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Existence results of nonlocal Robin mixed Hahn and q-difference boundary value problems

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Abstract

In this paper, we aim to study a nonlocal Robin boundary value problem for fractional sequential fractional Hahn-q-equation. The existence and uniqueness results for this problem are revealed by using the Banach fixed point theorem. In addition, the existence of at least one solution is studied by using Schauder's fixed point theorem. The theorems for existence results are obtained.

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1 Introduction

The quantum difference operator has been applied in many mathematical areas such as orthogonal polynomials, combinatorics, and the calculus of variations [1–4]. The research works related to the quantum difference operator have been published continuously.

q-difference operator is the one type of quantum calculus proposed by Jackson [1] which is defined by

$$D_q f(t) := \begin{cases} \frac{f(qt)-f(t)}{t(q-1)}, & t \neq 0, \\ f'(0), & t = 0, \end{cases}$$

where $q \in (0,1)$. For fractional q-difference operator, it was proposed by Al-Salam [5] and Agarwal [6]. Basic knowledge of fractional q-difference calculus can be found in [7] and [8]. The studies of q-difference operator can be found in [9–33].

Hahn difference operator proposed by Hahn [34] is another tool that can be employed to study families of orthogonal polynomials and approximation problems (see [35–37]) which is defined by

$$D_{q,\omega} f(t) := \begin{cases} \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, & t \neq \omega_0, \\ f'(\omega_0), & t = \omega_0, \end{cases}$$

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where $q \in (0,1)$, $\omega > 0$, and $\omega_0 := \frac{\omega}{1-q}$. The right inverse operator of Hahn's operator, which generalizes both the Nörlund sum and the Jackson q-integral, was proposed by Aldwoah [38, 39]. The new extensive results of Hahn difference operator can be seen in [40–47]. Recently, Brikshavana et al. [48] and Wang et al. [49] introduced the fractional Hahn difference operator. The studies of fractional Hahn difference calculus can be found in [50–56].

We observe that the study of the boundary value problem of mixed difference operators had not been studied until the work of Dumrongpokaphan et al. [57]. They studied sequential fractional q-Hahn-difference equation. In this paper, we propose a sequential fractional Hahn-q-difference equation where the difference operators are reverse. Our problem is a nonlocal Robin boundary value problem for sequential fractional Hahn-q-difference equation of the form

$$\begin{aligned} &D^{\alpha}_{q,\omega} D^{\beta}_{q} u(t) = F\big[t, u(t), D^{\theta}_{q} u(t), D^{\nu}_{q,\omega} u(t)\big], \quad t \in I^{[0,T]}_{q,\omega}, \\ &\lambda_1 u(\eta) + \lambda_2 D^{\gamma}_{q} u(\eta) = \phi_1(u), \quad \eta \in (0,T), \\ &\mu_1 u(T) + \mu_2 D^{\gamma}_{q,\omega} u(T) = \phi_2(u), \end{aligned} \tag{1.1}$$

where $I^{[0,T]}_{q,\omega} := \bigcup_{k=0}^{\infty} I^{q^k s}_{q,\omega}$, $s \in [0,T]$; $I^{x}_{q,\omega} := \{q^n x + \omega[n]_q : n \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $[n]_q := \frac{1-q^n}{1-q}$; $0 < q < 1$; $\omega > 0$; $T > \omega_0$; $\alpha, \beta, \gamma, \theta, \nu \in (0,1]$; $\alpha + \beta \in (1,2]$; $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$; $F \in C([0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function; and $\phi_1, \phi_2 : C([0,T], \mathbb{R}) \to \mathbb{R}$ are given functionals.

We organize the paper as follows. In Sect. 2, we provide some basic knowledge. In Sect. 3, we prove the existence and uniqueness of a solution to problem (1.1) by using the Banach fixed point theorem. In Sect. 4, we prove the existence of at least one solution to problem (1.1) by using Schauder's fixed point theorem. Finally, in the last section, we provide an example to show applications of our results.

2 Preliminaries

In this section, we recall some notations, definitions, and lemmas used in the main results.

For $q \in (0,1)$, $\omega > 0$, $n \in \mathbb{N}_0$, and $a, b \in \mathbb{R}$, we define the q-analogue of the power function $(a-b)^{\underline{n}}_{q}$ as

$$(a-b)^{\underline{0}}_{q} := 1, \qquad (a-b)^{\underline{n}}_{q} := \prod_{k=0}^{n-1} \big(a - bq^k\big),$$

and the q,ω-analogue of the power function $(a-b)^{\underline{n}}_{q,\omega}$ as

$$(a-b)^{\underline{0}}_{q,\omega} := 1, \qquad (a-b)^{\underline{n}}_{q,\omega} := \prod_{k=0}^{n-1} \big[a - \big(bq^k + \omega[k]_q\big)\big].$$

For $\alpha \in \mathbb{R}$, we define

$$(a-b)^{\underline{\alpha}}_{q} = a^{\alpha} \prod_{n=0}^{\infty} \frac{1 - (\frac{b}{a})q^n}{1 - (\frac{b}{a})q^{\alpha+n}}, \quad a \neq 0,$$

$$(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\underline{\alpha}} \prod_{n=0}^{\infty} \frac{1-(\frac{b-\omega_0}{a-\omega_0})q^n}{1-(\frac{b-\omega_0}{a-\omega_0})q^{\alpha+n}} = \big((a-\omega_0)-(b-\omega_0)\big)_q^{\underline{\alpha}}, \quad a \neq \omega_0.$$

In addition, we define $a_q^{\underline{\alpha}} = a^{\underline{\alpha}}$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^{\underline{\alpha}}$. For $\alpha > 0$, we let $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$.

For $k \in \mathbb{N}$, the q-analogue and q,ω-analogue of forward jump operator [58] are defined by

$$\sigma_q^k(t) := q^k t \quad \text{and} \quad \sigma_{q,\omega}^k(t) := q^k t + \omega[k]_q,$$

respectively. The q-analogue and q,ω-analogue of backward jump operator are defined by

$$\rho_q^k(t) := \frac{t}{q^k} \quad \text{and} \quad \rho_{q,\omega}^k(t) := \frac{t-\omega[k]_q}{q^k},$$

respectively.

Definition 2.1 For $q \in (0,1)$, the q-derivative of a real function f is defined by

$$D_q f(t) = \frac{f(t)-f(qt)}{(1-q)t} \quad \text{and} \quad D_q f(0) = f'(0).$$

The q-integral of a function f defined on the interval $[0,T]$ is defined by

$$\mathcal{I}_q f(t) = \int_0^t f(s)\,d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f\big(q^n t\big),$$

where the infinite series is convergent.

Definition 2.2 For $q \in (0,1)$, $\omega > 0$, and f defined on an interval $I \subseteq \mathbb{R}$ containing $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by

$$D_{q,\omega} f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega} \quad \text{for } t \neq \omega_0,$$

and $D_{q,\omega} f(\omega_0) = f'(\omega_0)$ whenever f is differentiable at ω_0.

