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Dynamical behaviors of a food-chain model with stage structure and time delays

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Abstract

Incorporating two delays (τ_1 represents the maturity of predator, τ_2 represents the maturity of top predator), we establish a novel delayed three-species food-chain model with stage structure in this paper. By analyzing the characteristic equations, constructing a suitable Lyapunov functional, using Lyapunov–LaSalle's principle, the comparison theorem and iterative technique, we investigate the existence of nonnegative equilibria and their stability. Some interesting findings show that the delays have great impacts on dynamical behaviors for the system: on one hand, if $\tau_1 \in (m_1, m_2)$ and $\tau_2 \in (m_4, +\infty)$, then the boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$ is asymptotically stable (AS), i.e., the prey species and the predator species will coexist, the top-predator species will go extinct; on the other hand, if $\tau_1 \in (m_2, +\infty)$, then the axial equilibrium $E_1(k, 0, 0, 0, 0)$ is AS, i.e., all predators will go extinct. Numerical simulations are great well agreement with the theoretical results.

Keywords: Food-chain model; Stage structure; Time delay; Stability

1 Introduction

Predator–prey type interaction is one of basic interspecies relations in the biology and ecology and it is also the basic block of the complicated food chain, food web and biochemical network structure [1–4]. Since the seminal work by Aiello and Freedman [5], species growth models with stage structure have drawn considerable attention (for more details as regards these studies, one can refer to [6, 7]). Incorporating stage structure for predator into the system, Xu [8] built a delayed Lotka–Volterra type predator–prey system. Further studies show that the stage structures for both predator and prey should be taken into consideration in modelling [9]. Some interesting results on the dynamical behaviors of predator–prey systems can be found in [10–15].

The 'prey–predator–top-predator' system (the top predator consumes only the predator trophic level), as one of the most important food-chain models [16–21], takes the form

$$\begin{cases} \frac{dP_1}{dt} = rP_1(1 - \frac{P_1}{k}) - h_1 P_1 P_2, \\ \frac{dP_2}{dt} = c_1 P_1 P_2 - h_2 P_2 T - d_1 P_2, \\ \frac{dT}{dt} = c_1 P_2 T - d_2 T, \end{cases} \tag{1}$$

where P_1, P_2 and T can be interpreted as the densities of prey species, predator species and top-predator species, respectively. The intrinsic growth rate of the prey species can be represented as r. k denotes the environmental carrying capacity of the prey species. h_1 and h_2 represent the hunting rate of the predator and top predator, respectively. c_1 and c_1 can be interpreted as the conversion rate of prey species to its predator species and predator species to the top-predator species, respectively. d_1 and d_2 represent the death rate of the predator and top predator, respectively.

A great deal of results on 'prey–predator–top-predator' type food-chain models have been reported in the literature. In [22], the dynamical behaviors of a three-species ratio-dependent food-chain model were investigated. Cui et al. [23] discussed the stability and bifurcation of periodic solutions for a three-species food-chain system. Pei et al. [24] established a delay three-species ecosystem with Holling functional response, the dynamical behaviors of the system were studied. In [25], Mbava et al. investigated the dynamics of a food-chain model with disease in species. To a large extent, the existing literature on theoretical studies of 'prey–predator–top-predator' systems is predominantly concerned with cases without stage structure. Literature dealing with the stage structure for both predator and prey appears to be scarce, such studies are, however, important for us to understand the dynamical characteristics of food-chain models. On the other hand, as we know, time delays do exist in many systems, such as population system [26, 27], economic system [28, 29], epidemic model [25, 30], neural network system [31–34], etc. Enlightened by the above discussions, in this paper, we intend to consider a new three-species food-chain model with stage structure and delays for both predator and top predator.

In the following, let us firstly introduce the parameters and a brief sketch of the construction of the model which may indicate the biological relevance of it.

(A1) There are three populations, namely, the prey species whose population density is denoted by $x(t)$, the predator whose immature and mature population densities are $y_1(t)$ and $y_2(t)$, respectively; the top predator whose immature and mature population densities are described by $z_1(t)$ and $z_2(t)$, respectively.

(A2) In the absence of predation, the prey population grow according to logistic laws of growth with intrinsic growth rate α_1, and the carrying capacity is k.

(A3) The mature predator consumes the prey with $c_1x(t)y_2(t)$ and contributes to its immature population growth rate $\alpha_2x(t)y_2(t)$; the mature top predator consumes the mature predator with $c_2y_2(t)z_2(t)$ and contributes to its immature population growth rate $\alpha_3y_2(t)z_2(t)$.

(A4) The mortality rate of predator is assumed to be proportional to the existing population. We also consider the density dependent mortality rate of the consumer specie as $\beta_1y_2^2(t)$ and $\beta_2z_2^2(t)$. If there is some other factor (other than food) which becomes limiting at high population densities, the self limitation will occur.

