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Monotonicity, concavity, and inequalities related to the generalized digamma function

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Abstract
In this paper, we establish a concave theorem and some inequalities for the generalized digamma function. Hence, we give complete monotonicity property of a determinant function involving all kinds of derivatives of the generalized digamma function.

MSC: 33B15

Keywords: Generalized digamma function; Complete monotonicity; Inequality; Concavity

1 Introduction

It is well known that the Euler gamma function is defined by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt,\quad x>0.$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is,

$$\begin{aligned}\psi(x)&=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}\\&=-\gamma-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{n(n+x)},\end{aligned}$$

where $\gamma=0.5772\ldots$ is the Euler–Mascheroni constant. The gamma, digamma, and polygamma functions play an important role in the theory of special function, and have many applications in many other branches such as statistics, fractional differential equations, mathematical physics, and theory of infinite series. The reader may see related references [5, 7, 9, 13, 16, 23–28].

In [6], the k-analogue of the gamma function is defined for $k>0$ and $x>0$ as follows:

$$\begin{aligned}\Gamma_k(x)&=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}\,dt\\&=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}},\end{aligned}$$

where $\lim_{k\to 1}\Gamma_k(x)=\Gamma(x)$. It is natural that the k-analogue of the digamma function is defined for $x>0$ by

$$\psi_k(x)=\frac{d}{dx}\log\Gamma_k(x)=\frac{\Gamma_k'(x)}{\Gamma_k(x)}.$$

It is worth noting that Nantomah et al. gave (p,k)-analogue of the gamma and the digamma functions in [15]. Further, they established some inequalities involving these new functions. The reader may see references [14, 15].

Very recently, Alzer and Jameson [2] presented a harmonic mean inequality for the digamma function, and also showed some interesting inequalities. It is natural to ask if one can generalize these results to the generalized digamma function with single parameters. This is the first object in this paper.

The second object of this paper came from the article of Ismail and Laforgia. In [11], they proved complete monotonicity of a determinant function involving the derivatives of the digamma function. Using their idea, we prove that their conclusion is also true for the generalized digamma function. In particular, some of the work about the complete monotonicity of these special functions may be found in [3, 4, 8, 10, 12, 17–22].

2 Lemmas

Lemma 2.1 *For $k>0$ and $x>0$, the following identities hold true*:

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{2.1}$$

$$\psi_k(x)=\frac{\ln k}{k}+\frac{1}{k}\psi\left(\frac{x}{k}\right). \tag{2.2}$$

Proof Using the substitution $\frac{t}{\sqrt[k]{k}}=u$ and $u^k=p$, we easily obtain

$$\begin{aligned}\Gamma_k(x)&=\left(\sqrt[k]{k}\right)^x\int_0^\infty\left(\frac{t}{\sqrt[k]{k}}\right)^{x-1}e^{-(\frac{t}{\sqrt[k]{k}})^k}\,d\left(\frac{t}{\sqrt[k]{k}}\right)\\&=k^{\frac{x}{k}}\int_0^\infty u^{x-1}e^{-u^k}\,du\\&=k^{\frac{x}{k}-1}\int_0^\infty p^{\frac{x}{k}-1}e^{-p}\,dp\\&=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right).\end{aligned}$$

So, we prove formula (2.1). To (2.2), direct computation yields

$$\begin{aligned}\psi_k(x)&=\frac{\Gamma_k'(x)}{\Gamma_k(x)}=\frac{[k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})]'}{[k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})]}\\&=\frac{\ln k}{k}+\frac{1}{k}\psi\left(\frac{x}{k}\right).\end{aligned}$$

□

Lemma 2.2 ([1, 2]) *For $x>0$, we have*

$$\psi'(x)<\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3} \tag{2.3}$$

and

$$\psi''(x) < -\frac{1}{x^2} - \frac{1}{x^3}. \tag{2.4}$$

Lemma 2.3 *For $k, x > 0, m \in \mathbb{N}$, we have*

$$\psi_k^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(nk+x)^{m+1}} \tag{2.5}$$

and

$$\psi_k^{(m)}(x) = (-1)^{m+1} \int_0^{\infty} \frac{t^m}{1-e^{-kt}} e^{-xt}\, dt. \tag{2.6}$$

