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Finite difference scheme for simulating a generalized two-dimensional multi-term time fractional non-Newtonian fluid model

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Abstract

A finite difference scheme, based upon the Crank–Nicolson scheme, is applied to the numerical approximation of a two-dimensional time fractional non-Newtonian fluid model. This model not only possesses a multi-term time derivative, but also contains a special time fractional operator on the spatial derivative. And a very important lemma is proposed and also proved, which plays a vital role in the proof of the unconditional stability. The stability and convergence of the finite difference scheme are discussed and theoretically proved by the energy method. Numerical experiments are given to validate the accuracy and efficiency of the scheme, and the results indicate that this Crank–Nicolson difference scheme is very effective for simulating the generalized non-Newtonian fluid diffusion model.

Keywords: Finite difference method; Crank–Nicolson difference scheme; Energy method; Caputo fractional derivative; Multi-term time derivative; Generalized non-Newtonian fluid

1 Introduction

Fractional partial differential equations have been applied to many anomalous phenomena and complex systems in natural science and engineering technology fields [1–3]. They have some advantages in describing real processes or phenomena with memory [4, 5] over the integer-order ones. Many existing models have used single-term time-fractional derivatives to describe anomalous diffusion phenomena [6, 7], optimum control [8, 9], ecological system [10, 11]. In recent years, the extension to multi-term time fractional diffusion equations has been considered. Liu et al. [12] derived the analytical solutions for the multi-time time-fractional diffusion-wave equations with nonhomogeneous boundary conditions by using the method of separating variables. Daftardar–Gejji et al. [13] obtained the solutions of multi-term time fractional diffusion wave equations under different homogeneous or non-homogeneous boundary conditions. Some numerical treatment of multi-term fractional diffusion equations is also an active area of current research. The Galerkin finite element method was discussed in [14]. Liu et al. [15] found numerical solutions for the multi-term fractional diffusion-wave diffusion whose fractional orders belong to the intervals $[0, 1)$, $[1, 2)$, $[0, 2)$, $[0, 3)$, $[2, 3)$, and $[2, 4)$, separately. In [16], they handled the multi-term fractional wave equations with a meshless collocation method

and applied the moving least squares reproducing kernel particle approximation to construct the shape functions for spatial approximation of the equations in two dimensions. An implicit compact difference scheme based on the L_2 approximation was constructed to simulate the solution of the two-dimensional multi-term fractional equations in [17].

Fractional equations also have many other applications, such as continuous time random walks in external fields [18], fractal mobile/immobile solute transport [19], attenuation coefficient for power-law biological tissue [20]. In the last few decades, non-Newtonian fluids have been widely applied in engineering and industry. The constitutive equation of non-Newtonian fluids is much more complex than its Newtonian counterparts, and the constitutive equations involving fractional derivative have been proved to be a valuable tool to handle viscoelastic properties [21], and the obtained results show that they are in good agreement with the experimental data [22].

The generalized Oldroyd-B fluid is a particular subclass of non-Newtonian fluids. Some recent papers about the generalized Oldroyd-B fluids can be found in references [23, 24]. The fundamental electromagnetic relations have been summarized by Sutton [25]. One important part is the incompressible Oldroyd-B fluid which is bounded by two infinite rigid plates, when a magnetic field is imposed on the above flow under the assumption of low magnetic Reynolds number. Khan et al. [26] considered the generalized Oldroyd-B fluid in a porous medium with the influence of Hall current. For the MHD flow, Zheng et al. [27] discussed the flow between two plates with slip boundary conditions and obtained the exact solution using some transform techniques. There are some literature works [27, 28] that give the exact solution of the generalized Oldroyd-B fluid, and they are typically given in generalized G or H -function. Fetecau et al. [29] considered the two-dimensional fluid model and also obtained the exact solution.

But these multi-term fractional fluid models are difficult to get their exact solution, therefore, many researchers look for other solutions [30, 31]; numerical method is a promising tool to solve these equations. And up to now, numerical methods to solve fractional equations mainly are finite difference methods [32–34], finite element methods [35, 36], finite volume methods [37, 38], and spectral methods [39–41]. For these multi-term fractional fluid models, Bazhlekova et al. [42] proposed a finite difference method to solve the viscoelastic flow with generalized fractional Oldroyd-B model, and they utilized the Grünwald–Letnikov formula to approximate the Riemann–Liouville time fractional derivative; the results were low accuracy and lacked theoretical analysis. Recently, Feng et al. [43] gave the numerical solution of these problems, but it was confined to a one-dimensional case. We use finite difference method to solve the generalized two-dimensional multi-term time fractional Oldroyd-B fluid equation. We get that the temporal convergence order is $\min\{3 - \gamma, 2 - \alpha, 2 - \beta\}$, this result is better than [43] where the convergence rate in time is only first order, and we also propose and prove a vital lemma. It will help us to prove the unconditional stability of the Crank–Nicolson finite difference scheme, but in literature [44], they only proved that this Crank–Nicolson difference scheme is conditionally stable.

Inspired by the above works and motivated by potential applications, we will consider the following generalized two-dimensional multi-term time fractional non-Newtonian fluid diffusion equation:

$$c_{10} {}^C D_t^\gamma u + c_2 \frac{\partial u}{\partial t} + c_{30} {}^C D_t^\alpha u + c_4 u = c_5 \Delta u + c_{60} {}^C D_t^\beta \Delta u + f(\mathbf{x}, t), \quad (1)$$

where $u = u(\mathbf{x}, t)$, $\mathbf{x} = (x, y) \in \Omega = (0, L_x) \times (0, L_y) \subset \mathbb{R}^2$, $0 < t \leq T$, and the following initial conditions:

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \quad u_t(\mathbf{x}, 0) = \phi(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

and Dirichlet boundary condition

$$u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega \subset \mathbb{R}^2, 0 < t \leq T,$$

where $c_i > 0$, $i = 1, \dots, 6$, and $1 < \gamma < 2$, $0 < \alpha, \beta < 1$, here $\partial\Omega$ is the boundary of Ω . Δ is the Laplace operator $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, $\varphi(\mathbf{x})$ and $\phi(\mathbf{x})$ are sufficiently smooth functions, $f(\mathbf{x}, t)$ is a known source term function. ${}_0^C D_t^\theta u$ is the Caputo time fractional derivative of order $\theta (\theta > 0, n - 1 \leq \theta < n)$ with respect to t [45, 46]

$${}_0^C D_t^\theta u(\mathbf{x}, t) = \begin{cases} \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-1-\theta} \frac{\partial^n u(\mathbf{x},s)}{\partial s^n} ds, & \theta \notin \mathbf{z}^+, \\ \frac{\partial^\theta u(\mathbf{x},t)}{\partial t^\theta}, & \theta \in \mathbf{z}^+, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function.