According to Table 1 and (A1)–(A4), we can build up the following stage-structured food-chain model:

$$\begin{cases} \dot{x}(t) = x(t)[\alpha_1(1 - \frac{x(t)}{k}) - c_1y_2(t)], \\ \dot{y}_1(t) = \alpha_2x(t)y_2(t) - d_{11}y_1(t) - \alpha_2e^{-d_{11}\tau_1}x(t-\tau_1)y_2(t-\tau_1), \\ \dot{y}_2(t) = \alpha_2e^{-d_{11}\tau_1}x(t-\tau_1)y_2(t-\tau_1) - d_{12}y_2(t) - \beta_1y_2^2(t) - c_2y_2(t)z_2(t), \\ \dot{z}_1(t) = \alpha_3y_2(t)z_2(t) - d_{21}z_1(t) - \alpha_3e^{-d_{21}\tau_2}y_2(t-\tau_2)z_2(t-\tau_2), \\ \dot{z}_2(t) = \alpha_3e^{-d_{21}\tau_2}y_2(t-\tau_2)z_2(t-\tau_2) - d_{22}z_2(t) - \beta_2z_2^2(t), \end{cases} \tag{2}$$

where all parameters are positive constants.

Table 1 Parameters for system (2)

Parameter	Description
α_1	Intrinsic growth rate of the prey
k	Environmental carrying capacity of the prey
c_1	Capture rate of the mature predator
$\frac{\alpha_2}{c_1}$	Conversion rate of nutrients into the reproduction of the mature predator
c_2	Capture rate of the mature top predator
$\frac{\alpha_3}{c_2}$	Conversion rate of nutrients into the reproduction of the mature top predator
d_{11}	Death rate of the immature predator
d_{12}	Death rate of the mature predator
d_{21}	Death rate of the immature top predator
d_{22}	Death rate of the mature top predator
β_1	Intra-specific competition rate of the mature predator species
β_2	Intra-specific competition rate of the mature top-predator species
τ_1	Maturity of the predator
τ_2	Maturity of the top predator

The remainder of this article is organized as follows. In Sect. 2, the preliminaries including the initial conditions, the positivity and boundedness of the solutions of system (2) are presented. In Sect. 3, we deal with the existence of various equilibria. By analyzing the corresponding characteristic equations, the local stability of the equilibria of system (2) are discussed in Sect. 4. In Sect. 5, we investigate the global stability of the interior equilibrium E^*, the boundary equilibrium E_2 and the axial equilibrium E_1. One illustrative example and simulations are shown in Sect. 6. Finally, a brief discussion is drawn in Sect. 7.

2 Preliminaries

Considering the biological interpretation of the model, the initial conditions for (2) are required to be

$$\begin{aligned} &x(\theta) = \phi(\theta), \qquad y_i(\theta) = \varphi_i(\theta), \qquad z_i(\theta) = \psi_i(\theta), \\ &\phi(0) > 0, \qquad \varphi_i(0) > 0, \qquad \psi_i(0) > 0, \quad i = 1,2, \qquad \theta \in [-\tau, 0], \end{aligned} \tag{3}$$

where

$$\begin{aligned} &\tau = \max\{\tau_1, \tau_2\}, \qquad \bigl(\phi(\cdot), \varphi_1(\cdot), \varphi_2(\cdot), \psi_1(\cdot), \psi_2(\cdot)\bigr) \in C\bigl([-\tau, 0], R^5_{+0}\bigr), \\ &R^5_{+0} = \bigl\{(x_1, x_2, x_3, x_4, x_5) : x_i \geq 0, i = 1,2,3,4,5\bigr\}. \end{aligned}$$

Theorem 1 *Let* $\Gamma(t) = (x(t), y_1(t), y_2(t), z_1(t), z_2(t))$ *be a solution of system* (2) *with initial conditions* (3), *then the solutions of system are strictly positive for all* $t \geq 0$.

Proof Firstly, we prioritize $y_2(t)$ for $t \in [0, \tau^*]$, where $\tau^* = \min\{\tau_1, \tau_2\}$. From the initial conditions (3), we can know that $\phi(\theta) \geq 0$, $\varphi_2(\theta) \geq 0$ for $\theta \in [-\tau, 0]$. Thus, we obtain the third equation of system (2), for $t \in [0, \tau^*]$,

$$\begin{aligned} \dot{y}_2(t) &= \alpha_2 e^{-d_{11}\tau_1} \phi(t-\tau_1)\varphi_2(t-\tau_1) - d_{12}y_2(t) - \beta_1 y_2^2(t) - c_2 y_2(t) z_2(t) \\ &\geq -d_{12}y_2(t) - \beta_1 y_2^2(t) - c_2 y_2(t) z_2(t). \end{aligned} \tag{4}$$

By the comparison theorem, we get

$$y_2(t) \geq y_2(0)e^{\int_0^t(-d_{12}-\beta_1 y_2(s)-c_2 z_2(s))\,ds} > 0.$$

Similarly, from the third equation of system (2), we obtain, for $t \in [0, \tau^*]$,

$$\begin{aligned}\dot{z}_2(t) &= \alpha_3 e^{-d_{21}\tau_2}\varphi_2(t-\tau_2)\psi_2(t-\tau_2) - d_{22}z_2(t) - \beta_2 z_2^2(t)\\ &\geq -d_{22}z_2(t) - \beta_2 z_2^2(t),\end{aligned} \tag{5}$$

since $\varphi_2(\theta) \geq 0$, $\psi_2(\theta) \geq 0$, $\theta \in [-\tau, 0]$.

