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The base of warped product submanifolds of Sasakian space forms characterized by differential equations

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Abstract

In the present paper, we find some characterization theorems. Under certain pinching conditions on the warping function satisfying some differential equation, we show that the base of warped product submanifolds of a Sasakian space form $\widetilde{M}^{2m+1}(\epsilon)$ is isometric either to a Euclidean space $\mathbb{R}^n$ or a warped product of a complete manifold N and the Euclidean line $\mathbb{R}$.

Keywords: C-totally real warped products; Sasakian space forms; Ordinary differential equation; Isometric immersions

1 Introduction and main results

One of the significant motives for the geometric analysis of Riemannian manifolds (Ω, g) is the study of the impact of differential equations on its geometry, as well as isometric properties. Furthermore, it is well-known that their classification has an extensive influence on the global analysis of a Riemannian manifold with differential equations. It should be noted that [21, 31, 33, 34] gave characterizations of Euclidean spaces by analyzing differential equations. They showed that a nonconstant function ψ on a complete manifold (Ω^n, g) satisfies the following equation:

$$\nabla^2 \psi + cg = 0 \tag{1}$$

if and only if (Ω^n, g) is isometric to some Euclidean space $\mathbb{R}^n$, where c is any positive constant. Another characterization using a differential equation has been discovered by Río, Kupeli, and Unal [21]. They demonstrated that the complete Riemannian manifold (Ω^n, g) is isometric to the warped product of a complete Riemannian manifold N and a Euclidean line $\mathbb{R}$ with warping function θ satisfying the differential equation

$$\frac{d^2\theta}{dt^2} + \lambda_1 \theta = 0 \tag{2}$$

if and only if there exists a real-valued nonconstant function ψ associated to a negative eigenvalue $\lambda_1 \leq 0$, which satisfies the following differential equation:

$$\Delta \psi + \lambda_1 \psi = 0. \tag{3}$$

Such complete space classifications are extremely attractive and have been studied by several mathematicians (see, e.g., [3, 4, 6, 7, 15, 16, 18–20]). For example, by using (1), Al, Dayel, Deshmukh, and Belova [1] showed that a connected and complete Riemannian manifold (Ω^n, g) is isometric to $\mathbb{R}^n$ if and only if the nontrivial concircular vector field **u** along the function ψ satisfies $R(\nabla \psi, \nabla \psi) = 0$ or $\Delta \mathbf{u} = 0$. In [13], Chen and Deshmukh proved that a complete Riemannian manifold admits a concurrent vector field if and only if it is isometric to a Euclidean space by (1). Similarly, in [14], it has been shown that (Ω^n, g) is isometric to a Euclidean space if and only if (Ω^n, g) permits a nontrivial gradient conformal vector field, that is, a Jacobi-type vector field. On the other hand, Matsuyama [24] derived a characterization stating that if the complete totally real submanifold Ω^n for the complex projective space $C\mathbb{P}^n$ with bounded Ricci curvature admits a function ψ satisfying (3), for $\lambda_1 \leq n$, then Ω^n is isometric to the hyperbolic space component that is connected if $(\nabla \psi)_x = 0$ or if it is isometric to the warped product of a complete Riemannian manifold and the Euclidean line if $\nabla \psi$ is nonvanishing, where the warping function θ on $\mathbb{R}$ satisfies equation (2). Furthermore, similar results have been obtained for generalized Sasakian space forms by Jamali and Shahid [22]. In this study, inspired by [1–3, 5, 7, 9–12, 22, 30, 35], we derive a similar characterization for C-totally real warped product submanifolds of Sasakian space forms as rigidity theorems. To prove our main result, the next lemma, which was proved in [26], will be stated.

Lemma 1.1 *Suppose $\widetilde{M}^{2m+1}(\epsilon)$ is a Sasakian space form and let $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \widetilde{M}^{2m+1}(\epsilon)$ be a C-totally real immersion of the warped product submanifold Ω^n into $\widetilde{M}^{2m+1}(\epsilon)$ such that the base $\mathbb{B}$ is minimal. Then, the Ricci inequality is given as*

$$\mathscr{R}ic(X) + q\Delta \ln f \leq \frac{n^2}{4}\|H\|^2 + q\|\nabla \ln f\|^2 + \frac{\epsilon+3}{4}\{pq + n - 1\}, \tag{4}$$

for every unit vector $X \in T_x\Omega^n$, where $p = \dim \mathbb{B}$ and $q = \dim \mathbb{F}$. The quantities in the above inequality have been discussed, in detail, in [26].

