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Entire solutions for a delayed lattice competitive system

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Abstract

In this paper, we investigate the existence of entire solutions for a delayed lattice competitive system. Here the entire solutions are the solutions that exist for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$. In order to prove the existence, we firstly embed the delayed lattice system into the corresponding larger system, of which the traveling front solutions are identical to those of the delayed lattice system. Then based on the comparison theorem and the sup–sub solutions method, we construct entire solutions which behave as two opposite traveling front solutions moving towards each other from both sides of x-axis and then annihilating. Moreover, our conclusions extend the invading way, which the superior species invade the inferior ones from both sides of x-axis and then the inferior ones extinct, into the lattice and delay case.

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Keywords: Entire solutions; Traveling front solutions; Delayed lattice competitive system; Asymptotic behavior

1 Introduction

In this paper, we study the following delayed lattice competitive system:

$$\begin{cases} \frac{du_n}{dt} = [u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)] + au_n[1 - u_n(t) - kv_n(t-\tau_1)], \\ \frac{dv_n}{dt} = d[v_{n+1}(t) - 2v_n(t) + v_{n-1}(t)] + bv_n[1 - v_n(t) - hu_n(t-\tau_2)], \end{cases} \tag{1.1}$$

where a, b, d, k, h are all positive numbers and $\tau_i > 0$ ($i = 1, 2$) are the maturation time for the species. Here $u_n = u_n(t)$ and $v_n = v_n(t)$, $t \in \mathbb{R}$, denote the population density of two competitive species at time t and niches n, respectively. Hence we consider that both $u_n(t)$ and $v_n(t)$ are nonnegative. It is obvious that there are four equilibria of (1.1),

$$(0,0), (0,1), (1,0), \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) := (\hat{k}_1, \hat{k}_2).$$

When $\tau_1 = \tau_2 = 0$, as stated in [5], the solution of (1.1) has the following asymptotic behaviors depending on h and k as $t \to \infty$:

(i) If $0 < h < 1 < k$, then $(u_n(t), v_n(t)) \to (0, 1)$ (v_n wins).
(ii) If $0 < k < 1 < h$, then $(u_n(t), v_n(t)) \to (1, 0)$ (u_n wins).
(iii) If $k, h > 1$, then $(u_n(t), v_n(t)) \to (1, 0)$ or $(0, 1)$ depending on the initial condition.
(iv) If $0 < k, h < 1$, then $(u_n(t), v_n(t)) \to (\hat{k}_1, \hat{k}_2)$ (u_n and v_n coexist).

By exchanging the roles of u and v, (i) will become (ii). Thus in this paper, we only consider the existence of entire solutions in case (i).

The theory of traveling wave solutions plays an important role in the study of lattice equations and systems. For system (1.1), a traveling wave solution connecting $(1,0)$ and $(0,1)$ is the solution with the form

$$\big(u_n(t), v_n(t)\big) = \big(\phi(\xi), \psi(\xi)\big), \quad \xi := n + ct,$$

satisfying

$$\begin{cases} c\phi'(\xi) = D[\phi(\xi)] + a\phi(\xi)[1 - \phi(\xi) - k\psi(\xi - c\tau_1)], \\ c\psi'(\xi) = dD[\psi(\xi)] + b\psi(\xi)[1 - \psi(\xi) - h\phi(\xi - c\tau_2)], \\ (\phi, \psi)(-\infty) = (1,0), \qquad (\phi, \psi)(+\infty) = (0,1), \\ 0 \le \phi, \psi \le 1 \quad \text{on } \mathbb{R}, \end{cases} \tag{1.2}$$

where $D[\omega(\xi)] := \omega(\xi + 1) + \omega(\xi - 1) - 2\omega(\xi)$ for $\omega = \phi, \psi$. Moreover, if ϕ and ψ are monotone, we call $(\phi(\xi), \psi(\xi))$ a traveling front solution. Very recently, in case (i), Li, Huang, Li and He in [8] proved the existence, asymptotic behavior, strict monotonicity and uniqueness of traveling front solutions of (1.1) with $c \ge c^*$, where c^* is the minimal speed.

Though the study of the traveling wave solutions is a significant topic for diffusive equations, see [1, 3, 8] and the references therein, it is not enough to consider the existence of the traveling wave solutions to study the global attractor of the diffusion equations. In order to understand the phenomenon of invasion between two species, the study of the entire solutions becomes a very important subject. Here an entire solution of (1.1) is a classical solution that exists for all $(n,t) \in \mathbb{Z} \times \mathbb{R}$. Thus the aim of this paper is to investigate the existence of an entire solution for (1.1) which converges to two monotone fronts with opposite speeds. In fact, Hamel and Nadirashvili [6] proved the existence of entire solutions of the famous Fisher-KPP equation (monostable case) by the comparison theorem, subsolution and super estimates. While for the bistable case, Yagisita [16] investigated that the annihilation process is approximated by a backward global solution, which is an entire solution. Later, by using the comparison theorem and the explicit expression of the traveling front solutions for the Allen–Cahn equation, Fukao, Morita and Ninomiya in [2] improved the proof of the existence of entire solutions which was already showed in [16]. In addition, Guo and Morita in [4] extended the conclusions in [6] and [16] to a more general case including the Fisher-KPP equation with the discrete diffusion.

For systems, Morita and Tachibana in [12] firstly extended the existence of entire solutions from scalar equations to the following Lotka–Volterra competition–diffusion system on $\mathbb{R}$:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + u(x,t)[1 - u(x,t) - kv(x,t)], \\ v_t(x,t) = dv_{xx}(x,t) + av(x,t)[1 - v(x,t) - hu(x,t)]. \end{cases} \tag{1.3}$$

