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A class of dynamic integral inequalities with mixed nonlinearities and their applications in partial dynamic systems

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Abstract

This paper investigates a class of nonlinear dynamic integral inequalities with mixed nonlinearities in two independent variables. The obtained results can be utilized to study the boundedness of partial dynamic systems on time scales. At the end, an example is presented to illustrate the main results.

Keywords: Time scale; Integral inequality; Mixed nonlinearity

1 Introduction

The theory and application of time scales was introduced by Hilger [1] and Bohner et al. [2]. At present, there exist various research branches of time scales theory such as oscillation [3–6], stability [7], and boundedness [8]. For the study of time scales theory, integral inequalities are usually used to investigate the boundedness of dynamic systems. In recent years, different types of integral inequalities have been widely studied [9–26]. For example, the sublinear integral inequality

$$u(t,s) \le a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \bigl[f(\xi,\tau)u(\xi,\tau) + h(\xi,\tau)u^{\lambda_1}\bigl(\sigma(\xi),\tau\bigr)\bigr]\Delta\tau\,\Delta\xi, \quad 0 < \lambda_1 < 1,$$

was investigated in [11]. Later, Sun et al. [12] studied the integral inequality

$$u(t) \le a(t) + b(t) \int_{t_0}^{t} \bigl[f(s)u(s) + h_1(s)u^{\lambda_1}\bigl(\sigma(s)\bigr) - h_2(s)u^{\lambda_2}\bigl(\sigma(s)\bigr)\bigr]\Delta s, \quad 0 < \lambda_1 < 1 < \lambda_2,$$

which was generalized to the more general nonlinear case by Tian et al. [13]. The following integral inequality

$$\begin{aligned} u^p(t) \le{}& a(t) + b(t) \int_{t_0}^{t} \bigl[f(s)u^p(s) + h_1(s)u^q(s) - h_2(s)u^r(s)\bigr]\,ds \\ &+ c(t) \sum_{t_0<t_i<t} \beta_i x^m(t_i - 0), \quad 0 < q < p < r \end{aligned}$$

was considered in [14], and the theoretical results can provide the bounds for a class of dynamic systems with mixed nonlinearities and impulsive effects. Very recently, Boudeliou

[25] investigated a class of nonlinear integral inequalities in two independent variables and their applications. Up to now, two dimensional integral inequalities with mixed nonlinearities have received less attention.

In this paper, we investigate the integral inequalities with mixed nonlinearities and forward jump operators, which can be used to estimate the bounds of the solutions to a class of partial dynamic systems on time scales. Consider the integral inequalities

$$\begin{aligned} u^p(t,s) \le{}& a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)u^q(\xi,\tau) + h_1(\xi,\tau)u^{\lambda_1}\bigl(\sigma(\xi),\tau\bigr) \\ &- h_2(\xi,\tau)u^{\lambda_2}\bigl(\sigma(\xi),\tau\bigr)\bigr]\Delta\tau\Delta\xi, \qquad (1.1) \\ u^p(t,s) \le{}& a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)u^q(\xi,\tau) + h_1(\xi,\tau)u^{\lambda_1}\bigl(\sigma(\xi),\tau\bigr) \\ &- h_2(\xi,\tau)u^{\lambda_2}\bigl(\sigma(\xi),\tau\bigr) + h_3(\xi,\tau)u^{\lambda_3}\bigl(\xi,\sigma(\tau)\bigr) \\ &- h_4(\xi,\tau)u^{\lambda_4}\bigl(\xi,\sigma(\tau)\bigr)\bigr]\Delta\tau\Delta\xi, \qquad (1.2) \end{aligned}$$

and

$$\begin{aligned} u^p(t,s) \le{}& a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)u^q(\xi,\tau) + h_1(\xi,\tau)u^{\lambda_1}\bigl(\sigma(\xi),\tau\bigr) \\ &- h_2(\xi,\tau)u^{\lambda_2}\bigl(\sigma(\xi),\tau\bigr) + h_3(\xi,\tau)u^{\lambda_3}\bigl(\xi,\sigma(\tau)\bigr) - h_4(\xi,\tau)u^{\lambda_4}\bigl(\xi,\sigma(\tau)\bigr) \\ &+ h_5(\xi,\tau)u^{\lambda_5}\bigl(\sigma(\xi),\sigma(\tau)\bigr) - h_6(\xi,\tau)u^{\lambda_6}\bigl(\sigma(\xi),\sigma(\tau)\bigr)\bigr]\Delta\tau\Delta\xi, \qquad (1.3) \end{aligned}$$

where $p \ge q > 0$ and $0 < \lambda_i < p < \lambda_{i+1}$ $(i = 1,3,5)$ are real constants, u,a,b,f,h_i $(i = 1,2,\ldots,6) : \mathbb{T}\times\tilde{\mathbb{T}} \to \mathbb{R}_+$ are rd-continuous functions.