The outline of this paper is organized as follows. First, preliminary knowledge is given, and the numerical discretization of the time fractional derivative is proposed. Then, we develop the finite difference method for the generalized non-Newtonian fluid model and derive the implicit scheme. After that, we proceed with the proof of the stability and convergence of the scheme by energy method and discuss the solvability of the numerical scheme. Finally, we present a numerical example to demonstrate the effectiveness of our method and draw some conclusions.

2 Preliminary

In the x -direction $[0, L_p]$, we take the mesh points $x_p = ph_1$, $p = 0, 1, \dots, M_1$, in the y -direction $[0, L_q]$, we take the mesh points $y_q = qh_1$, $q = 0, 1, \dots, M_1$, and $t_n = n\tau$, $n = 0, 1, \dots, N$, where $h_x = L_p/M_1$, $h_y = L_q/M_1$, $\tau = T/N$ are the uniform spatial step size and temporal step size, respectively. Denote $\Omega_\tau \equiv \{t_n \mid 0 \leq n \leq N\}$, $\Omega_h \equiv \{(x_p, y_q) \mid 0 \leq p \leq M_1, 0 \leq q \leq M_2\}$. Suppose $u_{pq}^n = u(x_p, y_q, t_n)$, u_{pq}^n is a grid function on $\Omega_h \times \Omega_\tau$. We introduce the following notations:

$$\begin{aligned} \nabla_t u_{pq}^n &= \frac{u_{pq}^n - u_{pq}^{n-1}}{\tau}, & u_{pq}^{n-\frac{1}{2}} &= \frac{u_{pq}^n + u_{pq}^{n-1}}{2}, & \nabla_x u_{pq}^n &= \frac{u_{pq}^n - u_{p-1q}^n}{h} \\ \delta_x^2 u_{pq}^n &= \frac{u_{p-1q}^n - 2u_{pq}^n + u_{p+1q}^n}{h_x^2}, & \delta_y^2 u_{pq}^n &= \frac{u_{pq-1}^n - 2u_{pq}^n + u_{pq+1}^n}{h_y^2}. \end{aligned}$$

We define $\mathcal{V}_h = \{v \mid v \text{ is a grid function on } \Omega_h \text{ and } v_{0q} = v_{M_xq} = v_{p0} = v_{pM_y} = 0\}$. For any $u, v \in \mathcal{V}_h$, we also define the following discrete inner products and induced norms:

$$\begin{aligned} (u, v) &= h_x h_y \sum_{p=1}^{M_x-1} \sum_{p=1}^{M_y-1} u_{pq} v_{pq}, \\ \langle (\nabla_x + \nabla_y)u, (\nabla_x + \nabla_y)v \rangle &= h_x h_y \sum_{p=1}^{M_x} \sum_{p=1}^{M_y} (\nabla_x + \nabla_y)u_{pq} \cdot (\nabla_x + \nabla_y)v_{pq}, \end{aligned}$$

$$\|v\|_0 = \sqrt{(v, v)}, \quad \|v\|_\infty = \max_{\substack{1 \leq p \leq M_x \\ 1 \leq q \leq M_y}} |v|, \quad |v|_1 = \sqrt{\langle (\nabla_x + \nabla_y)v, (\nabla_x + \nabla_y)v \rangle}.$$

The $\|\cdot\|_1$ norm of the grid function $v \in \mathcal{V}_h$ is given as

$$\|v\|_1 = \sqrt{c_4 \|v\|_0^2 + c_5 \|v\|_1^2}. \tag{2}$$

For the two-dimensional case, we can get the following properties:

$$\begin{aligned} ((\delta_x^2 + \delta_y^2)v^k, v^n) &= -\langle (\nabla_x + \nabla_y)v^k, (\nabla_x + \nabla_y)v^n \rangle, \\ ((\delta_x^2 + \delta_y^2)v^k, \nabla_t v^n) &= -\frac{1}{\tau} \langle (\nabla_x + \nabla_y)v^k, (\nabla_x + \nabla_y)v^n - (\nabla_x + \nabla_y)v^{n-1} \rangle \\ &= -\langle (\nabla_x + \nabla_y)v^k, \nabla_t(\nabla_x + \nabla_y)v^n \rangle, \end{aligned} \tag{3}$$

$$(\nabla_t((\delta_x^2 + \delta_y^2)v^k), \nabla_t v^n) = -\langle \nabla_t((\nabla_x + \nabla_y)v^k), \nabla_t((\nabla_x + \nabla_y)v^n) \rangle. \tag{4}$$

Proof We only give the proof of formula (4), similar to the one-dimensional case, it is easy to get

$$\begin{aligned} &(\nabla_t((\delta_x^2 + \delta_y^2)v^k), \nabla_t v^n) \\ &= \frac{1}{\tau} \langle (\nabla_x + \nabla_y)v^k - (\nabla_x + \nabla_y)v^{k-1}, \nabla_t v^n \rangle \\ &= -\frac{1}{\tau} \langle ((\nabla_x + \nabla_y)v^k, \nabla_t(\nabla_x + \nabla_y)v^n) - ((\nabla_x + \nabla_y)v^{k-1}, \nabla_t(\nabla_x + \nabla_y)v^n) \rangle \\ &= -\langle \nabla_t((\nabla_x + \nabla_y)v^k), \nabla_t((\nabla_x + \nabla_y)v^n) \rangle. \end{aligned} \quad \square$$

Next we will give the Crank–Nicolson scheme of fractional derivative.