For $a,b \in I \subseteq \mathbb{R}$, $a < \omega_0 < b$, and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we define the q,ω-interval by

$$\begin{aligned}[a,b]_{q,\omega} &:= \big\{q^k a + \omega[k]_q : k \in \mathbb{N}_0\big\} \cup \big\{q^k b + \omega[k]_q : k \in \mathbb{N}_0\big\} \cup \{\omega_0\} \\ &= [a,\omega_0]_{q,\omega} \cup [\omega_0,b]_{q,\omega} \\ &= (a,b)_{q,\omega} \cup \{a,b\} = [a,b)_{q,\omega} \cup \{b\} = (a,b]_{q,\omega} \cup \{a\}.\end{aligned}$$

For each $s \in [a,b]_{q,\omega}$, the sequence $\{\sigma_{q,\omega}^k(s)\}_{k=0}^{\infty} = \{q^k s + \omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to ω_0.

Definition 2.3 Let I be any closed interval of $\mathbb{R}$ containing a, b, and ω_0, and $f : I \to \mathbb{R}$ is a given function. We define q,ω-integral of f from a to b by

$$\int_a^b f(t)\,d_{q,\omega}t := \int_{\omega_0}^b f(t)\,d_{q,\omega}t - \int_{\omega_0}^a f(t)\,d_{q,\omega}t,$$

where $\int_{\omega_0}^x f(t)\,d_{q,\omega}t := [x(1-q)-\omega]\sum_{k=0}^{\infty} q^k f(xq^k+\omega[k]_q)$, $x \in I$, and the series converges at $x = a$ and $x = b$. The sum to the right-hand side of the above equation is called the Jackson–Nörlund sum.

We note that the actual domain of function f is $[a,b]_{q,\omega} \subset I$.

In what follows, we define fractional q-integral, fractional Hahn integral, fractional q-difference, and fractional Hahn difference of Riemann–Liouville type.

Definition 2.4 For $\alpha \geq 0$ and f defined on $[0,T]$, the fractional q-integral of Riemann–Liouville type is defined by

$$\begin{aligned}\left(\mathcal{I}_q^\alpha f\right)(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)_q^{\underline{\alpha-1}} f(t)\,d_q s \\ &= \frac{t(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(t-q^{n+1}t\right)_q^{\underline{\alpha-1}} f\left(q^n t\right) \\ &= \frac{t^\alpha(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(1-q^{n+1}\right)_q^{\underline{\alpha-1}} f\left(q^n t\right),\end{aligned}$$

and $(\mathcal{I}_q^0 f)(x) = f(x)$.

Definition 2.5 For $\alpha,\omega > 0$, $q \in (0,1)$, and f defined on $[\omega_0,T]_{q,\omega}$, the fractional Hahn integral is defined by

$$\begin{aligned}\mathcal{I}_{q,\omega}^\alpha f(t) &:= \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^t \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} f(s)\,d_{q,\omega}s \\ &= \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(t-\sigma_{q,\omega}^{n+1}(t)\right)_{q,\omega}^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^n(t)\right) \\ &= \frac{(1-q)(t-\omega_0)^\alpha}{\Gamma_q(\alpha)}\sum_{n=0}^{\infty} q^n\left(1-q^{n+1}\right)_q^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^n(t)\right),\end{aligned}$$

and $(\mathcal{I}_{q,\omega}^0 f)(t) = f(t)$.

Definition 2.6 For $N \in \mathbb{N}$, $\alpha \geq 0$, where $N-1 < \alpha \leq N$, and f defined on $[0,T]$, the fractional q-derivative of the Riemann–Liouville type of order α is defined by

$$\begin{aligned}\left(D_q^\alpha f\right)(t) &:= \left(D_q^N \mathcal{I}_q^{N-\alpha} f\right)(t) \\ &= \frac{1}{\Gamma_q(-\alpha)}\int_0^t \left(t-\sigma_q(s)\right)_q^{\underline{-\alpha-1}} f(s)\,d_q s,\end{aligned}$$

and $(D_q^0 f)(x) = f(x)$.

Definition 2.7 For $N \in \mathbb{N}$, $\alpha \geq 0$, where $N-1 < \alpha \leq N$, $q \in (0,1)$, $\omega > 0$, and f defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of Riemann–Liouville type of order α is defined by

$$\begin{aligned} D^{\alpha}_{q,\omega} f(t) &:= \left(D^{N}_{q,\omega} \mathcal{I}^{N-\alpha}_{q,\omega} f\right)(t) \\ &= \frac{1}{\Gamma_q(-\alpha)} \int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)^{\underline{-\alpha-1}}_{q,\omega} f(s)\, d_{q,\omega} s, \end{aligned}$$

and $D^{0}_{q,\omega} f(t) = f(t)$.

Lemma 2.1 ([17]) *For $N \in \mathbb{N}$, $\alpha \geq 0$, where $N-1 < \alpha \leq N$, $q \in (0,1)$, and $f : I^{T}_{q} \to \mathbb{R}$,*

$$\mathcal{I}^{\alpha}_{q} D^{\alpha}_{q} f(t) = f(t) + C_1 t^{\alpha-1} + \cdots + C_N t^{\alpha-N}$$

for some $C_i \in \mathbb{R}$, $i = \{1, 2, \ldots, N\}$.

Lemma 2.2 ([48]) *For $N \in \mathbb{N}$, $\alpha \geq 0$, where $N-1 < \alpha \leq N$, $q \in (0,1)$, $\omega > 0$, and $f : I^{T}_{q,\omega} \to \mathbb{R}$,*

$$\mathcal{I}^{\alpha}_{q,\omega} D^{\alpha}_{q,\omega} f(t) = f(t) + C_1 (t-\omega_0)^{\alpha-1} + \cdots + C_N (t-\omega_0)^{\alpha-N}$$

for some $C_i \in \mathbb{R}$, $i = \{1, 2, \ldots, N\}$.

The q-gamma and q-beta functions are defined by

$$\Gamma_q(x) := \frac{(1-q)^{\underline{x-1}}_{q}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},$$

$$B_q(x,s) := \int_0^1 t^{x-1} (1-qt)^{\underline{s-1}}_{q} \, d_q t = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)}, \quad \text{respectively.}$$

Next, we aim to find a solution of the linear variant of mixed problem (1.1) where the following auxiliary lemmas will be used for simplifying calculations.