Inequalities (1.1)–(1.3) can be applied to the following partial dynamic system:

$$u^{\Delta_t\Delta_s}(t,s) = F\bigl(t,s,u(t,s),u\bigl(\sigma(t),s\bigr),u\bigl(t,\sigma(s)\bigr),u\bigl(\sigma(t),\sigma(s)\bigr)\bigr) \qquad (1.4)$$

with boundary conditions $u(t,s_0) = \alpha(t)$, $u(t_0,s) = \beta(s)$, and $u(t_0,s_0) = u_0$. Integrating (1.4) yields

$$\begin{aligned} u(t,s) ={}& \alpha(t) + \beta(s) - u_0 \\ &+ \int_{t_0}^{t}\int_{s_0}^{s} F\bigl(\xi,\tau,u(\xi,\tau),u\bigl(\sigma(\xi),\tau\bigr),u\bigl(\xi,\sigma(\tau)\bigr),u\bigl(\sigma(\xi),\sigma(\tau)\bigr)\bigr)\Delta\tau\Delta\xi. \end{aligned}$$

It is not difficult to apply the theoretical results to estimate the bounds of the above system.

2 Preliminaries

Let $\mathbb{R} = (-\infty,+\infty)$ and $\mathbb{R}_+ = [0,+\infty)$. Both $\mathbb{T}$ and $\tilde{\mathbb{T}}$ are arbitrary time scales. $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. C_{rd} and C_{rd}^+ are the sets of all rd-continuous functions and positive rd-continuous functions, respectively. $\Re$ represents the set of all rd-continuous and regressive functions, and $\Re^+ = \{p \in \Re : 1 + \mu(t)p(t) > 0, t \in \mathbb{T}\}$. $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $\mu(t) = \sigma(t) - t$, and $\oplus$ is defined as $(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t)$, $t \in \mathbb{T}$.

Next, some lemmas are introduced.

Lemma 2.1 *Let u be a nonnegative function, $0 < \lambda_1 < p < \lambda_2$, $h_1 \geq 0$, $h_2 > 0$, $k_1 > 0$, and $k_2 \geq 0$. Then, for $i = 1, 2$,*

$$(-1)^{i+1} h_i u^{\lambda_i} + (-1)^i k_i u^p \leq \theta_i(\lambda_i, h_i, k_i, p),$$

where

$$\theta_i(\lambda_i, h_i, k_i, p) = (-1)^i \left(\frac{\lambda_i}{p} - 1\right)\left(\frac{\lambda_i}{p}\right)^{\frac{\lambda_i}{p-\lambda_i}} h_i^{\frac{p}{p-\lambda_i}} k_i^{\frac{\lambda_i}{\lambda_i - p}}.$$

Proof Define $F_i(u) = (-1)^{i+1} h_i u^{\lambda_i} + (-1)^i k_i u^p$. Then $F_i(u)$ reaches the maximum value at $u = (\frac{\lambda_i h_i}{k_i p})^{\frac{1}{p-\lambda_i}}$ and

$$(F_i)_{\max} = (-1)^i \left(\frac{\lambda_i}{p} - 1\right)\left(\frac{\lambda_i}{p}\right)^{\frac{\lambda_i}{p-\lambda_i}} h_i^{\frac{p}{p-\lambda_i}} k_i^{\frac{\lambda_i}{\lambda_i - p}} \quad \text{for } i = 1, 2.$$

This completes the proof. □

Lemma 2.2 ([2]) *Assume that $u, b \in C_{\mathrm{rd}}$, $a \in \Re^+$. Then*

$$u^\Delta(t) \leq a(t)u(t) + b(t), \quad t \geq t_0, t \in \mathbb{T}^k$$

implies

$$u(t) \leq u(t_0) e_a(t, t_0) + \int_{t_0}^t b(\tau) e_a\bigl(t, \sigma(\tau)\bigr)\Delta\tau, \quad t \geq t_0, t \in \mathbb{T}^k.$$

Lemma 2.3 ([10]) *Let $a \geq 0$ and $p \geq q > 0$. Then, for any $K > 0$,*

$$a^{q/p} \leq \frac{q}{p} K^{(q-p)/p} a + \frac{p-q}{p} K^{q/p}.$$

3 Main results

Theorem 3.1 *Suppose $k_1(t,s), k_2(t,s) \in C_{\mathrm{rd}}^+$ are defined on $\mathbb{T} \times \tilde{\mathbb{T}}$ satisfying $k_{12}(t,s) = k_1(t,s) - k_2(t,s) \geq 0$ and*

$$\mu(t) \int_{s_0}^s b\bigl(\sigma(t), \tau\bigr) k_{12}(t, \tau)\Delta\tau < 1.$$

Then inequality (1.1) yields

$$u(t,s) \leq \left\{ a(t,s) + b(t,s) \int_{t_0}^t \bigl(1 + \mu(\tau) B_1(\tau, s)\bigr) C_1(\tau, s) e_{(A_1 \oplus B_1)(\tau,s)}\bigl(t, \sigma(\tau)\bigr)\Delta\tau \right\}^{1/p} \tag{3.1}$$

for any $K > 0$, $(t,s) \in \mathbb{T} \times \tilde{\mathbb{T}}$, where

$$A_1(t,s) = \frac{q}{p} K^{(q-p)/p} \int_{s_0}^s b(t,\tau) f(t,\tau)\Delta\tau,$$

$$B_1(t,s) = \frac{\int_{s_0}^{s} b(\sigma(t),\tau)k_{12}(t,\tau)\Delta\tau}{1-\mu(t)\int_{s_0}^{s} b(\sigma(t),\tau)k_{12}(t,\tau)\Delta\tau},$$

and

$$\begin{aligned} C_1(t,s) = {}& \int_{s_0}^{s}\left[a\bigl(\sigma(t),\tau\bigr)k_{12}(t,\tau) + \left(\frac{q}{p}K^{(q-p)/p}a(t,\tau) + \frac{p-q}{p}K^{q/p}\right)f(t,\tau)\right]\Delta\tau \\ & + \sum_{i=1}^{2}\int_{s_0}^{s}\theta_i\bigl(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\bigr)\Delta\tau. \end{aligned}$$