The following abbreviations are used towards the end of this paper: 'SSF' stands for Sasakian space form, 'WF' for warping function, and 'WPS' for warped product submanifold. More precisely, we give the next theorem:

Theorem 1.1 *Let $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{M}^{2m+1}(\epsilon)$ be a C-totally real isometric immersion from a WPS Ω^n into the SSF $\mathbb{M}^{2m+1}(\epsilon)$ such that the Ricci curvature is bounded below by a positive constant $K > 0$. Then, a complete minimal base $\mathbb{B}$ is isometric to a Euclidean space $\mathbb{R}^p$ if the following equality holds:*

$$(\lambda_1 + q)K = \lambda_1 \left\{ \frac{q\lambda_1}{p} + \frac{n^2}{4}\|H\|^2 + \frac{(\epsilon+3)}{4}(pq + n - 1) \right\}. \tag{5}$$

The next result is motivated by the study of Río, Kupeli, and Unal [21].

Theorem 1.2 *Assuming that* $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{M}^{2m+1}(\epsilon)$ *is a C-totally real isometric immersion of a complete WPS* Ω^n *into the SSF* $\mathbb{M}^{2m+1}(\epsilon)$ *such that its Ricci curvature is bounded below by a positive constant* $K > 0$. *Let the complete base* $\mathbb{B}$ *be minimal in* $\mathbb{M}^{2m+1}(\epsilon)$ *and satisfy the following assumption*:

$$n^2|\mathbb{H}|^2 + \frac{4pq}{\lambda_1}\left|\mathrm{Hess}(\psi)\right|^2 = \frac{4p}{\lambda_1}\left(\frac{\epsilon+3}{4}(1 - pq - n) + K\right) \tag{6}$$

for $\lambda_1 < 0$. *Then,* $\mathbb{B}$ *is isometric to a warped product of the form* $\mathbb{R} \times_\theta N$ *with the warping function* θ *satisfying the following differential equation*:

$$\frac{d^2\theta}{dt^2} + \lambda_1\theta = 0.$$

Remark 1.1 The paper deals with ordinary differential equations on C-totally real warped product submanifolds. By optimizing the warping function of a C-totally real warped product submanifold of Sasakian space forms, we studied characterizations theorems for a C-totally real warped product submanifold of Sasakian space forms. Therefore, the paper exhibits an excellent combination of the theory of ordinary differential equations with Riemannian geometry.

2 Notation and formulas

The *almost contact metric manifold* $(\widetilde{M}, g)$ with Riemannian metric g preserves the following conditions:

$$\phi^2 = -I + \xi \otimes \eta,$$
$$\eta(\xi) = 1, \qquad \phi(\xi) = 0, \qquad \eta \circ \phi = 0,$$
$$g(\phi W_1, \phi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2),$$
$$\eta(W_1) = g(W_1, \xi), \tag{7}$$

for the almost contact structure (ϕ, η, ζ) and $\forall\ W_1, W_2 \in \Gamma(T\widetilde{M})$. A manifold $\widetilde{M}^{2m+1}$ is defined to be a *Sasakian manifold* if the following relation holds:

$$(\widetilde{\nabla}_{W_1}\phi)W_2 = g(W_1, W_2)\zeta - \eta(W_2)W_1. \tag{8}$$

It follows that

$$\widetilde{\nabla}_{W_1}\zeta = -\phi W_1, \tag{9}$$

for every $W_1, W_2 \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to the metric g. A *Sasakian space form* is a Sasakian manifold considering constant ϕ-sectional curvature ϵ, which is also defined as $\widetilde{M}^{2m+1}(\epsilon)$. Consequently, in [6, Eq. (6)], the Riemannian curvature tensor of $\widetilde{M}^{2m+1}(\epsilon)$ is defined in detail, which is also usually defined from $\widetilde{R}$. If the structure field ξ is perpendicular to the submanifold Ω^n in $\widetilde{M}^{2m+1}(\epsilon)$, then Ω^n is a C-totally real submanifold of $\widetilde{M}^{2m+1}(\epsilon)$. Furthermore, in this case, ϕ maps any tangent space of Ω^n into its corresponding normal space (see [2, 8, 23, 25, 32, 36]). Now, we recall