Based on the known results of traveling front solutions for (1.3) in [3, 7] and by using the similar methods in [2] and [4], they have proved the existence of two-front entire solutions to (1.3) for the cases (i) and (iii). This entire solution behave like two traveling front solutions moving towards each other from both sides of x-axis at $\xi \approx -\infty$, and one component converges to 0 while the other converges to 1 as $t \to \infty$. Moreover, there is a technical condition that there is an $\eta_0 > 0$ such that

$$\frac{\psi(\xi)}{1-\phi(\xi)} \ge \eta_0, \quad \xi \le 0 \tag{1.4}$$

in [12]. Then Wang and Li in [14] investigated this technical condition and gave some sufficient conditions to ensure this technical condition. In addition, they used the exact solutions, which do not satisfy the technical condition, to construct an entire solution for (1.3). For the delayed Lotka–Volterra competition–diffusion system, Lv in [10] obtained the existence of entire solutions by using the comparison principle and super–sub solutions methods, which is similar to that of [12]. Later, for the nonlocal competition–diffusion system, Li, Zhang and Zhang in [9] have proved the existence of entire solutions to it, in which the asymptotic behavior of the entire solutions was similar to that of [10] and [12]. Very recently, Wang, Liu and Li in [15] investigated the existence of an entire solution for the nonlocal competitive Lokta–Volterra system with delays which extend the results in [9, 10, 12]. While for a two-component competition system in a lattice, Guo and Wu [5] obtained the existence of the entire solutions of (1.1) when $\tau_1 = \tau_2 = 0$. Others paper for the existence of entire solutions for systems, one can refer to [11, 13] and the references therein. As is well known, there is no result for the existence of entire solutions of (1.1). Thus in this paper we will investigate the existence of entire solutions to the system (1.1).

In order to establish the two-front solutions for (1.1), we need to embed it into the following larger one:

$$\begin{cases} u_t(x,t) = D[u(x,t)] + au(x,t)[1-u(x,t)-kv(x,t-\tau_1)], \\ v_t(x,t) = dD[v(x,t)] + bv(x,t)[1-v(x,t)-hu(x,t-\tau_2)], \end{cases} \tag{1.5}$$

where $(x,t) \in \mathbb{R}^2$ and $D[\omega(x,t)] := \omega(x+1,t) + \omega(x-1,t) - 2\omega(x,t)$ for $\omega = u, v$. Obviously, the traveling front solution of (1.1) is equivalent to that of (1.5). For simplification, setting $\hat{u}(x,t) = 1-u(x,t)$ and replacing $\hat{u}$ by u, then (1.5) turns to the following system:

$$\begin{cases} u_t(x,t) = D[u(x,t)] + a[1-u(x,t)][kv(x,t-\tau_1)-u(x,t)], \\ v_t(x,t) = dD[v(x,t)] + bv(x,t)[1-h-v(x,t)+hu(x,t-\tau_2)]. \end{cases} \tag{1.6}$$

Similarly, by letting $\hat{\phi} = 1-\phi$ and replacing $\hat{\phi}$ by ϕ, then (1.2) turns to

$$\begin{cases} c\phi'(\xi) = D[\phi(\xi)] + a(1-\phi(\xi))[k\psi(\xi-c\tau_1)-\phi(\xi)], \\ c\psi'(\xi) = dD[\psi(\xi)] + b\psi(\xi)[1-h-\psi(\xi)+h\phi(\xi-c\tau_2)], \\ \lim_{\xi\to-\infty}(\phi(\xi),\psi(\xi)) = (0,0), \qquad \lim_{\xi\to+\infty}(\phi(\xi),\psi(\xi)) = (1,1), \\ \phi', \psi' > 0, \end{cases} \tag{1.7}$$

which is also a traveling front solution of (1.6). Similar to [5, 9–13, 15], in this paper, we also assume that there exists a $K > 0$ such that for the solution $(\phi(\xi),\psi(\xi))$ of (1.7)

$$\frac{\psi(\xi)}{\phi(\xi)} \ge K, \tag{1.8}$$

which is also a technical condition.

The rest of the paper is arranged as follows. In Sect. 2, we show the existence and the asymptotic behaviors of traveling front solutions of (1.1) and give some preparations that will be used to construct the sup–sub solutions. In Sect. 3, we firstly give three lemmas which are key steps for the construction of the supersolution. Then an entire solution to (1.1) can be obtained based on the constructed sup–sub solutions.

2 Preliminaries

In this section, we introduce some known results which will be used in the following section. Firstly, set

$$\begin{aligned}
&\Delta_1(\lambda,c)=d\bigl(e^{\lambda}+e^{-\lambda}-2\bigr)-c\lambda+b(1-h),\\
&\Delta_2(\lambda,c)=\bigl(e^{\lambda}+e^{-\lambda}-2\bigr)-c\lambda-a,\\
&\Delta_3(\lambda,c)=\bigl(e^{\lambda}+e^{-\lambda}-2\bigr)-c\lambda+a(1-k),\\
&\Delta_4(\lambda,c)=d\bigl(e^{\lambda}+e^{-\lambda}-2\bigr)-c\lambda-b.
\end{aligned}$$

For $\Delta_i(\lambda,c)$, $i=1,2,3,4$, from [8], we conclude the following results.

Lemma 1 ([8]) *Assume that* $0<h<1<k$ *holds. Then*:

(i) *There exist* $c^*>0$ *and* $\lambda^*>0$ *such that*

$$\Delta_1\bigl(\lambda^*,c^*\bigr)=0,\qquad \frac{\partial}{\partial\lambda}\Delta_1\bigl(\lambda^*,c^*\bigr)=0.$$

In addition, when $c\ge c^*$, *the equation* $\Delta_1(\lambda,c)=0$ *has two positive roots* λ_1, λ_2 *such that* $0<\lambda_1<\lambda^*<\lambda_2$ *if* $c>c^*$ *while* $\lambda_1=\lambda^*=\lambda_2$ *if* $c=c^*$ *and*

$$\Delta_1(\lambda,c)\begin{cases}>0, & \lambda<\lambda_1,\\ <0, & \lambda_1<\lambda<\lambda_2,\\ >0, & \lambda>\lambda_2.\end{cases}$$

For $0<c<c^*$, $\Delta_1(\lambda,c)>0$ *for* $\lambda\in\mathbb{R}$.

(ii) *The equation* $\Delta_2(\lambda,c)=0$ *has a unique positive root* λ_3 *for* $c>0$ *and* $\Delta_2(\lambda,c)<0$ *for* $\lambda\in(0,\lambda_3)$.

(iii) *For* $c>0$, *the equation* $\Delta_3(\lambda,c)=0$ *has a unique negative root* λ_4.

(iv) *For* $c>0$, *the equation* $\Delta_4(\lambda,c)=0$ *has a unique negative root* λ_5 *and* $\Delta_4(\lambda,c)<0$ *for* $\lambda\in(\lambda_5,0)$.

