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Periodic solutions for discrete $p(k)$-Laplacian systems with partially periodic potential

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Abstract

In this paper, we are concerned with the existence of periodic solutions for discrete $p(k)$-Laplacian systems with partially periodic potential. Some new existence results are obtained by using the generalized saddle point theorem in critical point theory, which extends and improves some known results in the literature.

MSC: 39A11; 58E50; 70H05; 37J45

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1 Introduction

The theory of nonlinear difference equations has been widely used to study discrete models in many fields, such as statistics, neural network, computer science, electrical circuit analysis, optimal control, biological models, data classification, and so on.

The existence results on periodic solutions were usually obtained by analytic techniques or various fixed point theorems. In [1, 2], Guo and Yu developed a new method to study the existence and multiplicity of periodic and subharmonic solutions of the second order difference equation via variational methods. In 2005, Zhou et al. [3] applied the same approach for subharmonic solutions of a class of subquadratic Hamiltonian systems. Here we also point out the contribution of Mawhin [4, 5] in the study of second order nonlinear difference systems with φ-Laplacian and periodic potential by using critical point theory.

During the past decade, periodic solutions, subharmonic solutions, and homoclinic orbits for second order discrete Hamiltonian systems have captured special attention, and some solvability conditions have been given under distinct hypotheses on potential function [6–13].

Especially, Yan et al. [12] considered the second order discrete Hamiltonian system

$$\Delta^2 x(k-1) + \nabla F\bigl(k, x(k)\bigr) = 0, \quad t \in \mathbb{Z}, \tag{1.1}$$

where $\mathbb{Z}$ is the set of all integers, $\Delta x(k) = x(k+1) - x(k)$ is the forward difference, $\Delta^2 x(k) = \Delta(\Delta x(k))$, $F : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$, and $\nabla F(k,x)$ denotes the gradient of $F(k,x)$ in x.

In [12], the authors obtained some existence results for system (1.1) with partially periodic potentials and sublinear nonlinearity.

Theorem A ([12]) *Suppose that F satisfies the following conditions*:

(F1) *There exists an integer $r \in [0,N]$ such that $F(k,x)$ is T_i-periodic in x_i, $1 \le i \le r$, where x_i is the ith component of $x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N$.*

(F2) *There exist constants $M_1 > 0$, $M_2 > 0$, and $0 \le \alpha < 1$ such that*

$$\left|\nabla F(k,x)\right| \le M_1 |x|^{\alpha} + M_2$$

for all $(k,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$, where $\mathbb{Z}[a,b] := \mathbb{Z} \cap [a,b]$ for every $a, b \in \mathbb{Z}$ with $a \le b$.

(F3) $\lim_{|x|\to\infty} |x|^{-2\alpha} \sum_{k=1}^{T} F(k,x) = +\infty$, $x \in \{0\} \times \mathbb{R}^{N-r}$.

Then problem (1.1) possesses at least $r+1$ distinct T-periodic solutions.

Recently, Jiang et al. [13] extended Theorem A, and they proved the same results under more general coercive condition:

(F4) $\liminf_{|x|\to\infty} |x|^{-2\alpha} \sum_{k=1}^{T} F(k,x) > L$, $x \in \{0\} \times \mathbb{R}^{N-r}$, where L is a positive constant.

Theorem B ([13]) *Suppose that F satisfies* (F1), (F2), *and* (F4). *Then problem* (1.1) *possesses at least $r+1$ distinct T-periodic solutions.*

In [10, 13], when F satisfies (F1) and $\nabla F(k,x)$ is growing linearly, that is, there exist constants $M_1 > 0$ and $M_2 > 0$, such that

$$\left|\nabla F(k,x)\right| \le M_1 |x| + M_2$$

for all $(k,x) \in \mathbb{Z}[1,T] \times \mathbb{R}^N$, the authors considered the multiple periodic solutions for system (1.1) and got some interesting results.

In recent years, many scholars were interested in difference equations involving the discrete variable exponent Laplacian operator. For instance, the case of homoclinic solutions of a class of $p(k)$-Laplacian difference systems was first considered by Chen et al. [14]. The existence of nontrivial homoclinic solutions was obtained by using the mountain pass theorem and the symmetric mountain pass theorem.

In [15], when $N = 1$, Bereanu et al. considered the existence of periodic or Neumann boundary value problems for the discrete $p(k)$-Laplacian equations of this type

$$-\Delta\left(\left|\Delta x(k-1)\right|^{p(k)-2} \Delta x(k-1)\right) = f(k),$$

through the use of Rabinowitz saddle point theorem, where $p(k) : \mathbb{Z}[0,T] \to (1,+\infty)$ and the nonlinear term $f(k) : \mathbb{Z}[0,T] \to \mathbb{R}$ is continuous and bounded.