Lemma 2.3 ([21]) *Let $\alpha, \beta \geq 0$ and $0 < p, q < 1$. Then the following formulas hold*:

$$\int_0^{\eta} (\eta - qt)^{\underline{\alpha-1}}_{q} t^{\beta} \, d_q t = \eta^{\alpha+\beta} B_q(\beta+1, \alpha),$$

$$\int_0^{\eta} \int_0^{s} (\eta - ps)^{\underline{\alpha-1}}_{p} (s - qt)^{\underline{\beta-1}}_{q} \, d_q t \, d_p s = \frac{\eta^{\alpha+\beta}}{[\beta]_q} B_q(\beta+1, \alpha).$$

Lemma 2.4 ([48]) *For $\alpha, \beta > 0$, $p, q \in (0,1)$, and $\omega > 0$,*

$$\int_{\omega_0}^{t} \left(t - \sigma_{q,\omega}(s)\right)^{\underline{\alpha-1}}_{q,\omega} (s-\omega_0)^{\beta} \, d_{q,\omega} s = (t-\omega_0)^{\alpha+\beta} B_q(\beta+1, \alpha),$$

$$\int_{\omega_0}^{t} \int_{\omega_0}^{x} \left(t - \sigma_{p,\omega}(x)\right)^{\underline{\alpha-1}}_{p,\omega} \left(x - \sigma_{q,\omega}(s)\right)^{\underline{\beta-1}}_{q,\omega} d_{q,\omega} s \, d_{p,\omega} x = \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q} B_q(\beta+1, \alpha).$$

Lemma 2.5 *Let $\alpha, \beta, \gamma \in (0,1]$, $\alpha + \beta \in (1,2]$; $0 < q < 1$; $\omega > 0$; $T > \omega_0$; $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^+$; $h \in C([0,T], \mathbb{R})$ be a given function; $\phi_1, \phi_2 : C([0,T], \mathbb{R}) \to \mathbb{R}$ be given functionals. Then the linear variant of problem (1.1) given by*

$$\begin{aligned} &D^{\alpha}_{q,\omega} D^{\beta}_{q} u(t) = h(t), \quad t \in I^{[0,T]}_{q,\omega}, \\ &\lambda_1 u(\eta) + \lambda_2 D^{\gamma}_{q} u(\eta) = \phi_1(u), \quad \eta \in (0,T), \\ &\mu_1 u(T) + \mu_2 D^{\gamma}_{q,\omega} u(T) = \phi_2(u) \end{aligned} \tag{2.1}$$

has the unique solution, which is in a form

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t \int_{\omega_0}^x \big(t - \sigma_q(s)\big)^{\frac{\beta-1}{q}} \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x \\ & + \big\{ \mathbf{A}_T \Phi_\eta\big[\phi_1(u), h\big] - \mathbf{A}_\eta \Phi_T\big[\phi_2(u), h\big] \big\} \frac{1}{\Omega \Gamma_q(\beta)} \int_0^t \big(t - \sigma_q(s)\big)^{\frac{\beta-1}{q}} (s - \omega_0)^{\alpha-1}\, d_q s \\ & - \big\{ \mathbf{B}_T \Phi_\eta\big[\phi_1(u), h\big] - \mathbf{B}_\eta \Phi_T\big[\phi_2(u), h\big] \big\} \frac{t^{\beta-1}}{\Omega} \end{aligned} \tag{2.2}$$

for $t \in [0,T]$, and the functionals $\Phi_\eta[\phi_1(u), h]$, $\Phi_T[\phi_2(u), h]$ are defined by

$$\begin{aligned} \Phi_\eta\big[\phi_1(u), h\big] := {} & \phi_1(u) - \frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^\eta \int_{\omega_0}^x \big(\eta - \sigma_q(x)\big)^{\frac{\beta-1}{q}} \\ & \times \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x \\ & - \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_0^\eta \int_0^r \int_{\omega_0}^x \big(\eta - \sigma_q(r)\big)^{\frac{-\gamma-1}{q}} \\ & \times \big(r - \sigma_q(x)\big)^{\frac{\beta-1}{q}} \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x\, d_q r, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \Phi_T\big[\phi_2(u), h\big] := {} & \phi_2(u) - \frac{\mu_1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^T \int_{\omega_0}^x \big(T - \sigma_q(x)\big)^{\frac{\beta-1}{q}} \\ & \times \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x \\ & - \frac{\mu_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_{\omega_0}^T \int_0^r \int_{\omega_0}^x \big(T - \sigma_{q,\omega}(r)\big)^{\frac{-\gamma-1}{q,\omega}} \\ & \times \big(r - \sigma_q(x)\big)^{\frac{\beta-1}{q}} \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x\, d_{q,\omega} r, \end{aligned} \tag{2.4}$$

and the constants $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, and Ω are defined by

$$\mathbf{A}_\eta := \lambda_1 \eta^{\beta-1} + \frac{\lambda_2}{\Gamma_q(-\gamma)} \int_0^\eta \big(\eta - \sigma_q(s)\big)^{\frac{-\gamma-1}{q}} s^{\beta-1}\, d_q s, \tag{2.5}$$

$$\mathbf{A}_T := \mu_1 T^{\beta-1} + \frac{\mu_2}{\Gamma_q(-\gamma)} \int_{\omega_0}^T \big(T - \sigma_{q,\omega}(s)\big)^{\frac{-\gamma-1}{q}} s^{\beta-1}\, d_{q,\omega} s, \tag{2.6}$$

$$\begin{aligned} \mathbf{B}_\eta := {} & \frac{\lambda_1}{\Gamma_q(\beta)} \int_0^\eta \big(\eta - \sigma_q(s)\big)^{\frac{\beta-1}{q}} (s - \omega_0)^{\alpha-1}\, d_q s \\ & + \frac{\lambda_2}{\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_0^\eta \int_0^x \big(\eta - \sigma_q(x)\big)^{\frac{-\gamma-1}{q}} \big(x - \sigma_q(s)\big)^{\frac{\beta-1}{q}} (s - \omega_0)^{\alpha-1}\, d_q s\, d_q x, \end{aligned} \tag{2.7}$$

$$\mathbf{B}_T := \frac{\mu_1}{\Gamma_q(\beta)} \int_0^T \big(T - \sigma_q(s)\big)_q^{\underline{\beta-1}} (s-\omega_0)^{\underline{\alpha-1}}\, d_q s + \frac{\mu_2}{\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_{\omega_0}^T \int_0^x \big(T - \sigma_{q,\omega}(x)\big)_{q,\omega}^{\underline{-\gamma-1}} \big(x - \sigma_q(s)\big)_q^{\underline{\beta-1}} (s-\omega_0)^{\underline{\alpha-1}}\, d_q s\, d_{q,\omega} x, \tag{2.8}$$

$$\Omega := \mathbf{A}_T \mathbf{B}_\eta - \mathbf{A}_\eta \mathbf{B}_T \neq 0. \tag{2.9}$$

Proof We first take fractional Hahn integral of order α for (2.1). Then the problem becomes fractional q-difference equation as follows:

$$\begin{aligned} D_q^\beta u(t) &= C_0 (t-\omega_0)^{\underline{\alpha-1}} + \frac{(1-q)(t-\omega_0)^{\underline{\alpha}}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k \big(1-q^{k+1}\big)_q^{\underline{\alpha-1}} h\big(\sigma_{q,\omega}^k(t)\big) \\ &= C_0 (t-\omega_0)^{\underline{\alpha-1}} + \frac{1}{\Gamma_q(\alpha)} \int_{\omega_0}^t \big(t - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} h(x)\, d_{q,\omega} s \end{aligned} \tag{2.10}$$

for $t \in I_q^{[0,T]} := \{q^n s : s \in [0,T], n \in \mathbb{N}_0\} \cup \{0\}$.