Also from [8], we can summarize the asymptotic behaviors of solutions for (1.7).

Lemma 2 *Suppose that* $0<h<1<k$ *holds. When* $c\ge c^*$, *for the solution* $(\phi(\xi),\psi(\xi))$ *of* (1.7), *then*

(i) *there are* $\theta_i=\theta_i(\phi,\psi)$ $(i=1,2)$ *such that* $\lim_{\xi\to-\infty}\frac{\psi(\xi+\theta_1)}{e^{\Lambda\xi}}=1$ *if* $c>c^*$, $\lim_{\xi\to-\infty}\frac{\psi(\xi+\theta_2)}{|\xi|^{\nu}e^{\Lambda\xi}}=1$ *if* $c=c^*$;

(ii) *for* $c\ge c^*$, $\lim_{\xi\to-\infty}\frac{\psi'(\xi)}{\psi(\xi)}=\Lambda$;

(iii) *for* $c>c^*$, *there are* $\theta_i=\theta_i(\phi,\psi)$ $(i=3,4,5)$ *such that* $\lim_{\xi\to-\infty}\frac{\phi(\xi+\theta_3)}{e^{\Lambda\xi}}=1$ *if* $\lambda_3>\Lambda$, $\lim_{\xi\to-\infty}\frac{\phi(\xi+\theta_4)}{|\xi|e^{\Lambda\xi}}=1$ *if* $\lambda_3=\Lambda$, $\lim_{\xi\to-\infty}\frac{\phi(\xi+\theta_5)}{e^{\lambda_3\xi}}=1$ *if* $\lambda_3<\Lambda$;

(iv) *for* $c = c^*$, *there are* $\theta_i = \theta_i(\phi, \psi)$ $(i = 6, 7, 8)$ *such that* $\lim_{\xi\to-\infty} \frac{\phi(\xi+\theta_6)}{|\xi|^{\nu} e^{\Lambda\xi}} = 1$ *if* $\lambda_3 > \Lambda$, $\lim_{\xi\to-\infty} \frac{\phi(\xi+\theta_7)}{|\xi|^{\nu+1} e^{\Lambda\xi}} = 1$ *if* $\lambda_3 = \Lambda$, $\lim_{\xi\to-\infty} \frac{\phi(\xi+\theta_8)}{e^{\lambda_3\xi}} = 1$ *if* $\lambda_3 < \Lambda$;

(v) *for* $c \geq c^*$, $\lim_{\xi\to-\infty} \frac{\phi'(\xi)}{\phi(\xi)} = \gamma_1$;

where $\Lambda \in \{\lambda_1, \lambda_2\}$, $\gamma_1 = \min\{\Lambda, \lambda_3\}$, $\nu = 1$ *if* $\int_{-\infty}^{\infty} \psi(\xi)[-h\phi(\xi - c\tau_2) + \psi(\xi)]e^{-\Lambda\xi}\,d\xi \neq 0$, $\nu = 0$ *if* $\int_{-\infty}^{\infty} \psi(\xi)[-h\phi(\xi - c\tau_2) + \psi(\xi)]e^{-\Lambda\xi}\,d\xi = 0$.

Lemma 3 *Suppose that* $0 < h < 1 < k$ *holds. When* $c \geq c^*$, *for the solution* $(\phi(\xi), \psi(\xi))$ *of* (1.7), *then*

(i) *for* $c \geq c^*$, $\lim_{\xi\to\infty} \frac{\phi'(\xi)}{1-\phi(\xi)} = -\lambda_4 > 0$;

(ii) *there is a* $\theta_9 = \theta_9(\phi, \psi)$ *such that* $\lim_{\xi\to\infty} \frac{1-\phi(\xi+\theta_9)}{e^{\lambda_4\xi}} = 1$;

(iii) *for* $b \leq d$, *there are* $\theta_i = \theta_i(\phi, \psi)$ $(i = 10, 11, 12)$ *such that* $\lim_{\xi\to\infty} \frac{1-\psi(\xi+\theta_{10})}{e^{\lambda_5\xi}} = 1$ *if* $\lambda_5 > \lambda_4$, $\lim_{\xi\to\infty} \frac{1-\psi(\xi+\theta_{11})}{\xi e^{\lambda_5\xi}} = 1$ *if* $\lambda_5 = \lambda_4$, $\lim_{\xi\to\infty} \frac{1-\psi(\xi+\theta_{12})}{e^{\lambda_4\xi}} = 1$ *if* $\lambda_5 < \lambda_4$;

(iv) *for* $c \geq c^*$, $\lim_{\xi\to\infty} \frac{\psi'(\xi)}{1-\psi(\xi)} = -\gamma_2 > 0$,

where $\gamma_2 = \max\{\lambda_4, \lambda_5\} < 0$.

Remark 2.1 From Lemma 2 (i) and (iii), when $\lambda_3 > \Lambda$, then $\frac{\psi}{\phi}$ has a positive low bound. Thus under this case, the additional condition (1.8) holds.

As a result of Lemmas 2 and 3, we can get the following estimates, which will be needed later for constructing the supersolutions and subsolutions.

Lemma 4 *Suppose that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1, 2)$ *be solutions of* (1.7). *Then there are positive constants* μ, m, M *and* M_ρ *such that*

$$m \leq \frac{\phi_i'(\xi)}{\phi_i(\xi)} \leq M, \quad \textit{for } \xi \leq 1, \tag{2.1}$$

$$m \leq \frac{\psi_i'(\xi)}{\psi_i(\xi)} \leq M, \quad \textit{for } \xi \leq 1, \tag{2.2}$$

$$0 < \phi_i(\xi) \leq Me^{\mu\xi}, \quad \textit{for } \xi \leq 1, \tag{2.3}$$

$$0 < \psi_i(\xi) \leq Me^{\mu\xi}, \quad \textit{for } \xi \leq 1, \tag{2.4}$$

$$m \leq \frac{\phi_i'(\xi)}{1-\phi_i(\xi)} \leq M, \quad \textit{for } \xi \geq -1, \tag{2.5}$$

$$m \leq \frac{\psi_i'(\xi)}{1-\psi_i(\xi)} \leq M, \quad \textit{for } \xi \geq -1, \tag{2.6}$$

$$\frac{1-\phi_i(\xi+\theta)}{1-\phi_i(\xi)} \leq M, \quad \xi \in \mathbb{R}, \theta \in [-1, 1], \tag{2.7}$$

$$\frac{\phi_i(\xi+\theta)}{\phi_i(\xi)} \leq M, \quad \xi \in \mathbb{R}, \theta \in [-1, 1], \tag{2.8}$$

$$\frac{1-\phi_i(\xi-\rho)}{1-\psi_i(\xi)} \leq M_\rho, \quad \xi \in \mathbb{R}, \tag{2.9}$$

where M_ρ *is a constant depending on* $\rho > 0$.

