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Ulam–Hyers–Mittag-Leffler stability for ψ-Hilfer fractional-order delay differential equations

Kui Liu[1], JinRong Wang[2,3]* and Donal O'Regan[4]

*Correspondence:
wjr9668@126.com
[2]Department of Mathematics, Guizhou University, Guiyang, China
[3]School of Mathematical Sciences, Qufu Normal University, Qufu, China
Full list of author information is available at the end of the article

Abstract

In this paper, we present results on the existence, uniqueness, and Ulam–Hyers–Mittag-Leffler stability of solutions to a class of ψ-Hilfer fractional-order delay differential equations. We use the Picard operator method and a generalized Gronwall inequality involved in a ψ-Riemann–Liouville fractional integral. Finally, we give two examples to illustrate our main theorems.

Keywords: ψ-Hilfer fractional-order delay differential equations; Solutions; Existence; Stability

1 Introduction

Fractional-order differential equations are important since their nonlocal property is suitable to characterize memory phenomena in economic, control, and materials sciences. Existence, stability, and control theory to fractional differential equations was investigated in [1–21]. In particular, the Ulam-type stability of delay differential equations was investigated in [22–30]. In [22], results for a delay differential equation were obtained using the Picard operator method, and in [23] the authors adopted a similar approach to establish the existence and uniqueness results for a Caputo-type fractional-order delay differential equation. In [31, 32], the authors gave stability and numerical schemes for two classes of fractional equations. Sousa and Oliveira [33] proposed the ψ-Hilfer fractional differentiation operator and established ψ-Hilfer fractional differential equations. In [24] the authors studied the Ulam–Hyers stability and the Ulam–Hyers–Rassias stability of ψ-Hilfer fractional integro-differential equations via the Banach fixed point method, and in [28] the author discussed the existence and uniqueness of solutions and Ulam–Hyers and Ulam–Hyers–Rassias stabilities for ψ-Hilfer nonlinear fractional differential equations via a generalized Gronwall inequality (see [34]).

Motivated by [23, 24, 28], we consider the ψ-Hilfer fractional differential equation

$$\begin{cases} {}^{H}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}x(\tau) = f(\tau,x(\tau),x(g(\tau))), \quad \tau\in I=(0,d], \\ I_{0^+}^{1-\gamma;\psi}x(0^+) = x_0\in\mathbb{R}, \\ x(\tau)=\varphi(\tau), \quad \tau\in[-h,0], \end{cases} \tag{1}$$

where ${}^{H}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative (see Definition 2.1) of order $0<\alpha\leq 1$ and type $0\leq\beta\leq 1$, $I_{0^+}^{1-\gamma;\psi}(\cdot)$ is the Riemann–Liouville fractional integral of order $1-\gamma$, $\gamma=\alpha+\beta(1-\alpha)$ with respect to the function ψ (see [2]), and $f: I\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given function.

We establish the existence and uniqueness of solutions for (1) using the Picard operator approach in a weight function space. We also introduce and present Ulam–Hyers–Mittag-Leffler stability of solutions to (1).

2 Preliminaries

We collect the basic definitions of the ψ-Riemann–Liouville fractional integral, the ψ-Hilfer fractional derivative, and the standard Picard operator and an abstract Gronwall lemma.

Let $[c,d]$ $(0<c<d<\infty)$ be a finite interval on $\mathbb{R}^+$, and let $C[c,d]$ be the space of continuous function $g:[c,d]\to\mathbb{R}$ with norm

$$\|g\|_{C[c,d]}=\max_{c\leq x\leq d}\bigl|g(x)\bigr|.$$

The weighted space $C_{1-\gamma;\psi}[c,d]$ of continuous g on $(c,d]$ is defined by (see [24])

$$C_{1-\gamma;\psi}[c,d]=\bigl\{g:(c,d]\to\mathbb{R};\bigl(\psi(x)-\psi(c)\bigr)^{1-\gamma}g(x)\in C[c,d]\bigr\},\quad 0\leq\gamma<1,$$

with norm

$$\|g\|_{C_{1-\gamma;\psi}[c,d]}=\max_{x\in[c,d]}\bigl|\bigl(\psi(x)-\psi(c)\bigr)^{1-\gamma}g(x)\bigr|$$

or

$$\|g\|_B:=\max_{\tau\in[c,d]}\bigl|\bigl(\psi(\tau)-\psi(c)\bigr)^{1-\gamma}g(\tau)\bigr|e^{-\theta(\psi(\tau)-\psi(c))},\quad\theta>0.$$