After taking fractional q-integral of order β for (2.10), we get the solution which is in the form

$$\begin{aligned} u(t) &= C_1 t^{\underline{\beta-1}} + \frac{C_0}{\Gamma_q(\beta)} (1-q) t^{\underline{\beta}} \sum_{k=0}^{\infty} q^k \big(1-q^{k+1}\big)_q^{\underline{\beta-1}} \big(\sigma_{q,\omega}^k(t) - \omega_0\big)^{\underline{\alpha-1}} \\ &\quad + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} (1-q)^2 (t-\omega_0)^{\underline{\beta}} \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} q^{h+k} \big(1-q^{h+1}\big)_q^{\underline{\beta-1}} \\ &\quad \times \big(1-q^{k+1}\big)_q^{\underline{\alpha-1}} \big(\sigma_{q,\omega}^h(t)\big)^{\underline{\alpha}} h\big(\sigma_{q,\omega}^k\big(\sigma_q^h(t)\big)\big) \\ &= C_1 t^{\underline{\beta-1}} + \frac{C_0}{\Gamma_q(\beta)} \int_0^t \big(t - \sigma_q(s)\big)_q^{\underline{\beta-1}} (s-\omega_0)^{\underline{\alpha-1}}\, d_q s \\ &\quad + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^t \int_{\omega_0}^x \big(t - \sigma_q(x)\big)_q^{\underline{\beta-1}} \big(x - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} h(s)\, d_{q,\omega} s\, d_q x \end{aligned} \tag{2.11}$$

for $t \in [0,T]$.

In order to find the unknown constants C_1 and C_0 that appeared in (2.11), we first take fractional q-difference and fractional Hahn difference of order γ for (2.11) to get

$$\begin{aligned} D_q^\gamma u(t) &= \frac{C_1}{\Gamma_q(-\gamma)} \int_0^t \big(t - \sigma_q(s)\big)_q^{\underline{-\gamma-1}} s^{\underline{\beta-1}}\, d_q s \\ &\quad + \frac{C_0}{\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_0^t \int_0^x \big(t - \sigma_q(x)\big)_q^{\underline{-\gamma-1}} \big(x - \sigma_q(s)\big)_q^{\underline{\beta-1}} (s-\omega_0)^{\underline{\alpha-1}}\, d_q s\, d_q x \\ &\quad + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_0^t \int_0^r \int_{\omega_0}^x \big(t - \sigma_q(r)\big)_q^{\underline{-\gamma-1}} \big(r - \sigma_q(x)\big)_q^{\underline{\beta-1}} \\ &\quad \times \big(x - \sigma_{q,\omega}(s)\big)_{q,\omega}^{\underline{\alpha-1}} h(s)\, d_q s\, d_q x\, d_{q,\omega} r \end{aligned} \tag{2.12}$$

for $t \in [0, T]$, and

$$\begin{aligned} D^{\gamma}_{q,\omega}u(t) &= \frac{C_1}{\Gamma_q(-\gamma)} \int_{\omega_0}^{t} \big(t - \sigma_{q,\omega}(s)\big)^{\frac{-\gamma-1}{q,\omega}} s^{\beta-1}\, d_{q,\omega}s \\ &\quad + \frac{C_0}{\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_{\omega_0}^{t} \int_{0}^{x} \big(t - \sigma_{q,\omega}(x)\big)^{\frac{-\gamma-1}{q,\omega}} \big(x - \sigma_q(s)\big)^{\frac{\beta-1}{q}} (s-\omega_0)^{\alpha-1}\, d_q s\, d_{q,\omega}x \\ &\quad + \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)} \int_{\omega_0}^{t} \int_{0}^{r} \int_{\omega_0}^{x} \big(t - \sigma_{q,\omega}(r)\big)^{\frac{-\gamma-1}{q,\omega}} \big(r - \sigma_q(x)\big)^{\frac{\beta-1}{q}} \\ &\quad \times \big(x - \sigma_{q,\omega}(s)\big)^{\frac{\alpha-1}{q,\omega}} h(s)\, d_{q,\omega}s\, d_q x\, d_{q,\omega}r \end{aligned} \tag{2.13}$$

for $t \in [\omega_0, T]$, respectively.

Substitute $t = \eta$ into (2.11) and (2.12) and use the first condition of (2.1). Then we get

$$\mathbf{A}_\eta C_1 + \mathbf{B}_\eta C_0 = \Phi_\eta[\phi_1, h]. \tag{2.14}$$

Similarly, substitute $t = T$ into (2.11) and (2.13) and employ the second condition of (2.1). We have

$$\mathbf{A}_T C_1 + \mathbf{B}_T C_0 = \Phi_T[\phi_2, h]. \tag{2.15}$$

We can solve the linear system for (2.14)–(2.15), and we find that

$$C_1 = \frac{\mathbf{B}_\eta \Phi_T[\phi_2(u), h] - \mathbf{B}_T \Phi_\eta[\phi_1(u), h]}{\Omega}$$

and

$$C_0 = \frac{\mathbf{A}_T \Phi_\eta[\phi_1(u), h] - \mathbf{A}_\eta \Phi_T[\phi_2(u), h]}{\Omega},$$

where $\Phi_\eta[\phi_1(u), h]$, $\Phi_T[\phi_2(u), h]$, $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, Ω are defined by (2.3)–(2.9), respectively.

Solution (2.2) can be exposed after substituting C_1 and C_0 into (2.11). □

3 Existence and uniqueness result

In this section, we employ the Banach fixed point theorems to consider the existence and uniqueness result for problem (1.1). Let $\mathcal{C} = C([0, T], \mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}} = \|u\| + \big\|D^{\theta}_q u\big\| + \big\|D^{v}_{q,\omega} u\big\|,$$

where

$$\|u\| = \max_{t\in[0,T]} \big|u(t)\big|, \qquad \big\|D^{\theta}_q u\big\| = \max_{t\in[0,T]} \big|D^{\theta}_q u(t)\big|,$$

and

$$\big\|D^{v}_{q,\omega} u\big\| = \max_{t\in[\omega_0,T]} \big|D^{v}_{q,\omega} u(t)\big|.$$

We define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}(\mathcal{F}u)(t) :=& \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^t\int_{\omega_0}^x \big(t-\sigma_q(x)\big)_q^{\frac{\beta-1}{}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}\\ &\times F\big[s,u(s),D_q^\theta u(s),D_{q,\omega}^\nu u(s)\big]\,d_{q,\omega}s\,d_qx\\ &+\big\{\mathbf{A}_T^*\Phi_\eta\big[\phi_1(u),F_u\big]-\mathbf{A}_\eta^*\Phi_T\big[\phi_2(u),F_u\big]\big\}\frac{1}{\Omega\Gamma_q(\beta)}\\ &\times\int_0^t\big(t-\sigma_q(s)\big)_q^{\frac{\beta-1}{}}(s-\omega_0)^{\alpha-1}\,d_qs\\ &-\big\{\mathbf{B}_T^*\Phi_\eta\big[\phi_1(u),F_u\big]-\mathbf{B}_\eta^*\Phi_T\big[\phi_2(u),F_u\big]\big\}\frac{t^{\beta-1}}{\Omega},\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\Phi_\eta^*\big[\phi_1(u),F_u\big] :=& \phi_1(u)-\frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^\eta\int_{\omega_0}^x\big(\eta-\sigma_q(x)\big)_q^{\frac{\beta-1}{}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}\\ &\times F\big[s,u(s),D_q^\theta u(s),D_{q,\omega}^\nu u(s)\big]\,d_{q,\omega}s\,d_qx-\frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)}\\ &\times\int_0^\eta\int_0^r\int_{\omega_0}^x\big(\eta-\sigma_q(r)\big)_q^{\frac{\gamma-1}{}}\big(r-\sigma_q(x)\big)_q^{\frac{\beta-1}{}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}\\ &\times F\big[s,u(s),D_q^\theta u(s),D_{q,\omega}^\nu u(s)\big]\,d_{q,\omega}s\,d_qx\,d_qr,\end{aligned}\tag{3.2}$$