Proof Based on (1.1) and Lemma 2.1, we obtain

$$\begin{aligned} u^p(t,s) \le {}& a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)u^q(\xi,\tau) + h_1(\xi,\tau)u^{\lambda_1}\bigl(\sigma(\xi),\tau\bigr) \\ & - h_2(\xi,\tau)u^{\lambda_2}\bigl(\sigma(\xi),\tau\bigr)\bigr]\Delta\tau\Delta\xi \\ \le {}& a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\Biggl[f(\xi,\tau)u^q(\xi,\tau) + k_{12}(\xi,\tau)u^p\bigl(\sigma(\xi),\tau\bigr) \\ & + \sum_{i=1}^{2}\theta_i\bigl(\lambda_i, h_i(\xi,\tau), k_i(\xi,\tau), p\bigr)\Biggr]\Delta\tau\Delta\xi. \end{aligned}$$

Define $v(t,s)$ by

$$\begin{aligned} v(t,s) = {}& \int_{t_0}^{t}\int_{s_0}^{s}\Biggl[f(\xi,\tau)u^q(\xi,\tau) + k_{12}(\xi,\tau)u^p\bigl(\sigma(\xi),\tau\bigr) \\ & + \sum_{i=1}^{2}\theta_i\bigl(\lambda_i, h_i(\xi,\tau), k_i(\xi,\tau), p\bigr)\Biggr]\Delta\tau\Delta\xi. \end{aligned}$$

It is easy to obtain that $v(t,s) \ge 0$ for $(t,s) \in \mathbb{T}\times\tilde{\mathbb{T}}$, $v(t,s)$ is nondecreasing with respect to t and s, and

$$u(t,s) \le \bigl(a(t,s) + b(t,s)v(t,s)\bigr)^{1/p}. \tag{3.2}$$

Taking the derivative of $v(t,s)$ with respect to t, we get

$$\begin{aligned} v^{\Delta_t}(t,s) = {}& \int_{s_0}^{s}\Biggl[f(t,\tau)u^q(t,\tau) + k_{12}(t,\tau)u^p\bigl(\sigma(t),\tau\bigr) \\ & + \sum_{i=1}^{2}\theta_i\bigl(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\bigr)\Biggr]\Delta\tau. \end{aligned} \tag{3.3}$$

Based on Lemma 2.3, for any $K > 0$,

$$\begin{aligned} u^q(t,\tau) \le {}& \bigl(a(t,\tau) + b(t,\tau)v(t,\tau)\bigr)^{q/p} \\ \le {}& \frac{q}{p}K^{(q-p)/p}\bigl(a(t,\tau) + b(t,\tau)v(t,\tau)\bigr) + \frac{p-q}{p}K^{q/p}. \end{aligned} \tag{3.4}$$

Inequalities (3.2)–(3.4) yield

$$\begin{aligned}
v^{\Delta_t}(t,s) &\le \int_{s_0}^{s}\Big[f(t,\tau)\Big(\frac{q}{p}K^{(q-p)/p}\big(a(t,\tau)+b(t,\tau)v(t,\tau)\big)+\frac{p-q}{p}K^{q/p}\Big)\\
&\quad + k_{12}(t,\tau)\big(a(\sigma(t),\tau)+b(\sigma(t),\tau)v(\sigma(t),\tau)\big)\Big]\Delta\tau\\
&\quad + \sum_{i=1}^{2}\int_{s_0}^{s}\theta_i\big(\lambda_i,h_i(t,\tau),k_i(t,\tau),p\big)\Delta\tau\\
&\le \Big(\frac{q}{p}K^{(q-p)/p}\int_{s_0}^{s}b(t,\tau)f(t,\tau)\Delta\tau\Big)v(t,s)\\
&\quad + \Big(\int_{s_0}^{s}b(\sigma(t),\tau)k_{12}(t,\tau)\Delta\tau\Big)v(\sigma(t),s)\\
&\quad + \int_{s_0}^{s}\Big[a(\sigma(t),\tau)k_{12}(t,\tau)+\Big(\frac{q}{p}K^{(q-p)/p}a(t,\tau)+\frac{p-q}{p}K^{q/p}\Big)f(t,\tau)\Big]\Delta\tau\\
&\quad + \sum_{i=1}^{2}\int_{s_0}^{s}\theta_i\big(\lambda_i,h_i(t,\tau),k_i(t,\tau),p\big)\Delta\tau\\
&= A_1(t,s)v(t,s)+\frac{B_1(t,s)}{1+\mu(t)B_1(t,s)}v(\sigma(t),s)+C_1(t,s),
\end{aligned}$$

which implies that

$$v^{\Delta_t}(t,s)\le (A_1\oplus B_1)(t,s)v(t,s)+\big(1+\mu(t)B_1(t,s)\big)C_1(t,s).$$

Based on Lemma 2.2 and $v(t_0,s)=0$, we can deduce that

$$v(t,s)\le \int_{t_0}^{t}\big(1+\mu(\tau)B_1(\tau,s)\big)C_1(\tau,s)e_{(A_1\oplus B_1)(\tau,s)}\big(t,\sigma(\tau)\big)\Delta\tau.$$

This combined with (3.2) yields (3.1). The proof is completed. □

Remark 3.1 Letting $p=q=1$ and $h_2(t,s)\equiv 0$, the inequality in Theorem 3.1 reduces to [11, Theorem 3.1].

