) - Dz, w_{n+1} - z\big\rangle \\
&\le (1 - \alpha_n \overline{\gamma}) \|w_n - z\|^2 - (1 - \alpha_n \overline{\gamma})\gamma(2 - \gamma L)\|Gw_n\|^2 \\
&\quad + 2\alpha_n \left\|\sigma f(w_n) - Dz\right\| \|w_{n+1} - z\|.
\end{aligned} \tag{23}$$

This implies that

$$\begin{aligned}
&(1 - \alpha_n \overline{\gamma})\gamma(2 - \gamma L)\|Gw_n\|^2 \\
&\quad \le \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + 2\alpha_n \left\|\sigma f(w_n) - Dz\right\| \|w_n - z\|.
\end{aligned} \tag{24}$$

Now, the rest of the proofs will be analyzed as two cases due to the monotone property of $\{\|w_n - z\|\}$.

Case 1: $\{\|w_n - z\|\}$ is a monotone sequence.

Since $\{\|w_n - z\|\}$ is bounded, $\{\|w_n - z\|\}$ is convergent. Take the limit on both sides for (24), in view of condition (i). We have

$$\|Gw_n\| \to 0$$

By the same argument as in the proof of Theorem 3.1, we derive that

$$\|v_n - z\|^2 \le \|w_n - z\|^2 - \|w_n - v_n\|^2 + 2\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\|. \tag{25}$$

Then, from (23), (25), and (10), we derive that

$$\begin{aligned}\|w_{n+1} - z\|^2 &\le (1 - \alpha_n\overline{\gamma})^2\|v_n - z\|^2 + 2\alpha_n\langle \sigma f(w_n) - Dz, w_{n+1} - z\rangle \\ &\le (1 - \alpha_n\overline{\gamma})\|w_n - z\|^2 - (1 - \alpha_n\overline{\gamma})\|w_n - v_n\|^2 \\ &\quad + 2(1 - \alpha_n\overline{\gamma})\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\| \\ &\quad + 2\alpha_n\|\sigma f(w_n) - Dz\|\|w_{n+1} - z\|.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}(1 - \alpha_n\overline{\gamma})\|w_n - v_n\|^2 \le{}& \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + 2(1 - \alpha_n\overline{\gamma})\gamma\sqrt{L}\|w_n - v_n\|\|Gw_n\| \\ &+ 2\alpha_n\|\sigma f(w_n) - Dz\|\|w_n - z\|.\end{aligned}$$

Since $\{w_n\}$, $\{v_n\}$, $\{f(w_n)\}$ are bounded, $\lim_{n\to\infty}\{\|w_n - z\|\}$ exists and $\alpha_n \to 0$, then $\|w_n - v_n\| \to 0$.

Indeed, $J_{u_n}^{(U,K)}(I - \gamma G^*G)$ is nonexpansive and by Lemma 2.5(iii) we derive that

$$\begin{aligned}\|v_{n+1} - v_n\| &= \|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_{n+1} - J_{u_n}^{(U,K)}(I - \gamma G^*G)w_n\| \\ &\le \|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_{n+1} - J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_n\| \\ &\quad + \|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_n - J_{u_n}^{(U,K)}(I - \gamma G^*G)w_n\| \\ &\le \|w_{n+1} - w_n\| + \frac{|u_{n+1} - u_n|}{u_{n+1}}\|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_n - w_n\|\end{aligned}$$

Since $\liminf_{n\to\infty} u_n > 0$, we may assume that there exists a real number m such that $u_n \ge m > 0$ for all $n \in N$. Then we have

$$\begin{aligned}\|v_{n+1} - v_n\| &\le \|w_{n+1} - w_n\| + \frac{|u_{n+1} - u_n|}{m}\|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_n - w_n\| \\ &\le \|w_{n+1} - w_n\| + M_1|u_{n+1} - u_n|,\end{aligned}$$

where $M_1 = \sup\{\frac{1}{m}\|J_{u_{n+1}}^{(U,K)}(I - \gamma G^*G)w_n - w_n\| : n \in N\}$.