In this paper, we further investigate the existence and multiplicity of periodic solution for the nonautonomous discrete $p(k)$-Laplacian system

$$-\Delta\left(\left|\Delta x(k-1)\right|^{p(k)-2} \Delta x(k-1)\right) = \nabla F\bigl(k, x(k)\bigr), \quad k \in \mathbb{Z}, \tag{1.2}$$

where the variable exponent $p(k) : \mathbb{Z}[0,T] \to (1,+\infty)$ satisfies $p(0) = p(T)$, T is a positive integer, and $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator, F: $\mathbb{Z} \times \mathbb{R}^N \mapsto \mathbb{R}$ is continuously differentiable in x for every $k \in \mathbb{Z}$ and T-periodic in k for all $x \in \mathbb{R}^N$.

We may think of (1.2) as being a discrete analogue of the following $p(t)$-Laplacian system:

$$-\frac{d}{dt}\big(\big|\dot{x}(t)\big|^{p(t)-2}\dot{x}(t)\big) = \nabla F\big(t, x(t)\big),$$

where $\Phi(x) = -\frac{d}{dt}(|\dot{x}(t)|^{p(t)-2}\dot{x}(t))$ is said to be $p(t)$-Laplacian.

During the last fifteen years, differential and partial differential equations with variable exponent growth conditions have become increasingly popular. This type of problems has very strong background, the $p(t)$-Laplacian systems provide a natural description of the physical phenomena with "pointwise different properties" which first arose from the nonlinear elasticity theory, see [16]. In [17], the authors proposed a framework for image restoration based on a nonhomogeneous $p(t)$-Laplacian operator.

In addition, problem (1.2) is also very interesting from a purely mathematical point of view. When the variable exponent $p(k) \equiv 2$, discrete $p(k)$-Laplacian system (1.2) becomes the second order discrete Hamiltonian system (1.1), problem (1.2) represents the extension to the variable exponent space setting. The $p(k)$-Laplacian operator possesses more complicated nonlinearity than the constant case, for example, it is inhomogeneous, which provokes some mathematical difficulties. We point out that commonly known methods and techniques for studying constant exponent equations fail in the setting of problems involving variable exponents, thus our problem (1.2) is more difficult and more delicate.

Inspired by the above-mentioned papers, the objective of this article is to use a control function $\omega(|x|)$ instead of $|x|^\alpha$ in conditions (F2), (F3), and (F4). By using the theory of variable exponent Sobolev spaces and the generalized saddle point theorem in [18], we will prove the existence of multiple periodic solutions for (1.2) for a new and large range of nonlinear terms.

Now, we state the assumptions on function F:

(F5) There exist constants $K_0 > 0$, $K_1 > 0$, $K_2 > 0$, $\alpha \in [0, p^- - 1)$ and a nonnegative function $\omega \in C([0,\infty), [0,\infty))$ such that

- (ω_1) $\omega(s) \le \omega(t)$, $\forall s \le t$, $s, t \in [0,\infty)$.
- (ω_2) $\omega(s+t) \le K_0(\omega(s) + \omega(t))$, $\forall s, t \in [0,\infty)$.
- (ω_3) $0 \le \omega(s) \le K_1 s^\alpha + K_2$, $\forall s, t \in [0,\infty)$.
- (ω_4) $\omega(s) \to \infty$, as $s \to \infty$.

Moreover, there exist $f, g : \mathbb{Z}[0, T] \to \mathbb{R}^+$ such that

$$\big|\nabla F(k,x)\big| \le f(k)\omega\big(|x|\big) + g(k)$$

for all $(k, x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$.

(F6) Let $\frac{1}{q^+} + \frac{1}{p^-} = 1$, and

$$\liminf_{|x|\to\infty} \frac{\sum_{k=1}^T F(k,x)}{\omega^{q^+}(|x|)} > \frac{p^-(2K_0C_0\sum_{k=1}^T f(k))^{q^+}}{q^+(p^- - 1)},$$

as $x \in \{0\} \times \mathbb{R}^{N-r}$, where C_0 is a positive constant.

Our main results are the following theorems.

Theorem 1.1 *Suppose that F satisfies* (F1), (F5), *and* (F6). *Then problem* (1.2) *possesses at least $r + 1$ distinct T-periodic solutions*.

Remark 1.1 Obviously, Theorem 1.1 generalizes Theorem A, which corresponds to the spacial case $p(k) = 2, f(k) = M_1, g(k) = M_2$ and control function $\omega(|x|) = |x|^{\alpha}$.

Comparing with the results in [6–11, 13–15], Theorem 1.1 is a different result even in the case $p(k) = 2$. For example, $x = (x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N$, let $p(k) = 2$ and

$$\begin{aligned} F(t,x) &= M_1 \ln^{\frac{3}{2}}\left[1 + \left(r + 1 + \sum_{j=1}^{r} \sin^2 x_j + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\right)\right] \\ &\quad + M_2 \ln\left[1 + \left(r + 1 + \sum_{j=1}^{r} \sin^2 x_j + \frac{1}{2}\sum_{j=r+1}^{N} x_j^2\right)\right], \end{aligned}$$

where M_1 and M_2 are positive constants. Then F satisfies (F1) with $T_i = \pi$, $i = 1, 2, \ldots, r$. Choose

$$f(t) = \frac{3}{2}M_1, \qquad g(t) = M_2, \qquad K_0 = 2$$

and control function

$$\omega\bigl(|x|\bigr) = \ln^{\frac{1}{2}}\bigl[1 + \bigl(r + 1 + |x|^2\bigr)\bigr],$$

it is easy to see that all the conditions of Theorem 1.1 hold, but F is not covered by the results in [6–13].