Definition 2.1 (see [33]) Let (c,d) $(-\infty\leq c<d\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$, and let $\alpha>0$. In addition, let $\psi(x)$ be an increasing and positive monotone function on $(c,d]$ having a continuous derivative $\psi'(x)$ on (c,d). The left- and right-sided fractional integrals of a function g with respect to a function ψ on $[c,d]$ are defined by

$$I_{c^+}^{\alpha:\psi}g(x)=\frac{1}{\Gamma(\alpha)}\int_c^x\psi'(\tau)\bigl(\psi(x)-\psi(\tau)\bigr)^{\alpha-1}g(\tau)\,dt,$$
$$I_{d^-}^{\alpha:\psi}g(x)=\frac{1}{\Gamma(\alpha)}\int_x^d\psi'(\tau)\bigl(\psi(\tau)-\psi(x)\bigr)^{\alpha-1}g(\tau)\,dt,$$

respectively; here Γ is the gamma function.

Definition 2.2 (see [33]) Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, and let $f,\psi\in C^n[c,d]$ be two functions such that ψ is increasing and $\psi'(x)\neq 0$ for all $x\in[c,d]$. The left-side ψ-Hilfer fractional derivative ${}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi}(\cdot)$ of a function g of order α and type $0\leq\beta\leq 1$ is defined by

$${}^{H}\mathbb{D}_{c^+}^{\alpha,\beta;\psi}g(x)=I_{c^+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{c^+}^{(1-\beta)(n-\alpha);\psi}g(x).$$

The right-sided ψ-Hilfer fractional derivative is defined in an analogous way.

Theorem 2.3 (see [33]) *If $g \in C^1[c,d]$, $0 < c < 1$, and $0 \le \beta \le 1$, then*

$$ {}^H\mathbb{D}_{c^+}^{\alpha,\beta;\psi} I_{c^+}^{\alpha;\psi} g(x) = g(x). $$

Theorem 2.4 (see [33]) *If $g \in C^1[c,d]$, $0 < c < 1$, and $0 \le \beta \le 1$, then*

$$ I_{c^+}^{\alpha;\psi}\, {}^H\mathbb{D}_{c^+}^{\alpha,\beta;\psi} g(x) = g(x) - \frac{(\psi(x) - \psi(c))^{\gamma-1}}{\Gamma(\gamma)} I_{c^+}^{(1-\beta)(1-\alpha);\psi} g(c). $$

Let $I = [c,d]$. For $f \in C(I \times \mathbb{R}^2, \mathbb{R})$ and $\varepsilon > 0$, we consider the equations

$$ {}^H\mathbb{D}_{0^+}^{\alpha,\beta;\psi} x(\tau) = f\bigl(\tau, x(\tau), x\bigl(g(\tau)\bigr)\bigr), \quad \tau \in (0,d], \tag{2} $$

$$ I_{0^+}^{1-\gamma;\psi} x\bigl(0^+\bigr) = x_0, \tag{3} $$

$$ x(\tau) = \varphi(\tau), \quad \tau \in [-h,0], \tag{4} $$

and the inequality

$$ \bigl|{}^H\mathbb{D}_{0^+}^{\alpha,\beta;\psi} x(\tau) - f\bigl(\tau, x(\tau), x\bigl(g(\tau)\bigr)\bigr)\bigr| \le \varepsilon \mathbb{E}_\alpha \bigl(\psi(\tau) - \psi(0)\bigr)^\alpha, \quad \tau \in (0,d], \tag{5} $$

where $\mathbb{E}_\alpha$ is the Mittag-Leffler function [2] defined by

$$ \mathbb{E}_\alpha(x) := \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(i\alpha + 1)}, \quad x \in \mathbb{C}, \Re(\alpha) > 0. \tag{6} $$

Motivated by [23, Lemma 2.4], we introduce the following concept.

Definition 2.5 Equation (2) is Ulam–Hyers–Mittag-Leffler stable with respect to $\mathbb{E}_\alpha((\psi(\tau) - \psi(0))^\alpha)$ if there exists $c_{\mathbb{E}_\alpha} > 0$ such that, for each $\varepsilon > 0$ and each solution $y \in C([-h,d],\mathbb{R})$ to (5), there exists a solution $x \in C([-h,d],\mathbb{R})$ to (2) with

$$ \bigl|y(\tau) - x(\tau)\bigr| \le c_{\mathbb{E}_\alpha} \varepsilon \mathbb{E}_\alpha \bigl(\bigl(\psi(\tau) - \psi(0)\bigr)^\alpha\bigr), \quad \tau \in [-h,d]. $$

Remark 2.6 A function $x \in C([-h,d],\mathbb{R})$ is a solution of inequality (5) if and only if there exists a function $\tilde{h} \in C([-h,d],\mathbb{R})$ (which depends on x) such that

(i) $|\tilde{h}(\tau)| \le \varepsilon \mathbb{E}_\alpha((\psi(\tau) - \psi(0))^\alpha)$, $\tau \in [-h,d]$,

(ii) ${}^H\mathbb{D}_{0^+}^{\alpha,\beta;\psi} x(\tau) = f(\tau, x(\tau), x(g(\tau))) + \tilde{h}(\tau)$, $\tau \in (0,d]$.