Proof Formula (2.5) may be found in reference [15]. By using formula (9) in reference [15], we have

$$\begin{aligned}
\psi_k^{(m)}(x) &= \lim_{p\to\infty} \psi_{p,k}^{(m)}(x) \\
&= \lim_{p\to\infty} (-1)^{m+1} \int_0^{\infty} \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}} \right) t^m e^{-xt}\, dt \\
&= (-1)^{m+1} \int_0^{\infty} \frac{1}{1-e^{-kt}} t^m e^{-xt}\, dt.
\end{aligned}$$

So, we prove formula (2.6). □

3 Main results

Theorem 3.1 *For $k > 0$, the function $x^2 \psi_k'(x)$ is strictly increasing on $(0, \infty)$.*

Proof Using Lemma 2.1, we have

$$\psi_k'(x) = \frac{1}{k^2} \psi'\left(\frac{x}{k}\right) \quad \text{and} \quad \psi_k''(x) = \frac{1}{k^3} \psi''\left(\frac{x}{k}\right).$$

Combining with the identity $\psi^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(n+x)^{m+1}}$, we get

$$\begin{aligned}
\frac{d}{dx}\left(x^2 \psi_k'(x)\right) &= \frac{2x}{k^2} \psi'\left(\frac{x}{k}\right) + \frac{x^2}{k^3} \psi''\left(\frac{x}{k}\right) \\
&= 2x \sum_{n=0}^{\infty} \frac{nk}{(nk+x)^3} > 0.
\end{aligned}$$

□

Theorem 3.2 *For $k > 0$, the function $\psi_k(\frac{1}{x})$ is strictly concave on $(0, \infty)$.*

Proof Easy computation results in

$$\frac{d}{dx}\left(\psi_k\left(\frac{1}{x}\right)\right) = -\frac{1}{x^2} \psi_k'\left(\frac{1}{x}\right).$$

Considering Theorem 3.1, we complete the proof. □

Theorem 3.3 *For* $k \geq \frac{1}{\sqrt[3]{3}} = 0.693361\ldots$, *the function*

$$\lambda_k(x) = \psi_k(x) + \psi_k\left(\frac{1}{x}\right)$$

is strictly concave on $(0, \infty)$.

Proof By differentiation and applying Lemma 2.1, we easily obtain

$$\lambda_k'(x) = \psi_k'(x) - \frac{1}{x^2}\psi_k'\left(\frac{1}{x}\right),$$
$$\lambda_k''(x) = \psi_k''(x) + \frac{2}{x^3}\psi_k'\left(\frac{1}{x}\right) + \frac{1}{x^4}\psi_k''\left(\frac{1}{x}\right),$$

and

$$k^3x^4\lambda_k''(x) = x^4\psi''\left(\frac{x}{k}\right) + 2kx\psi'\left(\frac{1}{kx}\right) + \psi''\left(\frac{1}{kx}\right).$$

Applying Lemma 2.2, $k \geq \frac{1}{\sqrt[3]{3}}$, and the recurrence relations

$$\psi'\left(\frac{1}{kx} + 1\right) = \psi'\left(\frac{1}{kx}\right) - k^2x^2,$$
$$\psi''\left(\frac{1}{kx} + 1\right) = \psi''\left(\frac{1}{kx}\right) + 2k^3x^3,$$

we have

$$\begin{aligned}
k^3x^4\lambda_k''(x) &= x^4\psi''\left(\frac{x}{k}\right) + 2kx\psi'\left(\frac{1}{kx}\right) + \psi''\left(\frac{1}{kx}\right) \\
&< x^4\left(-\frac{k^2}{x^2} - \frac{k^3}{x^3}\right) + 2kx\left[\frac{kx}{1+kx} + \frac{k^2x^2}{2(1+kx)^2} + \frac{k^3x^3}{6(1+kx)^3}\right] \\
&\quad - \frac{k^2x^2}{(1+kx)^2} - \frac{k^3x^3}{(1+kx)^3} \\
&= -\frac{kx}{3(1+kx)^3}\left[3k^2 + 9k^3x + 9k^2x^2 + k^2\left(3k^3 - 1\right)x^3 + 3k^4x^4\right] \\
&< 0.
\end{aligned}$$