Remark 1.2 When $p(k) \equiv 2$, (F5) was introduced in [19], which is an extension of the usual sublinear growth condition, that is, there exist $\alpha \in [0, 1)$ and $f, g : \mathbb{Z}[0, T] \to \mathbb{R}^+$ such that

$$\bigl|\nabla F(k,x)\bigr| \le f(k)|x|^{\alpha} + g(k)$$

for all $(k,x) \in \mathbb{Z}[1, T] \times \mathbb{R}^N$. From (F5), we can see that the nonlinearity $\nabla F(k,x)$ grows slightly slower than $|x|^{\alpha}$. Comparing with the results in [19], the periodicity and coercivity conditions in our theorems are only in a part of variables of potentials, and

$$\liminf_{|x|\to\infty} \frac{\sum_{k=1}^{T} F(k,x)}{\omega^{q^+}(|x|)}$$

has appropriate lower bound.

By Theorem 1.1, it is easy to obtain the following corollary.

Corollary 1.1 *Suppose that* (F1), (F5) *hold and*

$$\lim_{|x|\to\infty} \frac{\sum_{k=1}^{T} F(k,x)}{\omega^{q^+}(|x|)} = +\infty,$$

as $x \in \{0\} \times \mathbb{R}^{N-r}$, *where* $\frac{1}{q^+} + \frac{1}{p^-} = 1$. *Then problem* (1.2) *possesses at least* $r + 1$ *distinct* T*-periodic solutions*.

2 Preliminaries

For the reader's convenience, we first give some necessary background knowledge and propositions concerning the generalized Lebesgue–Sobolev spaces. We can refer the reader to [14, 15, 20].

Let $p(k):\mathbb{Z}[0,T]\to(1,+\infty)$ satisfy $p(0)=p(T)$. From now on, we shall employ the usual notations: $p^-=\min_{k\in[0,T]}p(k)$, $p^+=\max_{k\in[0,T]}p(k)$.

Define

$$l^{p(k)}=\left\{x(k):\mathbb{Z}[0,T+1]\to\mathbb{R}^N\,\middle|\,\sum_{k=1}^{T+1}|x|^{p(k)}<\infty\right\}$$

with the norm

$$|x|_{p(k)}=\inf\left\{\lambda>0\,\middle|\,\sum_{k=1}^{T+1}\left|\frac{x}{\lambda}\right|^{p(k)}\le 1\right\}.$$

Define

$$E=\left\{x(k)\in l^{p(k)}\mid \Delta x(k-1)\in l^{p(k)},x(0)=x(T+1)\right\},$$

and

$$\widetilde{E}=\left\{x(k)\in E\,\middle|\,\overline{x}:=\frac{1}{T}\sum_{k=1}^{T}x(k)=0\right\}.$$

For $x\in E$, we write

$$\|x\|=|\overline{x}|+\|\widetilde{x}\|_{p(k)},\tag{2.1}$$

then $\|\cdot\|$ is an equivalent norm on E, where $\overline{x}=\frac{1}{T}\sum_{k=1}^{T}x(k)\in\mathbb{R}^N$ and $\widetilde{x}(k):=x(k)-\overline{x}\in\widetilde{E}$. Obviously, E and $\widetilde{E}$ are finite dimensional, and

$$\|\widetilde{x}\|=\|\widetilde{x}\|_{p(k)}.\tag{2.2}$$

This enables us to split

$$E=\mathbb{R}^N\oplus\widetilde{E}.$$

Proposition 2.1 ([14]) *If we denote*

$$\rho(x)=\sum_{k=1}^{T+1}|x|^{p(k)},\quad \forall x\in l^{p(k)},$$

then

(i) $|x|_{p(k)}<1\ (=1;>1)\Leftrightarrow\rho(x)<1\ (=1;>1)$;
(ii) $|x|_{p(k)}>1\Rightarrow|x|_{p(k)}^{p^-}\le\rho(x)\le|x|_{p(k)}^{p^+}$;
(iii) $|x|_{p(k)}<1\Rightarrow|x|_{p(k)}^{p^+}\le\rho(x)\le|x|_{p(k)}^{p^-}$.

Proposition 2.2 ([15]) *For all $\widetilde{x} \in \widetilde{E}$ and $x \in E$, one has*

(i) $\|\widetilde{x}\| < 1 \Rightarrow \|\widetilde{x}\|^{p^+} \leq \sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} \leq \|\widetilde{x}\|^{p^-}$;

(ii) $\|\widetilde{x}\| > 1 \Rightarrow \|\widetilde{x}\|^{p^-} \leq \sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} \leq \|\widetilde{x}\|^{p^+}$;

(iii) $\|\widetilde{x}\| = 1 \Rightarrow \sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} = 1$.