Definition 2.7 (see Definition 3.1 of [23]) Let (Y,ρ) be a metric space. Now $T : Y \to Y$ is a Picard operator if there exists $y^* \in Y$ such that

(i) $F_T = y^*$ where $F_T = \{y \in Y : T(y) = y\}$ is the fixed point set of T;

(ii) the sequence $(T^n(y_0))_{n\in\mathbb{N}}$ converges to y^* for all $y_0 \in Y$.

Lemma 2.8 (see Lemma 3.2 of [23]) *Let $(Y,\rho,\le)$ be an ordered metric space, and let $T : Y \to Y$ be an increasing Picard operator with $F_T = \{y_T^*\}$. Then for $y \in Y$, $y \le T(y)$ implies $y \le y_T^*$.*

From Theorems 2.3 and 2.4 we have the following:

Lemma 2.9 (see [24]) *Let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then (2) is equivalent to*

$$x(\tau) = \frac{(\psi(\tau) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} f\big(s, x(s), x\big(g(s)\big)\big)\, ds. \qquad (7)$$

Remark 2.10 Let $y \in C(I, \mathbb{R})$ be a solution of inequality (5). Then y is a solution of the following integral inequality:

$$\begin{aligned}
&\left| y(\tau) - \frac{(\psi(\tau) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} f\big(s, y(s), y\big(g(s)\big)\big)\, ds \right| \\
&\quad \le \frac{\varepsilon}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \mathbb{E}_\alpha\big(\big(\psi(s) - \psi(0)\big)^\alpha\big)\, ds \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \sum_{k=0}^\infty \frac{(\psi(s) - \psi(0))^{k\alpha}}{\Gamma(k\alpha + 1)}\, ds \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \int_0^\tau \big(\psi(\tau) - \psi(s)\big)^{\alpha-1} \big(\psi(s) - \psi(0)\big)^{k\alpha}\, d\psi(s) \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \int_0^{\psi(\tau)-\psi(0)} \big(\psi(\tau) - \psi(0) - u\big)^{\alpha-1} u^{k\alpha}\, du \\
&\quad \big(\text{let } u = \psi(s) - \psi(0)\big) \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{\alpha-1} \int_0^{\psi(\tau)-\psi(0)} \left(1 - \frac{u}{\psi(\tau) - \psi(0)}\right)^{\alpha-1} u^{k\alpha}\, du \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{(k+1)\alpha} \int_0^1 (1 - v)^{\alpha-1} v^{k\alpha}\, dv \\
&\quad \left(\text{let } v = \frac{u}{\psi(\tau) - \psi(0)}\right) \\
&\quad = \frac{\varepsilon}{\Gamma(\alpha)} \sum_{k=0}^\infty \frac{1}{\Gamma(k\alpha + 1)} \big(\psi(\tau) - \psi(0)\big)^{(k+1)\alpha} \frac{\Gamma(k\alpha + 1)\Gamma(\alpha)}{\Gamma((k+1)\alpha + 1)} \\
&\quad \le \varepsilon \sum_{n=0}^\infty \frac{(\psi(\tau) - \psi(0))^{n\alpha}}{\Gamma(n\alpha + 1)} \\
&\quad = \varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau) - \psi(0)\big)^\alpha\big).
\end{aligned}$$

Lemma 2.11 (see [34]) *Let $\alpha > 0$, and let $\psi \in C^1((0, d], \mathbb{R})$ be a function such that ψ is increasing and $\psi'(\tau) \neq 0$ for all $\tau \in (0, d]$. Suppose that $d \geq 0$, z is a nonnegative function locally integrable on $(0, d]$, and w is nonnegative and locally integrable on $(0, d]$ with*

$$w(\tau) \le z(\tau) + k \int_0^\tau \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} w(s)\, ds, \quad \tau \in (0, d].$$

Then

$$w(\tau) \le z(\tau) + \int_0^\tau \sum_{n=1}^\infty \frac{[k\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \psi'(s)\big(\psi(\tau) - \psi(s)\big)^{\alpha-1} z(s)\, ds, \quad \tau \in (0, d].$$

Remark 2.12 (see [34]) Under the hypotheses of Lemma 2.11, let z be a nondecreasing function on $(0,d]$. Then we have

$$w(\tau) \le z(\tau)\mathbb{E}_\alpha\bigl(k\Gamma(\alpha)\bigl[\psi(\tau)-\psi(0)\bigr]^\alpha\bigr), \quad \tau \in (0,d],$$

where $\mathbb{E}_\alpha$ is the Mittag-Leffler function defined by (6).