Proof The proof of (2.1)–(2.6) and (2.9) follows from Lemmas 2 and 3 directly. Noting that

$$\frac{1-\phi(\xi+\theta)}{1-\phi(\xi)} = \exp\left\{-\int_{\xi}^{\xi+\theta} \frac{\phi'(z)}{1-\phi(z)}\,\mathrm{d}z\right\}$$

and

$$\frac{\phi(\xi+\theta)}{\phi(\xi)} = \exp\left\{\int_{\xi}^{\xi+\theta} \frac{\phi'(z)}{\phi(z)}\,\mathrm{d}z\right\},$$

then (2.7) and (2.8) can be proved. □

3 Existence of entire solutions

In this section, we will prove the existence of entire solutions. Firstly, we give the definition of the supersolution and subsolution (1.1). Now let

$$\begin{cases} F_1(u,v) = u_t - D[u] - f_1(u,v), \\ F_2(u,v) = v_t - dD[v] - f_2(u,v), \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_1(u,v) &= a\big(1-u(x,t)\big)\big[kv(x,t-\tau_1) - u(x,t)\big], \\ f_2(u,v) &= bv(x,t)\big[1-h-v(x,t)+hu(x,t-\tau_2)\big]. \end{aligned}$$

If there are two functions $\bar{u}$ and $\bar{v}$ satisfying $F_1(\bar{u},\bar{v}) \geq 0$ and $F_2(\bar{u},\bar{v}) \geq 0$ for all $(x,t) \in \mathbb{R} \times [T_1,T_2]$, then we call $(\bar{u},\bar{v})$ a supersolution of (3.1) for $(x,t) \in \mathbb{R} \times [T_1,T_2]$, where $T_1 > T_2$. Similarly, we define a subsolution $(\underline{u},\underline{v})$ by reversing the above inequalities. In order to construct a supersolution of (3.1), we introduce the following ordinary differential system:

$$\begin{cases} p_1'(t) = c_1 + Le^{\mu p_1}, & t<0, \\ p_2'(t) = c_2 + Le^{\mu p_1}, & t<0, \\ p_2(0) \leq p_1(0) \leq 0, \end{cases} \tag{3.2}$$

where $c_2 \geq c_1 \geq c^*$, μ is defined in Lemma 4 and L will be determined later. From direct calculation, for $i=1,2$,

$$p_i(t) = p_i(0) + c_i t - \frac{1}{\mu}\ln\left(1 + \frac{L}{c_1}e^{\mu p_1(0)}\big(1-e^{c_1\mu t}\big)\right) < 0, \tag{3.3}$$

and $p_2(t) \leq p_1(t)$, for $t \leq 0$, since $p_2(0) \leq p_1(0)$. Then let

$$v_1 = p_1(0) - \frac{1}{\mu}\ln\left(1+\frac{L}{c_1}e^{\mu p_1(0)}\right), \qquad v_2 = p_2(0) - \frac{1}{\mu}\ln\left(1+\frac{L}{c_1}e^{\mu p_1(0)}\right). \tag{3.4}$$

For $i=1,2$, since

$$p_i(t) - c_i t - v_i = -\frac{1}{\mu}\ln\left(1 - \frac{\eta}{1+\eta}e^{c_1\mu t}\right), \quad \eta = \frac{L}{c_1}e^{\mu p_1(0)},$$

there exists a constant $R_0 > 0$ such that, for $t \le 0$,

$$0 < p_1(t) - c_1 t - \nu_1 = p_2(t) - c_2 t - \nu_2 \le R_0 e^{c_1 \mu t}.$$

Next, we will prove three lemmas which are key steps for the construction of the super-solution.

Lemma 5 *Assume that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1, 2)$ *be solutions of* (1.7) *with* $c_2 \ge c_1 \ge c^*$ *and define*

$$\begin{aligned} A(x,t) &:= \bigl[\phi_1(x+1+p_1) - \phi_1(x+p_1)\bigr]\bigl[\phi_2(-x+p_2) - \phi_2(-x-1+p_2)\bigr], \\ B(x,t) &:= \bigl[\phi_1(x+p_1) - \phi_1(x-1+p_1)\bigr]\bigl[\phi_2(-x+1+p_2) - \phi_2(-x+p_2)\bigr], \\ C(x,t) &:= \phi_1'(x+p_1)\bigl[1 - \phi_2(-x+p_2)\bigr] + \phi_2'(-x+p_2)\bigl[1 - \phi_1(x+p_1)\bigr], \end{aligned}$$

then

$$A(x,t) \le N_1 e^{\mu p_1} C(x,t), \tag{3.5}$$

$$B(x,t) \le N_1 e^{\mu p_1} C(x,t), \tag{3.6}$$

where

$$\begin{aligned} N_1 = \max\biggl\{ &\frac{M^4}{m(1-\phi_2(0))} e^{\mu p_1}, \frac{M^4}{m(1-\phi_1(0))} e^{\mu p_1}, \\ &\frac{M^4 e^{\mu}}{m(1-\phi_1(0))} e^{\mu p_1}, \frac{M^4 e^{\mu}}{m(1-\phi_2(0))} e^{\mu p_1} \biggr\} > 0. \end{aligned}$$

Proof Firstly, we prove (3.5).