2.1 Crank–Nicolson discretization of $D_t^\beta u$ ($0 < \beta < 1$)

From the Crank–Nicolson scheme in [44, 47], we give the discretization scheme at grid points $(x_p, y_q, t_{n-\frac{1}{2}})$ directly

$$\begin{aligned} D_t^\beta u(x_p, y_q, t_{n-\frac{1}{2}}) &= \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \left[d_0^{(\beta)} u_{pq}^n - \sum_{k=1}^{n-1} (d_{n-k-1}^{(\beta)} - d_{n-k}^{(\beta)}) u_{pq}^k - d_{n-1}^{(\beta)} u_{pq}^0 \right] + R_1 \\ &= \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{k=1}^n d_{n-k}^{(\beta)} \nabla_t u_{pq}^k + R_1, \end{aligned} \tag{5}$$

where $R_1 \leq C(\tau^{2-\beta})$, and

$$d_k^{(\beta)} = \begin{cases} (k + \frac{1}{2})^{1-\beta} - (k - \frac{1}{2})^{1-\beta}, & k = 1, 2, 3, \dots, \\ 2^{\beta-1}, & k = 0. \end{cases} \tag{6}$$

Lemma 1 *The coefficients $d_k^{(\beta)}$ satisfy the following properties:*

- (1) $d_k^{(\beta)} > 0, \lim_{k \rightarrow \infty} d_k^{(\beta)} = 0,$
- (2) $d_k^{(\beta)} > d_{k+1}^{(\beta)}, k \geq 1,$
- (3) $d_{k+1}^{(\beta)} - 2d_k^{(\beta)} + d_{k-1}^{(\beta)} > 0, k \geq 2.$

Remark 1 When $0 < \beta < 0.4811$, $d_2^{(\beta)} - 2d_1^{(\beta)} + d_0^{(\beta)} < 0$. Regarding these coefficients' property, literature [44] proved that this Crank–Nicolson difference scheme is only conditionally stable. Next, we will give the following important lemma, which is a valuable tool to prove the unconditional stability of this Crank–Nicolson difference scheme.

Lemma 2 For $0 < \beta < 1$, $d_k^{(\beta)}$ are defined as (5), (6), and for any positive integer N and real vector $\mathbf{Q} = (v^1, v^2, \dots, v^{N-1}, v^N) \in \mathbb{R}^{N+1}$, we have

$$\sum_{n=1}^N \sum_{k=1}^n d_{n-k}^{(\beta)} v^k v^n \geq 0. \tag{7}$$

Proof From Remark 1, when $0 < \beta < 0.4811$, the coefficient $d_2^{(\beta)} - 2d_1^{(\beta)} + d_0^{(\beta)} < 0$. Next, we will give the proof that it does not affect the results. Then, similar to Feng's proof [48], we will first prove the following results. We can rewrite (7) in the following form:

$$\sum_{n=1}^N \sum_{k=1}^n d_{n-k}^{(\beta)} v^k v^n = \sum_{n=1}^N \sum_{k=0}^{n-1} d_{n-k}^{(\beta)} v^{n-k} v^n = \mathbf{Q} \mathbf{W} \mathbf{Q}^T,$$

where

$$\mathbf{W} = \begin{bmatrix} d_0^{(\beta)} & 0 & 0 & \cdots & 0 & 0 \\ d_1^{(\beta)} & d_0^{(\beta)} & 0 & \cdots & 0 & 0 \\ d_2^{(\beta)} & d_1^{(\beta)} & d_0^{(\beta)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{N-2}^{(\beta)} & d_{N-3}^{(\beta)} & d_{N-4}^{(\beta)} & \cdots & d_0^{(\beta)} & 0 \\ d_{N-1}^{(\beta)} & d_{N-2}^{(\beta)} & d_{N-3}^{(\beta)} & \cdots & d_1^{(\beta)} & d_0^{(\beta)} \end{bmatrix}.$$

To prove (7) is equivalent to proving that Matrix \mathbf{W} is positive definite. So we only need to prove $\mathbf{T}_N = \frac{\mathbf{W} + \mathbf{W}^T}{2}$ is positive definite [43]. \mathbf{T}_N is a real symmetric Toeplitz matrix and has the form

$$\mathbf{T}_N = \frac{1}{2} \begin{bmatrix} 2d_0^{(\beta)} & d_1^{(\beta)} & d_2^{(\beta)} & \cdots & d_{N-2}^{(\beta)} & d_{N-1}^{(\beta)} \\ d_1^{(\beta)} & 2d_0^{(\beta)} & d_1^{(\beta)} & \cdots & d_{N-3}^{(\beta)} & d_{N-2}^{(\beta)} \\ d_2^{(\beta)} & d_1^{(\beta)} & 2d_0^{(\beta)} & \cdots & d_{N-4}^{(\beta)} & d_{N-3}^{(\beta)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{N-2}^{(\beta)} & d_{N-3}^{(\beta)} & d_{N-4}^{(\beta)} & \cdots & 2d_0^{(\beta)} & d_1^{(\beta)} \\ d_{N-1}^{(\beta)} & d_{N-2}^{(\beta)} & d_{N-3}^{(\beta)} & \cdots & d_1^{(\beta)} & 2d_0^{(\beta)} \end{bmatrix}.$$

Next, we will prove $\det(\mathbf{T}_N) > 0$. It is easy to verify that $\det(\mathbf{T}_1) = 2d_0^{(\beta)} > 0$, and $\det(\mathbf{T}_2) = 2(d_0^{(\beta)})^2 - \frac{1}{2}d_1^{(\beta)} > 0$, for a finite integer N , we explicitly calculate the value of $\det(\mathbf{T}_N) > 0$. When N is sufficiently large, according to [49], we have

$$\frac{\det(\mathbf{T}_N)}{\det(\mathbf{T}_{N+1})} > 0,$$

then we can conclude that $\det(\mathbf{T}_{N+1}) > 0$. As matrix $\mathbf{T}_k, k = 1, 2, \dots, N$ are the principal minors of matrix H_{N+1} and $\det(\mathbf{T}_k) > 0, k = 1, 2, \dots, N + 1$, then the real symmetric Toeplitz matrix \mathbf{T}_{N+1} is positive definite, matrix \mathbf{W} is positive definite, so (7) is proved. \square

In this part, we will give the scheme of the fractional derivative $D_t^\beta \Delta u, 0 < \beta < 1$. Since

$$\begin{aligned} \Delta u(x_p, y_q, t_{n-\frac{1}{2}}) &= (\delta_x^2 + \delta_y^2)u(x_p, y_q, t_{n-\frac{1}{2}}) \\ &\quad - \frac{h_x^2}{12} \frac{\partial^4 u(\xi_p, y_q, t_{n-\frac{1}{2}})}{\partial x^4} - \frac{h_y^2}{12} \frac{\partial^4 u(x_p, \zeta_q, t_{n-\frac{1}{2}})}{\partial y^4}, \end{aligned}$$

where $x_{p-1} \leq \xi_p \leq x_p, y_{q-1} \leq \zeta_q \leq y_q$, then

$$D_t^\beta \Delta u(x_p, y_q, t_{n-\frac{1}{2}}) = \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{k=1}^n d_{n-k}^{(\beta)} \nabla_t (\delta_x^2 + \delta_y^2) u_{pq}^k + R_2,$$

where $|R_2| \leq C\tau^{2-\beta} + h_x^2 + h_y^2$.