Proposition 2.3 ([20]) *For all $x \in E$, there exists a constant $C_0 > 0$ such that*

$$\|x\|_\infty := \max_{k \in [0,T+1]} \left|x(k)\right| \leq C_0 \|x\|.$$

Combining Proposition 2.2 with Proposition 2.3, we can obtain the following.

Proposition 2.4 *For all $\widetilde{x} \in \widetilde{E}$ and $x \in E$, we have*

$$\|\widetilde{x}\| \leq \left(\sum_{k=1}^{T+1} \left|\Delta x(k-1)\right|^{p(k-1)} \right)^{\frac{1}{p^-}} + 1 \tag{2.3}$$

and

$$\|\widetilde{x}\|_\infty \leq C_0 \left[\left(\sum_{k=1}^{T+1} \left|\Delta x(k-1)\right|^{p(k-1)} \right)^{\frac{1}{p^-}} + 1 \right]. \tag{2.4}$$

Applying Proposition 2.2, from (2.1) and (2.2), it is easy to prove the following.

Proposition 2.5 *For all $x \in E$, we have*

$$\|x\| \to \infty \Rightarrow |\overline{x}| + \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^{p(k-1)} \right)^{\frac{1}{p^-}} \to \infty.$$

The functional on E given by

$$\varphi(x) = \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} - \sum_{k=1}^{T} F\bigl(k, x(k)\bigr), \quad \forall x \in E,$$

is continuously differentiable and weakly semicontinuous on E. Moreover, we have

$$\bigl\langle \varphi'(x), y \bigr\rangle = \sum_{k=1}^{T+1} \bigl(\left|\Delta x(k-1)\right|^{p(k-1)-2} \Delta x(k-1), \Delta y(k-1) \bigr) - \sum_{k=1}^{T} \bigl(\nabla F\bigl(k, x(k)\bigr), y(k) \bigr)$$

for all $x, y \in E$. Then the critical points of φ correspond to the solutions of system (1.2).

Take

$$\widehat{x}(t) = P\overline{x} + Q\overline{x} + \widetilde{x}(k),$$

where

$$P\overline{x} = \sum_{i=r+1}^{N} (\overline{x}, e_i) e_i,$$

and

$$Q\overline{x} = \sum_{i=1}^{r} \bigl[(\bar{u}, e_i) - l_i T_i\bigr] e_i,$$

and for $1 \le i \le r$, l_i is the unique integer such that

$$0 \le (\overline{x}, e_i) - l_i T_i < T_i,$$

and $\{e_1, e_2, \dots, e_N\}$ is the canonical basis of $\mathbb{R}^N$. Hence, $|Q\overline{x}|$ is bounded and

$$|Q\overline{x}| \le \left(\sum_{i=1}^{r} T_i^2\right)^{\frac{1}{2}}. \tag{2.5}$$

Define $G = \{\sum_{i=1}^{r} k_i T_i e_i \mid k_i \in \mathbb{Z}, 1 \le i \le r\}$, then G is a discrete subgroup of E. Let $E/G = X \times V$, $X = Y \oplus W$, where

$$W = \widetilde{E} = \left\{ x \in E \;\middle|\; \overline{x} := \frac{1}{T} \sum_{k=1}^{T} x(k) = 0 \right\},$$

$$Y = \operatorname{span}\{e_{r+1}, \dots, e_N\},$$

and

$$V = \operatorname{span}\{e_1, \dots, e_r\}/G,$$

and V is isomorphic to the torus T^r. Let $\pi : E \to E/G$ be the canonical surjection and $\psi : X \times V \to \mathbb{R}$ by $\psi(\pi(x)) = \varphi(x)$. By (F1), we have

$$F\bigl(t, x(t)\bigr) = F\left(t, \widehat{x}(t) + \sum_{i=1}^{r} k_i T_i e_i\right) = F\bigl(t, \widehat{x}(t)\bigr),$$

$$\nabla F\bigl(t, x(t)\bigr) = \nabla F\left(t, \widehat{x}(t) + \sum_{i=1}^{r} k_i T_i e_i\right) = \nabla F\bigl(t, \widehat{x}(t)\bigr)$$

and

$$\varphi(x) = \varphi(\widehat{x}), \qquad \varphi'(x) = \varphi'(\widehat{x}).$$

Then

$$\psi\bigl(\pi(x)\bigr) = \psi\bigl(\pi(\widehat{x})\bigr), \qquad \psi'\bigl(\pi(u)\bigr) = \psi'\bigl(\pi(\hat{u})\bigr).$$

Definition 2.1 ([21]) Suppose that ψ satisfies the (PS) condition, that is, every sequence $\{x_n\}$ of $X \times V$ such that $\psi\{x_n\}$ is bounded and $\psi'\{x_n\} \to 0$ as $n \to \infty$ possesses a convergent subsequence.