Case 1. For $x \ge -p_1$. In this case, $x + p_1 \ge 0$, $-x + p_2 \le 0$. Then there exist $\omega_1(x), \omega_2(x) \in (0,1)$ and combining with (2.1), (2.3), (2.5), (2.7) such that

$$\begin{aligned} \frac{A(x,t)}{C(x,t)} &\le \frac{\phi_1'(x+\omega_1+p_1)\phi_2'(-x-\omega_2+p_2)}{\phi_1'(x+p_1)[1-\phi_2(-x+p_2)]} \\ &\le \biggl[\frac{\phi_1'(x+\omega_1+p_1)}{1-\phi_1(x+\omega_1+p_1)} \frac{1-\phi_1(x+\omega_1+p_1)}{1-\phi_1(x+p_1)} \frac{1-\phi_1(x+p_1)}{\phi_1'(x+p_1)}\biggr] \\ &\quad \times \frac{\phi_2'(-x-\omega_2+p_2)}{1-\phi_2(0)} \\ &\le \frac{M^4}{m(1-\phi_2(0))} e^{\mu(-x-\omega_2+p_2)} \\ &\le \frac{M^4}{m(1-\phi_2(0))} e^{\mu p_1}. \end{aligned}$$

Case 2. For $0 \le x \le -p_1$. In this case, $x + p_1 \le 0$, $-x + p_2 \le 0$. From (2.1), (2.3), (2.8), we obtain

$$\frac{A(x,t)}{C(x,t)} \le \frac{\phi_1'(x+\omega_1+p_1)\phi_2'(-x-\omega_2+p_2)}{\phi_1'(x+p_1)[1-\phi_2(-x+p_2)]}$$

$$
\begin{aligned}
&\le \left[\frac{\phi_1'(x+\omega_1+p_1)}{\phi_1(x+\omega_1+p_1)}\frac{\phi_1(x+\omega_1+p_1)}{\phi_1(x+p_1)}\frac{\phi_1(x+p_1)}{\phi_1'(x+p_1)}\right]\frac{\phi_2'(-x-\omega_2+p_2)}{1-\phi_2(0)} \\
&\le \frac{M^4}{m(1-\phi_2(0))}e^{\mu p_1}.
\end{aligned}
$$

Case 3. For $p_2 \le x \le 0$. In this case, $x+p_1 \le 0$, $-x+p_2 \le 0$. From (2.1), (2.3), (2.8), we obtain

$$
\begin{aligned}
\frac{A(x,t)}{C(x,t)} &\le \frac{\phi_1'(x+\omega_1+p_1)\phi_2'(-x-\omega_2+p_2)}{\phi_2'(-x+p_2)[1-\phi_1(x+p_1)]} \\
&\le \left[\frac{\phi_2'(-x-\omega_2+p_2)}{\phi_2(-x-\omega_2+p_2)}\frac{\phi_2(-x-\omega_2+p_2)}{\phi_2(-x+p_2)}\frac{\phi_2(-x+p_2)}{\phi_2'(-x+p_2)}\right]\frac{\phi_1'(x+\omega_1+p_1)}{1-\phi_1(0)} \\
&\le \frac{M^4e^{\mu}}{m(1-\phi_1(0))}e^{\mu p_1}.
\end{aligned}
$$

Case 4. For $x \le p_2$. In this case, $x+p_1 \le 0$, $-x+p_2 \ge 0$. From (2.1), (2.3), (2.5) and (2.7), we obtain

$$
\begin{aligned}
\frac{A(x,t)}{C(x,t)} &\le \frac{\phi_1'(x+\omega_1+p_1)\phi_2'(-x-\omega_2+p_2)}{\phi_2'(-x+p_2)[1-\phi_1(x+p_1)]} \\
&\le \left[\frac{\phi_2'(-x-\omega_2+p_2)}{1-\phi_2(-x-\omega_2+p_2)}\frac{1-\phi_2(-x-\omega_2+p_2)}{1-\phi_2(-x+p_2)}\frac{1-\phi_2(-x+p_2)}{\phi_2'(-x+p_2)}\right] \\
&\quad\times\frac{\phi_1'(x+\omega_1+p_1)}{1-\phi_1(0)} \\
&\le \frac{M^4e^{\mu}}{m(1-\phi_1(0))}e^{\mu p_1}.
\end{aligned}
$$

Thus (3.5) holds for all $x \in \mathbb{R}$. By similar statements, we find that (3.6) holds for all $x \in \mathbb{R}$. □

Lemma 6 *Assume that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1, 2)$ *be solutions of* (1.7) *with* $c_2 \ge c_1 \ge c^*$ *and define*

$$
R(x,t) := a\bigl(1-\phi_1(x+p_1)\bigr)\bigl(1-\phi_2(-x+p_2)\bigr)\phi_1(x+p_1)\phi_2(-x+p_2),
$$

then

$$
R(x,t) \le \frac{aM}{m}e^{\mu p_1}C(x,t). \tag{3.7}
$$

Proof We prove (3.7) by dividing $\mathbb{R}$ to several cases.

Case 1. For $x \ge -p_1$, from (2.3), (2.5), it follows that

$$
\frac{R(x,t)}{C(x,t)} \le \frac{a(1-\phi_1(x+p_1))\phi_1(x+p_1)\phi_2(-x+p_2)}{\phi_1'(x+p_1)} \le \frac{aM}{m}e^{\mu p_1}.
$$

Case 2. For $0 \le x \le -p_1$, from (2.1), (2.3), it follows that

$$
\frac{R(x,t)}{C(x,t)} \le \frac{a(1-\phi_2(-x+p_2))\phi_1(x+p_1)\phi_2(-x+p_2)}{\phi_1'(x+p_1)} \le \frac{aM}{m}e^{\mu p_1}.
$$

We can obtain a similar result for $x \le 0$. □

Lemma 7 *Assume that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1, 2)$ *be solutions of* (1.7) *with* $c_2 \geq c_1 \geq c^*$ *and define*

$$\begin{aligned} D(x,t) &:= bh(\psi_1 + \psi_2)\big[\phi_1\big(x + p_1(t - \tau_2)\big) + \phi_2\big(-x + p_2(t - \tau_2)\big)\big] \\ &\quad - bh(\psi_1 + \psi_2)\big[\phi_1\big(x + p_1(t - \tau_2)\big)\phi_2\big(-x + p_2(t - \tau_2)\big)\big] \\ &\quad - 2b\psi_1\psi_2 - bh\psi_1\phi_1\big(x + p_1(t) - c_1\tau_2\big) - bh\psi_2\phi_2\big(-x + p_2(t) - c_2\tau_2\big), \\ E(x,t) &:= \psi_1'\big(x + p_1(t)\big) + \psi_2'\big(-x + p_2(t)\big), \end{aligned}$$

where $\psi_1 = \psi_1(x + p_1(t))$ *and* $\psi_2 = \psi_2(-x + p_2(t))$. *Then*

$$D(x,t) \leq N_2 e^{\mu p_1} E(x,t), \tag{3.8}$$

where

$$N_2 = \max\left\{\frac{bhM(1+K)}{Km}, \frac{bhM(1+M_{\rho_1})}{m}, \frac{bhM(1+M_{\rho_2})}{m}\right\} > 0.$$