$$\begin{aligned}\Phi_T^*\big[\phi_2(u),F_u\big] :=& \phi_2(u)-\frac{\mu_1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^T\int_{\omega_0}^x\big(T-\sigma_q(x)\big)_q^{\frac{\beta-1}{}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}\\ &\times F\big[s,u(s),D_q^\theta u(s),D_{q,\omega}^\nu u(s)\big]\,d_{q,\omega}s\,d_qx-\frac{\mu_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)}\\ &\times\int_0^T\int_0^r\int_{\omega_0}^x\big(T-\sigma_q(r)\big)_q^{\frac{\gamma-1}{}}\big(r-\sigma_q(x)\big)_q^{\frac{\beta-1}{}}\big(x-\sigma_{q,\omega}(s)\big)_{q,\omega}^{\alpha-1}\\ &\times F\big[s,u(s),D_q^\theta u(s),D_{q,\omega}^\nu u(s)\big]\,d_{q,\omega}s\,d_qx\,d_qr,\end{aligned}\tag{3.3}$$

and the constants $\mathbf{A}_\eta$, $\mathbf{A}_T$, $\mathbf{B}_\eta$, $\mathbf{B}_T$, Ω are defined by (2.5)–(2.9), respectively.

If one can prove that $\mathcal{F}$ has a fixed point, we can conclude that problem (1.1) has a solution.

Theorem 3.1 *Let $F:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a continuous function, and assume that the following conditions hold*:

(H_1) *There exist constants $\ell_1,\ell_2,\ell_3>0$ such that, for each $t\in[0,T]$ and $u_i,v_i\in\mathbb{R}$, $i=1,2,3$,*

$$\big|F[t,u_1,u_2,u_3]-F[t,v_1,v_2,v_3]\big|\le\ell_1|u_1-v_1|+\ell_2|u_2-v_2|+\ell_3|u_3-v_3|.$$

(H_2) *There exist constants $\xi_1,\xi_2>0$ such that, for each $u,v\in\mathcal{C}$,*

$$\big|\phi_1(u)-\phi_1(v)\big|\le\xi_1\|u-v\|_{\mathcal{C}}\quad\text{and}\quad\big|\phi_2(u)-\phi_2(v)\big|\le\xi_2\|u-v\|_{\mathcal{C}}.$$

(H_3) $\mathcal{X}:=(\ell_1+\ell_2+\ell_3)\Theta+\xi_1\Upsilon_T+\xi_2\Upsilon_\eta<\frac{1}{3}$,

where

$$\Theta := \frac{(T-\omega_0)^{\alpha}T^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} + \mathcal{O}_1\Upsilon_T + \mathcal{O}_2\Upsilon_\eta, \tag{3.4}$$

$$\mathcal{O}_1 := \frac{(\eta-\omega_0)^{\alpha}\eta^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}\left|\lambda_1 - \frac{\lambda_2\eta^{-\gamma}}{\Gamma_q(1-\gamma)}\right|, \tag{3.5}$$

$$\mathcal{O}_2 := \frac{(T-\omega_0)^{\alpha}T^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}\left|\mu_1 - \frac{\mu_2(T-\omega_0)^{-\gamma}}{\Gamma_q(1-\gamma)}\right|, \tag{3.6}$$

$$\Upsilon_T := \frac{1}{|\Omega|}\left\{|\mathbf{A}_T|\frac{(T-\omega_0)^{\alpha-1}T^{\beta}}{\Gamma_q(\beta+1)} + |\mathbf{B}_T|T^{\beta-1}\right\}, \tag{3.7}$$

$$\Upsilon_\eta := \frac{1}{|\Omega|}\left\{|\mathbf{A}_\eta|\frac{(T-\omega_0)^{\alpha-1}T^{\beta}}{\Gamma_q(\beta+1)} + |\mathbf{B}_\eta|T^{\beta-1}\right\}. \tag{3.8}$$

Then problem (1.1) *has a unique solution.*

Proof Let $\mathcal{H}|u-v|(t) := |F[t,u,D_q^{\theta}u,D_{q,\omega}^{\nu}u] - F[t,v,D_q^{\theta}v,D_{q,\omega}^{\nu}v]|$. For each $t\in[0,T]$ and $u,v\in\mathcal{C}$, from (3.2), we see that

$$\begin{aligned}
&\left|\Phi_\eta^*\left[\phi_1(u),F_u\right] - \Phi_\eta^*\left[\phi_1(u),F_v\right]\right| \\
&\quad\le \left|\phi_1(u)-\phi_1(v)\right| + \left|\frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^{\eta}\int_{\omega_0}^{x}\left(\eta-\sigma_q(x)\right)_q^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\right. \\
&\qquad\times\mathcal{H}\left|u-v\right|(s)\,d_{q,\omega}s\,d_qx - \frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)}\int_0^{\eta}\int_0^{r}\int_{\omega_0}^{x}\left(\eta-\sigma_q(r)\right)_q^{\underline{\gamma-1}} \\
&\qquad\left.\times\left(r-\sigma_q(x)\right)_q^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\mathcal{H}\left|u-v\right|(s)\,d_{q,\omega}s\,d_qx\,d_qr\right| \\
&\quad\le \xi_1\|u-v\|_{\mathcal{C}} + \left(\ell_1|u-v| + \ell_2\left|D_q^{\theta}u - D_q^{\theta}v\right| + \ell_3|D_{q,\omega}^{\nu}u - D_{q,\omega}^{\nu}v|\right) \\
&\qquad\times\left|\frac{\lambda_1(\eta-\omega_0)^{\alpha}\eta^{\beta}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} - \frac{\lambda_2(\eta-\omega_0)^{\alpha}\eta^{\beta-\gamma}}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)\Gamma_q(1-\gamma)}\right| \\
&\quad\le \xi_1\|u-v\|_{\mathcal{C}} + \left(\ell_1|u-v| + \ell_2\left|D_q^{\theta}u - D_q^{\theta}v\right| + \ell_3\left|D_{q,\omega}^{\nu}u - D_{q,\omega}^{\nu}v\right|\right)\mathcal{O}_1 \\
&\quad\le \left[\xi_1 + (\ell_1+\ell_2+\ell_3)\mathcal{O}_1\right]\|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Similarly, we see from (3.3) that

$$\left|\Phi_T^*\left[\phi_2(u),F_u\right] - \Phi_T^*\left[\phi_2(u),F_v\right]\right| \le \left[\xi_2 + (\ell_1+\ell_2+\ell_3)\mathcal{O}_2\right]\|u-v\|_{\mathcal{C}}.$$