the Bochner formula [8] for a differentiable function on a Riemannian manifold Ω^n, that is, $\psi : \Omega^n \to \mathbb{R}$. Then, we have that

$$\frac{1}{2}\Delta|\nabla\psi|^2 = \mathrm{Ric}_{\Omega^n}(\nabla\psi, \nabla\psi) + \left|\mathrm{Hess}(\psi)\right|^2 + g\bigl(\nabla(\Delta\psi), \nabla\psi\bigr), \tag{10}$$

where the Ricci tensor of Ω^n is denoted by Ric.

3 The main results
3.1 Proof of Theorem 1.1
Equation (4) gives

$$\mathscr{R}ic(X) + q\Delta\psi \le \frac{n^2}{4}\|H\|^2 + q\|\nabla\psi\|^2 + \frac{\epsilon+3}{4}\{pq + n - 1\}.$$

Assuming the Ricci curvature is bounded below by a positive constant $K > 0$ (i.e., $\mathscr{R}ic(X) \ge K$), we get

$$K + q\Delta\psi \le \frac{n^2}{4}\|H\|^2 + q\|\nabla\psi\|^2 + \frac{\epsilon+3}{4}\{pq + n - 1\}. \tag{11}$$

One of the most famous results connecting the curvature and topology of the complete Riemannian manifold Ω^n is a famous theorem of Myers [29], which states that if the Ricci curvature with respect to unit vectors on $\mathbb{B}$ is bounded by a positive constant $K > 0$, then $\mathbb{B}$ is compact. Then, integrating (11) and using Green's lemma, we find that

$$\mathrm{Vol}(\mathbb{B})K \le \frac{n^2}{4}\int_{\mathbb{B}\times\{q\}}|\mathbb{H}|^2\,dV + q\int_{\mathbb{B}\times\{q\}}|\nabla\psi|^2\,dV + \int_{\mathbb{B}\times\{q\}}\frac{\epsilon+3}{4}\{pq + n - 1\}\,dV.$$

This can be written as

$$\int_{\mathbb{B}\times\{q\}}|\nabla\psi|^2\,dV \ge \frac{K}{q}\mathrm{Vol}(\mathbb{B}) - \frac{n^2}{4q}\int_{\mathbb{B}\times\{q\}}|\mathbb{H}|^2\,dV - \frac{1}{q}\int_{\mathbb{B}\times\{q\}}\frac{\epsilon+3}{4}\{pq + n - 1\}\,dV. \tag{12}$$

On the other hand, we have

$$\left|\mathrm{Hess}(\psi) - tI\right|^2 = \left|\mathrm{Hess}(\psi)\right|^2 + t^2|I|^2 - 2tg\bigl(I, \mathrm{Hess}(\psi)\bigr),$$

which leads to

$$\left|\mathrm{Hess}(\psi) - tI\right|^2 = 2t\Delta\psi + t^2 p + \left|\mathrm{Hess}(\psi)\right|^2.$$

Substituting $t = \frac{\lambda_1}{p}$ and integrating the preceding equation with respect to the volume element dV, we obtain

$$\int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi) - \frac{\lambda_1}{p}I\right|^2\,dV = \int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi)\right|^2\,dV + \int_{\mathbb{B}\times\{q\}}\frac{\lambda_1^2}{p}\,dV. \tag{13}$$

Using the Bochner formula (10), along with the fact that $\Delta\psi = \lambda_1\psi$, we have

$$\int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi)\right|^2 dV = -\lambda_1\int_{\mathbb{B}\times\{q\}}\|\nabla\psi\|^2 dV - \int_{\mathbb{B}\times\{q\}}\mathrm{Ric}(\nabla\psi,\nabla\psi)\,dV. \tag{14}$$

Combining Eqs. (13) and (14), we derive

$$\int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi) - \frac{\lambda_1}{p}I\right|^2 dV = \int_{\mathbb{B}\times\{q\}}\frac{\lambda_1^2}{p}\,dV - \lambda_1\int_{\mathbb{B}\times\{q\}}\|\nabla\psi\|^2 dV$$
$$- \int_{\mathbb{B}\times\{q\}}\mathrm{Ric}(\nabla\psi,\nabla\psi)\,dV. \tag{15}$$