By the comparison theorem, one has

$$z_2(t) \geq z_2(0)e^{\int_0^t(-d_{22}-\beta_2 z_2(s))\,ds} > 0.$$

Repeat the process above, it is obvious to derive that $y_2(t) > 0$, $z_2(t) > 0$ on the intervals $[\tau^*, 2\tau^*], \ldots, [n\tau^*, (n+1)\tau^*]$, $n \in N$.

The first equation of system (2) together with initial conditions (3) gives

$$x(t) = x(0)e^{\int_0^t(\alpha_1(1-\frac{x(s)}{k})-c_1 y_2(s))\,ds} > 0.$$

By the second equation of system (2), we can get

$$y_1(t) = \int_{t-\tau_1}^t \alpha_2 e^{-d_{11}(t-s)}x(s)y_2(s)\,ds > 0. \tag{6}$$

With the fourth equation of system (2), one has

$$z_1(t) = \int_{t-\tau_2}^t \alpha_3 e^{-d_{21}(t-s)}y_2(s)z_2(s)\,ds > 0. \tag{7}$$

This completes the proof. □

Remark 1 Taking account for the maturity of predator and top predator, we incorporate two delays in model (2), which is more general than system (1.2) in [8]. To investigate the positivity of system (2), we extend and improve the method in [8]. Specifically, we define a new τ^* satisfying $\tau^* = \min\{\tau_1, \tau_2\}$. If $t \in [0, \tau^*]$, then $t - \tau_i \in [-\tau, 0]$ $(i = 1, 2)$, where $\tau = \max\{\tau_1, \tau_2\}$.

Theorem 2 *Let* $\Gamma(t) = (x(t), y_1(t), y_2(t), z_1(t), z_2(t))$ *be a solution of system* (2), *then the solutions of system* (2) *with initial conditions* (3) *are ultimately bounded.*

Proof Define $\rho(t)$ associated with (2) as

$$\rho(t) = \alpha_2 x(t) + c_1 y_1(t) + c_1 y_2(t) + \frac{c_1 c_2}{\alpha_3}z_1(t) + \frac{c_1 c_2}{\alpha_3}z_2(t).$$

Denote $d = \min\{d_{11}, d_{12}, d_{21}, d_{22}\}$, by calculating the derivative of $\rho(t)$ with respect to system (2), we derive

$$\begin{aligned}\dot{\rho}(t) &= \alpha_1\alpha_2\left(1-\frac{x(t)}{k}\right)x(t) - c_1d_{11}y_1(t) - c_1\bigl(d_{12}+\beta_1y_2(t)\bigr)\\ &\quad - \frac{c_1c_2}{\alpha_3}d_{21}z_1(t) - \frac{c_1c_2}{\alpha_3}\bigl(d_{22}+\beta_2z_2(t)\bigr)z_2(t)\\ &\le -d\rho(t) + (\alpha_1+d)\alpha_2x(t) - \alpha_1\alpha_2\frac{1}{k}x^2(t)\\ &\le -d\rho(t) + \frac{\alpha_2k}{4\alpha_1}(\alpha_1+d)^2.\end{aligned}$$

Hence, one obtains

$$\limsup_{t\to+\infty}\rho(t) \le \frac{\alpha_2k(\alpha_1+d)^2}{4\alpha_1d}.$$

This completes the proof. □

3 Existence of equilibria

In this section, we consider the existence of equilibria. From system (2), $(x, y_1, y_2, z_1, z_2) \in R^5_{+0}$ is an equilibrium if and only if:

$$\begin{cases} x[\alpha_1(1-\frac{x}{k}) - c_1y_2] = 0,\\ \alpha_2xy_2 - d_{11}y_1 - \alpha_2e^{-d_{11}\tau_1}xy_2 = 0,\\ \alpha_2e^{-d_{11}\tau_1}xy_2 - d_{12}y_2 - \beta_1y_2^2 - c_2y_2z_2 = 0,\\ \alpha_3y_2z_2 - d_{21}z_1 - \alpha_3e^{-d_{21}\tau_2}y_2z_2 = 0,\\ \alpha_3e^{-d_{21}\tau_2}y_2z_2 - d_{22}z_2 - \beta_2z_2^2 = 0.\end{cases} \tag{8}$$

Therefore, there are four equilibria of system (2):

(i) The trivial equilibrium $E_0(0,0,0,0,0)$ and the axial equilibrium $E_1(k,0,0,0,0)$ of system (2) exist irrespective of any parametric restriction.