Thus, we get

$$\begin{aligned}&\|w_{n+2} - w_{n+1}\| \\ &\quad = \|\alpha_{n+1}\sigma f(w_{n+1}) + (I - \alpha_{n+1}D)S_{n+1}v_{n+1} - \alpha_n\sigma f(w_n) - (I - \alpha_n D)S_n v_n\|\end{aligned}$$

$$\begin{aligned}
&= \big\| \alpha_{n+1}\sigma f(w_{n+1}) + (I-\alpha_{n+1}D)S_{n+1}v_{n+1} - \alpha_n \sigma f(w_n) - (I-\alpha_n D)S_n v_n \\
&\quad - (I-\alpha_{n+1}D)S_{n+1}v_n + (I-\alpha_{n+1}D)S_{n+1}v_n - (I-\alpha_n D)S_{n+1}v_n \\
&\quad + (I-\alpha_n D)S_{n+1}v_n - \alpha_{n+1}\sigma f(w_n) + \alpha_{n+1}\sigma f(w_n)\big\| \\
&= \big\| (I-\alpha_{n+1}D)(S_{n+1}v_{n+1} - S_{n+1}v_n) + (\alpha_n - \alpha_{n+1})DS_{n+1}v_n \\
&\quad + (I-\alpha_n D)(S_{n+1}v_n - S_n v_n) + (\alpha_{n+1} - \alpha_n)\sigma f(w_n) + \alpha_{n+1}\sigma\big(f(w_{n+1}) - f(w_n)\big)\big\| \\
&\leq (I-\alpha_{n+1}\overline{\gamma})\|v_{n+1} - v_n\| + |\alpha_n - \alpha_{n+1}|\|DS_{n+1}v_n\| + (I-\alpha_n\overline{\gamma})\|S_{n+1}v_n - S_n v_n\| \\
&\quad + |\alpha_n - \alpha_{n+1}|\big\|\sigma f(w_n)\big\| + \alpha_{n+1}\sigma k\|w_{n+1} - w_n\| \\
&= (I-\alpha_{n+1}\overline{\gamma})\|v_{n+1} - v_n\| + \alpha_{n+1}\sigma k\|w_{n+1} - w_n\| \\
&\quad + |\alpha_n - \alpha_{n+1}|\big(\|DS_{n+1}v_n\| + \big\|\sigma f(w_n)\big\|\big) + (I-\alpha_n\overline{\gamma})\|S_{n+1}v_n - S_n v_n\| \\
&\leq \big(1 - \alpha_{n+1}(\overline{\gamma} - \sigma k)\big)\|w_{n+1} - w_n\| + M_1|u_{n+1} - u_n| \\
&\quad + |\alpha_n - \alpha_{n+1}|\big(\|DS_{n+1}v_n\| + \big\|\sigma f(w_n)\big\|\big) + \|S_{n+1}v_n - S_n v_n\| \\
&\leq \big(1 - \alpha_{n+1}(\overline{\gamma} - \sigma k)\big)\|w_{n+1} - w_n\| + M_1|u_{n+1} - u_n| + 2M_2|\alpha_n - \alpha_{n+1}| + L_n \\
&\leq \big(1 - \alpha_{n+1}(\overline{\gamma} - \sigma k)\big)\|w_{n+1} - w_n\| + M_3\big(|u_{n+1} - u_n| + |\alpha_n - \alpha_{n+1}|\big) + L_n,
\end{aligned}$$

where $M_2 = \max\{\sup_{n\in N}\|DS_{n+1}v_n\|, \sup_{n\in N}\|\sigma f(w_n)\|\}$, $M_3 = \max\{M_1, 2M_2\}$, $L_n = \sup\{\|S_{n+1}v - S_n v\| : v \in v_n\}$.

By Lemma 2.4, we have $\|w_{n+1} - w_n\| \to 0$.

Then, from condition (i) and $|u_{n+1} - u_n| \to 0$, we have $\|v_{n+1} - v_n\| \to 0$.

Indeed,

$$\begin{aligned}
\|w_n - S_n v_n\| &\leq \|w_n - S_{n-1}v_{n-1}\| + \|S_{n-1}v_{n-1} - S_{n-1}v_n\| + \|S_{n-1}v_n - S_n v_n\| \\
&\leq \alpha_{n-1}\big\|\sigma f(w_{n-1}) - DS_{n-1}v_{n-1}\big\| + \|v_{n-1} - v_n\| \\
&\quad + \sup\big\{\|S_{n-1}w - S_n w\| : w \in v_n\big\}.
\end{aligned}$$

Since $\|v_{n+1} - v_n\| \to 0$, by Lemma 2.3 and condition (i), we have $\|w_n - S_n v_n\| \to 0$.