To prove the main theorem of the paper, we need the following generalized saddle point theorem due to Liu.

Lemma 2.1 (Theorem 1.7 in [18]) *Let X be a Banach space with a decomposition $X = Y + W$, where Y and W are two subspaces of X with $\dim Y < +\infty$. Let V be a finite-dimensional, compact C^2-manifold without boundary. Let $\psi : X \times V \to \mathbb{R}$ be a C^1-function and satisfy the (PS) condition. Suppose that there exist constants $\rho > 0$ and $\gamma < \beta$ such that*

$$\text{(a)} \quad \inf_{x \in W \times V} \psi(x) \geq \beta,$$

$$\text{(b)} \quad \sup_{x \in S \times V} \psi(x) \leq \gamma,$$

where $S = \partial D$, $D = \{z \in Y \mid |z| \leq \rho\}$. Then the functional ψ has at least cuplength$(V) + 1$ *critical points.*

3 Proof of Theorem 1.1

Now, we give the proof of Theorem 1.1. For the sake of convenience, we denote by C_i ($i = 1, 2, \ldots, 27$) various positive constants.

Proof of Theorem 1.1 Now, we use Lemma 2.1 to prove this theorem. Firstly, we prove that ψ satisfies the (PS) condition. Suppose that $\{\pi(x_n)\}$ is a (PS) sequence for ψ, that is, $\psi(\pi(x_n))$ is bounded and $\psi'(\pi(x_n)) \to 0$. Then $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$.

By properties (ω_1)–(ω_3) of (F5), we have

$$\begin{aligned}
&\omega\bigl(|P\bar{x}| + |Q\bar{x}| + |\tilde{x}(k)|\bigr) \\
&\quad \leq K_0\bigl[\omega\bigl(|P\bar{x}|\bigr) + \omega\bigl(|Q\bar{x}| + |\tilde{x}(k)|\bigr)\bigr] \\
&\quad \leq K_0\bigl[\omega\bigl(|P\bar{x}|\bigr) + K_0\omega\bigl(|Q\bar{x}|\bigr) + K_0\omega\bigl(|\tilde{x}(k)|\bigr)\bigr] \\
&\quad \leq K_0\bigl[\omega\bigl(|P\bar{x}|\bigr) + K_0\bigl(K_1|Q\bar{x}|^\alpha + K_2\bigr) + K_0\bigl(K_1|\tilde{x}(k)|^\alpha + K_2\bigr)\bigr] \\
&\quad \leq K_0\bigl[\omega\bigl(|P\bar{x}|\bigr) + K_0K_1|Q\bar{x}|^\alpha + K_0K_1\|\tilde{x}\|_\infty^\alpha + 2K_0K_2\bigr].
\end{aligned}$$

Then one has

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\bigl[F\bigl(k, \hat{x}(k)\bigr) - F(k, P\bar{x})\bigr]\right| \\
&\quad = \left|\sum_{k=1}^{T}\int_0^1 \bigl(\nabla F\bigl(k, P\bar{x} + s\bigl(Q\bar{x} + \tilde{x}(k)\bigr)\bigr), Q\bar{x} + \tilde{x}(k)\bigr)\, ds\right| \\
&\quad \leq \sum_{k=1}^{T}\int_0^1 f(k)\omega\bigl(\bigl|P\bar{x} + s\bigl(Q\bar{x} + \tilde{x}(k)\bigr)\bigr|\bigr)\bigl|Q\bar{x} + \tilde{x}(k)\bigr|\, ds + \sum_{k=1}^{T}\int_0^1 g(k)\bigl|Q\bar{x} + \tilde{x}(k)\bigr|\, ds \\
&\quad \leq \sum_{k=1}^{T}\int_0^1 f(k)\omega\bigl(|P\bar{x}| + |Q\bar{x}| + |\tilde{x}(k)|\bigr)\bigl(|Q\bar{x}| + |\tilde{x}(k)|\bigr)\, ds \\
&\qquad + \sum_{k=1}^{T}\int_0^1 g(k)\bigl(|Q\bar{x}| + |\tilde{x}(k)|\bigr)\, ds.
\end{aligned}$$

So, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\bigl[F\bigl(k,\hat{x}(k)\bigr)-F(k,P\bar{x})\bigr]\right| \\
&\quad\leq \sum_{k=1}^{T} f(k)K_0\omega\bigl(|P\bar{x}|\bigr)\bigl(|Q\bar{x}|+\|\tilde{x}\|_\infty\bigr) \\
&\qquad+\sum_{k=1}^{T} f(k)K_0^2K_1\|\tilde{x}\|_\infty^\alpha\bigl(|Q\bar{x}|+\|\tilde{x}\|_\infty\bigr) \\
&\qquad+\sum_{k=1}^{T} f(k)K_0\bigl[K_0K_1|Q\bar{x}|^\alpha+2K_0K_2\bigr]\bigl(|Q\bar{x}|+\|\tilde{x}\|_\infty\bigr) \\
&\qquad+\sum_{k=1}^{T} g(k)\bigl(|Q\bar{x}|+\|\tilde{x}\|_\infty\bigr).
\end{aligned}$$