Consider

$$\begin{aligned}
&\left|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\right| \\
&\quad\le \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^{T}\int_{\omega_0}^{x}\left(T-\sigma_q(x)\right)_q^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\,d_{q,\omega}s\,d_qx \\
&\qquad+\left\{|\mathbf{A}_T|\left|\Phi_\eta^*\left[\phi_1(u),F_u\right] - \Phi_\eta^*\left[\phi_1(u),F_v\right]\right| + |\mathbf{A}_\eta|\left|\Phi_T^*\left[\phi_2(u),F_u\right] - \Phi_T^*\left[\phi_2(u),F_v\right]\right|\right\} \\
&\qquad\times\frac{(T-\omega_0)^{\alpha-1}T^{\beta}}{|\Omega|\Gamma_q(\beta+1)}
\end{aligned}$$

$$
\begin{aligned}
&+ \left\{ |\mathbf{B}_T| \left| \Phi_\eta^* \left[\phi_1(u), F_u \right] - \Phi_\eta^* \left[\phi_1(u), F_v \right] \right| + |\mathbf{B}_\eta| \left| \Phi_T^* \left[\phi_2(u), F_u \right] - \Phi_T^* \left[\phi_2(u), F_v \right] \right| \right\} \\
&\times \frac{T^{\beta-1}}{\Omega} \\
&\leq \|u - v\|_{\mathcal{C}} \left[\frac{(\ell_1 + \ell_2 + \ell_3)(T - \omega_0)^\alpha T^\beta}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)} + \frac{[\xi_1 + (\ell_1 + \ell_2 + \ell_3)\mathcal{O}_1]}{|\Omega|} \right. \\
&\times \left\{ |\mathbf{A}_T| \frac{(T - \omega_0)^{\alpha-1} T^\beta}{\Gamma_q(\beta+1)} + |\mathbf{B}_T| T^{\beta-1} \right\} \\
&\left. + \frac{[\xi_2 + (\ell_1 + \ell_2 + \ell_3)\mathcal{O}_2]}{|\Omega|} \left\{ |\mathbf{A}_\eta| \frac{(T - \omega_0)^{\alpha-1} T^\beta}{\Gamma_q(\beta+1)} + |\mathbf{B}_\eta| T^{\beta-1} \right\} \right] \\
&\leq \left[(\ell_1 + \ell_2 + \ell_3)\Theta + \xi_1 \Upsilon_T + \xi_2 \Upsilon_\eta \right] \|u - v\|_{\mathcal{C}} \\
&= \mathcal{X} \|u - v\|_{\mathcal{C}}. \qquad (3.9)
\end{aligned}
$$

Taking fractional fractional q-difference of order θ and fractional Hahn difference of order ν for (3.1), we get

$$
\begin{aligned}
D_q^\theta u(t) = {} & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_p(-\theta)} \int_0^t \int_0^r \int_{\omega_0}^x \left(t - \sigma_q(r)\right)_q^{\frac{-\theta-1}{}} \left(r - \sigma_q(x)\right)_q^{\frac{\beta-1}{}} \\
& \times \left(x - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\frac{\alpha-1}{}} F\left[s, u(s), D_q^\theta u(s), D_{q,\omega}^\phi u(s)\right] d_q s \, d_q x \, d_{q,\omega} r \\
& + \frac{\{\mathbf{A}_T^* \Phi_\eta[\phi_1(u), F_u] - \mathbf{A}_\eta^* \Phi_T[\phi_2(u), F_u]\}}{\Omega \Gamma_q(\beta)\Gamma_p(-\theta)} \int_0^t \int_0^x \left(t - \sigma_q(x)\right)_q^{\frac{-\theta-1}{}} \\
& \times \left(x - \sigma_q(s)\right)_q^{\frac{\beta-1}{}} (s - \omega_0)^{\alpha-1} \, d_q s \, d_q x, \\
& - \frac{\{\mathbf{B}_T^* \Phi_\eta[\phi_1(u), F_u] - \mathbf{B}_\eta^* \Phi_T[\phi_2(u), F_u]\}}{\Omega \Gamma_q(-\theta)} \int_0^t \left(t - \sigma_q(s)\right)_q^{\frac{-\theta-1}{}} s^{\beta-1} \, d_q s \qquad (3.10)
\end{aligned}
$$

for $t \in [0, T]$, and

$$
\begin{aligned}
D_{q,\omega}^\nu u(t) = {} & \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_p(-\nu)} \int_{\omega_0}^t \int_0^r \int_{\omega_0}^x \left(t - \sigma_{q,\omega}(r)\right)_{q,\omega}^{\frac{-\nu-1}{}} \left(r - \sigma_q(x)\right)_q^{\frac{\beta-1}{}} \\
& \times \left(x - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\frac{\alpha-1}{}} F\left[s, u(s), D_q^\theta u(s), D_{q,,\omega}^\nu u(s)\right] d_{q,\omega} s \, d_q x \, d_{q,\omega} r \\
& + \frac{\{\mathbf{A}_T^* \Phi_\eta[\phi_1(u), F_u] - \mathbf{A}_\eta^* \Phi_T[\phi_2(u), F_u]\}}{\Omega \Gamma_q(\beta)\Gamma_p(-\nu)} \int_{\omega_0}^t \int_0^x \left(t - \sigma_{q,\omega}(x)\right)_{q,\omega}^{\frac{-\nu-1}{}} \\
& \times \left(x - \sigma_q(s)\right)_q^{\frac{\beta-1}{}} (s - \omega_0)^{\alpha-1} \, d_q s \, d_{q,\omega} x, \\
& - \frac{\{\mathbf{B}_T^* \Phi_\eta[\phi_1(u), F_u] - \mathbf{B}_\eta^* \Phi_T[\phi_2(u), F_u]\}}{\Omega \Gamma_q(-\nu)} \int_{\omega_0}^t \left(t - \sigma_{q,\omega}(s)\right)_{q,\omega}^{\frac{-\nu-1}{}} s^{\beta-1} \, d_{q,\omega} s \qquad (3.11)
\end{aligned}
$$

for $t \in [\omega_0, T]$, respectively.

Similarly, we have

$$
\left| \left(D_q^\theta \mathcal{F} u\right)(t) - \left(D_q^\theta \mathcal{F} v\right)(t) \right| < \mathcal{X} \|u - v\|_{\mathcal{C}}, \qquad (3.12)
$$

$$
\left| \left(D_{q,\omega}^\nu \mathcal{F} u\right)(t) - \left(D_{q,\omega}^\nu \mathcal{F} v\right)(t) \right| < \mathcal{X} \|u - v\|_{\mathcal{C}}. \qquad (3.13)
$$

From (3.9), (3.12), and (3.13), it implies that

$$\begin{aligned}\|\mathcal{F}u-\mathcal{F}v\|_{\mathcal{C}} &= \|\mathcal{F}u-\mathcal{F}v\|+\left\|D_q^{\theta}\mathcal{F}u-D_q^{\theta}\mathcal{F}v\right\|+\left\|D_{q,\omega}^{v}\mathcal{F}u-D_{q,\omega}^{v}\mathcal{F}v\right\|\\ &< 3\mathcal{X}\|u-v\|_{\mathcal{C}}.\end{aligned}$$

Therefore, $\mathcal{F}$ is a contraction by (H_3). Thus, $\mathcal{F}$ has a fixed point, which is a unique solution of problem (1.1) by using the Banach fixed point theorem. □

4 Existence of at least one solution

In this section, we also prove the existence of at least one solution of (1.1) by using Schauder's fixed point theorem. Firstly, we provide some basic knowledge that is used in this section as follows.