3 Main results

In this section, we establish the existence, uniqueness, and Ulam–Hyers–Mittag-Leffler stability.

We impose the following conditions.

(H_1) $f \in C(I \times \mathbb{R}^2, \mathbb{R})$, $g \in C(I, [-h,d])$, $g(\tau) \le \tau$, $h > 0$.

(H_2) There exists $L_f > 0$ such that

$$\bigl|f(\tau,u_1,u_2)-f(\tau,v_1,v_2)\bigr| \le L_f \sum_{i=1}^{2} |u_i - v_i| \quad \text{for all } \tau \in I, u_i, v_i \in \mathbb{R}, i = 1,2.$$

(H_3) We have the inequality

$$\frac{2L_f\Gamma(\gamma)(\psi(d)-\psi(0))^\alpha}{\Gamma(\gamma+\alpha)} < 1.$$

Theorem 3.1 *Assume that* (H_1), (H_2), *and* (H_3) *are satisfied. Then*

(i) (2)–(4) *has a unique solution in* $C[-h,d] \cap C_{1-\gamma;\psi}[c,d]$.

(ii) (2) *is Ulam–Hyers–Mittag-Leffler stable.*

Proof From Lemma 2.9 we get that (2)–(4) is equivalent to the following system:

$$y(\tau) = \begin{cases} \varphi(\tau), & \tau \in [-h,0], \\ \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} f(s,y(s),y(g(s)))\,ds, & \tau \in (0,d]. \end{cases} \tag{8}$$

The existence of a solution for (8) can be turned into a fixed point problem in $X := C[-h,d]$ for the operator T_f defined by

$$T_f(x)(\tau) = \begin{cases} \varphi(\tau), & \tau \in [-h,0], \\ \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} f(s,y(s),y(g(s)))\,ds, & \tau \in (0,d]. \end{cases} \tag{9}$$

Note that for any continuous function f, T_f is also continuous. Indeed,

$$\begin{aligned} &\bigl|T_f(x)(\tau)-T_f(x)(\tau_0)\bigr| \\ &\quad = \biggl|\frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 + \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} f\bigl(s,y(s),y\bigl(g(s)\bigr)\bigr)\,ds \end{aligned}$$

$$- \frac{(\psi(\tau_0) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^{\tau_0} \psi'(s)\bigl(\psi(\tau_0) - \psi(s)\bigr)^{\alpha-1} f\bigl(s, y(s), y\bigl(g(s)\bigr)\bigr)\, ds \Bigr|$$
$$\to 0$$

as $\tau \to \tau_0$.

Next, we show T_f defined in (9) is a contraction mapping on $X := C[-h,d]$ with respect to $\|\cdot\|_{C_{1-\gamma;\psi}[0,d]}$. Consider $T_f : X \to X$ defined in (9). For $\tau \in [-h,0]$, we have

$$\bigl|T_f(x)(\tau) - T_f(y)(\tau)\bigr| = 0, \quad x, y \in C\bigl([-h,0], \mathbb{R}\bigr).$$