As we assumed that $\mathrm{Ric}(\nabla\psi,\nabla\psi) \geq K$ for $K > 0$, we have

$$\int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi) - \frac{\lambda_1}{p}I\right|^2 dV \leq \int_{\mathbb{B}\times\{q\}}\frac{\lambda_1^2}{p}\,dV - \lambda_1\int_{\mathbb{B}\times\{q\}}\|\nabla\psi\|^2 dV - K\mathrm{Vol}(\mathbb{B}).$$

Inserting Eq. (12) into the above equation, we derive

$$\int_{\mathbb{B}\times\{q\}}\left|\mathrm{Hess}(\psi) - \frac{\lambda_1}{p}I\right|^2 dV \leq \int_{\mathbb{B}\times\{q\}}\frac{\lambda_1^2}{p}\,dV - \int_{\mathbb{B}\times\{q\}}\left(\frac{\lambda_1 K}{q} + K\right)dV$$
$$+ \frac{\lambda_1 n^2}{4q}\int_{\mathbb{B}\times\{q\}}|\mathbb{H}|^2 dV + \frac{\lambda_1}{q}\int_{\mathbb{B}\times\{q\}}\frac{\epsilon+3}{4}(pq+n-1)\,dV. \tag{16}$$

If (5) is satisfied, then (16) implies that

$$\left|\mathrm{Hess}(\psi) - \frac{\lambda_1}{p}I\right|^2 = 0.$$

Hence, we get

$$\mathrm{Hess}(\psi)(V,V) = cg(V,V), \tag{17}$$

for any $V \in \Gamma(\mathbb{B})$ with constant $c = \frac{\lambda_1}{p}$. Therefore, by applying the Tashiro theorems [31, 34], we obtain that $\mathbb{B}$ is isometric to a Euclidean space $\mathbb{R}^p$.

3.2 Proof of Theorem 1.2

Let us define the following equation with $\psi = \ln f$. We have

$$\left|t\psi I + \mathrm{Hess}(\psi)\right|^2 = t^2(\psi)^2|I|^2 + \left|\mathrm{Hess}(\psi)\right|^2 + 2t\psi g\big(I,\mathrm{Hess}(\psi)\big).$$

However, it is well-known that $|I|^2 = \mathrm{tr}(II^*) = p$, as well as $g(\mathrm{Hess}(\psi),I^*) = \mathrm{tr}(I^*\mathrm{Hess}(\psi)) = \mathrm{tr}(\mathrm{Hess}(\psi))$. Thus, the preceding equation takes the form

$$\left|t\psi I + \mathrm{Hess}(\psi)\right|^2 = \left|\mathrm{Hess}(\psi)\right|^2 + pt^2(\psi)^2 - 2t\psi\Delta\psi. \tag{18}$$

If ψ is an eigenfunction associated to the eigenvalue λ_1 such that $\Delta\psi = \lambda_1\psi$, then we get

$$\left|t\psi I + \mathrm{Hess}(\psi)\right|^2 = \left|\mathrm{Hess}(\psi)\right|^2 + \left(pt^2 - 2t\lambda_1\right)(\psi)^2. \tag{19}$$

On the other hand, we obtain

$$\Delta\frac{\psi^2}{2} = \psi\Delta\psi - |\nabla\psi|^2.$$

Using $\Delta\psi = \lambda_1\psi$ again, we have

$$\Delta\frac{\psi^2}{2} = \psi\lambda_1\psi - |\nabla\psi|^2,$$

which implies that

$$\int_{\mathbb{B}\times\{q\}} (\psi)^2\, dV = \frac{1}{\lambda_1}\int_{\mathbb{B}\times\{q\}} |\nabla\psi|^2\, dV. \tag{20}$$

It follows, from (19) and (20), that

$$\int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi) + t\psi I\right|^2 dV = \int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi)\right|^2 dV + \left(\frac{pt^2}{\lambda_1} - 2t\right)\int_{\mathbb{B}\times\{q\}} |\nabla\psi|^2\, dV. \tag{21}$$