Lemma 4.1 ([59] (Arzelá–Ascoli theorem)) *A set of functions in $C[a,b]$ with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on $[a,b]$.*

Lemma 4.2 ([59]) *If a set is closed and relatively compact, then it is compact.*

Lemma 4.3 ([60] (Schauder's fixed point theorem)) *Let (D,d) be a complete metric space, U be a closed convex subset of D, and $T:D\to D$ be the map such that the set $Tu:u\in U$ is relatively compact in D. Then the operator T has at least one fixed point $u^*\in U$: $Tu^*=u^*$.*

Based on the above lemmas, we prove the existence of at least one solution of (1.1) as shown in the following theorem.

Theorem 4.1 *Suppose that (H_1) and (H_3) hold. Then problem (1.1) has at least one solution.*

Proof The proof is divided into three steps as follows.

Step I. We verify that $\mathcal{F}$ maps bounded sets into bounded sets in $B_R=\{u\in\mathcal{C}:\|u\|_{\mathcal{C}}\le R\}$. We let $\max_{t\in[0,T]}|F(t,0,0,0)|=M$, $\sup_{u\in\mathcal{C}}|\phi_1(u)|=N_1$, $\sup_{u\in\mathcal{C}}|\phi_2(u)|=N_2$ and choose a constant

$$R\ge\frac{M\Theta+N_1\Upsilon_T+N_2\Upsilon_\eta}{\frac{1}{3}-(\ell_1+\ell_2+\ell_3)\Theta}. \tag{4.1}$$

We denote that

$$\left|\mathcal{S}(t,u,0)\right|=\left|F\left[t,u,D_q^{\theta}u,D_{q,\omega}^{v}u\right]-F[t,0,0,0]\right|+\left|F[t,0,0,0]\right|.$$

For each $t\in[0,T]$ and $u\in B_R$, we have

$$\begin{aligned}&\left|\Phi_\eta^*\left[\phi_1(u),F_u\right]\right|\\ &\quad\le N_1+\left|\frac{\lambda_1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^\eta\int_{\omega_0}^x\left(\eta-\sigma_q(x)\right)_q^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\left|\mathcal{S}(s,u,0)\right|\right|d_{q,\omega}s\,d_qx\\ &\qquad-\frac{\lambda_2}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(-\gamma)}\int_0^\eta\int_0^r\int_{\omega_0}^x\left(\eta-\sigma_q(r)\right)_q^{\underline{\gamma-1}}\left(r-\sigma_q(x)\right)_q^{\underline{\beta-1}}\left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}}\end{aligned}$$

$$\begin{aligned}
&\times \big|\mathcal{S}(s,u,0)\big|\, d_{q,\omega}s\, d_q x\, d_q r\Big| \\
&\leq N_1 + \big[(\ell_1+\ell_2+\ell_3)\|u\|_{\mathcal{C}} + M\big]\mathcal{O}_1 \\
&\leq N_1 + \big[(\ell_1+\ell_2+\ell_3)R + M\big]\mathcal{O}_1. \qquad (4.2)
\end{aligned}$$

Similarly, we find that

$$\big|\Phi_T^*\big[\phi_2(u), F_u\big]\big| \leq N_2 + \big[(\ell_1+\ell_2+\ell_3)R + M\big]\mathcal{O}_2. \qquad (4.3)$$

From (4.2)–(4.3), we find that

$$\big|(\mathcal{F}u)(t)\big| \leq \Theta\big[(\ell_1+\ell_2+\ell_3)R + M\big] + N_1\Upsilon_T + N_2\Upsilon_\eta \leq \frac{R}{3} \qquad (4.4)$$

and

$$\big|\big(D_q^\theta \mathcal{F}u\big)(t)\big| < \frac{R}{3} \quad \text{and} \quad \big|\big(D_{q,\omega}^\nu \mathcal{F}u\big)(t)\big| < \frac{R}{3}. \qquad (4.5)$$

Hence,

$$\|\mathcal{F}u\|_{\mathcal{C}} = \|\mathcal{F}u\| + \big\|D_q^\theta \mathcal{F}u\big\| + \big\|D_{q,\omega}^\nu \mathcal{F}u\big\| < \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R,$$

which implies that $\mathcal{F}$ is uniformly bounded.

Step II. It is obvious that the operator $\mathcal{F}$ is continuous on B_R due to the continuity of F.

Step III. We prove that $\mathcal{F}$ is equicontinuous on B_R. For any $t_1, t_2 \in I_{q,\omega}^T$ with $t_1 < t_2$, we find that

$$\begin{aligned}
\big|(\mathcal{F}u)(t_2) - (\mathcal{F}u)(t_1)\big| &\leq \frac{\|F\|(T-\omega_0)^\alpha}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}\big|t_2^\beta - t_1^\beta\big| \\
&\quad + \frac{(T-\omega_0)^{\alpha-1}|t_2^\beta - t_1^\beta|}{|\Omega|\Gamma_q(\beta+1)}\big\{|\mathbf{A}_T|\big|\Phi_\eta^*[\phi_1,F]\big| + |\mathbf{A}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\} \\
&\quad + \frac{|t_2^{\beta-1} - t_1^{\beta-1}|}{|\Omega|}\big\{|\mathbf{B}_T|\big|\Phi_\eta^*[\phi_1,F]\big| + |\mathbf{B}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\}, \qquad (4.6)
\end{aligned}$$

$$\begin{aligned}
&\big|\big(D_q^\theta \mathcal{F}u\big)(t_2) - \big(D_q^\theta \mathcal{F}u\big)(t_1)\big| \\
&\leq \frac{\|F\|(T-\omega_0)^\alpha T^\beta}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)\Gamma_q(1-\theta)}\big|t_2^{-\theta} - t_1^{-\theta}\big| \\
&\quad + \frac{(T-\omega_0)^{\alpha-1}T^\beta|t_2^{-\theta} - t_1^{-\theta}|}{|\Omega|\Gamma_q(\beta+1)\Gamma_q(1-\theta)}\big\{|\mathbf{A}_T|\big|\Phi_\eta^*[\phi_1,F]\big| + |\mathbf{A}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\} \\
&\quad + \frac{T^{\beta-1}|t_2^{-\theta} - t_1^{-\theta}|}{|\Omega|\Gamma_q(1-\theta)}\big\{|\mathbf{B}_T|\big|\Phi_\eta^*[\phi_1,F]\big| + |\mathbf{B}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\}, \qquad (4.7)
\end{aligned}$$

and

$$\begin{aligned}
&\big|\big(D_{q,\omega}^\nu \mathcal{F}u\big)(t_2) - \big(D_{q,\omega}^\nu \mathcal{F}u\big)(t_1)\big| \\
&\leq \frac{\|F\|(T-\omega_0)^\alpha T^\beta}{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)\Gamma_q(1-\nu)}\big|(t_2-\omega_0)^{-\nu} - (t_1-\omega_0)^{-\nu}\big|
\end{aligned}$$

$$+\frac{(T-\omega_0)^{\alpha-1}T^{\beta}|(t_2-\omega_0)^{-\nu}-(t_1-\omega_0)^{-\nu}|}{|\Omega|\Gamma_q(\beta+1)\Gamma_q(1-\nu)}\big\{|\mathbf{A}_T|\big|\Phi_\eta^*[\phi_1,F]\big|+|\mathbf{A}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\}$$
$$+\frac{T^{\beta-1}|(t_2-\omega_0)^{-\nu}-(t_1-\omega_0)^{-\nu}|}{|\Omega|\Gamma_q(1-\nu)}\big\{|\mathbf{B}_T|\big|\Phi_\eta^*[\phi_1,F]\big|+|\mathbf{B}_\eta|\big|\Phi_T^*[\phi_2,F]\big|\big\}. \tag{4.8}$$