(ii) If the following inequality (C1) holds:

$$\text{(C1)}\quad \alpha_2ke^{-d_{11}\tau_1} - d_{12} > 0,$$

then there exists the boundary equilibrium boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$, where

$$\begin{aligned} x^0 &= \frac{k(\alpha_1\beta_1 + c_1d_{12})}{\alpha_1\beta_1 + \alpha_2c_1ke^{-d_{11}\tau_1}},\\ y_1^0 &= \frac{\alpha_1\alpha_2k(\alpha_1\beta_1 + c_1d_{12})(1-e^{-d_{11}\tau_1})(\alpha_2ke^{-d_{11}\tau_1} - d_{12})}{d_{11}(\alpha_1\beta_1 + \alpha_2c_1ke^{-d_{11}\tau_1})^2},\\ y_2^0 &= \frac{\alpha_1(\alpha_2ke^{-d_{11}\tau_1} - d_{12})}{\alpha_1\beta_1 + \alpha_2c_1ke^{-d_{11}\tau_1}}.\end{aligned}$$

(iii) If the following inequalities (C2), (C3) and (C4) hold:

$$\text{(C2)}\quad \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \beta_2 c_1 d_{12} - c_1 c_2 d_{22} > 0,$$

$$\text{(C3)}\quad \alpha_2\beta_2 k e^{-d_{11}\tau_1} - \beta_2 d_{12} + c_2 d_{22} > 0,$$

$$\text{(C4)}\quad \alpha_1\alpha_3 e^{-d_{21}\tau_2}\bigl(\alpha_2 k e^{-d_{11}\tau_1} - d_{12}\bigr) - d_{22}\bigl(\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1}\bigr) > 0,$$

then, apart from the axial and boundary equilibria, there exists a unique interior equilibrium $E^*(x^*, y_1^*, y_2^*, z_1^*, z_2^*)$, where

$$x^* = \frac{k\Lambda_1}{\Lambda_2}, \qquad y_1^* = \frac{\alpha_1\alpha_2 k(1-e^{-d_{11}\tau_1})}{d_{11}}\frac{\Lambda_1\Lambda_3}{\Lambda_2^2},$$

$$y_2^* = \frac{\alpha_1\Lambda_3}{\Lambda_2}, \qquad z_1^* = \frac{\alpha_1\alpha_3(1-e^{-d_{21}\tau_2})}{d_{21}}\frac{\Lambda_3\Lambda_4}{\Lambda_2^2},$$

$$z_2^* = \frac{\Lambda_4}{\Lambda_2},$$

and

$$\Lambda_1 = \alpha_1\beta_1\beta_2 + \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_1} + \beta_2 c_1 d_{12} - c_1 c_2 d_{22},$$

$$\Lambda_2 = \alpha_1\beta_1\beta_2 + \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \alpha_2\beta_2 c_1 k e^{-d_{11}\tau_1},$$

$$\Lambda_3 = \alpha_2\beta_2 k e^{-d_{11}\tau_1} - \beta_2 d_{12} + c_2 d_{22},$$

$$\Lambda_4 = \alpha_1\alpha_3 e^{-d_{21}\tau_2}\bigl(\alpha_2 k e^{-d_{11}\tau_1} - d_{12}\bigr) - d_{22}\bigl(\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1}\bigr).$$

Remark 2 Since we consider a three-species-food-chain model, the dynamical behaviors are more complicated and the system has more equilibria than those in [4, 10, 12]. Although these conditions of (C2), (C3) and (C4) seem to be intricate, take *Case I* (please see the section of *Numerical simulation* (Sect. 6)) as an example, one can find that these conditions can achieve.

4 Local stability analysis of the equilibria

In this section, we study the local stability of system (2) at equilibria. For this purpose, we first introduce the following lemma.

Lemma 1 ([6]) *For the equation*

$$\lambda^2 + a_1\lambda + a_2 + (b_1\lambda + b_2)e^{-\lambda\tau} = 0, \tag{9}$$

assume that $a_2 + b_2 \neq 0$, $a_1^2 + b_1^2 + b_2^2 \neq 0$, *the number of different positive* (*negative*) *imaginary roots of* (9) *can be zero, one, or two only.*

If $a_2^2 > b_2^2$ *and* $b_1^2 + 2a_2 - a_1^2 < 2\sqrt{a_2^2 - b_2^2}$, *then* (9) (*for* $\tau > 0$) *has the same stability or instability as when* $\tau = 0$.

4.1 The local stability of the trivial equilibrium $E_0(0,0,0,0,0)$

Theorem 3 *The trivial equilibrium* E_0 *is unstable.*

Proof The characteristic equation for the linearized system of (2) about $E_0(0,0,0,0,0)$ is given by

$$\begin{vmatrix} \lambda-\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda+d_{11} & 0 & 0 & 0 \\ 0 & 0 & \lambda+d_{12} & 0 & 0 \\ 0 & 0 & 0 & \lambda+d_{21} & 0 \\ 0 & 0 & 0 & 0 & \lambda+d_{22} \end{vmatrix} = 0. \tag{10}$$

Then the characteristic equation (10) about the equilibrium E_0 is

$$(\lambda-\alpha_1)(\lambda+d_{11})(\lambda+d_{12})(\lambda+d_{21})(\lambda+d_{22}) = 0.$$

Since $\lambda_1 = \alpha_1$ is a positive root, the trivial equilibrium $E_0(0,0,0,0,0)$ is unstable. □

4.2 The local stability of the axial equilibrium

Theorem 4 *Basing on the existence of equilibria which has been presented in Sect.* 3, *we have the following results*:

(i) *If* $\alpha_2 k e^{-d_{11}\tau_1} < d_{12}$, *then the axial equilibrium* E_1 *is locally asymptotically stable* (*LAS*).