Next we give the following lemmas [50].

Lemma 3 *Discretization of the time fractional derivative $D_t^\gamma u(x, y, t) (1 < \gamma < 2)$. From the results in [50], at mesh points $(x_p, y_q, t_{n-\frac{1}{2}})$ we get*

$$\begin{aligned} D_t^\gamma u(x_p, y_q, t_{n-\frac{1}{2}}) &\approx \frac{1}{2} [D_t^\gamma u(x_p, y_q, t_n) + D_t^\gamma u(x_p, y_q, t_{n-1})] \\ &= \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \left[a_0^{(\gamma)} \nabla_t u_{pq}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) \nabla_t u_{pq}^k \right. \\ &\quad \left. - a_{n-1}^{(\gamma)} \frac{\partial u(x_p, y_q, 0)}{\partial t} \right] + R_3, \end{aligned} \tag{8}$$

where $R_3 = O(\tau^{3-\gamma})$, the coefficients $a_k^{(\gamma)} = (k + 1)^{2-\gamma} - k^{2-\gamma}, k = 0, 1, 2, \dots$, and they satisfy the following properties:

- (1) $a_k^{(\gamma)} > 0, a_0^{(\gamma)} = 1, a_k^{(\gamma)} > a_{k+1}^{(\gamma)}, \lim_{k \rightarrow \infty} a_k^{(\gamma)} = 0$,
- (2) $\sum_{k=0}^{n-1} (a_k^{(\gamma)} - a_{k+1}^{(\gamma)}) + a_n^{(\gamma)} = 1$,
- (3) $(2 - \gamma)(k + 1)^{1-\gamma} \leq a_k^{(\gamma)} \leq (2 - \gamma)k^{1-\gamma}$.

Lemma 4 ([43, 50]) *For $1 < \gamma < 2$, define $a_k^{(\gamma)} = (k + 1)^{2-\gamma} - k^{2-\gamma}, k = 0, 1, 2, \dots, n$, and $S = \{S_1, S_2, S_3, \dots\}$ and P , then it holds that*

$$\begin{aligned} &\frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)} \sum_{n=1}^N \left[a_0^{(\gamma)} S_n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) S_k - a_{n-1}^{(\gamma)} P \right] S_n \\ &\geq \frac{T^{1-\gamma}}{2\Gamma(2-\gamma)} \sum_{n=1}^N S_n^2 - \frac{T^{2-\gamma}}{2\tau\Gamma(3-\gamma)} P^2, \quad N = 1, 2, 3, \dots \end{aligned}$$

3 The derivation of the difference scheme

We assume $u(x, y, t) \in C^{4,4,3}_{x,y,t}(\Omega \times (0, T])$ and define the grid function $f_{pq}^n = f(x_p, y_q, t_n)$, $\varphi_{pq} = \varphi(x_p, y_q)$, $\phi_{pq} = \phi(x_p, y_q)$, $(x_p, y_q) \in \Omega_h$, $0 \leq n \leq N$. Now, we will present the finite difference scheme for the two-dimensional generalized non-Newtonian fluid model (1)

$$\begin{aligned}
 & c_1 D_t^\gamma u(x_p, y_q, t_{n-\frac{1}{2}}) + c_2 \frac{\partial u(x_p, y_q, t_{n-\frac{1}{2}})}{\partial t} + c_3 D_t^\alpha u(x_p, y_q, t_{n-\frac{1}{2}}) + c_4 u(x_p, y_q, t_{n-\frac{1}{2}}) \\
 & = c_5 \Delta u(x_p, y_q, t_{n-\frac{1}{2}}) + c_6 D_t^\beta \Delta u(x_p, y_q, t_{n-\frac{1}{2}}) + f(x_p, y_q, t_{n-\frac{1}{2}}).
 \end{aligned} \tag{9}$$

From Eqs. (5) and (8), one gets

$$\begin{aligned}
 & c_1 \mu_1 \left[a_0^{(\gamma)} \nabla_t u_{pq}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) \nabla_t u_{pq}^k - a_{n-1}^{(\gamma)} \phi_{pq} \right] \\
 & + c_2 \nabla_t u_{pq}^n + c_3 \mu_2 \sum_{k=1}^n d_{n-k}^{(\alpha)} \nabla_t u_{pq}^k + c_4 u_{pq}^{n-\frac{1}{2}} \\
 & = c_5 (\delta_x^2 + \delta_y^2) u_{pq}^{n-\frac{1}{2}} + c_6 \mu_3 \sum_{k=1}^n d_{n-k}^{(\beta)} \nabla_t (\delta_x^2 + \delta_y^2) u_{pq}^k \\
 & + f_{pq}^{n-\frac{1}{2}} + R_{pq}^n,
 \end{aligned} \tag{10}$$

where $\mu_1 = \frac{\tau^{1-\gamma}}{\Gamma(3-\gamma)}$, $\mu_2 = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}$, $\mu_3 = \frac{\tau^{1-\beta}}{\Gamma(2-\beta)}$, and $u(x_p, y_q, t_{n-\frac{1}{2}}) = \frac{u(x_p, y_q, t_n) + u(x_p, y_q, t_{n-1})}{2} + O(\tau^2)$, $\frac{\partial}{\partial t} u(x_p, y_q, t_{n-\frac{1}{2}}) = \frac{u(x_p, y_q, t_n) - u(x_p, y_q, t_{n-1})}{\tau} + O(\tau^2)$, so $|R_{pq}^n| \leq C(\tau^{\min(3-\gamma, 2-\alpha, 2-\beta)} + h_1^2 + h_2^2)$, in which C is independent of τ , h_1 , and h_2 . Omitting the error term, we use U_{pq}^n as the numerical solution, then we obtain the implicit finite difference scheme for generalized non-Newtonian fluid Eq. (1)