From (2.4) and (2.5), we obtain

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\bigl[F\bigl(k,\hat{x}(k)\bigr)-F(k,P\bar{x})\bigr]\right| \\
&\quad\leq \sum_{k=1}^{T} f(k)K_0\omega\bigl(|P\bar{x}|\bigr)\|\tilde{x}\|_\infty+C_1\|\tilde{x}\|_\infty^{\alpha+1}+C_2\|\tilde{x}\|_\infty \\
&\qquad+C_3\,|\tilde{x}\,|_\infty^\alpha+C_4\omega\bigl(|P\bar{x}|\bigr)+C_5 \\
&\quad\leq K_0C_0\sum_{k=1}^{T} f(k)\omega\bigl(|P\bar{x}|\bigr)\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{1}{p^-}} \\
&\qquad+C_6\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} \\
&\qquad+C_7\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{1}{p^-}} \\
&\qquad+C_8\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha}{p^-}} \\
&\qquad+C_9\omega\bigl(|P\bar{x}|\bigr)+C_{10}.
\end{aligned} \tag{3.1}$$

By Young's inequality, one has that

$$\begin{aligned}
&K_0C_0\sum_{k=1}^{T} f(k)\omega\bigl(|P\bar{x}|\bigr)\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{1}{p^-}} \\
&\quad\leq \frac{1}{q^+}\left(K_0C_0\sum_{k=1}^{T} f(k)\right)^{q^+}\omega^{q^+}\bigl(|P\bar{x}|\bigr)+\frac{1}{p^-}\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)},
\end{aligned} \tag{3.2}$$

where $\frac{1}{q^+} + \frac{1}{p^-} = 1$. Hence, by (3.1) and (3.2), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\bigl[F\bigl(k,\widehat{x}(k)\bigr) - F(k,P\bar{x})\bigr]\right| \\
&\quad\le \frac{1}{q^+}\left(K_0C_0\sum_{k=1}^{T}f(k)\right)^{q^+}\omega^{q^+}\bigl(|P\bar{x}|\bigr) + \frac{1}{p^-}\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)} \\
&\qquad + C_6\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} + C_7\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{1}{p^-}} \\
&\qquad + C_8\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha}{p^-}} + C_9\omega\bigl(|P\bar{x}|\bigr) + C_{10}. \qquad (3.3)
\end{aligned}$$

In a way similar to the proof of (3.3), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\bigl(\nabla F\bigl(k,\widehat{x}(k)\bigr),\widetilde{x}(k)\bigr)\right| \\
&\quad\le \sum_{k=1}^{T}f(k)\omega\bigl(\bigl|\widehat{x}(k)\bigr|\bigr)\bigl|\widetilde{x}(k)\bigr| + \sum_{k=1}^{T}g(k)\bigl|\widetilde{x}(k)\bigr| \\
&\quad\le \sum_{k=1}^{T}f(k)\omega\bigl(|P\bar{x}| + |Q\bar{x}| + \bigl|\tilde{x}(k)\bigr|\bigr)\bigl|\widetilde{x}(k)\bigr| + \sum_{k=1}^{T}g(k)\bigl|\widetilde{x}(k)\bigr| \\
&\quad\le \frac{1}{q^+}\left(K_0C_0\sum_{k=1}^{T}f(k)\right)^{q^+}\omega^{q^+}\bigl(|P\bar{x}|\bigr) + \frac{1}{p^-}\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)} \\
&\qquad + C_{11}\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} \\
&\qquad + C_{12}\left(\sum_{k=1}^{T+1}\bigl|\Delta x(k-1)\bigr|^{p(k-1)}\right)^{\frac{1}{p^-}} + C_{13}. \qquad (3.4)
\end{aligned}$$

Then for n large enough, by (3.4), we have

$$\begin{aligned}
\|\widetilde{x}_n\| &\ge \bigl\langle\varphi'(x_n),\widetilde{x}_n\bigr\rangle \\
&= \bigl\langle\varphi'(\widehat{x}_n),\widetilde{x}_n\bigr\rangle \\
&= \sum_{k=1}^{T+1}\bigl|\Delta x_n(k-1)\bigr|^{p(k-1)} - \sum_{k=1}^{T}\bigl(\nabla F\bigl(k,\widehat{x}_n(k)\bigr),\widetilde{x}_n(k)\bigr) \\
&\ge \left(1-\frac{1}{p^-}\right)\sum_{k=1}^{T+1}\bigl|\Delta x_n(k-1)\bigr|^{p(k-1)} \\
&\quad - C_{11}\left(\sum_{k=1}^{T+1}\bigl|\Delta x_n(k-1)\bigr|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}}
\end{aligned}$$

$$
\begin{aligned}
&- C_{12}\left(\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)}\right)^{\frac{1}{p^-}} - C_{13} \\
&- \frac{1}{q^+}\left(K_0C_0\sum_{k=1}^{T}f(k)\right)^{q^+}\omega^{q^+}\left(|P\bar{x}_n|\right).
\end{aligned} \tag{3.5}
$$