This implies that $\lambda_k(x)$ is strictly concave on $(0, \infty)$. □

Theorem 3.4 *For* $x \in (0, \infty)$ *and* $k \geq \frac{1}{\sqrt[3]{3}}$, *we have*

$$\psi_k(x) + \psi_k\left(\frac{1}{x}\right) \leq \frac{2\ln k + 2\psi(\frac{1}{k})}{k}. \tag{3.1}$$

Proof Since the function $\lambda_k(x) = \psi_k(x) + \psi_k(\frac{1}{x})$ is strictly concave on $(0, \infty)$, we get

$$\lambda_k'(x) \geq \lambda_k'(1) = 0, \quad x \in (0, 1],$$

and

$$\lambda_k'(x) \le \lambda_k'(1) = 0, \quad x \in [1,\infty).$$

It follows that λ_k is increasing on $(0,1]$ and decreasing on $[1,\infty)$. Hence, $\lambda_k(x) \le \lambda_k(1)$ for $x > 0$. The proof is complete. □

Remark 3.1 Let $\gamma_k = -\psi_k(1) = -\frac{\ln k}{k} - \frac{1}{k}\psi(\frac{1}{k})$ be the k-analogue of the Euler–Mascheroni constant. It is obvious that $\lim_{k\to 1}\gamma_k = \gamma$.

Definition 3.1 It is known that the generalized digamma function $\psi_k(x)$ is strictly increasing on $(0,\infty)$ with $\psi_k(0^+)\psi_k(\infty) < 0$. So, the function has a sole positive root in $(0,\infty)$. We define this positive root for x_k. That is,

$$\ln k + \psi\left(\frac{x_k}{k}\right) = 0.$$

Theorem 3.5 *For* $x \in (0,1)$ *and* $\frac{1}{\sqrt[3]{3}} \le k \le 1$, *we have*

$$\psi_k(1+x)\psi_k(1-x) \le \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}. \tag{3.2}$$

Proof Considering $\frac{1}{\sqrt[3]{3}} \le k \le 1$ and the definition of x_k, we have

$$\frac{1}{\sqrt[3]{3}}x_0 \le x_k \le x_0,$$

where x_0 satisfies $\psi(x_0) = 0$ with $x_0 = 1.46163\ldots$.

Case 1. If $x \in [x_k - 1, 1)$, then we have $\psi_k(1-x) \le 0 \le \psi_k(1+x)$. This implies that formula (3.2) holds.

Case 2. If $x \in (0, x_k - 1]$, using the power series expansion

$$\psi(1+z) = -\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)z^{k-1}, \quad |z| < 1, \tag{3.3}$$

we obtain

$$\begin{aligned}\psi_k(1+x) \ge \psi_k(k+x) &= \frac{\ln k}{k} + \frac{1}{k}\psi\left(1+\frac{x}{k}\right)\\ &= \frac{\ln k}{k} + \frac{1}{k}\left[-\gamma + \sum_{k=2}^{\infty}(-1)^k\zeta(k)x^{k-1}\right],\end{aligned}$$

where $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$ is the Riemann zeta function.