For all $\tau \in (0,d]$, we have

$$\begin{aligned}
&\bigl|T_f(x)(\tau) - T_f(y)(\tau)\bigr| \\
&\quad \le \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s)\bigl(\psi(\tau) - \psi(s)\bigr)^{\alpha-1} \bigl|f\bigl(s, x(s), x\bigl(g(s)\bigr)\bigr) - f\bigl(s, y(s), y\bigl(g(s)\bigr)\bigr)\bigr|\, ds \\
&\quad \le \frac{L_f}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s)\bigl(\psi(\tau) - \psi(s)\bigr)^{\alpha-1} \bigl(\psi(s) - \psi(0)\bigr)^{\gamma-1} \bigl\{\bigl(\psi(s) - \psi(0)\bigr)^{1-\gamma} \bigl[\bigl|x(s) - y(s)\bigr| \\
&\qquad + \bigl|x\bigl(g(s) - y\bigl(g(s)\bigr)\bigr)\bigr|\bigr]\bigr\}\, ds \\
&\quad \le \frac{L_f}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s)\bigl(\psi(\tau) - \psi(s)\bigr)^{\alpha-1} \bigl(\psi(s) - \psi(0)\bigr)^{\gamma-1} \\
&\qquad \times \Bigl[\max_{s\in[0,d]} \bigl|\bigl(\psi(s) - \psi(0)\bigr)^{1-\gamma} \bigl(x(s) - y(s)\bigr)\bigr| \\
&\qquad + \max_{s\in[0,d]} \bigl|\bigl(\psi(s) - \psi(0)\bigr)^{1-\gamma} x\bigl(g(s) - y\bigl(g(s)\bigr)\bigr)\bigr|\Bigr]\, ds \\
&\quad \le \frac{2L_f}{\Gamma(\alpha)} \|x - y\|_{C_{1-\gamma;\psi}[0,d]} \int_0^{\tau} \psi'(s)\bigl(\psi(\tau) - \psi(s)\bigr)^{\alpha-1} \bigl(\psi(s) - \psi(0)\bigr)^{\gamma-1}\, ds \\
&\quad = \frac{2L_f(\psi(\tau) - \psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha)} \frac{\Gamma(\gamma)\Gamma(\alpha)}{\Gamma(\gamma+\alpha)} \|x - y\|_{C_{1-\gamma;\psi}[0,d]},
\end{aligned}$$

which implies that

$$\bigl\|T_f(x) - T_f(y)\bigr\|_{C_{1-\gamma;\psi}[0,d]} \le \frac{2L_f\Gamma(\gamma)(\psi(d) - \psi(0))^{\alpha}}{\Gamma(\alpha+\gamma)} \|x - y\|_{C_{1-\gamma;\psi}[0,d]}.$$

Thus T_f is a contraction (via the norm $\|\cdot\|_{C_{1-\gamma;\psi}[0,d]}$ on X). Now apply the Banach contraction principle to establish (i).

Now we prove (ii). Let $y \in C[-h,0] \cap C_{1-\gamma;\psi}[0,d]$ be a solution to (2). We denote by $x \in C[-h,0] \cap C_{1-\gamma;\psi}[0,d]$ the unique solution to problem (1). Now

$$x(\tau) = \begin{cases} \varphi(\tau), & \tau \in [-h,0], \\ \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 & \\ \quad + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \psi'(s)(\psi(\tau) - \psi(s))^{\alpha-1} f(s, y(s), y(g(s)))\, ds, & \tau \in (0,d]. \end{cases}$$

From Remark 2.10 we have

$$\begin{aligned} &\left| y(\tau) - \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} f\bigl(s, y(s), y(g(s))\bigr)\,ds \right| \\ &\quad \le \varepsilon \mathbb{E}_\alpha\bigl(\bigl(\psi(\tau)-\psi(0)\bigr)^\alpha\bigr) \end{aligned} \tag{10}$$

for $\tau \in (0,d]$ and note that $|y(\tau)-x(\tau)| = 0$ for $\tau \in [-h,0]$.

For all $\tau \in (0,d]$, it follows from (H_2) and (10) that

$$\begin{aligned} &\bigl|y(\tau)-x(\tau)\bigr| \\ &\quad \le \left| y(\tau) - \frac{(\psi(\tau)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} y_0 - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} f\bigl(s, y(s), y(g(s))\bigr)\,ds \right| \\ &\qquad + \left| \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} f\bigl(s, y(s), y(g(s))\bigr)\,ds \right. \\ &\qquad \left. - \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} f\bigl(s, x(s), x(g(s))\bigr)\,ds \right| \\ &\quad \le \varepsilon \mathbb{E}_\alpha\bigl(\bigl(\psi(\tau)-\psi(0)\bigr)^\alpha\bigr) \\ &\qquad + \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} \bigl[\bigl|y(s)-x(s)\bigr| + \bigl|y(g(s))-x(g(s))\bigr|\bigr]\,ds. \end{aligned} \tag{11}$$

For all $w \in C([-h,d],\mathbb{R}_+)$, consider the operator

$$T_1 : C\bigl([-h,d],\mathbb{R}_+\bigr) \to C\bigl([-h,d],\mathbb{R}_+\bigr)$$

defined by

$$T_1(w)(\tau) = \begin{cases} 0, & \tau \in [-h,0], \\ \varepsilon \mathbb{E}_\alpha((\psi(\tau)-\psi(0))^\alpha) & \\ \quad + \frac{L_f}{\Gamma(\alpha)}(\int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} w(s)\,ds & \\ \quad + \int_0^\tau \psi'(s)(\psi(\tau)-\psi(s))^{\alpha-1} w(g(s))\,ds), & \tau \in (0,d]. \end{cases}$$