$$\begin{aligned}
 & c_1 \mu_1 \left[a_0^{(\gamma)} \nabla_t U_{pq}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) \nabla_t U_{pq}^k - a_{n-1}^{(\gamma)} \phi_{pq} \right] \\
 & + c_2 \nabla_t U_{pq}^n + c_3 \mu_2 \sum_{k=1}^n d_{n-k}^{(\alpha)} \nabla_t U_{pq}^k + c_4 U_{pq}^{n-\frac{1}{2}} \\
 & = c_5 (\delta_x^2 + \delta_y^2) U_{pq}^{n-\frac{1}{2}} + c_6 \mu_3 \sum_{k=1}^n c_{n-k}^{(\beta)} \nabla_t (\delta_x^2 + \delta_y^2) U_{pq}^k + f_{pq}^{n-\frac{1}{2}},
 \end{aligned} \tag{11}$$

with the initial and boundary conditions

$$\begin{aligned}
 & U_{pq}^0 = \varphi_{pq}, \quad (x_p, y_q) \in \Omega_h, \\
 & U_{pq}^n = 0, \quad (x_p, y_q) \in \mathcal{V}_h, 1 \leq n \leq N.
 \end{aligned}$$

4 Analysis of the numerical scheme

4.1 Solvability of the scheme

Theorem 1 *The implicit finite difference scheme (11) is uniquely solvable.*

Proof At each time level, the coefficient matrix A is

$$A = \begin{bmatrix} B & C & 0 & \cdots & 0 & 0 \\ C & B & C & \cdots & 0 & 0 \\ 0 & C & B & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B & C \\ 0 & 0 & 0 & \cdots & C & B \end{bmatrix},$$

where B and C are block matrices,

$$B = \begin{bmatrix} r_1 + 2r_2 + 2r_3 & -r_3 & 0 & \cdots & 0 & 0 \\ -r_3 & r_1 + 2r_2 + 2r_3 & -r_3 & \cdots & 0 & 0 \\ 0 & -r_3 & r_1 + 2r_2 + 2r_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r_1 + 2r_2 + 2r_3 & -r_3 \\ 0 & 0 & 0 & \cdots & -r_3 & r_1 + 2r_2 + 2r_3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} -r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -r_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -r_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -r_2 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -r_2 \end{bmatrix},$$

where $r_1 = \frac{c_1 k_1 + c_2 + c_3 k_2 d_0^{(\alpha)}}{\tau} + \frac{c_4}{2} > 0$, $r_2 = \frac{c_5 + c_6 k_3 d_0^{(\beta)}}{h_1^2} > 0$, $r_3 = \frac{c_5 + c_6 k_3 d_0^{(\beta)}}{h_2^2} > 0$, $r_4 = \frac{c_1 k_1 + c_2}{\tau} - \frac{b_2}{2}$, and $r_5 = \frac{c_5}{2}$. Then A is a strictly diagonally dominant matrix. Therefore A is nonsingular, which means that the numerical scheme (11) is uniquely solvable. \square

4.2 Stability

Theorem 2 *The implicit finite difference scheme (11) is unconditionally stable.*

Proof Multiplying Eq. (11) by $h_1 h_2 \tau \nabla_t U_{pq}^n$ and summing p from 1 to $M_1 - 1$, q from 1 to $M_2 - 1$ and summing n from 1 to N , we obtain

$$\begin{aligned} & c_1 k_1 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \left[a_0^{(\gamma)} \nabla_t U_{pq}^n - \sum_{k=1}^{n-1} (a_{n-k-1}^{(\gamma)} - a_{n-k}^{(\gamma)}) \nabla_t U_{pq}^k - a_{n-1}^{(\gamma)} \phi_{pq} \right] \nabla_t U_{pq}^n \\ & + c_2 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\nabla_t U_{pq}^n)^2 \\ & + c_3 k_2 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \sum_{k=1}^n a_{n-k}^{(\alpha)} \nabla_t U_{pq}^k \nabla_t U_{pq}^n \\ & + c_4 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 U_{pq}^{n-\frac{1}{2}} \nabla_t U_{pq}^n \end{aligned}$$

$$\begin{aligned}
 &= c_5 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\delta_x^2 + \delta_y^2) U_{pq}^{n-\frac{1}{2}} \nabla_t U_{pq}^n \\
 &\quad + c_6 k_3 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \sum_{k=1}^n a_{n-k}^{(\beta)} \nabla_t (\delta_x^2 + \delta_y^2) U_{pq}^k \nabla_t U_{pq}^n \\
 &\quad + \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 f_{pq}^{n-\frac{1}{2}} \nabla_t U_{pq}^n. \tag{12}
 \end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned}
 &c_1 k_1 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \left[a_0^{(\gamma)} \nabla_t U_{pq}^n - \sum_{k=1}^{n-1} (a_{n-k}^{(\gamma)} - a_{n-k}^{(\gamma)}) \nabla_t U_{pq}^k - a_{n-1}^{(\gamma)} \phi_{pq} \right] \nabla_t U_{pq}^n \\
 &\geq c_1 \frac{\tau T^{1-\gamma}}{2\Gamma(2-\gamma)} \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\nabla_t U_{pq}^n)^2 - c_1 \frac{a_1 T^{2-\gamma}}{2\Gamma(3-\gamma)} \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \phi_{pq}^2 \\
 &= c_1 \frac{\tau T^{1-\gamma}}{2\Gamma(2-\gamma)} \sum_{n=1}^N \|\nabla_t U^n\|_0^2 - c_1 \frac{T^{2-\gamma}}{2\Gamma(3-\gamma)} \|\phi\|_0^2. \tag{13}
 \end{aligned}$$

For the second term, we obtain

$$c_2 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\nabla_t U_{pq}^n)^2 = c_2 \tau \sum_{n=1}^N \|\nabla_t U^n\|_0^2. \tag{14}$$

Using Lemma 2, we get

$$\begin{aligned}
 &c_3 k_2 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \sum_{k=1}^n a_{n-k}^{(\alpha)} \nabla_t U_{pq}^k \nabla_t U_{pq}^n \\
 &= c_3 k_2 \sum_{n=1}^N \sum_{k=1}^n a_{n-k}^{(\alpha_m)} (\nabla U^k, \nabla U^n) \geq 0. \tag{15}
 \end{aligned}$$