Furthermore, we have

$$0 < -\psi_k(1+x) \le -\frac{\ln k}{k} + \frac{1}{k}\left[\gamma - \zeta(2)x + \zeta(3)x^2\right]. \tag{3.4}$$

Completely similar to (3.4), we have

$$
\begin{aligned}
0 < -\psi_k(1-x) &\le -\frac{\ln k}{k} + \frac{1}{k}\left[\gamma + \zeta(2)y + \zeta(3)\sum_{k=2}^{\infty} x^k\right] \\
&\le -\frac{\ln k}{k} + \frac{1}{k}\left[\gamma + \zeta(2)x + \zeta(3)x^2\right]. \qquad (3.5)
\end{aligned}
$$

Combining (3.4) with (3.5), we obtain

$$
\psi_k(1+x)\psi_k(1-x) \le \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}
$$

by using $\zeta(3)x^2 < 1$. □

Theorem 3.6 *For $x \in (0,\infty)$ and $\frac{1}{\sqrt[3]{3}} \le k \le 1$, we have*

$$
\psi_k(x)\cdot\psi_k\left(\frac{1}{x}\right) \le \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}. \qquad (3.6)
$$

Proof We only need to prove (3.6) for $x \ge 1$. If $x \ge x_k$, then we get $\psi_k(\frac{1}{x}) \le 0 \le \psi_k(x)$. It follows that inequality (3.6) holds true.

If $x \in (1, x_k]$ and setting $x = 1+z$, we get

$$
\psi_k(1-z) \le \psi_k\left(\frac{1}{x}\right).
$$

Therefore, we have

$$
\begin{aligned}
\psi_k(x)\cdot\psi_k\left(\frac{1}{x}\right) &= \psi_k(1+z)\psi_k\left(\frac{1}{x}\right) \\
&\le \psi_k(1+z)\psi_k(1-z) \\
&\le \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}
\end{aligned}
$$

by using Theorem 3.5. □

Corollary 3.1 *For $x \in (0,\infty)$ and $\frac{1}{\sqrt[3]{3}} \le k \le 1$, we have*

$$
\frac{2\psi_k(x)\psi_k(\frac{1}{x})}{\psi_k(x)+\psi_k(\frac{1}{x})} \ge \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k[\ln k + \psi(\frac{1}{k})]}. \qquad (3.7)
$$

Proof Applying Theorems 3.4 and 3.6, we obtain

$$
\begin{aligned}
\frac{2\psi_k(x)\psi_k(\frac{1}{x})}{\psi_k(x)+\psi_k(\frac{1}{x})} &\ge 2\cdot\frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}\frac{1}{\psi_k(x)+\psi_k(\frac{1}{x})} \\
&\ge \frac{\ln^2 k + \gamma^2 - 2(\gamma+1)\ln k}{k^2}\frac{k}{\ln k + \psi(\frac{1}{k})}.
\end{aligned}
$$

The proof is complete. □

Next, for $m, n, j \in \mathbb{N}$, we define the function μ_n by

$$\mu_n(x) = \begin{vmatrix} \psi_k^{(m)}(x) & \psi_k^{(m+j)}(x) & \cdots & \psi_k^{(m+nj)}(x) \\ \psi_k^{(m+j)}(x) & \psi_k^{(m+2j)}(x) & \cdots & \psi_k^{[m+(n+1)j]}(x) \\ \vdots & \vdots & & \vdots \\ \psi_k^{(m+nj)}(x) & \psi_k^{(m+(n+1)j)}(x) & \cdots & \psi_k^{(m+2nj)}(x) \end{vmatrix}.$$

Completely similar to the method in [11], the following Theorem 3.7 can be proved.

Theorem 3.7 *For $m, n, j \in \mathbb{N}$, then $(-1)^{(n+1)(m+1)}\mu_n(x)$ is completely monotonic on $(0, \infty)$.*