Then

$$\|S_n v_n - v_n\| \leq \|S_n v_n - w_n\| + \|w_n - v_n\|.$$

Since $\|w_n - S_n v_n\| \to 0$ and $\|w_n - v_n\| \to 0$, we get $\|S_n v_n - v_n\| \to 0$.

Moreover, we note that

$$\begin{aligned}
\|Sv_n - v_n\| &\leq \|Sv_n - S_n v_n\| + \|S_n v_n - v_n\| \\
&\leq \sup\big\{\|Sw - S_n w\| : w \in \{v_n\}\big\} + \|S_n v_n - v_n\|.
\end{aligned}$$

By Lemma 2.3 we have

$$\|Sv_n - v_n\| \to 0. \tag{26}$$

Since $\{v_n\}$ is bounded, we may assume that there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ which converges weakly to a point $\widetilde{w}$, *i.e.* $v_{n_i} \rightharpoonup \widetilde{w}$ as $i \to \infty$, $\{w_n\}$ and $\{v_n\}$ are bounded, $\{w_n\}$ and $\{v_n\}$ have the same asymptotical behavior.

We may assume that there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ which also converges weakly to the point $\widetilde{w}$, *i.e.* $w_{n_i} \rightharpoonup \widetilde{w}$ as $i \to \infty$. By the same argument as in the proof of Theorem 3.1, we derive that $\widetilde{w}$ is a solution of SEVIP, *i.e.* $\widetilde{w} \in \Omega = \mathrm{Fix}(S) \cap \Gamma$.

Next, by the same argument as in the proof of Theorem 3.1, we derive that

$$\limsup_{n\to\infty} \langle \sigma f(w^*) - Dw^*, w_n - w^* \rangle \le 0,$$

where $w^* = P_\Omega(I - D + \sigma f)(w^*)$ is the unique solution of VI (22).

Finally, we show that w_n converges strongly to w^* as $n \to \infty$.

$$\begin{aligned}
&\|w_{n+1} - w^*\|^2 \\
&\quad = \|\alpha_n \sigma f(w_n) + (I - \alpha_n D) S_n v_n - w^*\|^2 \\
&\quad = \|(I - \alpha_n D)(S_n v_n - w^*) + \alpha_n(\sigma f(w_n) - Dw^*)\|^2 \\
&\quad \le \|(I - \alpha_n D)(S_n v_n - w^*)\|^2 + 2\alpha_n \langle \sigma f(w_n) - Dw^*, w_{n+1} - w^* \rangle \\
&\quad \le (1 - \alpha_n \overline{\gamma})^2 \|v_n - w^*\|^2 + 2\alpha_n \sigma \langle f(w_n) - f(w^*), w_{n+1} - w^* \rangle \\
&\qquad + 2\alpha_n \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle \\
&\quad \le (1 - \alpha_n \overline{\gamma})^2 \|w_n - w^*\|^2 + 2\alpha_n \sigma k \|w_n - w^*\| \|w_{n+1} - w^*\| \\
&\qquad + 2\alpha_n \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle \\
&\quad \le (1 - \alpha_n \overline{\gamma})^2 \|w_n - w^*\|^2 + \alpha_n \sigma k (\|w_n - w^*\|^2 + \|w_{n+1} - w^*\|^2) \\
&\qquad + 2\alpha_n \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|w_{n+1} - w^*\|^2 &\le \frac{1 - 2\alpha_n \overline{\gamma} + \alpha_n^2 \overline{\gamma}^2 + \alpha_n \sigma k}{1 - \alpha_n \sigma k} \|w_n - w^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \sigma k} \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle \\
&= \left(1 - \frac{2\alpha_n(\overline{\gamma} - \sigma k)}{1 - \alpha_n \sigma k}\right) \|w_n - w^*\|^2 + \frac{\alpha_n^2 \overline{\gamma}^2}{1 - \alpha_n \sigma k} \|w_n - w^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \sigma k} \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle,
\end{aligned}$$

where $\gamma_n = \frac{2\alpha_n(\overline{\gamma} - \sigma k)}{1 - \alpha_n \sigma k}$ and $\delta_n = \frac{\alpha_n^2 \overline{\gamma}^2}{1 - \alpha_n \sigma k} \|w_n - w^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n \sigma k} \langle \sigma f(w^*) - Dw^*, w_{n+1} - w^* \rangle$.