For the forth term, we have

$$\begin{aligned}
 &c_4 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 U_{pq}^{n-\frac{1}{2}} \nabla_t U_{pq}^n = \frac{c_4}{2} \sum_{n=1}^N (U^n + U^{n-1}, U^n - U^{n-1}) \\
 &= \frac{c_4}{2} \sum_{n=1}^N (\|U^n\|_0^2 - \|U^{n-1}\|_0^2) = \frac{c_4}{2} (\|U^N\|_0^2 - \|U^0\|_0^2). \tag{16}
 \end{aligned}$$

Applying (3) and using the inequality $a(a - b) \geq \frac{1}{2}(a^2 - b^2)$, we obtain

$$\begin{aligned}
 &c_5 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\delta_x^2 + \delta_y^2) U_{pq}^n \nabla_t U_{pq}^n \\
 &= c_5 \tau \sum_{n=1}^N ((\delta_x^2 + \delta_y^2) U^n, \nabla_t U^n)
 \end{aligned}$$

$$\begin{aligned}
 &= -c_5 \sum_{n=1}^N \langle (\nabla_x + \nabla_y)U^n, (\nabla_x + \nabla_y)U^n - (\nabla_x + \nabla_y)U^{n-1} \rangle \\
 &\leq -\frac{c_5}{2} \sum_{n=1}^N (|U^n|_1^2 - |U^{n-1}|_1^2) = \frac{c_5}{2} (|U^0|_1^2 - |U^N|_1^2). \tag{17}
 \end{aligned}$$

Combining (4) and Lemma 2, we have

$$\begin{aligned}
 &c_6 k_3 \tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 \sum_{k=1}^n c_{n-k}^{(\beta)} \nabla_t (\delta_x^2 + \delta_y^2) U_{pq}^k \nabla_t U_{pq}^n \\
 &= c_6 k_3 \tau \sum_{n=1}^N \sum_{k=1}^n c_{n-k}^{(\beta)} \langle \nabla_t ((\delta_x^2 + \delta_y^2)U^k), \nabla_t U^n \rangle \\
 &= -c_6 k_3 \tau \sum_{n=1}^N \sum_{k=1}^n c_{n-k}^{(\beta)} \langle \nabla_t ((\nabla_x + \nabla_y)U^k), \nabla_t ((\nabla_x + \nabla_y)U^n) \rangle \leq 0. \tag{18}
 \end{aligned}$$

For the last term and using the important inequality $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$, we have

$$\begin{aligned}
 &\tau \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 f_{pq}^{n-\frac{1}{2}} \nabla_t U_{pq}^n \\
 &\leq \tau \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right) \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (\nabla_t U_{pq}^{n-\frac{1}{2}})^2 \\
 &\quad + \frac{\tau}{4 \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right)} \sum_{n=1}^N \sum_{p=1}^{M_1-1} \sum_{q=1}^{M_2-1} h_1 h_2 (f_{pq}^{n-\frac{1}{2}})^2 \\
 &= \tau \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right) \sum_{n=1}^N \|\nabla_t U^n\|_0^2 + \frac{\tau}{4 \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right)} \sum_{n=1}^N \|f^{n-\frac{1}{2}}\|_0^2. \tag{19}
 \end{aligned}$$

Substituting (13)–(19) into (12), we have

$$\begin{aligned}
 &\tau \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right) \sum_{n=1}^N \|\nabla_t U^n\|_0^2 - \frac{c_1 T^{2-\gamma_l}}{2\Gamma(3-\gamma_l)} \|\phi\|_0^2 + \frac{c_4}{2} (\|U^N\|_0^2 - \|u^0\|_0^2) \\
 &\leq \frac{c_5}{2} (|U^0|_1^2 - |U^N|_1^2) \\
 &\quad + \tau \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right) \sum_{n=1}^N \|\nabla_t U^n\|_0^2 + \frac{\tau}{4 \left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + c_2 \right)} \sum_{n=1}^N \|f^{n-\frac{1}{2}}\|_0^2,
 \end{aligned}$$

then we have

$$\begin{aligned}
 &c_4 \|U^N\|_0^2 + c_5 |U^N|_1^2 \\
 &\leq c_4 \|U^0\|_0^2 + c_5 |U^0|_1^2 \\
 &\quad + \frac{c_1 T^{2-\gamma_l}}{\Gamma(3-\gamma_l)} \|\phi\|_0^2 + \frac{T}{\left(\frac{c_1 T^{1-\gamma_l}}{2\Gamma(2-\gamma_l)} + 2c_2 \right)} \max_{1 \leq n \leq N} \|f^{n-\frac{1}{2}}\|_0^2. \tag{20}
 \end{aligned}$$

Then we have

$$\begin{aligned} \|U^N\|_1^2 &\leq \|U^0\|_1^2 \\ &+ c_1 \frac{T^{2-\gamma_i}}{\Gamma(3-\gamma_i)} \|\phi\|_0^2 + \frac{T}{\left(\frac{c_1 T^{1-\gamma_i}}{\Gamma(2-\gamma_i)} + 2c_2\right)} \max_{1 \leq n \leq N} \|f^{n-\frac{1}{2}}\|_0^2, \end{aligned} \tag{21}$$

which means that scheme (11) is unconditionally stable. □

4.3 Convergence

Theorem 3 Define u^n and U^n as the exact solution and numerical solution vectors of scheme (11), respectively. Suppose that the solution $u(x, y, t) \in C_{x,y,t}^{4,4,3}(\Omega)$, then there exists a positive constant C independent of h and τ such that

$$\|u^n - U^n\|_1 \leq C \sqrt{\frac{TL_p L_q}{\frac{2c_1 T^{1-\gamma}}{\Gamma(2-\gamma)} + 4b_1}} (\tau^{\min\{3-\gamma, 2-\alpha, 2-\beta\}} + h_1^2 + h_2^2).$$