(ii) *If* $\alpha_2 k e^{-d_{11}\tau_1} > d_{12}$, *then the axial equilibrium* E_1 *is unstable*.

Proof The characteristic equation for the linearized system of (2) about $E_1(k,0,0,0,0)$ takes the form

$$\begin{vmatrix} \lambda+\alpha_1 & 0 & c_1 k & 0 & 0 \\ 0 & \lambda+d_{11} & (e^{-(\lambda+d_{11})\tau_1}-1)\alpha_2 k & 0 & 0 \\ 0 & 0 & \lambda+d_{12}-\alpha_2 k e^{-d_{11}\tau_1}e^{-\lambda\tau_1} & 0 & 0 \\ 0 & 0 & 0 & \lambda+d_{21} & 0 \\ 0 & 0 & 0 & 0 & \lambda+d_{22} \end{vmatrix} = 0. \tag{11}$$

Hence, the characteristic equation (11) about the equilibrium E_1 can reduce to

$$(\lambda+\alpha_1)(\lambda+d_{11})\bigl(\lambda+d_{12}-\alpha_2 k e^{-d_{11}\tau_1}e^{-\lambda\tau_1}\bigr)(\lambda+d_{21})(\lambda+d_{22}) = 0.$$

It is obvious that $\lambda_1 = -\alpha_1$, $\lambda_2 = -d_{11}$, $\lambda_3 = -d_{21}$, $\lambda_4 = -d_{22}$ are all negative eigenvalues, thus the stability of axial equilibrium E_1 is determined by the equation of $\lambda + d_{12} - \alpha_2 k e^{-d_{11}\tau_1}e^{-\lambda\tau_1} = 0$. Let $f(\lambda)$ have the following form:

$$f(\lambda) = \lambda + d_{12} - \alpha_2 k e^{-d_{11}\tau_1}e^{-\lambda\tau_1}.$$

By analyzing, one can obtain the following cases.

If $\alpha_2 k e^{-d_{11}\tau_1} < d_{12}$, we assume that $\operatorname{Re}\lambda \geq 0$. By calculating, we get

$$\begin{aligned} \operatorname{Re}\lambda &= -d_{12} + \alpha_2 k e^{-d_{11}\tau_1}e^{-\tau_1 \operatorname{Re}\lambda}\cos(\tau_1 \operatorname{Im}\lambda) \\ &\leq -d_{12} + \alpha_2 k e^{-d_{11}\tau_1} < 0, \end{aligned}$$

which is a contradiction. Hence, $\operatorname{Re}\lambda < 0$. Consequently, the result (i) of Theorem 4 holds.

If $\alpha_2 k e^{-d_{11}\tau_1} > d_{12}$, then we have

$$\begin{cases} f(0) = d_{12} - \alpha_2 k e^{-d_{11}\tau_1} < 0, \\ \lim_{\lambda\to+\infty} f(\lambda) = +\infty, \\ f'(\lambda) = 1 + \tau_1\alpha_2 k e^{-d_{11}\tau_1} e^{-\lambda\tau_1} > 0. \end{cases}$$

Therefore, for $f(\lambda) = 0$ there must exist a positive root. Thus, the result (ii) of Theorem 4 holds as well. This completes the proof. □

4.3 The local stability of the boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$

Theorem 5 *Under the condition* (C1), *we get the following results*:

(i) *If* $\alpha_3 y_2^0 e^{-d_{21}\tau_2} < d_{22}$ *and* $\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1} > 0$, *then the boundary equilibrium* E_2 *is LAS.*

(ii) *If* $\alpha_3 y_2^0 e^{-d_{21}\tau_2} > d_{22}$, *then the boundary equilibrium* E_2 *is unstable.*

Proof The characteristic equation for the linearized system of (2) about $E_2(x^0, y_1^0, y_2^0, 0, 0)$ is given as below

$$\begin{vmatrix} \lambda + \frac{\alpha_1 x^0}{k} & 0 & c_1 x^0 & 0 & 0 \\ \alpha_2\Delta_3 y_2^0 & \lambda + d_{11} & \alpha_2\Delta_3 x^0 & 0 & 0 \\ -\alpha_2 e^{-(\lambda+d_{11})\tau_1} y_2^0 & 0 & \lambda + \Delta_1 & 0 & c_2 y_2^0 \\ 0 & 0 & 0 & \lambda + d_{21} & \alpha_3\Delta_4 y_2^0 \\ 0 & 0 & 0 & 0 & \lambda + \Delta_2 \end{vmatrix} = 0, \tag{12}$$

where

$$\begin{aligned} \Delta_1 &= \beta_1 y_2^0 + \alpha_2 x^0 e^{-d_{11}\tau_1}\left(1 - e^{-\lambda\tau_1}\right), \\ \Delta_2 &= d_{22} - \alpha_3 y_2^0 e^{-d_{21}\tau_2} e^{-\lambda\tau_2}, \\ \Delta_3 &= e^{-(\lambda+d_{11})\tau_1} - 1, \\ \Delta_4 &= e^{-(\lambda+d_{21})\tau_2} - 1. \end{aligned}$$