Theorem 3.2 *Assume $k_i(t,s)\in C_{\mathrm{rd}}^{+}$, $i=1,2,3,4$, are defined on $\mathbb{T}\times\tilde{\mathbb{T}}$ satisfying $k_{12}(t,s)=k_1(t,s)-k_2(t,s)\ge 0$, $k_{34}(t,s)=k_3(t,s)-k_4(t,s)\ge 0$, and*

$$\mu(t)\int_{s_0}^{s}b(\sigma(t),\tau)k_{12}(t,\tau)\Delta\tau<1.$$

Then inequality (1.2) implies

$$u(t,s)\le\Big\{a(t,s)+b(t,s)\int_{t_0}^{t}\big(1+\mu(\tau)B_2(\tau,s)\big)C_2(\tau,s)e_{(A_2\oplus B_2)(\tau,s)}\big(t,\sigma(\tau)\big)\Delta\tau\Big\}^{1/p} \quad (3.5)$$

for any $K>0$, $(t,s)\in\mathbb{T}\times\tilde{\mathbb{T}}$, where

$$\bar{k}_{34}^{\Delta_\tau}(t,\tau)=\max\big\{0,k_{34}^{\Delta_\tau}(t,\tau)\big\},$$

$$A_2(t,s) = \int_{s_0}^{s} \left(\frac{q}{p} K^{(q-p)/p} b(t,\tau) f(t,\tau) + \frac{k_{34}(t,\sigma(\tau)) b(t,\sigma(\tau))}{1+\mu(\tau) b(t,\sigma(\tau))} + \bar{k}_{34}^{\Delta_\tau}(t,\tau) \right) \Delta\tau + k_{34}(t,s),$$

$$B_2(t,s) = \frac{\int_{s_0}^{s} b(\sigma(t),\tau) k_{12}(t,\tau) \Delta\tau}{1-\mu(t)\int_{s_0}^{s} b(\sigma(t),\tau) k_{12}(t,\tau)\Delta\tau},$$

and

$$\begin{aligned} C_2(t,s) = {} & \int_{s_0}^{s} \Big[a\big(\sigma(t),\tau\big) k_{12}(t,\tau) + \frac{a(t,\sigma(\tau)) k_{34}(t,\sigma(\tau))}{1+\mu(\tau) b(t,\sigma(\tau))} \\ & + \left(\frac{q}{p} K^{(q-p)/p} a(t,\tau) + \frac{p-q}{p} K^{q/p} \right) f(t,\tau) \Big] \Delta\tau \\ & + \sum_{i=1}^{2} \int_{s_0}^{s} \theta_i\big(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\big) \Delta\tau \\ & + \sum_{i=3}^{4} \int_{s_0}^{s} \theta_i\left(\lambda_i, h_i(t,\tau), \frac{k_i(t,\sigma(\tau))}{1+\mu(\tau) b(t,\sigma(\tau))}, p\right) \Delta\tau. \end{aligned}$$

Proof Based on (1.2) and Lemma 2.1, we obtain

$$\begin{aligned} u^p(t,s) \le {} & a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \big[f(\xi,\tau) u^q(\xi,\tau) + h_1(\xi,\tau) u^{\lambda_1}\big(\sigma(\xi),\tau\big) \\ & - h_2(\xi,\tau) u^{\lambda_2}\big(\sigma(\xi),\tau\big) + h_3(\xi,\tau) u^{\lambda_3}\big(\xi,\sigma(\tau)\big) - h_4(\xi,\tau) u^{\lambda_4}\big(\xi,\sigma(\tau)\big) \big] \Delta\tau \Delta\xi \\ \le {} & a(t,s) + b(t,s) \int_{t_0}^{t} \int_{s_0}^{s} \Big[f(\xi,\tau) u^q(\xi,\tau) + k_{12}(\xi,\tau) u^p\big(\sigma(\xi),\tau\big) \\ & + \frac{k_{34}(\xi,\sigma(\tau))}{1+\mu(\tau) b(\xi,\sigma(\tau))} u^p\big(\xi,\sigma(\tau)\big) + \sum_{i=1}^{2} \theta_i\big(\lambda_i, h_i(\xi,\tau), k_i(\xi,\tau), p\big) \\ & + \sum_{i=3}^{4} \theta_i\left(\lambda_i, h_i(\xi,\tau), \frac{k_i(\xi,\sigma(\tau))}{1+\mu(\tau) b(\xi,\sigma(\tau))}, p\right) \Big] \Delta\tau \Delta\xi. \end{aligned}$$