In particular, setting $t = \frac{\lambda_1}{p}$ in (21) and integrating, we get

$$\int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi) + \frac{\lambda_1}{p}\psi I\right|^2 dV = \int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi)\right|^2 dV - \frac{\lambda_1}{p}\int_{\mathbb{B}\times\{q\}} |\nabla\psi|^2\, dV. \tag{22}$$

Again taking the integral of (4) and involving the Green lemma, we have

$$\int_{\mathbb{B}\times\{q\}} \mathscr{R}ic_M(X)\, dV \leq \frac{n^2}{4}\int_{\mathbb{B}\times\{q\}} |\mathbb{H}|^2\, dV + q\int_{\mathbb{B}\times\{q\}} |\nabla\psi|^2\, dV$$

$$+ \int_{\mathbb{B}\times\{q\}} \frac{\epsilon + 3}{4}(pq + n - 1)\, dV. \tag{23}$$

From (22) and (23), we can obtain

$$\frac{1}{q}\int_{\mathbb{B}\times\{q\}} \mathscr{R}ic_M(X)\, dV \leq \frac{n^2}{4q}\int_{\mathbb{B}\times\{q\}} |\mathbb{H}|^2\, dV - \frac{p}{\lambda_1}\int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi) + \frac{\lambda_1}{n}\psi I\right|^2 dV$$

$$+ \frac{p}{\lambda_1}\int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi)\right|^2 dV + \int_{\mathbb{B}\times\{q\}} \frac{\epsilon + 3}{4}\left(p + 1 + \frac{p-1}{q}\right) dV.$$

As we considered that the Ricci curvature is bounded (i.e., $\mathscr{R}ic(X) \geq K$ for some $K > 0$), the preceding equation implies that

$$\int_{\mathbb{B}\times\{q\}} \left|\mathrm{Hess}(\psi) + \frac{\lambda_1}{p}\psi I\right|^2 dV$$

$$\leq \frac{n^2 \lambda_1}{4pq} \int_{\mathbb{B} \times \{q\}} |\mathbb{H}|^2 \, dV + \int_{\Omega^n} \left| \text{Hess}(\psi) \right|^2 dV - \frac{\lambda_1}{pq} \int_{\mathbb{B} \times \{q\}} K \, dV$$
$$+ \frac{\lambda_1}{p} \int_{\mathbb{B} \times \{q\}} \frac{\epsilon + 3}{4} \left(p + 1 + \frac{p-1}{q} \right) dV,$$

which is equivalent to the following:

$$\int_{\mathbb{B} \times \{q\}} \left| \text{Hess}(\psi) + \frac{\lambda_1}{p} \psi I \right|^2 dV$$
$$\leq \int_{\mathbb{B} \times \{q\}} \left| \text{Hess}(\psi) \right|^2 dV$$
$$+ \frac{\lambda_1}{p} \int_{\mathbb{B} \times \{q\}} \left\{ \frac{n^2}{4q} |\mathbb{H}|^2 + \frac{(\epsilon + 3)}{4} \left(p + 1 + \frac{p-1}{q} \right) - \frac{K}{q} \right\} dV. \tag{24}$$

This gives us the following inequality:

$$\int_{\mathbb{B} \times \{q\}} \left| \text{Hess}(\psi) + \frac{\lambda_1}{p} \psi I \right|^2 dV$$
$$\leq \int_{\mathbb{B} \times \{q\}} \left\{ \frac{\lambda_1}{p} \left(\frac{n^2}{4q} |\mathbb{H}|^2 + \frac{(\epsilon + 3)}{4} \left(p + 1 + \frac{p-1}{q} \right) - \frac{K}{q} \right) + \left| \text{Hess}(\psi) \right|^2 \right\} dV. \tag{25}$$

Our assumption is satisfied, that is,

$$n^2 |\mathbb{H}|^2 + \frac{4pq}{\lambda_1} \left| \text{Hess}(\psi) \right|^2 = \frac{4pq}{\lambda_1} \left(\frac{(\epsilon + 3)}{4q} (1 - pq - n) + \frac{K}{q} \right). \tag{26}$$