Proof Using Lemma 2.3, we have

$$\begin{aligned} \mu_n(x) &= (-1)^{n+1} \underbrace{\int_{-\infty}^{0} \cdots \int_{-\infty}^{0}}_{n+1 \text{ times}} \begin{vmatrix} u_0^m & u_0^{m+j} & \cdots & u_0^{m+nj} \\ u_1^{m+j} & u_1^{m+2j} & \cdots & u_1^{m+(n+1)j} \\ \vdots & \vdots & & \vdots \\ u_n^{m+nj} & u_n^{m+(n+1)j} & \cdots & u_n^{m+2nj} \end{vmatrix} \\ &\quad \cdot \frac{e^{\frac{x}{k}(u_0+u_1+\cdots+u_n)}}{\prod_{i=0}^{n}(1-e^{u_i})}\,du_0\,du_1\cdots du_n \\ &= (-1)^{n+1} \underbrace{\int_{-\infty}^{0} \cdots \int_{-\infty}^{0}}_{n+1 \text{ times}} \begin{vmatrix} u_{\delta(0)}^m & u_{\delta(0)}^{m+j} & \cdots & u_{\delta(0)}^{m+nj} \\ u_{\delta(1)}^{m+j} & u_{\delta(1)}^{m+2j} & \cdots & u_{\delta(1)}^{m+(n+1)j} \\ \vdots & \vdots & & \vdots \\ u_{\delta(n)}^{m+nj} & u_{\delta(n)}^{m+(n+1)j} & \cdots & u_{\delta(n)}^{m+2nj} \end{vmatrix} \\ &\quad \cdot \frac{e^{\frac{x}{k}(u_0+u_1+\cdots+u_n)}}{\prod_{i=0}^{n}(1-e^{u_i})}\,du_0\,du_1\cdots du_n, \end{aligned}$$

where δ is a permutation on $0, 1, 2, \ldots, n$.

Let $\operatorname{sgn}(\delta)$ be the sign of δ, we can obtain

$$\begin{aligned} u_n(x) &= (-1)^{n+1} \underbrace{\int_{-\infty}^{0} \cdots \int_{-\infty}^{0}}_{n+1 \text{ times}} \frac{e^{\frac{x}{k}(u_0+u_1+\cdots+u_n)}}{\prod_{i=0}^{n}(1-e^{u_i})} \operatorname{sgn}(\delta) \prod_{i=0}^{n} u_i^m \\ &\quad \cdot \begin{vmatrix} u_0^0 & u_0^j & \cdots & u_0^{nj} \\ u_1^j & u_1^{2j} & \cdots & u_1^{(n+1)j} \\ \vdots & \vdots & & \vdots \\ u_n^{nj} & u_n^{(n+1)j} & \cdots & u_n^{2nj} \end{vmatrix} du_0\,du_1\cdots du_n \\ &= \frac{(-1)^{n+1}}{(n+1)!} \underbrace{\int_{-\infty}^{0} \cdots \int_{-\infty}^{0}}_{n+1 \text{ times}} \frac{e^{\frac{x}{k}(u_0+u_1+\cdots+u_n)}}{\prod_{i=0}^{n}(1-e^{u_i})} (u_0u_1\cdots u_n)^m \\ &\quad \cdot \prod_{0\le i<l\le n} \left(u_i^j - u_l^j\right) du_0\,du_1\cdots du_n. \end{aligned}$$

Replacing $u_0, u_1, \dots, u_n$ by $-u_0, -u_1, \dots, -u_n$, we get

$$\mu_n(x) = (-1)^{(n+1)(m+1)} \underbrace{\int_0^\infty \cdots \int_0^\infty}_{n+1 \text{ times}} e^{-\frac{x}{k}(u_0+u_1+\cdots+u_n)} \cdot \prod_{0\le i<l\le n} (u_i^j - u_l^j) \prod_{i=0}^{n} \frac{u_i^n}{1-e^{-u_i}} \, du_0 \, du_1 \cdots du_n.$$

This implies that $(-1)^{(n+1)(m+1)}\mu_n(x)$ is completely monotonic. □

By taking $n = 1$, the following Corollary 3.2 can be easily obtained.

Corollary 3.2 *For $m, j \in \mathbb{N}$, and $x > 0$, we have*

$$\begin{vmatrix} \psi_k^{(m)}(x) & \psi_k^{(m+j)}(x) \\ \psi_k^{(m+j)}(x) & \psi_k^{(m+2j)}(x) \end{vmatrix} > 0.$$

4 Conclusions

We established a concave theorem and some monotonic properties for the generalized digamma function, and some interesting inequalities were obtained. These conclusions generalize Alzer's results. On the other hand, we prove a completely monotonic property for the generalized digamma function by using Ismail and Laforgia's idea.

5 Methods and experiment

Not applicable.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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