Thus, the characteristic equation (12) about the equilibrium E_2 is

$$(\lambda + d_{11})(\lambda + d_{21})(\lambda + \Delta_2)\left[(\lambda + \Delta_1)\left(\lambda + \frac{\alpha_1 x^0}{k}\right) + c_1\alpha_2 x^0 y_2^0 e^{-(\lambda+d_{11})\tau_1}\right] = 0.$$

Clearly, $\lambda_1 = -d_{11}$, $\lambda_2 = -d_{21}$, which are always negative. Hence, the stability of the boundary equilibrium E_2 is determined by the following equations:

$$\lambda + \Delta_2 = 0,$$

and

$$(\lambda + \Delta_1)\left(\lambda + \frac{\alpha_1 x^0}{k}\right) + c_1\alpha_2 x^0 y_2^0 e^{-(\lambda+d_{11})\tau_1} = 0.$$

For $\lambda + \Delta_2 = 0$, that is, $\lambda + d_{22} - \alpha_3 y_2^0 e^{-d_{21}\tau_2} e^{-\lambda\tau_2} = 0$, let $f(\lambda)$ have the following form:

$$f(\lambda) = \lambda + d_{22} - \alpha_3 y_2^0 e^{-d_{21}\tau_2} e^{-\lambda\tau_2}.$$

By analyzing, one can obtain the following cases.

If $\alpha_3 y_2^0 e^{-d_{21}\tau_2} > d_{22}$, then we have

$$\begin{cases} f(0) = d_{22} - \alpha_3 y_2^0 e^{-d_{21}\tau_2} < 0, \\ \lim_{\lambda\to+\infty} f(\lambda) = +\infty, \\ f'(\lambda) = 1 + \tau_2\alpha_3 y_2^0 e^{-d_{21}\tau_2} e^{-\lambda\tau_2} > 0. \end{cases}$$

Thus, for $f(\lambda) = 0$ there must exist a positive root, thereby, the result (ii) of Theorem 5 holds.

If $\alpha_3 y_2^0 e^{-d_{21}\tau_2} < d_{22}$, we assume that $\operatorname{Re}\lambda \geq 0$. By calculating, we get

$$\begin{aligned} \operatorname{Re}\lambda &= -d_{22} + \alpha_3 y_2^0 e^{-d_{21}\tau_2} e^{-\tau_2 \operatorname{Re}\lambda} \cos(\tau_2 \operatorname{Im}\lambda) \\ &\leq -d_{22} + \alpha_3 y_2^0 e^{-d_{21}\tau_2} < 0, \end{aligned}$$

which is a contradiction. Hence, $\operatorname{Re}\lambda < 0$.

For $(\lambda + \Delta_1)(\lambda + \frac{\alpha_1 x^0}{k}) + c_1\alpha_2 x^0 y_2^0 e^{-(\lambda+d_{11})\tau_1} = 0$, by calculating, we can obtain

$$\lambda^2 + a_1\lambda + a_2 + (b_1\lambda + b_2)e^{-\lambda\tau_1} = 0, \tag{13}$$

where

$$\begin{aligned} a_1 &= \frac{\alpha_1 x^0}{k} + \beta_1 y_2^0 + \alpha_2 x^0 e^{-d_{11}\tau_1}, \qquad a_2 = \frac{\alpha_1 x^0}{k}\bigl(\beta_1 y_2^0 + \alpha_2 x^0 e^{-d_{11}\tau_1}\bigr), \\ b_1 &= -\alpha_2 x^0 e^{-d_{11}\tau_1}, \qquad b_2 = \alpha_2 x^0 e^{-d_{11}\tau_1}\left(c_1 y_2^0 - \frac{\alpha_1 x^0}{k}\right). \end{aligned}$$

When $\tau_1 = 0$, Eq. (13) can reduce to

$$\lambda^2 + \left(\frac{\alpha_1 x^0}{k} + \beta_1 y_2^0\right)\lambda + x^0 y_2^0\left(\frac{\alpha_1\beta_1}{k} + \alpha_2 c_1\right) = 0.$$

Obviously, there only exist negative eigenvalues. Hence, the boundary equilibrium E_2 is LAS when $\tau_1 = 0$ and $\alpha_3 y_2^0 e^{-d_{21}\tau_2} < d_{22}$.

When $\tau_1 \neq 0$, one can derive that

$$b_1^2 + 2a_2 - a_1^2 = -\left(\frac{\alpha_1 x^0}{k}\right)^2 - \bigl(\beta_1 y_2^0\bigr)^2 - 2\alpha_2\beta_1 x^0 y_2^0 e^{-d_{11}\tau_1} < 0,$$

and

$$\begin{aligned} a_2^2 - b_2^2 = \frac{(x^0)^2 y_2^0}{k^2}\bigl[&2k\alpha_1\alpha_2 e^{-d_{11}\tau_1}(\alpha_1\beta_1 + c_1 d_{12}) \\ &+ \bigl(\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1}\bigr)\alpha_1\bigl(\alpha_2 k e^{-d_{11}\tau_1} - d_{12}\bigr)\bigr]. \end{aligned}$$