Hence, all conditions of Lemma 2.4 are satisfied.

Therefore, we immediately deduce that $w_n \to w^*$.

Case 2: The sequence $\{\|w_n - z\|\}$ is not monotone.

By Lemma 2.6, there exists a sequence of positive integers: $\{\tau(n)\}$, $n \ge n_0$, where n_0 is large enough such that

$$\tau(n) = \max\{k \le n : \|w_k - w^*\| \le \|w_{k+1} - w^*\|\}.$$

It is easy to see that $\{\tau(n)\}$ is nondecreasing and $\tau(n) \to \infty$ as $n \to \infty$.
We have $\|w_{\tau(n)} - w^*\| < \|w_{\tau(n)+1} - w^*\|$; $\|w_n - w^*\| < \|w_{\tau(n)+1} - w^*\|$.
Just as the argument of Case 1, we have

$$\begin{cases} \lim_{n\to\infty} \|w_{\tau(n)} - v_{\tau(n)}\| = 0; \\ \lim_{n\to\infty} \|v_{\tau(n)+1} - v_{\tau(n)}\| = \lim_{n\to\infty} \|w_{\tau(n)+1} - w_{\tau(n)}\| = 0; \\ \lim_{n\to\infty} \|Sv_{\tau(n)} - v_{\tau(n)}\| = 0; \\ \limsup_{n\to\infty} \langle \sigma f(w^*) - Dw^*, w_{\tau(n)} - w^* \rangle \le 0. \end{cases}$$

According to Case 1, we have $\lim_{n\to\infty} \|w_{\tau(n)} - w^*\| = 0$ and $\lim_{n\to\infty} \|w_{\tau(n)+1} - w^*\| = 0$.
Finally, from Lemma 2.6, we get

$$0 \le \|w_n - w^*\| \le \max\{\|w_n - w^*\|, \|w_{\tau(n)} - w^*\|\} \le \|w_{\tau(n)+1} - w^*\| \to 0, \quad n \to \infty.$$

Therefore, the sequence $\{w_n\}$ converges strongly to w^*.
This completes the proof. □

Corollary 3.1 *Let* $H_1, H_2, H_3, A, B, A^*, B^*, U, K, J_{u_n}^{(U,K)}, G, G^*, f, S_n, S$ *be the same as them of Theorem* 3.1. *Let* w_n *be generated by*

$$\begin{cases} v_n = J_{u_n}^{(U,K)}(I - \gamma G^* G)w_n; \\ w_{n+1} = \alpha_n f(w_n) + (1 - \alpha_n)S_n v_n. \end{cases}$$

Suppose S_n *satisfies the AKTT condition*, $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$. *If the solution set* $\Omega = \mathrm{Fix}(S) \cap \Gamma$ *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\sum_{n=0}^{\infty} |u_{n+1} - u_n| < \infty$,

then the sequence w_n *converges strongly to* w^*, *where* $w^* = P_\Omega f(w^*)$.

Corollary 3.2 *Let* H_1, H_2, H_3, A, B, A^*, B^*, U, K, $J_{u_n}^{(U,K)}$, G, G^*, f, S_n, S *be the same as them of Theorem* 3.1. *Let* $\{\omega_k\}$ *be a sequence of positive real numbers with* $\sum_{k=1}^{\infty} \omega_k = 1$, $S = \sum_{k=1}^{\infty} \omega_k S_k$, $L_n = \sum_{k=1}^{n} \frac{\omega_k}{M_n} S_k$, *and* $M_n = \sum_{k=1}^{n} \omega_k$. *Let* w_n *be generated by*

$$\begin{cases} v_n = J_{u_n}^{(U,K)}(I - \gamma G^* G)w_n; \\ w_{n+1} = \alpha_n f(w_n) + (1 - \alpha_n)L_n v_n. \end{cases}$$

Suppose S_n *satisfies the AKTT condition*, $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$. *If the solution set* $\Omega = \mathrm{Fix}(S) \cap \Gamma$ *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n \in (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iii) $\sum_{n=0}^{\infty} |u_{n+1} - u_n| < \infty$.