Proof Because of $\tau_2 \geq 0$ and from (3.3), we have

$$\begin{aligned} p_i(t - \tau_2) &= p_i(0) + c_i(t - \tau_2) - \frac{1}{\mu}\ln\left(1 + \frac{L}{c_1}e^{\mu p_1(0)}\big(1 - e^{c_1\mu(t-\tau_2)}\big)\right) \\ &= p_i(t) - c_i\tau_2 + \frac{1}{\mu}\ln\left(\frac{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu t})}{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu(t-\tau_2)})}\right) \\ &\leq p_i(t) - c_i\tau_2, \end{aligned} \tag{3.9}$$

then $\phi_1(x + p_1(t - \tau_2)) \leq \phi_1(x + p_1(t) - c_1\tau_2)$ and $\phi_2(-x + p_2(t - \tau_2)) \leq \phi_2(-x + p_2(t) - c_2\tau_2)$. Hence,

$$\begin{aligned} D(x,t) &\leq -2b\psi_1\psi_2 + bh\psi_1\phi_2\big(-x + p_2(t - \tau_2)\big) + bh\psi_2\phi_1\big(x + p_1(t - \tau_2)\big) \\ &\quad - bh(\psi_1 + \psi_2)\phi_1\big(x + p_1(t - \tau_2)\big)\phi_2\big(-x + p_2(t - \tau_2)\big). \end{aligned}$$

Moreover, by direct calculation we can see that, for $t \in (-\infty, -T]$ $(T > 0)$,

$$\begin{aligned} &\frac{1}{\mu}\ln\left(\frac{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu t})}{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu(t-\tau_2)})}\right) \\ &\quad = \frac{1}{\mu}\ln\frac{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu t})}{1 + \frac{L}{c_1}e^{\mu p_1(0)}(1 - e^{c_1\mu t}) + \frac{L}{c_1}e^{\mu p_1(0)}e^{c_1\mu t}(1 - e^{-c_1\mu\tau_2})} \\ &\quad := \frac{1}{\mu}\ln\frac{1}{1 + \rho(t)}, \end{aligned}$$

where

$$\rho(t) = \frac{\frac{L}{c_1}e^{c_1 p_1(0)}e^{c_1\mu t}(1 - e^{-c_1\mu\tau_2})}{1 + \frac{L}{c_1}e^{c_1 p_1(0)}(1 - e^{c_1\mu t})} \leq \frac{e^{-c_1\mu T}(1 - e^{-c_1\mu\tau_2})}{1 - e^{-c_1\mu T}} := \rho_0.$$

Set

$$\rho_i = c_i\tau_2 - \frac{1}{\mu}\ln\left(\frac{1}{1+\rho_0}\right),$$

then, for $t \in (-\infty, -T]$, we have $p_i(t-\tau_2) \geq p_i(t) - \rho_i$ $(i = 1, 2)$.

Next we prove (3.8) by dividing $\mathbb{R}$ into three parts.

Case 1. $p_2 \leq x \leq -p_1$.

For $p_2 \leq x \leq 0$, from (1.8), (2.2)–(2.4), we have

$$\begin{aligned}
\frac{D(x,t)}{E(x,t)} &\leq \frac{bh\psi_1(x+p_1)\phi_2(-x+p_2)}{\psi_2'(-x+p_2)} + \frac{bh\psi_2(-x+p_2)\phi_1(x+p_1)}{\psi_2'(-x+p_2)} \\
&= \frac{bh\psi_1(x+p_1)\phi_2(-x+p_2)}{\psi_2(-x+p_2)}\frac{\psi_2(-x+p_2)}{\psi_2'(-x+p_2)} + \frac{bh\psi_2(-x+p_2)\phi_1(x+p_1)}{\psi_2'(-x+p_2)} \\
&\leq \frac{bhM}{Km}e^{\mu(x+p_1)} + \frac{bhM}{m}e^{\mu(x+p_1)} \\
&\leq \frac{bhM(1+K)}{Km}e^{\mu p_1}.
\end{aligned} \tag{3.10}$$

We can obtain a similar result for $0 \leq x \leq -p_1$.

Case 2. $x \geq -p_1$.

Because of the increasing of p_i, $p_i(t-\tau_2) \geq p_i(t) - \rho_i$ $(i = 1, 2)$ and

$$-2b\psi_1\psi_2 = -(2b - bh)\psi_1\psi_2 - bh\psi_1\psi_2,$$

it follows that

$$\begin{aligned}
D(x,t) &\leq bh\psi_1(x+p_1)\phi_2\bigl(-x+p_2(t-\tau_2)\bigr)\bigl[1-\phi_1\bigl(x+p_1(t-\tau_2)\bigr)\bigr] \\
&\quad + bh\psi_2(-x+p_2)\bigl[\phi_1\bigl(x+p_1(t-\tau_2)\bigr) - \psi_1(x+p_1)\bigr] \\
&\leq bh\phi_2\bigl(-x+p_2(t-\tau_2)\bigr)\bigl[1-\phi_1(x+p_1-\rho_1)\bigr] \\
&\quad + bh\psi_2(-x+p_2)\bigl[1-\psi_1(x+p_1)\bigr].
\end{aligned}$$

Hence, from (2.3), (2.4), (2.6), (2.9), we obtain

$$\begin{aligned}
\frac{D(x,t)}{E(x,t)} &\leq \frac{bh\phi_2(-x+p_2)[1-\phi_1(x+p_1-\rho_1)]}{\psi_1'(x+p_1)} \\
&\quad + \frac{bh\psi_2(-x+p_2)[1-\psi_1(x+p_1)]}{\psi_1'(x+p_1)} \\
&\leq \frac{bhMM_{\rho_1}}{m}e^{\mu(-x+p_2)} + \frac{bhM}{m}e^{\mu(-x+p_2)} \\
&\leq \frac{bhM(1+M_{\rho_1})}{m}e^{\mu p_1}.
\end{aligned} \tag{3.11}$$

Case 3. $x \leq p_2 < 0$.