When $|t_2-t_1|\to 0$, we find that the right-hand side of (4.6)–(4.8) tends to be zero. Hence, $\mathcal{F}$ is relatively compact on B_R and the set $\mathcal{F}(B_R)$ is an equicontinuous set. From Steps I to III and the Arzelá–Ascoli theorem, we can conclude that $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is completely continuous. Therefore, problem (1.1) has at least one solution by Schauder's fixed point theorem. □

5 Example

Consider the nonlocal Robin boundary value problems for sequential fractional Hahn-q-difference equation as given by

$$\begin{aligned}
&D^{\frac{1}{3}}_{\frac{1}{2},\frac{2}{3}}D^{\frac{3}{4}}_{\frac{1}{2}}u(t)=\frac{1}{(1000\pi^2+t^3)(1+|u(t)|)}\big[e^{-(t+\frac{e}{3})}\big(u^2+2|u|\big)+e^{-(\frac{\pi}{2}+\cos^2\pi t)}\big|D^{\frac{1}{2}}_{\frac{1}{2}}u(t)\big|\\
&\qquad\qquad+e^{-(2+\sin^2\pi t)}\big|D^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}}u(t)\big|\big],\quad t\in I^{[0,10]}_{\frac{1}{2},\frac{2}{3}},\\
&\frac{1}{10e}u(5)+\frac{1}{20\pi}D^{\frac{1}{5}}_{\frac{1}{2}}u(5)=\sum_{i=0}^{\infty}\frac{C_i|u(t_i)|}{1+|u(t_i)|},\quad t_i=10\bigg(\frac{1}{2}\bigg)^i,\\
&\frac{1}{20e}u(10)+\frac{1}{10\pi}D^{\frac{1}{5}}_{\frac{1}{2},\frac{2}{3}}u(10)=\sum_{i=0}^{\infty}\frac{D_i|u(t_i)|}{1+|u(t_i)|},\quad t_i=10\bigg(\frac{1}{2}\bigg)^i+\frac{2}{3}[i]_{\frac{1}{2}},
\end{aligned}\tag{5.1}$$

where C_i, D_i are given constants with $\frac{1}{2000e^2}\le\sum_{i=0}^{\infty}C_i\le\frac{1}{2000e}$ and $\frac{1}{1000\pi^3}\le\sum_{i=0}^{\infty}D_i\le\frac{1}{1000\pi^2}$.

We let $\alpha=\frac{1}{3}$, $\beta=\frac{3}{4}$, $\gamma=\frac{1}{5}$, $\theta=\frac{1}{2}$, $\nu=\frac{2}{5}$, $q=\frac{1}{2}$, $\omega=\frac{2}{3}$, $\omega_0=\frac{\omega}{1-q}=\frac{4}{3}$, $T=10$, $\eta=5$, $\lambda_1=\frac{1}{10e}$, $\lambda_2=\frac{1}{20\pi}$, $\mu_1=\frac{1}{20e}$, $\mu_2=\frac{1}{10\pi}$, and

$$\begin{aligned}
F\big[t,u(t),D^{\theta}_qu(t),D^{\nu}_{q,\omega}u(t)\big]&=\frac{1}{(1000\pi^2+t^3)(1+|u(t)|)}\big[e^{-(t+\frac{e}{3})}\big(u^2+2|u|\big)\\
&\quad+e^{-(\frac{\pi}{2}+\cos^2\pi t)}\big|D^{\frac{1}{2}}_{\frac{1}{2}}u(t)\big|+e^{-(2+\sin^2\pi t)}\big|D^{\frac{2}{5}}_{\frac{1}{2},\frac{2}{3}}u(t)\big|\big].
\end{aligned}$$

We find that

$$|\mathbf{A}_\eta|=0.308,\qquad |\mathbf{A}_T|\approx 0.0207,\qquad |\mathbf{B}_\eta|\approx 0.0723,\qquad |\mathbf{B}_T|\approx 0.0308,$$

and

$$|\Omega|\approx 0.00799.$$

For all $t\in[0,10]$ and $u,v\in\mathbb{R}$, we find that

$$\begin{aligned}
&\big|F\big[t,u,D^{\theta}_qu,D^{\nu}_{q,\omega}u\big]-F\big[t,v,D^{\theta}_qv,D^{\nu}_{q,\omega}v\big]\big|\\
&\quad\le\frac{1}{1000\pi^2e^{\frac{e}{3}}}|u-v|+\frac{1}{1000\pi^2e^{\frac{\pi}{2}}}\big|D^{\theta}_qu-D^{\theta}_qv\big|+\frac{1}{1000\pi^2e^2}\big|D^{\nu}_{q,\omega}u-D^{\nu}_{q,\omega}v\big|.
\end{aligned}$$

Thus, (H_1) holds with $\ell_1=0.0000409$, $\ell_2=0.0000211$, and $\ell_3=0.0000137$.

For all $u, v \in \mathcal{C}$,

$$\left|\phi_1(u) - \phi_1(v)\right| \leq \frac{1}{2000e}\|u - v\|_{\mathcal{C}},$$

$$\left|\phi_2(u) - \phi_2(v)\right| \leq \frac{1}{1000\pi^2}\|u - v\|_{\mathcal{C}}.$$

Thus, (H_2) holds with $\xi_1 = 0.000184$ and $\xi_2 = 0.000101$.

From

$$\mathcal{O}_1 = 0.2778, \qquad \mathcal{O}_2 = 0.2898, \qquad \Upsilon_1 = 2.2278, \qquad \Upsilon_2 = 0.5169,$$

and

$$\Theta = 1.0549,$$

we find that (H_3) holds with

$$\mathcal{X} \approx 0.000542 < \frac{1}{3}.$$

Hence, by Theorem 3.1 problem (5.1) has a unique solution.

6 Conclusion

A nonlocal Robin boundary value problem for fractional sequential fractional Hahn-q-equation (1.1) is studied. Our problem contains both fractional Hahn and fractional q-difference operators, which is a new idea. We establish the conditions for the existence and uniqueness of solution for problem (1.1) by using the Banach fixed point theorem, and the conditions of at least one solution by using Schauder's fixed point theorem.

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