Define $\omega(t,s)$ by

$$\begin{aligned} \omega(t,s) = {} & \int_{t_0}^{t} \int_{s_0}^{s} \Big[f(\xi,\tau) u^q(\xi,\tau) + k_{12}(\xi,\tau) u^p\big(\sigma(\xi),\tau\big) + \frac{k_{34}(\xi,\sigma(\tau))}{1+\mu(\tau) b(\xi,\sigma(\tau))} u^p\big(\xi,\sigma(\tau)\big) \\ & + \sum_{i=1}^{2} \theta_i\big(\lambda_i, h_i(\xi,\tau), k_i(\xi,\tau), p\big) \\ & + \sum_{i=3}^{4} \theta_i\left(\lambda_i, h_i(\xi,\tau), \frac{k_i(\xi,\sigma(\tau))}{1+\mu(\tau) b(t,\sigma(\tau))}, p\right) \Big] \Delta\tau \Delta\xi. \end{aligned}$$

Then $\omega(t,s) \ge 0$ is nondecreasing with respect to t and s, and

$$u(t,s) \le \big(a(t,s) + b(t,s)\omega(t,s) \big)^{1/p}. \tag{3.6}$$

Taking the derivative of $\omega(t,s)$ with respect to t, we have

$$\begin{aligned}\omega^{\Delta_t}(t,s) = \int_{s_0}^{s}\Bigg[& f(t,\tau)u^q(t,\tau) + k_{12}(t,\tau)u^p\big(\sigma(t),\tau\big) + \frac{k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}u^p\big(t,\sigma(\tau)\big) \\ &+ \sum_{i=1}^{2}\theta_i\big(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\big) \\ &+ \sum_{i=3}^{4}\theta_i\left(\lambda_i, h_i(t,\tau), \frac{k_i(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}, p\right)\Bigg]\Delta\tau. \end{aligned} \tag{3.7}$$

By Lemma 2.3,

$$\begin{aligned} u^q(t,\tau) &\le \big(a(t,\tau) + b(t,\tau)\omega(t,\tau)\big)^{q/p} \\ &\le \frac{q}{p}K^{(q-p)/p}\big(a(t,\tau) + b(t,\tau)\omega(t,\tau)\big) + \frac{p-q}{p}K^{q/p} \end{aligned} \tag{3.8}$$

for any $K > 0$. It follows from (3.6)–(3.8) that

$$\begin{aligned} \omega^{\Delta_t}(t,s) \le{}& \int_{s_0}^{s}\Bigg[f(t,\tau)\left(\frac{q}{p}K^{(q-p)/p}\big(a(t,\tau)+b(t,\tau)\omega(t,\tau)\big) + \frac{p-q}{p}K^{q/p}\right) \\ &+ k_{12}(t,\tau)\big(a\big(\sigma(t),\tau\big) + b\big(\sigma(t),\tau\big)\omega\big(\sigma(t),\tau\big)\big) \\ &+ \frac{k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\big(a\big(t,\sigma(\tau)\big) + b\big(t,\sigma(\tau)\big)\omega\big(t,\sigma(\tau)\big)\big)\Bigg]\Delta\tau \\ &+ \sum_{i=1}^{2}\int_{s_0}^{s}\theta_i\big(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\big)\Delta\tau \\ &+ \sum_{i=3}^{4}\int_{s_0}^{s}\theta_i\left(\lambda_i, h_i(t,\tau), \frac{k_i(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}, p\right)\Delta\tau \\ \le{}& \left(\frac{q}{p}K^{(q-p)/p}\int_{s_0}^{s} b(t,\tau)f(t,\tau)\Delta\tau\right)\omega(t,s) \\ &+ \left(\int_{s_0}^{s} b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)\omega\big(\sigma(t),s\big) \\ &+ \int_{s_0}^{s}\frac{b(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\omega\big(t,\sigma(\tau)\big)\Delta\tau \\ &+ \int_{s_0}^{s}\Bigg[a\big(\sigma(t),\tau\big)k_{12}(t,\tau) + \frac{a(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))} \\ &+ \left(\frac{q}{p}K^{(q-p)/p}a(t,\tau) + \frac{p-q}{p}K^{q/p}\right)f(t,\tau)\Bigg]\Delta\tau \\ &+ \sum_{i=1}^{2}\int_{s_0}^{s}\theta_i\big(\lambda_i, h_i(t,\tau), k_i(t,\tau), p\big)\Delta\tau \\ &+ \sum_{i=3}^{4}\int_{s_0}^{s}\theta_i\left(\lambda_i, h_i(t,\tau), \frac{k_i(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}, p\right)\Delta\tau \\ ={}& \left(\frac{q}{p}K^{(q-p)/p}\int_{s_0}^{s} b(t,\tau)f(t,\tau)\Delta\tau\right)\omega(t,s) \end{aligned}$$

$$+\left(\int_{s_0}^{s} b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)\omega\big(\sigma(t),s\big)$$
$$+\int_{s_0}^{s}\frac{b(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\omega\big(t,\sigma(\tau)\big)\Delta\tau+C_2(t,s).$$

Note that

$$\omega\big(t,\sigma(\tau)\big)=\omega(t,\tau)+\mu(\tau)\omega^{\Delta_\tau}(t,\tau).$$