Similar to Case 2, we obtain

$$D(x,t) \leq bh\psi_1(x+p_1)\bigl[\phi_2\bigl(-x+p_2(t-\tau_2)\bigr) - \psi_2(-x+p_2)\bigr]$$

$$
\begin{aligned}
&\quad + bh\psi_2(-x+p_2)\phi_1\big(x+p_1(t-\tau_2)\big)\big[1-\phi_2\big(-x+p_2(t-\tau_2)\big)\big] \\
&\le bh\psi_1(x+p_1)\big[1-\psi_2(-x+p_2)\big] \\
&\quad + bh\phi_1\big(x+p_1(t-\tau_2)\big)\big[1-\phi_2\big(-x+p_2(t)-\rho_2\big)\big].
\end{aligned}
$$

Hence, from (2.3), (2.4), (2.6), (2.9), we obtain

$$
\begin{aligned}
\frac{D(x,t)}{E(x,t)} &\le \frac{bh\psi_1(x+p_1)[1-\psi_2(-x+p_2)]}{\psi_2'(-x+p_2)} \\
&\quad + \frac{bh\phi_1(x+p_1(t-\tau_2))[1-\phi_2(-x+p_2(t)-\rho_2)]}{\psi_2'(-x+p_2)} \\
&\le \frac{bhM}{m}e^{\mu(x+p_1)} + \frac{bhMM_{\rho_2}}{m}e^{\mu(x+p_1)} \\
&\le \frac{bhM(1+M_{\rho_2})}{m}e^{\mu p_1}.
\end{aligned}
\tag{3.12}
$$

Therefore, combining with (3.10)–(3.12), (3.8) holds for all $x \in \mathbb{R}$. □

Lemma 8 *Assume* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1,2)$ *be solutions of* (1.7) *with* $c_2 \ge c_1 \ge c^*$. *Let* L *in* (3.2) *be*

$$
L \ge \max\left\{N_1, N_2, \frac{aM}{m}\right\},
$$

where N_i, $i = 1,2$, *are defined in Lemmas* 5 *and* 7. *Then*

$$
\begin{cases}
\bar{u}(x,t) = \phi_1(x+p_1(t)) + \phi_2(-x+p_2(t)) - \phi_1(x+p_1(t))\phi_2(-x+p_2(t)), \\
\bar{v}(x,t) = \min\{1, \psi_1(x+p_1(t)) + \psi_2(-x+p_2(t))\},
\end{cases}
$$

is a supersolution of (3.1) *for* $(x,t) \in \mathbb{R} \times (-\infty, -T]$ *with* $T > 0$.

Proof Let

$$
\begin{aligned}
H_1^+ &= \big\{(x,t) : \psi_1\big(x+p_1(t-\tau_1)\big) + \psi_2\big(-x+p_2(t-\tau_1)\big) \ge 1\big\}, \\
H_1^- &= H_{11}^- \cup H_{12}^-, \\
H_{11}^- &= \big\{(x,t) : \psi_1\big(x+p_1(t-\tau_1)\big) + \psi_2\big(-x+p_2(t-\tau_1)\big) < 1, \\
&\qquad \psi_1\big(x+p_1(t)\big) + \psi_2\big(-x+p_2(t)\big) \ge 1\big\}, \\
H_{12}^- &= \big\{(x,t) : \psi_1\big(x+p_1(t)\big) + \psi_2\big(-x+p_2(t)\big) < 1\big\}.
\end{aligned}
$$

Firstly, we show that $F_1(\bar{u},\bar{v}) \ge 0$ for $(x,t) \in \mathbb{R} \times (-\infty,-T]$ with $T > 0$. Noticing that $\bar{u}(x,t) = \phi_1 + \phi_2 - \phi_1\phi_2$ and combining with (1.7), we have

$$
\begin{aligned}
F_1(\bar{u},\bar{v}) &= \big[\phi_1'(1-\phi_2) + \phi_2'(1-\phi_1)\big]Le^{\mu p_1} \\
&\quad - \big[\phi_1(x+1+p_1) - \phi_1(x+p_1)\big]\big[\phi_2(-x+p_2) - \phi_2(-x-1+p_2)\big] \\
&\quad - \big[\phi_1(x+p_1) - \phi_1(x-1+p_1)\big]\big[\phi_2(-x+1+p_2) - \phi_2(-x+p_2)\big]
\end{aligned}
$$

$$
\begin{aligned}
&-f_1(\bar{u},\bar{v}) + (1-\phi_2)f_1(\phi_1,\psi_1) + (1-\phi_1)f_1(\phi_2,\psi_2) \\
&= C(x,t)Le^{\mu p_1(t)} - A(x,t) - B(x,t) - G(x,t),
\end{aligned}
$$

where $A(x,t)$, $B(x,t)$, $C(x,t)$ are defined in Lemma 5 and

$$G(x,t) = f_1(\bar{u},\bar{v}) - (1-\phi_2)f_1(\phi_1,\psi_1) - (1-\phi_1)f_1(\phi_2,\psi_2).$$

Now we consider $G(x,t)$. If $(x,t) \in H_1^+$, then $\bar{v}(x,t-\tau_1) \equiv 1$. From (3.9), we have

$$\psi_1(x+p_1-c_1\tau_1) + \psi_2(-x+p_2-c_2\tau_1) \geq \psi_1\bigl(x+p_1(t-\tau_1)\bigr) + \psi_2\bigl(-x+p_2(t-\tau_1)\bigr) \geq 1.$$

Thus

$$
\begin{aligned}
G(x,t) &= a(1-\phi_1)(1-\phi_2)\bigl[k + \phi_1\phi_2 - k\bigl(\psi_1(x+p_1-c_1\tau_1) + \psi_2(-x+p_2-c_2\tau_1)\bigr)\bigr] \\
&\leq a(1-\phi_1)(1-\phi_2)\phi_1\phi_2.
\end{aligned}
$$

If $(x,t) \in H_1^-$, then $\bar{v}(x,t-\tau_1) = \psi_1(x+p_1(t-\tau_1)) + \psi_2(-x+p_2(t-\tau_1))$. From (3.9), we have

$$\psi_1(x+p_1-c_1\tau_1) + \psi_2(-x+p_2-c_2\tau_1) \geq \psi_1\bigl(x+p_1(t-\tau_1)\bigr) + \psi_2\bigl(-x+p_2(t-\tau_1)\bigr).$$