Proof Similar to Theorem 2, it is easy to prove. □

5 Numerical example

In this section, we carry out numerical experiments using the proposed finite difference schemes to illustrate our theoretical statements. Consider the following multi-term time fractional diffusion equation:

$$\begin{cases} c_1 D_t^\gamma u(x, y, t) + c_2 \frac{\partial u(x,y,t)}{\partial t} + c_3 D_t^\alpha u(x, y, t) + c_4 u(x, y, t) \\ \quad = c_5 \Delta u(x, y, t) + c_6 D_t^\beta \Delta u(x, y, t) + f(x, y, t), \\ u(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad u_t(x, y, 0) = 0, \quad 0 \leq x \leq a, 0 \leq y \leq b, \\ u(0, y, t) = 0, \quad u(a, y, t) = 0; \quad u(x, 0, t) = 0, \quad u(x, b, t) = 0, \quad 0 \leq t \leq T, \end{cases}$$

where $(x, y, t) \in [0, a] \times [0, b] \times [0, T]$, $0 < \alpha, \beta < 1$, $1 < \gamma < 2$, the source term is $f(x, y, t) = \sin(\pi x) \sin(\pi y) \left[\frac{\Gamma(p+1)t^{p-\gamma}}{\Gamma(p+1-\gamma)} + pt^p + \frac{\Gamma(p+1)t^{p-\alpha}}{\Gamma(p+1-\alpha)} + (1 + 2\pi^2)(t^p + 1) + \frac{2\pi^2 \Gamma(p+1)t^{p-\delta}}{\Gamma(p+1-\delta)} \right]$. And the exact solution is $u(x, y, t) = (t^p + 1) \sin(\pi x) \sin(\pi y)$, $p \geq 2$.

Case 1: In this simulation, we choose $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$, $a = b = 1$, $h_1 = h_2 = h$. Firstly, we use the implicit finite difference scheme (11) to solve the equation, and the numerical results are given in Tables 1 and 2. Table 1 shows the L_2 error and L_∞ error and the convergence order of h for different γ, α, β with $\tau = 1/1000$ at $t = 1$. Table 2 shows the L_2 error and L_∞ error and the convergence order of τ for different γ, α, β with $\tau^{\min\{3-\gamma, 2-\alpha, 2-\beta\}} \approx h^2$ at $t = 1$.

From the tables, we can find that the numerical results are in good agreement with the exact solution and reach the accuracy of $\tau^{\min\{3-\gamma, 2-\alpha, 2-\beta\}} + h^2$ order, which demonstrates the effectiveness of our numerical method and confirms the theoretical analysis.

Case 2: In this calculation, we choose $a = b = 1$, $h_1 = h_2 = h = 1/20$, $\tau = 1/100$. In order to observe the effects of different physical parameters on the velocity field, we plot some fig-

Table 1 The spatial error and convergence order of the scheme for different γ, α, β with $\tau = 1/1000$

$\gamma = 1.7, \alpha = 0.8, \beta = 0.6$	L_2 error	Order	L_∞ error	Order
$h = 1/4$	3.5954E-02		7.1908E-02	
$h = 1/8$	8.8281E-03	2.03	1.7656E-02	2.03
$h = 1/16$	2.1957E-03	2.01	4.3914E-03	2.01
$h = 1/32$	5.4676E-04	2.01	1.0935E-03	2.01
$h = 1/64$	1.3510E-04	2.02	2.7020E-04	2.02
$\gamma = 1.5, \alpha = 0.6, \beta = 0.4$	L_2 error	Order	L_∞ error	Order
$h = 1/4$	3.5339E-02		7.0678E-02	
$h = 1/8$	8.6774E-03	2.03	1.7355E-02	2.03
$h = 1/16$	2.1590E-03	2.01	4.3181E-03	2.01
$h = 1/32$	5.3849E-04	2.00	1.0770E-03	2.00
$h = 1/64$	1.3392E-04	2.01	2.6784E-04	2.01
$\gamma = 1.2, \alpha = 0.4, \beta = 0.2$	L_2 error	Order	L_∞ error	Order
$h = 1/4$	3.4841E-02		6.9682E-02	
$h = 1/8$	7.7561E-03	2.03	1.7108E-02	2.03
$h = 1/16$	1.9248E-03	2.01	4.2575E-03	2.01
$h = 1/32$	4.7517E-04	2.02	1.0628E-03	2.00
$h = 1/64$	1.1326E-04	2.00	2.6526E-04	2.00

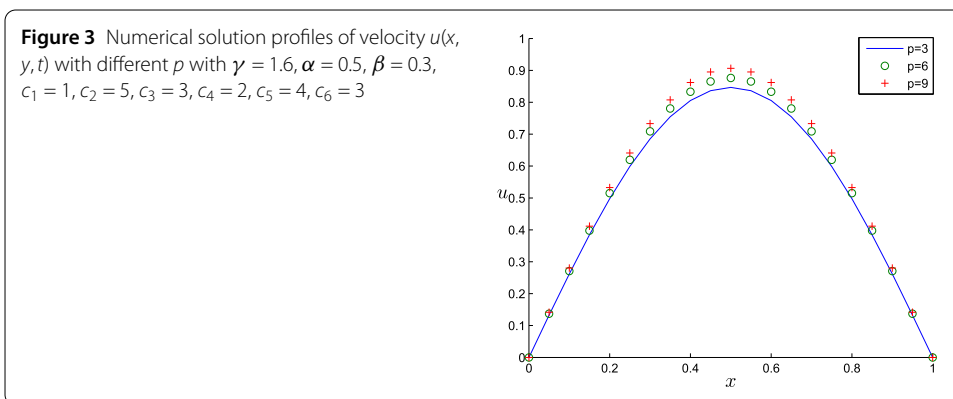
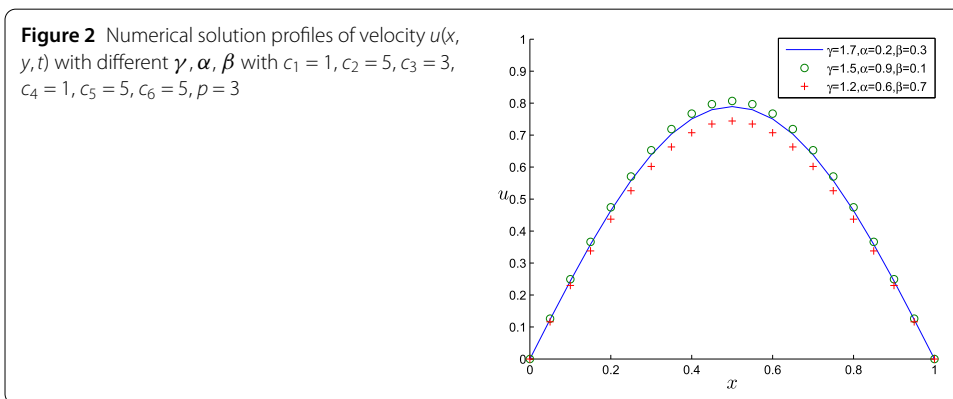
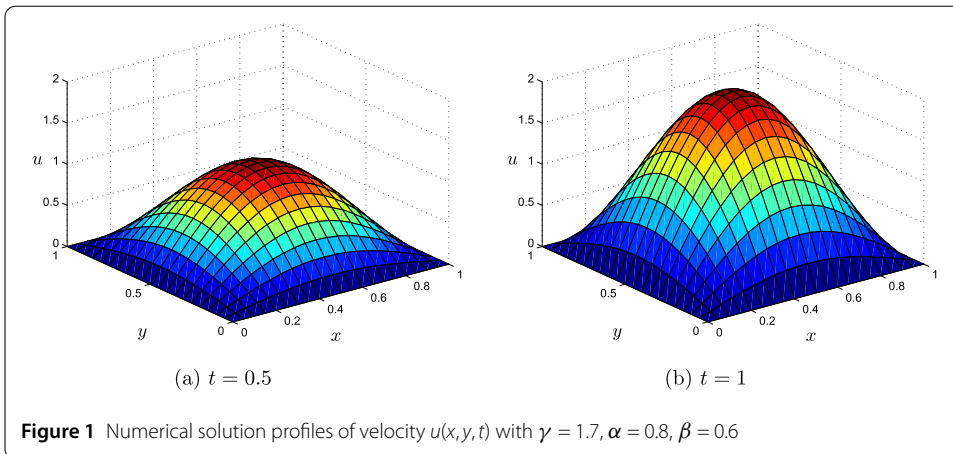
Table 2 The temporal error and convergence order of the scheme for different γ, α, β with $\tau^{\min\{3-\gamma, 2-\alpha, 2-\beta\}} \approx h^2$