Therefore

$$\begin{aligned}\omega^{\Delta_t}(t,s)\le{}&\left(\frac{q}{p}K^{(q-p)/p}\int_{s_0}^{s}b(t,\tau)f(t,\tau)\Delta\tau\right)\omega(t,s)\\&+\left(\int_{s_0}^{s} b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)\omega\big(\sigma(t),s\big)\\&+\int_{s_0}^{s}\frac{b(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\big(\omega(t,\tau)+\mu(\tau)\omega^{\Delta_\tau}(t,\tau)\big)\Delta\tau+C_2(t,s)\\\le{}&\left(\int_{s_0}^{s}\left(\frac{q}{p}K^{(q-p)/p}b(t,\tau)f(t,\tau)+\frac{b(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\right)\Delta\tau\right)\omega(t,s)\\&+\left(\int_{s_0}^{s} b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)\omega\big(\sigma(t),s\big)\\&+\int_{s_0}^{s}k_{34}\big(t,\sigma(\tau)\big)\omega^{\Delta_\tau}(t,\tau)\Delta\tau+C_2(t,s).\end{aligned}$$

Since

$$k_{34}\big(t,\sigma(\tau)\big)\omega^{\Delta_\tau}(t,\tau)=\big(k_{34}\big(t,\sigma(\tau)\big)\omega(t,\tau)\big)^{\Delta_\tau}-k_{34}^{\Delta_\tau}(t,\tau)\omega(t,\tau),$$

we get

$$\begin{aligned}\omega^{\Delta_t}(t,s)\le{}&\left(\int_{s_0}^{s}\left(\frac{q}{p}K^{(q-p)/p}b(t,\tau)f(t,\tau)+\frac{b(t,\sigma(\tau))k_{34}(t,\sigma(\tau))}{1+\mu(\tau)b(t,\sigma(\tau))}\right)\Delta\tau\right)\omega(t,s)\\&+\left(\int_{s_0}^{s} b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)\omega\big(\sigma(t),s\big)\\&+\left(k_{34}(t,s)+\int_{s_0}^{s}\bar{k}_{34}^{\Delta_\tau}(t,\tau)\Delta\tau\right)\omega(t,s)+C_2(t,s)\\={}&A_2(t,s)\omega(t,s)+\frac{B_2(t,s)}{1+\mu(t)B_2(t,s)}\omega\big(\sigma(t),s\big)+C_2(t,s),\end{aligned}$$

which implies that

$$w^{\Delta_t}(t,s)\le(A_2\oplus B_2)(t,s)w(t,s)+\big(1+\mu(t)B_2(t,s)\big)C_2(t,s).$$

By Lemma 2.2 and $w(t_0,s)=0$,

$$w(t,s)\le\int_{t_0}^{t}\big(1+\mu(\tau)B_2(\tau,s)\big)C_2(\tau,s)e_{(A_2\oplus B_2)(\tau,s)}\big(t,\sigma(\tau)\big)\Delta\tau.$$

This combined with (3.6) yields (3.5), which completes the proof. □

Theorem 3.3 *If there exist $k_i(t,s) \in C^+_{\mathrm{rd}}$, $i = 1,2,\dots,6$, defined on $\mathbb{T} \times \tilde{\mathbb{T}}$ such that $k_{ij}(t,s) = k_i(t,s) - k_j(t,s) \geq 0$, $j = i+1$, $i = 1,3,5$, and*

$$\mu(t)\Lambda(t,s) < 1,$$

then inequality (1.3) yields

$$u(t,s) \leq \left\{ a(t,s) + b(t,s)\int_{t_0}^{t} \big(1 + \mu(\tau)B_3(\tau,s)\big)C_2(\tau,s)e_{(A_2\oplus B_2)(\tau,s)}\big(t,\sigma(\tau)\big)\Delta\tau \right\}^{1/p} \tag{3.9}$$

for any $K > 0$, $(t,s) \in \mathbb{T} \times \tilde{\mathbb{T}}$, where $\bar{k}^{\Delta_\tau}_{56}(t,\tau) = \max\{0, k^{\Delta_\tau}_{56}(t,\tau)\}$,

$$\Lambda(t,s) = \int_{s_0}^{s} \left(b\big(\sigma(t),\tau\big)k_{12}(t,\tau) + \frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(t),\sigma(\tau))} + \bar{k}^{\Delta_\tau}_{56}(t,\tau) \right)\Delta\tau + k_{56}(t,s),$$

$$A_3(t,s) = A_2(t,s), \qquad B_3(t,s) = \frac{\Lambda(t,s)}{1 - \mu(t)\Lambda(t,s)},$$

$$\begin{aligned} C_3(t,s) = {} & C_2(t,s) + \int_{s_0}^{s} \frac{a(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(t),\sigma(\tau))}\Delta\tau \\ & + \sum_{i=5}^{6}\int_{s_0}^{s} \theta_i\left(\lambda_i, h_i(t,\tau), \frac{k_i(t,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(t),\sigma(\tau))}, p\right)\Delta\tau, \end{aligned}$$

$A_2(t,s)$ and $C_2(t,s)$ are defined by Theorem 3.2.