Under the condition (C1) $\alpha_2 k e^{-d_{11}\tau_1} - d_{12} > 0$, if $\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1} > 0$, then $a_2^2 > b_2^2$, by Lemma 1, the boundary equilibrium E_2 is LAS. Therefore, the result (i) of Theorem 5 holds as well. This completes the proof. □

4.4 The stability of the interior equilibrium $E^*(x^*, y_1^*, y_2^*, z_1^*, z_2^*)$

Theorem 6 *Under the conditions* (C2), (C3) *and* (C4), *if* $2\alpha_1 > \alpha_2 k e^{-d_{11}\tau_1}$, $2\beta_1 > \alpha_2 e^{-d_{11}\tau_1} + \alpha_3 e^{-d_{21}\tau_2}$ *and* $2\beta_2 > \alpha_3 e^{-d_{21}\tau_2}$, *then the interior equilibrium* E^* *is stable.*

Proof The linearized system of (2) about $E^*(x^*, y_1^*, y_2^*, z_1^*, z_2^*)$ is

$$\begin{cases} \dot{x}(t) = -\frac{\alpha_1}{k}x^*x(t) - c_1x^*y_2(t), \\ \dot{y}_1(t) = \alpha_2y_2^*x(t) + \alpha_2x^*y_2(t) - d_{11}y_1(t) - \alpha_2e^{-d_{11}\tau_1}y_2^*x(t-\tau_1) \\ \qquad - \alpha_2e^{-d_{11}\tau_1}x^*y_2(t-\tau_1), \\ \dot{y}_2(t) = \alpha_2e^{-d_{11}\tau_1}y_2^*x(t-\tau_1) + \alpha_2e^{-d_{11}\tau_1}x^*y_2(t-\tau_1) \\ \qquad - (\alpha_2e^{-d_{11}\tau_1}x^* + \beta_1y_2^*)y_2(t) - c_2y_2^*z_2(t), \\ \dot{z}_1(t) = \alpha_3z_2^*y_2(t) + \alpha_3y_2^*z_2(t) - d_{21}z_1(t) - \alpha_3e^{-d_{21}\tau_2}z_2^*y_2(t-\tau_2) \\ \qquad - \alpha_3e^{-d_{21}\tau_2}y_2^*z_2(t-\tau_2), \\ \dot{z}_2(t) = \alpha_3e^{-d_{21}\tau_2}z_2^*y_2(t-\tau_2) + \alpha_3e^{-d_{21}\tau_2}y_2^*z_2(t-\tau_2) - (\alpha_3e^{-d_{21}\tau_2}y_2^* + \beta_2z_2^*)z_2^2(t). \end{cases} \tag{14}$$

Define $V(x_t, y_{1t}, y_{2t}, z_{1t}, z_{2t})$ associated with (14) as

$$\begin{aligned} V(x_t, y_{1t}, y_{2t}, z_{1t}, z_{1t}) = {}& \frac{1}{2x^*}x^2(t) + \frac{1}{2y_2^*}y_2^2(t) + \frac{1}{2z_2^*}z_2^2(t) \\ & + \frac{x^*}{2y_2^*}\alpha_2e^{-d_{11}\tau_1}\int_{t-\tau_1}^t y_2^2(s)\,ds + \frac{y_2^*}{2z_2^*}\alpha_3e^{-d_{21}\tau_2}\int_{t-\tau_2}^t z_2^2(s)\,ds \\ & + \frac{\alpha_2e^{-d_{11}\tau_1}}{2}\int_{t-\tau_1}^t x^2(s)\,ds + \frac{\alpha_3e^{-d_{21}\tau_2}}{2}\int_{t-\tau_2}^t y_2^2(s)\,ds. \end{aligned}$$

By calculating the derivative of $V(x_t, y_{1t}, y_{2t}, z_{1t}, z_{2t})$ with respect to system (14), we derive

$$\begin{aligned} \dot{V}(x_t, y_{1t}, y_{2t}, z_{1t}, z_{2t}) = {}& \frac{1}{x^*}x(t)\dot{x}(t) + \frac{1}{y_2^*}y_2(t)\dot{y}_2(t) + \frac{1}{z_2^*}z_2(t)\dot{z}_2(t) \\ & + \frac{x^*}{2y_2^*}\alpha_2e^{-d_{11}\tau_1}\big[y_2^2(t) - y_2^2(t-\tau_1)\big] \\ & + \frac{y_2^*}{2z_2^*}\alpha_3e^{-d_{21}\tau_2}\big[z_2^2(t) - z_2^2(t-\tau_2)\big] \\ & + \frac{\alpha_2e^{-d_{11}\tau_1}}{2}\big[x^2(t) - x^2(t-\tau_1)\big] + \frac{\alpha_3e^{-d_{21}\tau_2}}{2}\big[y_2^2(t) - y_2^2(t-\tau_2)\big] \\ = {}& -\frac{\alpha_1}{k}x^2(t) - c_1x(t)y_2(t) \\ & + \alpha_2e^{-d_{11}\tau_1}x(t-\tau_1)y_2(t) + \frac{x^*}{y_2^*}\alpha_2e^{-d_{11}\tau_1}y_2(t-\tau_1)y_2(t) - \beta_1y_2^2(t) \\ & - c_2y_2(t)z_2(t) - \frac{x^*}{y_2^*}\alpha_2e^{-d_{11}\tau_1}y_2^2(t) - \frac{y_2^*}{z_2^*}\alpha_3e^{-d_{21}\tau_2}z_2^2(t) \\ & + \alpha_3e^{-d_{21}\tau_2}y_2(t-\tau_2)z_2(t) + \frac{y_2^*}{z_2^*}\alpha_3e^{-d_{21}\tau_2}z_2(t-\tau_2)z_2(t) - \beta_2z_2^2(t) \end{aligned}$$