Thus

$$
\begin{aligned}
G(x,t) &= a(1-\phi_1)(1-\phi_2)\bigl\{\phi_1\phi_2 + k\bigl[\psi_1\bigl(x+p_1(t-\tau_1)\bigr) + \psi_2\bigl(-x+p_2(t-\tau_1)\bigr) \\
&\qquad - \psi_1(x+p_1-c_1\tau_1) - \psi_2(-x+p_2-c_2\tau_1)\bigr]\bigr\} \\
&\leq a(1-\phi_1)(1-\phi_2)\phi_1\phi_2.
\end{aligned}
$$

Therefore, combining with (3.5)–(3.7), we can prove that $F_1(\bar{u},\bar{v}) \geq 0$ for $(x,t) \in \mathbb{R} \times (-\infty,-T]$.

Secondly we show that $F_2(\bar{u},\bar{v}) \geq 0$ for $(x,t) \in \mathbb{R} \times (-\infty,-T]$. For $(x,t) \in H_1^+$, then $\bar{v} \equiv 1$ because of $\psi_1(x+p_1(t-\tau_1)) + \psi_2(-x+p_2(t-\tau_1)) \leq \psi_1(x+p_1(t)) + \psi_2(-x+p_2(t))$. Thus $F_2(\bar{u},\bar{v}) = bh[1-\bar{u}(x,t-\tau_2)] \geq 0$. For $(x,t) \in H_{11}^-$, then $\bar{v} \equiv 1$. Similarly, we have $F_2(\bar{u},\bar{v}) \geq 0$. For $(x,t) \in H_{12}^-$, then $\bar{v} = \psi_1(x+p_1(t)) + \psi_2(-x+p_2(t))$ and

$$
\begin{aligned}
F_2(\bar{u},\bar{v}) &= \bigl(p_1'-c_1\bigr)\bigl(\psi_1'+\psi_2'\bigr) + 2b\psi_1\psi_2 \\
&\qquad + bh\bigl[\psi_1\phi_1\bigl(x+p_1(t)-c_1\tau_2\bigr) + \psi_2\phi_2\bigl(-x+p_2(t)-c_2\tau_2\bigr)\bigr] \\
&\qquad - bh(\psi_1+\psi_2)\bigl[\phi_1\bigl(x+p_1(t-\tau_2)\bigr) + \phi_2\bigl(-x+p_2(t-\tau_2)\bigr)\bigr] \\
&\qquad - bh\bigl[\phi_1\bigl(x+p_1(t-\tau_2)\bigr)\phi_2\bigl(-x+p_2(t-\tau_2)\bigr)\bigr] \\
&= E(x,t)Le^{\mu p_1(t)} - D(x,t),
\end{aligned}
$$

where $D(x,t)$ and $E(x,t)$ are defined in Lemma 7. Therefore, from (3.8), it follows that $F_2(\bar{u},\bar{v}) \geq 0$ for $(x,t) \in \mathbb{R} \times (-\infty,-T]$. This completes the proof. □

The following is easy to see.

Lemma 9 $(1,1)$ *is a supersolution to* (3.1) *for* $(x,t) \in \mathbb{R} \times (-T,+\infty)$.

Lemma 10 *Assume that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1,2)$ *be solutions of* (1.7) *with* $c_2 \geq c_1 \geq c^*$. *Then the pairing*

$$\begin{cases} \underline{u}(x,t) = \max\{\phi_1(x + c_1 t + \nu_1), \phi_2(-x + c_2 t + \nu_2)\}, \\ \underline{v}(x,t) = \max\{\psi_1(x + c_1 t + \nu_1), \psi_2(-x + c_2 t + \nu_2)\}, \end{cases}$$

is a subsolution of (3.1), *where L is defined in Lemma* 8, ν_1 *and* ν_2 *are defined in* (3.4).

Combining with Lemmas 8–10, similar to [5, 9–15], we can obtain the following main result of this paper.

Theorem 1 *Assume that* $0 < h < 1 < k$ *holds. Let* (c_i, ϕ_i, ψ_i) $(i = 1,2)$ *be solutions of* (1.7) *with* $c_2 \geq c_1 \geq c^*$. *Thus, for any given constants* ι_1, ι_2, *there is an entire solution* $(u(x,t), v(x,t)) \in (0,1) \times (0,1)$ *of* (1.6) *such that*

$$\lim_{t\to-\infty}\Bigl\{\sup_{x\geq 0}\bigl|u(x,t) - \phi_1(x + c_1 t + \iota_1)\bigr| + \sup_{x\leq 0}\bigl|u(x,t) - \phi_2(-x + c_2 t + \iota_2)\bigr|\Bigr\} = 0,$$

$$\lim_{t\to-\infty}\Bigl\{\sup_{x\geq 0}\bigl|v(x,t) - \psi_1(x + c_1 t + \iota_1)\bigr| + \sup_{x\leq 0}\bigl|v(x,t) - \psi_2(-x + c_2 t + \iota_2)\bigr|\Bigr\} = 0,$$

and

$$\lim_{t\to+\infty}\Bigl\{\sup_{x\in\mathbb{R}}\bigl|u(x,t) - 1\bigr| + \sup_{x\in\mathbb{R}}\bigl|v(x,t) - 1\bigr|\Bigr\} = 0.$$

In addition, this solution satisfies

(i) $\frac{\partial u(x,t)}{\partial t} > 0$ *and* $\frac{\partial v(x,t)}{\partial t} > 0$ *for all* $(x,t) \in \mathbb{R} \times \mathbb{R}$;

(ii) $\lim_{t\to-\infty}\{\sup_{x\in[\alpha,\beta]} |u(x,t)| + \sup_{x\in[\alpha,\beta]} |v(x,t)|\} = 0$ *with* $\alpha, \beta \in \mathbb{R}$ *and* $\alpha < \beta$;

(iii) $\lim_{|x|\to\infty}\{\sup_{t\in[S,\infty)} |u(x,t) - 1| + \sup_{t\in[S,\infty)} |v(x,t) - 1|\} = 0$ *with* $S \in \mathbb{R}$.

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