Proof Combining (1.3) and Lemma 2.1, we get

$$\begin{aligned} u^p(t,s) \leq {} & a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} \big[f(\xi,\tau)u^q(\xi,\tau) + h_1(\xi,\tau)u^{\lambda_1}\big(\sigma(\xi),\tau\big) \\ & - h_2(\xi,\tau)u^{\lambda_2}\big(\sigma(\xi),\tau\big) + h_3(\xi,\tau)u^{\lambda_3}\big(\xi,\sigma(\tau)\big) - h_4(\xi,\tau)u^{\lambda_4}\big(\xi,\sigma(\tau)\big) \\ & + h_5(\xi,\tau)u^{\lambda_5}\big(\sigma(\xi),\sigma(\tau)\big) - h_6(\xi,\tau)u^{\lambda_6}\big(\sigma(\xi),\sigma(\tau)\big)\big]\Delta\tau\Delta\xi, \\ \leq {} & a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} \bigg[f(\xi,\tau)u^q(\xi,\tau) + k_{12}(\xi,\tau)u^p\big(\sigma(\xi),\tau\big) \\ & + \frac{k_{34}(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\xi,\sigma(\tau))}u^p\big(\xi,\sigma(\tau)\big) + \frac{k_{56}(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(\xi),\sigma(\tau))}u^p\big(\sigma(\xi),\sigma(\tau)\big) \\ & + \sum_{i=1}^{2}\theta_i\big(\lambda_i, h_i(\xi,\tau), k_i(\xi,\tau), p\big) + \sum_{i=3}^{4}\theta_i\left(\lambda_i, h_i(\xi,\tau), \frac{k_i(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\xi,\sigma(\tau))}, p\right) \\ & + \sum_{i=5}^{6}\theta_i\left(\lambda_i, h_i(\xi,\tau), \frac{k_i(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(\xi),\sigma(\tau))}, p\right)\bigg]\Delta\tau\Delta\xi. \end{aligned}$$

Define $z(t,s)$ by

$$\begin{aligned} z(t,s) = {} & \int_{t_0}^{t}\int_{s_0}^{s} \bigg[f(\xi,\tau)u^q(\xi,\tau) + k_{12}(\xi,\tau)u^p\big(\sigma(\xi),\tau\big) \\ & + \frac{k_{34}(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\xi,\sigma(\tau))}u^p\big(\xi,\sigma(\tau)\big) + \frac{k_{56}(\xi,\sigma(\tau))}{1 + \mu(\tau)b(\sigma(\xi),\sigma(\tau))}u^p\big(\sigma(\xi),\sigma(\tau)\big) \end{aligned}$$

$$+\sum_{i=1}^{2}\theta_i\big(\lambda_i,h_i(\xi,\tau),k_i(\xi,\tau),p\big)+\sum_{i=3}^{4}\theta_i\left(\lambda_i,h_i(\xi,\tau),\frac{k_i(\xi,\sigma(\tau))}{1+\mu(\tau)b(\xi,\sigma(\tau))},p\right)$$
$$+\sum_{i=5}^{6}\theta_i\left(\lambda_i,h_i(\xi,\tau),\frac{k_i(\xi,\sigma(\tau))}{1+\mu(\tau)b(\sigma(\xi),\sigma(\tau))},p\right)\Bigg]\Delta\tau\Delta\xi.$$

Then $z(t,s)\geq 0$ is nondecreasing with respect to t and s, and

$$u(t,s)\leq\big(a(t,s)+b(t,s)z(t,s)\big)^{1/p}. \tag{3.10}$$

Similar to the procedure of Theorem 3.2, we get

$$\begin{aligned}
z^{\Delta_t}(t,s)&\leq A_2(t,s)z(t,s)+\left(\int_{s_0}^{s}b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)z\big(\sigma(t),s\big)+C_2(t,s)\\
&\quad+\int_{s_0}^{s}\frac{a(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}\Delta\tau\\
&\quad+\int_{s_0}^{s}\frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}z\big(\sigma(t),\sigma(\tau)\big)\Delta\tau\\
&\quad+\sum_{i=5}^{6}\int_{s_0}^{s}\theta_i\left(\lambda_i,h_i(t,\tau),\frac{k_i(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))},p\right)\Delta\tau\\
&=A_2(t,s)z(t,s)+\left(\int_{s_0}^{s}b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)z\big(\sigma(t),s\big)+C_3(t,s)\\
&\quad+\int_{s_0}^{s}\frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}z\big(\sigma(t),\sigma(\tau)\big)\Delta\tau.
\end{aligned}$$

Note that

$$z\big(\sigma(t),\sigma(\tau)\big)=z\big(\sigma(t),\tau\big)+\mu(\tau)z^{\Delta_\tau}\big(\sigma(t),\tau\big).$$

Therefore

$$\begin{aligned}
z^{\Delta_t}(t,s)&\leq A_2(t,s)z(t,s)+\left(\int_{s_0}^{s}b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)z\big(\sigma(t),s\big)+C_3(t,s)\\
&\quad+\left(\int_{s_0}^{s}\frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}\Delta\tau\right)z\big(\sigma(t),s\big)\\
&\quad+\int_{s_0}^{s}k_{56}\big(t,\sigma(\tau)\big)z^{\Delta_\tau}\big(\sigma(t),\tau\big)\Delta\tau\\
&=A_2(t,s)z(t,s)+\left(\int_{s_0}^{s}b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau\right)z\big(\sigma(t),s\big)+C_3(t,s)\\
&\quad+\left(\int_{s_0}^{s}\frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}\Delta\tau\right)z\big(\sigma(t),s\big)+k_{56}(t,s)z\big(\sigma(t),s\big)\\
&\quad-\int_{s_0}^{s}k_{56}^{\Delta_\tau}(t,\tau)z\big(\sigma(t),\tau\big)\Delta\tau\\
&\leq A_2(t,s)z(t,s)+\left(\int_{s_0}^{s}b\big(\sigma(t),\tau\big)k_{12}(t,\tau)\Delta\tau+\int_{s_0}^{s}\frac{b(\sigma(t),\sigma(\tau))k_{56}(t,\sigma(\tau))}{1+\mu(\tau)b(\sigma(t),\sigma(\tau))}\Delta\tau\right.
\end{aligned}$$