Note (2.3), one has that

$$
\|\widetilde{x}_n\| \le \left(\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)}\right)^{\frac{1}{p^-}} + 1. \tag{3.6}
$$

Consequently, combining (3.5) with (3.6), we obtain that

$$
\begin{aligned}
&\frac{1}{q^+}\left(K_0C_0\sum_{k=1}^{T}f(k)\right)^{q^+}\omega^{q^+}\left(|P\bar{x}_n|\right) \\
&\ge \left(1-\frac{1}{p^-}\right)\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)} \\
&\quad - C_{11}\left(\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} \\
&\quad - C_{14}\left(\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)}\right)^{\frac{1}{p^-}} - C_{15} \\
&\ge \frac{1}{2}\left(1-\frac{1}{p^-}\right)\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)} - C_{16},
\end{aligned} \tag{3.7}
$$

where $C_{16} = \min_{S\in[0,+\infty)}\{\frac{1}{2}(1-\frac{1}{p^-})S^{p^-} - C_{11}S^{\alpha+1} - C_{14}S - C_{15}\}$. Thus, we derive that

$$
\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)} \le \frac{2p^-(K_0C_0\sum_{k=1}^{T}f(k))^{q^+}}{q^+(p^- - 1)}\omega^{q^+}\left(|P\bar{x}_n|\right) + C_{17}. \tag{3.8}
$$

According to (3.3) and (3.8), we can obtain

$$
\begin{aligned}
\varphi(x_n) &= \varphi(\widehat{x}_n) \\
&= \sum_{k=1}^{T+1}\frac{|\Delta x_n(k-1)|^{p(k-1)}}{p(k-1)} - \sum_{k=1}^{T}F\left(k,\widehat{x}_n(k)\right) \\
&\le \frac{1}{p^-}\sum_{k=1}^{T+1}\left|\Delta x_n(k-1)\right|^{p(k-1)} - \sum_{k=1}^{T}F(k,P\bar{x}_n) \\
&\quad - \left[\sum_{k=1}^{T}F\left(k,\widehat{x}_n(k)\right) - \sum_{k=1}^{T}F(k,P\bar{x}_n)\right] \\
&\le \left[\frac{p^-(K_0C_0\sum_{k=1}^{T}f(k))^{q^+}}{q^+(p^- - 1)} - \frac{\sum_{k=1}^{T}F(k,P\bar{x}_n)}{\omega^{q^+}(|P\bar{x}_n|)}\right]\omega^{q^+}\left(|P\bar{x}_n|\right)
\end{aligned}
$$

$$+ C_{18}\omega^{\frac{q^+(\alpha+1)}{p^-}}\big(|P\bar{x}_n|\big) + C_{19}\omega^{\frac{q^+\alpha}{p^-}}\big(|P\bar{x}_n|\big)$$
$$+ C_{20}\omega^{\frac{q^+}{p^-}}\big(|P\bar{x}_n|\big) + C_{21}. \tag{3.9}$$

We claim that the sequence $|P\bar{x}_n|$ is bounded. Otherwise, we assume $|P\bar{x}_n| \to +\infty$ as $n \to \infty$. Note that (ω_4) of (F5), we have $\omega(|P\bar{x}_n|) \to +\infty$, as $n \to \infty$. This together with (F6), $\alpha \in [0, p^- - 1)$, and (3.9) yields $\varphi(x_n) \to -\infty$ as $n \to \infty$, this contradicts the boundedness of $\{\varphi(x_n)\}$, so $|P\bar{x}_n|$ is bounded. Combining the property (ω_4) of (F5), (2.3), and (3.8), we conclude that $\|\widetilde{x}_n\|$ is bounded. Notice that $|Q\bar{x}_n|$ is bounded, so $\{\widehat{x}_n\}$ is bounded in E. Since E is a finite dimensional space, then $\{\widehat{x}_n\}$ has a convergent subsequence. By $\pi(\widehat{x}_n) = \pi(x_n)$, we conclude that ψ satisfies the (PS) condition.