We prove that T_1 is a Picard operator. For all $\tau \in [0,d]$, it follows from (H_2) that

$$\bigl|T_1(w)(\tau) - T_1\bigl(z(\tau)\bigr)\bigr| \le \frac{2L_f \Gamma(\gamma)(\psi(\tau)-\psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} \|w-z\|_{C_{1-\gamma;\psi}[0,d]}$$

for all $w, z \in C([-h,d],\mathbb{R})$. Then we obtain

$$\bigl\|T_1(w) - T_1(z)\bigr\|_{C_{1-\gamma;\psi}[0,d]} \le \frac{2L_f \Gamma(\gamma)(\psi(d)-\psi(0))^{\alpha}}{\Gamma(\alpha+\gamma)} \|w-z\|_{C_{1-\gamma;\psi}[0,d]}$$

for all $w, z \in C([-h,d],\mathbb{R})$. Thus T_1 is a contraction (via the norm $\|\cdot\|_{C_{1-\gamma;\psi}[0,d]}$ on $C([-h,d],\mathbb{R})$).

Applying the Banach contraction principle to T_1, we see that T_1 is a Picard operator and $F_{T_1} = w^*$. Then, for all $\tau \in [0,d]$, we have

$$w^*(\tau) \bigl(= T_1 w^*(\tau)\bigr)$$

$$
\begin{aligned}
&= \varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big) \\
&\quad + \frac{L_f}{\Gamma(\alpha)}\bigg(\int_0^\tau \psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\alpha-1} w^*(s)\,ds \\
&\quad + \int_0^\tau \psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\alpha-1} w^*\big(g(s)\big)\,ds\bigg).
\end{aligned}
$$

Next, we show that the solution w^* is increasing. For all $0 \le \tau_1 < \tau_2 \le d$ (letting $m := \min_{s\in[0,d]}[w^*(s) + w^*(g(s))] \in \mathbb{R}_+$), we have

$$
\begin{aligned}
&w^*(\tau_2) - w^*(\tau_1) \\
&\quad = \varepsilon\big[\mathbb{E}_\alpha\big(\big(\psi(\tau_2)-\psi(0)\big)^\alpha\big) - \mathbb{E}_\alpha\big(\big(\psi(\tau_1)-\psi(0)\big)^\alpha\big)\big] \\
&\qquad + \frac{L_f}{\Gamma(\alpha)}\int_0^{\tau_1} \psi'(s)\big[\big(\psi(\tau_2)-\psi(s)\big)^{\alpha-1} - \big(\psi(\tau_1)-\psi(s)\big)^{\alpha-1}\big]\big(w^*(s) + w^*\big(g(s)\big)\,ds \\
&\qquad + \frac{L_f}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \psi'(s)\big(\psi(\tau_2)-\psi(s)\big)^{\alpha-1}\big(w^*(s) + w^*\big(g(s)\big)\,ds \\
&\quad \ge \varepsilon\big[\mathbb{E}_\alpha\big(\big(\psi(\tau_2)-\psi(0)\big)^\alpha\big) - \mathbb{E}_\alpha\big(\big(\psi(\tau_1)-\psi(0)\big)^\alpha\big)\big] \\
&\qquad + \frac{mL_f}{\Gamma(\alpha)}\int_0^{\tau_1} \psi'(s)\big[\big(\psi(\tau_2)-\psi(s)\big)^{\alpha-1} - \big(\psi(\tau_1)-\psi(s)\big)^{\alpha-1}\big]\,ds \\
&\qquad + \frac{mL_f}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} \psi'(s)\big(\psi(\tau_2)-\psi(s)\big)^{\alpha-1}\,ds \\
&\quad = \varepsilon\big[\mathbb{E}_\alpha\big(\big(\psi(\tau_2)-\psi(0)\big)^\alpha\big) - \mathbb{E}_\alpha\big(\big(\psi(\tau_1)-\psi(0)\big)^\alpha\big)\big] \\
&\qquad + \frac{mL_f}{\Gamma(\alpha+1)}\big[\big(\psi(\tau_2)-\psi(0)\big)^\alpha\big) - \big(\psi(\tau_1)-\psi(0)\big)^\alpha\big)\big] \\
&\quad > 0.
\end{aligned}
$$

Thus w^* is increasing, so $w^*(g(\tau)) \le w^*(\tau)$ since $g(\tau) \le \tau$ and

$$
w^*(\tau) \le \varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big) + \frac{2L_f}{\Gamma(\alpha)}\int_0^\tau \psi'(s)\big(\psi(\tau)-\psi(s)\big)^{\alpha-1} w^*(s)\,ds.
$$