$$+ k_{56}(t,s) + \int_{s_0}^{s} \bar{k}_{56}^{\Delta_\tau}(t,\tau)\Delta\tau\bigg)z\big(\sigma(t),s\big) + C_3(t,s)$$

$$= A_3(t,s)z(t,s) + \frac{B_3(t,s)}{1+\mu(t)B_3(t,s)}z\big(\sigma(t),s\big) + C_3(t,s),$$

i.e.,

$$z^{\Delta_t}(t,s) \le (A_3 \oplus B_3)(t,s)z(t,s) + \big(1+\mu(t)B_3(t,s)\big)C_3(t,s).$$

It follows from Lemma 2.2 that

$$z(t,s) \le \int_{t_0}^{t} \big(1+\mu(\tau)B_3(\tau,s)\big)C_3(\tau,s)e_{(A_3\oplus B_3)(\tau,s)}\big(t,\sigma(\tau)\big)\Delta\tau$$

due to $z(t_0,s)=0$. This together with (3.10) yields (3.9). The proof is completed. □

Remark 3.2 The inequalities in Theorems 3.1–3.3 generalize the results in [12–14] to two independent variables, which can be used to study the boundedness of dynamic systems.

Remark 3.3 The explicit bounds for inequalities (1.1)–(1.3) can be obtained by choosing proper $k_i(t,s)$ $(i=1,2,\ldots,6)$. For example, letting $k_1(t,s)=k_2(t,s)>0$ and $k_5(t,s)=k_6(t,s)>0$ yields $B_i(t,s)=0$ in Theorems 3.1–3.3. Under this case, Theorems 3.1–3.3 possess simpler forms.

4 Application

In this part, an example is presented to state the main results.

Example 4.1 Consider the partial dynamic system with positive and negative coefficients

$$\begin{cases} u^{\Delta_t\Delta_s}(t,s) = f(t,s)u(t,s) + h_1(t,s)u^{1/3}(\sigma(t),s) - h_2(t,s)u^2(\sigma(t),s), \\ u(t,s_0)=\alpha(t), \qquad u(t_0,s)=\beta(s), \qquad u(t_0,s_0)=u_0, \end{cases} \tag{4.1}$$

where $f,h_1,h_2 : \mathbb{T}\times\tilde{\mathbb{T}} \to \mathbb{R}_+$ are right-dense continuous functions. System (4.1) possesses sublinear and superlinear terms, which can be regarded as a class of dynamic systems with mixed nonlinearities. By simple calculation, the solution of System (4.1) satisfies

$$\big|u(t,s)\big| \le a(t,s) + \int_{t_0}^{t} e_{A(\tau,s)}\big(t,\sigma(\tau)\big)C(\tau,s)\Delta\tau, \tag{4.2}$$

where

$$a(t,s) = \big|\alpha(t)\big| + \big|\beta(s)\big| + |u_0|, \qquad A(t,s) = \frac{1}{3}K^{-2/3}\int_{s_0}^{s} f(t,\tau)\Delta\tau,$$

$$C(t,s) = \int_{s_0}^{s}\bigg(\frac{1}{3}K^{-2/3}a(t,\tau)f(t,\tau) + \frac{2}{3}K^{1/3}f(t,\tau)\bigg)\Delta\tau$$

$$+ \int_{s_0}^{s}\bigg[\theta_1\bigg(\frac{1}{3},h_1(t,\tau),k_1(t,\tau),1\bigg) + \theta_2\big(2,h_2(t,\tau),k_2(t,\tau),1\big)\bigg]\Delta\tau$$

for any $K > 0$ and any rd-continuous functions $k_1(t,s) > 0$ and $k_2(t,s) \geq 0$ satisfying $k_{12}(t,s) = k_1(t,s) - k_2(t,s) = 0$ for $(t,s) \in \mathbb{T} \times \tilde{\mathbb{T}}$.

Actually, integrating (4.1) generates

$$u(t,s) = \alpha(t) + \beta(s) - u_0 + \int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)u(\xi,\tau) + h_1(\xi,\tau)u^{1/3}\bigl(\sigma(\xi),\tau\bigr) - h_2(\xi,\tau)u^2\bigl(\sigma(\xi),\tau\bigr)\bigr]\Delta\tau\,\Delta\xi.$$

Therefore,

$$\bigl|u(t,s)\bigr| \leq a(t,s) + \int_{t_0}^{t}\int_{s_0}^{s}\bigl[f(\xi,\tau)\bigl|u(\xi,\tau)\bigr| + h_1(\xi,\tau)\bigl|u\bigl(\sigma(\xi),\tau\bigr)\bigr|^{1/3} - h_2(\xi,\tau)\bigl|u\bigl(\sigma(\xi),\tau\bigr)\bigr|^{2}\bigr]\Delta\tau\,\Delta\xi. \tag{4.3}$$

By Theorem 3.1, (4.3) yields (4.2).

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All three authors contributed equally to this work. They read and approved the final version of the manuscript.

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