Combining (25) and (26), we get

$$\left| \text{Hess}(\psi) + \frac{\lambda_1}{p} \psi I \right|^2 \leq 0.$$

The above equation gives us

$$\text{Hess}(\psi) + \frac{\lambda_1}{p} \psi I = 0. \tag{27}$$

Taking the trace of the preceding equation, we can derive

$$\Delta \psi + \lambda_1 \psi = 0. \tag{28}$$

According to [21], the base $\mathbb{B}$ is isometric to the connected components of a hyperbolic space if $(\nabla \psi)_x = 0$. However, $(\nabla \psi)_x = 0$ leads to a contradiction, as Ω^n is a nontrivial warped product. Hence, $\mathbb{B}$ is isometric to a warped product of the type $\mathbb{R} \times_\theta N$, where N is a complete Riemannian manifold and $\mathbb{R}$ is the Euclidean line. Moreover, the warping function θ satisfies the following differential equation:

$$\frac{d^2 \theta}{dt^2} + \lambda_1 \theta = 0.$$

Thus, the proof is completed.

Remark 3.1 It is well-known that $\mathbb{R}^{2m+1}(-3)$ and $\mathbb{S}^{2m+1}(1)$, considering standard Sasakian structures, may be seen as classical examples of Sasakian space forms with constant sectional curvature $\epsilon = -3$ and $\epsilon = 1$, respectively [25, 27].

We produce a striking application of Theorem 1.1 and Remark 3.1 by selecting $\epsilon = 1$ (see [2]):

Corollary 3.1 *Suppose* $\Psi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{S}^{2m+1}(1)$ *is a C-totally real isometric immersion of a complete WPS* Ω^n *into the SSF* $\mathbb{S}^{2m+1}(1)$ *with Ricci curvature bounded below by a positive constant* $K > 0$ *and the base* $\mathbb{B}$ *is minimal in* $\mathbb{S}^{2m+1}(1)$*. Then,* $\mathbb{B}$ *is isometric to the Euclidean space* $\mathbb{R}^p$ *if the following equality holds*:

$$(\lambda_1 + q)K = \lambda_1 \left\{ \frac{q\lambda_1}{p} + \frac{n^2}{4} \|H\|^2 + (pq + n - 1) \right\}. \tag{29}$$

For the minimal case (i.e., $\|H\|^2 = 0$), we give the following corollary:

Corollary 3.2 *Let* $\Psi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{S}^{2m+1}(1)$ *be a C-totally real minimal isometric immersion of a complete WPS* Ω^n *into the SSF* $\mathbb{S}^{2m+1}(1)$ *such that the Ricci curvature is bounded below by a positive constant* $K > 0$ *satisfying the condition*

$$(\lambda_1 + q)K = \lambda_1 \left\{ \frac{q\lambda_1}{p} + (pq + n - 1) \right\}. \tag{30}$$

Then, $\mathbb{B}$ *is isometric to a Euclidean space* $\mathbb{R}^p$*.*

Following Theorem 1.2, we give the following corollary:

Corollary 3.3 *Assume that* $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{S}^{2m+1}(1)$ *is a C-totally real isometric immersion of the complete WPS* Ω^n *into the SSF* $\mathbb{S}^{2m+1}(1)$ *such that Ricci curvature is bounded below by a positive constant* $K > 0$ *and the base* $\mathbb{B}$ *is minimal in* $\mathbb{S}^{2m+1}(1)$*, satisfying the assumption*

$$n^2 |\mathbb{H}|^2 + \frac{4pq}{\lambda_1} \left| \text{Hess}(\psi) \right|^2 = \frac{4p}{\lambda_1} \left\{ (1 - pq - n) + K \right\}. \tag{31}$$

Then, $\mathbb{B}$ *is isometric to a warped product of the form* $\mathbb{R} \times_\theta N$ *with the warping function* θ *satisfying the following differential equation*:

$$\frac{d^2\theta}{dt^2} + \lambda_1 \theta = 0.$$

Corollary 3.4 *Let* $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{S}^{2m+1}(1)$ *be a C-totally real minimal isometric immersion of a complete WPS* Ω^n *into the SSF* $\mathbb{S}^{2m+1}(1)$ *such that Ricci curvature is bounded below by a positive constant* $K > 0$ *satisfying the assumption*

$$q \left| \text{Hess}(\psi) \right|^2 = \left\{ (1 - pq - n) + K \right\}. \tag{32}$$

Then, $\mathbb{B}$ *is isometric to a warped product of the form* $\mathbb{R} \times_\theta N$ *with warping function* θ *satisfying the differential equation* (2).