Using Lemma 2.11 and Remark 2.12, we obtain

$$
\begin{aligned}
w^*(\tau) &\le \varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big)\mathbb{E}_\alpha\big(2L_f\big(\psi(\tau)-\psi(0)\big)^\alpha\big) \quad \big(\tau \in [0,d]\big) \\
&\le \varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big)\mathbb{E}_\alpha\big(2L_f\big(\psi(d)-\psi(0)\big)^\alpha\big) \\
&= c_{\mathbb{E}_\alpha}\varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big),
\end{aligned}
$$

where $c_{\mathbb{E}_\alpha} := \mathbb{E}_\alpha(2L_f(\psi(d)-\psi(0))^\alpha)$.

In particular, if $w = |y-x|$, from (11), $w \le T_1 w$ by Lemma 2.8 we obtain $w \le w^*$, where T_1 is an increasing Picard operator. As a result, we get

$$
\big|y(\tau)-x(\tau)\big| \le c_{E_\alpha}\varepsilon \mathbb{E}_\alpha\big(\big(\psi(\tau)-\psi(0)\big)^\alpha\big), \quad \tau \in [-h,d],
$$

and thus (2) is Ulam–Hyers–Mittag-Leffler stable. □

Now we change (H_3) to

(H_4) We have the inequality

$$\frac{2L_f \Gamma(\gamma) e^{\theta(\psi(d)-\psi(0))}(\psi(d)-\psi(0))^{\alpha}}{\Gamma(\alpha+\gamma)} < 1, \quad \theta > 0.$$

Theorem 3.2 *Assume that* (H_1), (H_2), *and* (H_4) *are satisfied. Then*

(i) (2)–(4) *has a unique solution in* $C[-h,d] \cap C_{1-\gamma;\psi}[0,d]$.

(ii) (2) *is Ulam–Hyers–Mittag-Leffler stable.*

Proof As in Theorem 3.1, we need only prove that T_f defined as before is a contraction on X (via the norm $\|\cdot\|_B$). Since the process is standard, we only give the main difference in the proof as follows: For all $\tau \in (0,d]$, we have

$$\begin{aligned}
&\bigl|T_f(x)(\tau) - T_f(y)(\tau)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} \bigl|f\bigl(s,x(s),x\bigl(g(s)\bigr)\bigr) - f\bigl(s,y(s),y\bigl(g(s)\bigr)\bigr)\bigr|\,ds \\
&\quad\le \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} \bigl(\bigl|x(s)-y(s)\bigr| + \bigl|x\bigl(g(s)\bigr) - y\bigl(g(s)\bigr)\bigr|\bigr) \\
&\quad\le \frac{L_f}{\Gamma(\alpha)} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1} \bigl(\psi(s)-\psi(0)\bigr)^{\gamma-1} e^{\theta(\psi(s)-\psi(0))} \\
&\qquad \times \Bigl\{\max_{0\le s\le d} \bigl(\psi(s)-\psi(0)\bigr)^{1-\gamma} e^{-\theta(\psi(s)-\psi(0))}\bigl(\bigl|x(s)-y(s)\bigr| + \bigl|x\bigl(g(s)\bigr)-y\bigl(g(s)\bigr)\bigr|\bigr)\Bigr\}\,ds \\
&\quad\le \frac{2L_f}{\Gamma(\alpha)} \|x-y\|_B \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1}\bigl(\psi(s)-\psi(0)\bigr)^{\gamma-1} e^{\theta(\psi(s)-\psi(0))}\,ds \\
&\quad\le \frac{2L_f}{\Gamma(\alpha)} \|x-y\|_B e^{\theta(\psi(d)-\psi(0))} \int_0^\tau \psi'(s)\bigl(\psi(\tau)-\psi(s)\bigr)^{\alpha-1}\bigl(\psi(s)-\psi(0)\bigr)^{\gamma-1}\,ds \\
&\quad\le \frac{2L_f \Gamma(\gamma) e^{\theta(\psi(d)-\psi(0))}(\psi(d)-\psi(0))^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} \|x-y\|_B.
\end{aligned}$$

Then

$$\bigl\|T_f(x) - T_f(y)\bigr\|_B \le \frac{2L_f \Gamma(\gamma) e^{\theta(\psi(d)-\psi(0))}(\psi(d)-\psi(0))^{\alpha}}{\Gamma(\alpha+\gamma)} \|x-y\|_B.$$

Thus T_f is a contraction (via the norm $\|\cdot\|_B$ on X). □

4 Examples

In this section, we give two examples illustrating our main results.