$\gamma = 1.7, \alpha = 0.8, \beta = 0.6$	L_2 error	Order	L_∞ error	Order
$\tau = 1/20$	1.5314E-02		3.0629E-02	
$\tau = 1/40$	6.7733E-03	1.18	1.3138E-02	1.22
$\tau = 1/80$	2.7946E-03	1.28	5.5893E-03	1.23
$\tau = 1/160$	1.2456E-03	1.17	2.4773E-03	1.17
$\tau = 1/320$	5.3792E-04	1.21	1.0758E-03	1.20
$\gamma = 1.5, \alpha = 0.6, \beta = 0.4$	L_2 error	Order	L_∞ error	Order
$\tau = 1/20$	8.3425E-03		1.6685E-02	
$\tau = 1/40$	3.1492E-03	1.41	6.2069E-03	1.43
$\tau = 1/80$	1.2085E-03	1.38	2.4035E-03	1.37
$\tau = 1/160$	4.6230E-04	1.39	9.2459E-04	1.38
$\tau = 1/320$	1.7080E-04	1.44	3.4159E-04	1.44
$\gamma = 1.2, \alpha = 0.4, \beta = 0.2$	L_2 error	Order	L_∞ error	Order
$\tau = 1/20$	4.3678E-03		8.5587E-03	
$\tau = 1/40$	1.4595E-03	1.58	2.8991E-03	1.56
$\tau = 1/80$	4.8352E-04	1.59	9.6484E-04	1.59
$\tau = 1/160$	1.5651E-04	1.63	3.1303E-04	1.62
$\tau = 1/320$	5.2751E-05	1.57	1.0550E-04	1.57

ures to demonstrate the dynamic characteristics of the generalized non-Newtonian fluid. Figure 1 shows the influence of time on the velocity and the flow velocity increase with $t = 0.5$ and $t = 1$ respectively, and coefficients $c_i = 1, i = 1, \dots, 6$. In order to show the difference clearly, we choose $\gamma = 1$, and the variations of $u(x, y, t)$ with x for different values of $\lambda, \alpha, \beta, p, c_i = 1, i = 1, \dots, 6$ at a fixed time ($t = 1$) are illustrated in Figs. 2–5. From the figures, we can conclude that the coefficients $c_i = 1, i = 1, \dots, 6$, parameter p , and the fractional order γ, α, β have effects on the velocity function $u(x, y, t)$.

6 Conclusion

In this paper, we proposed the finite difference method, based on the Crank–Nicolson method, to solve the multi-term time fractional generalized non-Newtonian fluid equa-



tion. This is a finite difference scheme with accuracy of $O(\tau^{\min\{3-\gamma, 2-\alpha, 2-\beta\}} + h_x^2 + h_y^2)$. And we established the unconditional stability and convergence analysis for this implicit difference scheme. Numerical experiments were exhibited to verify the effectiveness and reliability of this method. We can conclude that this numerical method is robust and can be extended to other multi-term time fractional diffusion equations, multi-term time fractional diffusion-wave equations.

Figure 4 Numerical solution profiles of velocity $u(x, y, t)$ with different c_1, c_3, c_6 with $\gamma = 1.2, \alpha = 0.5, \beta = 0.8, c_2 = c_4 = c_5 = 1, \rho = 3$

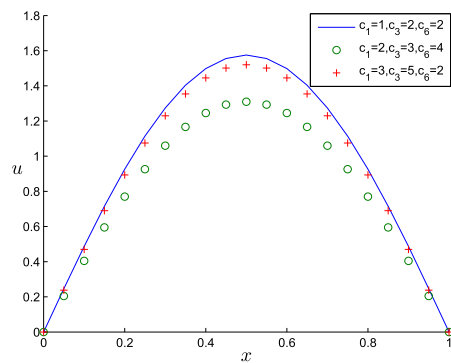
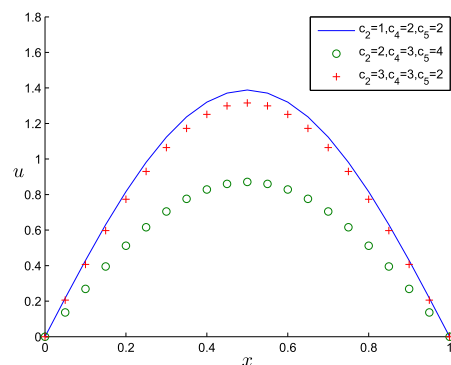


Figure 5 Numerical solution profiles of velocity $u(x, y, t)$ with different c_2, c_4, c_5 with $\gamma = 1.2, \alpha = 0.5, \beta = 0.8, c_1 = c_3 = c_6 = 1, \rho = 3$



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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to each part of this work. All authors read and approved the final manuscript.

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