Substituting the constant sectional curvature $\epsilon = -3$ into Theorems 1.1 and 1.2, we can directly derive the following:

Corollary 3.5 *Let* $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \mathbb{R}^{2m+1}(-3)$ *be a C-totally real isometric immersion for the complete WPS* Ω^n *into the SSF* $\mathbb{R}^{2m+1}(-3)$ *such that Ricci curvature is bounded below by a positive constant* $K > 0$.

(i) *If the base* $\mathbb{B}$ *is minimal in* $\mathbb{R}^{2m+1}(-3)$, *then* $\mathbb{B}$ *is isometric to the Euclidean space* $\mathbb{R}^p$ *if the following equality holds*:

$$(\lambda_1 + q)K = \lambda_1 \left\{ \frac{q\lambda_1}{p} + \frac{n^2}{4} \|H\|^2 \right\}. \tag{33}$$

(ii) *If* Φ *is a minimal isometric immersion in* $\mathbb{R}^{2m+1}(-3)$ *and* $p(\lambda_1 + q)K = q\lambda_1^2$ *is satisfied, then* $\mathbb{B}$ *is isometric to the Euclidean space* $\mathbb{R}^p$.

Using Theorem 1.2, we obtain the following:

Corollary 3.6 *Under the same assumptions of Corollary 3.5, we have the following*:

(i) *If the base* $\mathbb{B}$ *is minimal in* $\mathbb{R}^{2m+1}(-3)$ *and the following equality holds*:

$$\lambda_1 n^2 |\mathbb{H}|^2 + 4pq \left| \mathrm{Hess}(\psi) \right|^2 = 4pK, \tag{34}$$

then $\mathbb{B}$ *is isometric to a warped product of the form* $\mathbb{R} \times_\theta N$ *with warping function* θ *satisfying the differential equation* (2).

(ii) *If* Φ *is the minimal isometric immersion in* $\mathbb{R}^{2m+1}(-3)$ *and* $|\mathrm{Hess}(\psi)|^2 = \frac{K}{q}$ *is satisfied, then* $\mathbb{B}$ *is isometric to the warped product of* $\mathbb{R} \times_\theta N$ *with the warping function* θ *satisfying the differential equation* (2).

Classifying the Dirichlet energy of smooth functions is treated as an integral procedure in the fields of physics and engineering. Moreover, the Dirichlet energy is formulated as an equivalent of kinetic energy. Let ψ be any real-valued smooth function on a compact manifold. Then, the Dirichlet energy of ψ is defined by

$$\mathcal{E}(\psi) = \frac{1}{2} \int \|\nabla \psi\|^2 \, dV. \tag{35}$$

Using the above formula and Lemma 1.1, we obtain the following theorem:

Theorem 3.1 *Suppose* $\widetilde{M}^{2m+1}(\epsilon)$ *is a Sasakian space form and let* $\Phi : \Omega^n = \mathbb{B} \times_f \mathbb{F} \to \widetilde{M}^{2m+1}(\epsilon)$ *be a C-totally real immersion of the warped product submanifold* Ω^n *into* $\widetilde{M}^{2m+1}(\epsilon)$ *such that the base* $\mathbb{B}$ *is minimal. Then, the Dirichlet energy inequality is given by*

$$\int_{\mathbb{B} \times \{q\}} \mathscr{R}ic(X) \, dV \leq \frac{n^2}{4} \int_{\mathbb{B} \times \{q\}} \|\mathbb{H}\|^2 \, dV + 2q\mathcal{E}(\psi) + \{pq + n - 1\} \int_{\mathbb{B} \times \{q\}} \frac{\epsilon + 3}{4} \, dV, \tag{36}$$

for every unit vector $X \in T_x \Omega^n$, *where* $p = \dim \mathbb{B}$ *and* $q = \dim \mathbb{F}$.

Proof Taking the integral of Eq. (4) and using the Green lemma, we get the required result (36). This completes the proof of the theorem.   □

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this manuscript. All authors read and approved the final manuscript.

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