Secondly, we only need to verify the linking conditions of the generalized saddle point theorem. For $\pi(x) \in W \times V$, $x(k) = \tilde{x}(k) + Q\bar{x}$. By the proof of (3.3), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{T}\big[F\big(k,\widehat{x}(k)\big) - F(k,0)\big]\right| \\
&\quad\le \sum_{k=1}^{T}\left|\int_0^1 \big(\nabla F\big(k, s\big(Q\bar{x} + \widetilde{x}(k)\big)\big), Q\bar{x} + \widetilde{x}(k)\big)\,ds\right| \\
&\quad\le \sum_{k=1}^{T}\int_0^1 f(k)\omega\big(\big|Q\bar{x} + \widetilde{x}(k)\big|\big)\big|Q\bar{x} + \widetilde{x}(k)\big|\,ds \\
&\qquad + \sum_{k=1}^{T}\int_0^1 g(k)\big|Q\bar{x} + \widetilde{x}(k)\big|\,ds \\
&\quad\le C_{22}\left(\sum_{k=1}^{T+1}\big|\Delta x(k-1)\big|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} \\
&\qquad + C_{23}\left(\sum_{k=1}^{T+1}\big|\Delta x(k-1)\big|^{p(k-1)}\right)^{\frac{\alpha}{p^-}} \\
&\qquad + C_{24}\left(\sum_{k=1}^{T+1}\big|\Delta x(k-1)\big|^{p(k-1)}\right)^{\frac{1}{p^-}} + C_{25}.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi\big(\pi(x)\big) &= \psi\big(\pi\big(\widetilde{x}(k) + Q\bar{x}\big)\big) \\
&= \varphi\big(\widetilde{x}(k) + Q\bar{x}\big) \\
&= \sum_{k=1}^{T+1}\frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} - \sum_{k=1}^{T}F(k,0) \\
&\quad - \left[\sum_{k=1}^{T}F\big(k,\widehat{x}(k)\big) - \sum_{k=1}^{T}F(k,0)\right] \\
&\ge \frac{1}{p^+}\sum_{k=1}^{T+1}\big|\Delta x(k-1)\big|^{p(k-1)}
\end{aligned}$$

$$
\begin{aligned}
&- C_{22}\left(\sum_{k=1}^{T+1}\left|\Delta x(k-1)\right|^{p(k-1)}\right)^{\frac{\alpha+1}{p^-}} \\
&- C_{23}\left(\sum_{k=1}^{T+1}\left|\Delta x(k-1)\right|^{p(k-1)}\right)^{\frac{\alpha}{p^-}} \\
&- C_{24}\left(\sum_{k=1}^{T+1}\left|\Delta x(k-1)\right|^{p(k-1)}\right)^{\frac{1}{p^-}} - C_{25}.
\end{aligned}
$$

By Proposition 2.5 and the boundedness of $|Q\bar{x}|$, one has that

$$
\|x\| \to +\infty \Rightarrow \left(\sum_{k=1}^{T+1}\left|\Delta x(k-1)\right|^{p(k-1)}\right)^{\frac{1}{p^-}} \to +\infty
$$

on $W \times V$. Notice $\alpha \in [0, p^- - 1)$, we obtain that

$$
\psi\bigl(\pi(x)\bigr) \to +\infty
$$

as $\|x\| \to -\infty$, for all $\pi(x) \in W \times V$, which implies that there exists $\beta \in \mathbb{R}$ such that

$$
\psi\bigl(\pi(x)\bigr) \geq \beta,
$$

on $W \times V$. Thus part (a) of Lemma 2.1 is verified.

For $\pi(x) \in Y \times V$, $x = P\bar{x} + Q\bar{x}$. By (F5) and (2.5), we have

$$
\begin{aligned}
\psi\bigl(\pi(x)\bigr) &= \varphi(x) \\
&= \varphi(\widehat{x}) \\
&= -\sum_{k=1}^{T} F(k, P\bar{x} + Q\bar{x}) \\
&= -\sum_{k=1}^{T} F(k, P\bar{x}) - \sum_{k=1}^{T}\int_0^1 \bigl(\nabla F(k, P\bar{x} + sQ\bar{x}), Q\bar{x}\bigr)\, ds \\
&\leq -\sum_{k=1}^{T} F(k, P\bar{x}) + \sum_{k=1}^{T} f(k)\omega\bigl(|P\bar{x} + Q\bar{x}|\bigr)|Q\bar{x}| + \sum_{k=1}^{T} g(k)|Q\bar{x}| \\
&\leq \left[-\frac{p^-(K_0C_0\sum_{k=1}^{T} f(k))^{q^+}}{q^+(p^- - 1)} + \varepsilon\right]\omega^{q^+}\bigl(|P\bar{x}|\bigr) + C_{26}\omega\bigl(|P\bar{x}|\bigr) + C_{27}.
\end{aligned}
$$

Note that $\omega(|P\bar{x}|) \to +\infty$ as $|P\bar{x}| \to \infty$, and $q^+ > 1$, for sufficiently small ε, we can obtain that

$$
\psi\bigl(\pi(x)\bigr) \to -\infty \quad \text{as } |P\bar{x}| \to \infty
$$

uniformly for $\pi(Q\bar{x}) \in V$, where $x \in \mathbb{R}^N$. So part (b) of Lemma 2.1 holds.

Now, the functional ψ satisfies all the hypotheses of the generalized saddle point theorem, so it has at least cuplength$(V) + 1$ critical points, and since V is the torus T^r, it implies that cuplength$(V) = r$. Hence φ has at least $r + 1$ critical points. Therefore, problem (1.2) has at least $r + 1$ distinct solutions in E. The proof of Theorem 1.1 is completed. □

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