Then the sequence w_n *converges strongly to a point* w^*, *where* $w^* = P_\Omega f(w^*)$.

It should be noted that by Bruck's lemma [16] and He-Guo's lemma [17] each L_n is also nonexpansive mapping and $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$.

Table 1 Numerical results for some initial points $(x_0, y_0) = (0.01, 0.01), (1, 1), (15, 15)$

Initial point	ε	Iter.	Time
(0.01, 0.01)	0.00001	2	0.00251
(0.01, 0.01)	0.00001	2	0.00268
(1, 1)	0.00001	5	0.00452
(1, 1)	0.00001	6	0.00550
(15, 15)	0.00001	8	0.00606
(15, 15)	0.00001	8	0.00790

Table 2 Numerical results for some different $u_n = 1, 0.5, 0.2, 0.1$

u_n	ε	Iter.	Time
1	0.00001	22	0.02175
1	0.00001	21	0.01469
0.5	0.00001	36	0.02851
0.5	0.00001	37	0.03045
0.2	0.00001	125	0.11522
0.2	0.00001	113	0.08003
0.1	0.00001	1223	0.94707
0.1	0.00001	757	0.53070

4 Numerical example

In this section, we give an example and numerical results to illustrate our algorithms and the main result of this paper. All the experiment are performed on a personal Lenovo computer with Intel Core i3-2485M CPU 2.30 GHz and RAM 2.00 GB.

Example 4.1 Let $H_1 = R^4$, $H_2 = R^4$, $H_3 = R^4$, two operators of matrix multiplication $U : R^4 \to R^4$, $K : R^4 \to R^4$ defined by $U(x) = T_1(x)$, $K(x) = T_2(x)$, where

$$T_1 = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

put $S_n(x) = \frac{1}{1+2n}x$, $\sigma = 1$, $D = I$, then U, K, S_n satisfy all conditions of Theorem 3.1 and Corollary 3.1. We know (12) is equivalent to the following step:

$$\begin{cases} x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)S_n J_{u_n}^U(x_n - \gamma A^T(Ax_n - By_n)); \\ y_{n+1} = \alpha_n f_2(y_n) + (1-\alpha_n)S_n J_{u_n}^K(y_n + \gamma B^T(Ax_n - By_n)). \end{cases}$$

Note that if T_1, T_2 are positive linear operators, then they are maximal monotone. We defined the resolvent mappings $J_{u_n}^U = (I + u_n U)^{-1}$, $J_{u_n}^K = (I + u_n K)^{-1}$, where $u_n > 0$. Then we present the following algorithm.

Algorithm 4.1

Step 0. Choose initial point $(x_0, y_0) \in (0, 1 \times 10^5) \times (0, 1 \times 10^5)$, $c > 0$, $\gamma \in (0, \frac{2}{\lambda_A + \lambda_A})$ arbitrarily and put $n = 1$.

Step 1. Compute (x_{n+1}, y_{n+1}) as follows:

$$\begin{cases} x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)S_n J_{u_n}^U(x_n - \gamma A^T(Ax_n - By_n)); \\ y_{n+1} = \alpha_n f_2(y_n) + (1-\alpha_n)S_n J_{u_n}^K(y_n + \gamma B^T(Ax_n - By_n)). \end{cases}$$

Step 2. Set $\|Ax_n - By_n\| < \varepsilon$ as the stop criterion, else set $n = n + 1$ and go to step 1.

Table 1 shows the numerical results of Algorithm 4.1 with different initial points.

Table 2 shows that decreasing of u_n has an effect on the number of iterations, that is, u_n will converge faster to a solution when u_n is increased.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Acknowledgements

We wish to thank the referees for their helpful comments and suggestions. This research was supported by NSFC Grant No. 11071279 and Tian Yuan Special Foundation (No. 11426167).

Received: 8 August 2015 Accepted: 16 November 2015 Published online: 01 December 2015

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