Example 4.1 Consider the fractional-order system

$$\begin{cases} {}^H\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{2};e^\tau} x(\tau) = \frac{1}{4}\frac{x^2(\tau-1)}{1+x^2(\tau-1)} + \frac{1}{4}\arctan(x(\tau)), & \tau \in (0,1], \\ I_{0^+}^{1-\gamma;e^\tau} x(0^+) = x_0, \\ x(\tau) = 0, & \tau \in [-h,0], \end{cases} \tag{12}$$

and the following inequality

$$\left|{}^{H}\mathbb{D}_{0^+}^{\frac{1}{2},\frac{1}{2};e^\tau} x(\tau) - f\bigl(\tau, y(\tau), y(\tau-1)\bigr)\right| \le \varepsilon E_{\frac{1}{2}}\bigl(\bigl(e^\tau - 1\bigr)^{\frac{1}{2}}\bigr).$$

Let $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. Then $\gamma = \alpha + \beta(1-\alpha) = \frac{3}{4}$, $d = 1$, $\psi(\cdot) = e^{\cdot}$, $g(\cdot) = \cdot - 1$, $f(\cdot, x(\cdot), g(x(\cdot))) = \frac{1}{4}\frac{x^2(\cdot-1)}{1+x^2(\cdot-1)} + \frac{1}{4}\arctan(x(\cdot))$, and $L_f = \frac{1}{4}$. Thus,

$$\frac{2L_f\Gamma(\gamma)(\psi(d)-\psi(0))^\alpha}{\Gamma(\alpha+\gamma)} \approx 0.8861 < 1.$$

Now all the assumptions in Theorem 3.1 are satisfied, so problem (12) has a unique solution, and the first equation in (12) is Ulam–Hyers–Mittag-Leffler stable with

$$\bigl|y(\tau) - x(\tau)\bigr| \le c_{\mathbb{E}_{\frac{1}{2}}} \varepsilon \mathbb{E}_{\frac{1}{2}}\bigl(\bigl(e^\tau - 1\bigr)^{\frac{1}{2}}\bigr), \quad \tau \in [-1, 1],$$

where $c_{\mathbb{E}_{\frac{1}{2}}} = \mathbb{E}_{\frac{1}{2}}(\frac{\sqrt{e-1}}{2})$.

Example 4.2 Consider the fractional-order system

$$\begin{cases} {}^{H}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{1}{4};\tau^2} x(\tau-2) = \frac{1}{5}\frac{x^2(\tau-2)}{1+x^2(\tau-2)} + \frac{1}{5}\sin(x(\tau-2)), \quad \tau \in I = (0,1], \\ I_{0^+}^{1-\gamma;\tau^2} x(0^+) = x_0, \\ x(\tau) = 0, \quad \tau \in [-1, 0], \end{cases} \tag{13}$$

and the inequality

$$\left|{}^{H}\mathbb{D}_{0^+}^{\frac{1}{3},\frac{1}{4};\tau^2} x(\tau) - f\bigl(\tau, y(\tau), y(\tau-1)\bigr)\right| \le \varepsilon \mathbb{E}_{\frac{1}{3}}\bigl(\tau^{\frac{2}{3}}\bigr).$$

Following Theorem 3.2, let $\alpha = \frac{1}{3}$ and $\beta = \frac{1}{4}$. Then $\gamma = \alpha + \beta(1-\alpha) = \frac{1}{2}$. Let $d = 1$, $\theta = \frac{1}{3}$, $\psi(\cdot) = \cdot^2$, and $L_f = \frac{1}{5}$. Thus

$$\frac{2L_f\Gamma(\gamma)e^{\theta(\psi(d)-\psi(0))}(\psi(d)-\psi(0))^\alpha}{\Gamma(\alpha+\gamma)} \approx 0.8766 < 1.$$

Now all the assumptions in Theorem 3.2 are satisfied, so (13) has a unique solution, and the first equation in (13) is Ulam–Hyers–Mittag-Leffler stable with

$$\bigl|y(\tau) - x(\tau)\bigr| \le c_{\mathbb{E}_{\frac{1}{3}}} \varepsilon \mathbb{E}_{\frac{1}{3}}\bigl(\tau^{\frac{2}{3}}\bigr), \quad \tau \in [-1, 1],$$

where $c_{\mathbb{E}_{\frac{1}{3}}} = \mathbb{E}_{\frac{1}{3}}(\frac{2}{5})$.

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Competing interests
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Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]College of Science, Guizhou Institute of Technology, Guiyang, China. [2]Department of Mathematics, Guizhou University, Guiyang, China. [3]School of Mathematical Sciences, Qufu Normal University, Qufu, China. [4]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

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