$$\begin{aligned}&+\frac{x^*}{2y_2^*}\alpha_2 e^{-d_{11}\tau_1}\bigl[y_2^2(t)-y_2^2(t-\tau_1)\bigr]\\&+\frac{y_2^*}{2z_2^*}\alpha_3 e^{-d_{21}\tau_2}\bigl[z_2^2(t)-z_2^2(t-\tau_2)\bigr]\\&+\frac{\alpha_2 e^{-d_{11}\tau_1}}{2}\bigl[x^2(t)-x^2(t-\tau_1)\bigr]+\frac{\alpha_3 e^{-d_{21}\tau_2}}{2}\bigl[y_2^2(t)-y_2^2(t-\tau_2)\bigr].\end{aligned}$$

Applying fundamental inequality, one has

$$\begin{aligned}\dot V(x_t,y_{1t},y_{2t},z_{1t},z_{2t})\le{}&-\frac{\alpha_1}{k}x^2(t)-c_1x(t)y_2(t)\\&+\frac{\alpha_2 e^{-d_{11}\tau_1}}{2}\bigl[x^2(t-\tau_1)+y_2^2(t)\bigr]\\&+\frac{x^*}{2y_2^*}\alpha_2 e^{-d_{11}\tau_1}\bigl[y_2^2(t-\tau_1)+y_2^2(t)\bigr]\\&-\beta_1y_2^2(t)-c_2y_2(t)z_2(t)-\frac{x^*}{y_2^*}\alpha_2 e^{-d_{11}\tau_1}y_2^2(t)\\&+\frac{\alpha_3 e^{-d_{21}\tau_2}}{2}\bigl[y_2^2(t-\tau_2)+z_2^2(t)\bigr]\\&+\frac{y_2^*}{2z_2^*}\alpha_3 e^{-d_{21}\tau_2}\bigl[z_2^2(t-\tau_2)+z_2^2(t)\bigr]\\&-\beta_2z_2^2(t)-\frac{y_2^*}{z_2^*}\alpha_3 e^{-d_{21}\tau_2}z_2^2(t)\\&+\frac{x^*}{2y_2^*}\alpha_2 e^{-d_{11}\tau_1}\bigl[y_2^2(t)-y_2^2(t-\tau_1)\bigr]\\&+\frac{y_2^*}{2z_2^*}\alpha_3 e^{-d_{21}\tau_2}\bigl[z_2^2(t)-z_2^2(t-\tau_2)\bigr]\\&+\frac{\alpha_2 e^{-d_{11}\tau_1}}{2}\bigl[x^2(t)-x^2(t-\tau_1)\bigr]\\&+\frac{\alpha_3 e^{-d_{21}\tau_2}}{2}\bigl[y_2^2(t)-y_2^2(t-\tau_2)\bigr]\\={}&-\left(\frac{\alpha_1}{k}-\frac{\alpha_2 e^{-d_{11}\tau_1}}{2}\right)x^2(t)-c_1x(t)y_2(t)\\&-\left(\beta_1-\frac{\alpha_2 e^{-d_{11}\tau_1}}{2}-\frac{\alpha_3 e^{-d_{21}\tau_2}}{2}\right)y_2^2(t)-c_2y_2(t)z_2(t)\\&-\left(\beta_2-\frac{\alpha_3 e^{-d_{21}\tau_2}}{2}\right)z_2^2(t).\end{aligned}$$

If $2\alpha_1>\alpha_2ke^{-d_{11}\tau_1}$, $2\beta_1>\alpha_2e^{-d_{11}\tau_1}+\alpha_3e^{-d_{21}\tau_2}$ and $2\beta_2>\alpha_3e^{-d_{21}\tau_2}$, then $\dot V(t)\le 0$. With the help of Lyapunov–LaSalle's principle, the equilibrium $(0,0,0,0,0)$ of linearized system (14) is asymptotically stable. Therefore, the interior equilibrium E^* of system (2) is stable. This completes the proof. □

Remark 3 Incorporating two delays in system (2), the dynamical behaviors are more complicated than the system with one delay (for example, see [8, 10, 14, 24]). Obviously, the method applied in the mentioned papers cannot be applied to system (2) directly. For example, when deal with the distribution of characteristic roots for the transcendental equation like $\lambda^3 + c\lambda^2 + a_1\lambda + a_2 + (b_1\lambda + b_2)e^{-\lambda\tau_1} = 0$, the local stability of the interior equilibrium E^* cannot be derived by Lemma 3.1 [8]. As for this problem, we investigate the stability of the interior equilibrium E^* by constructing a suitable Lyapunov functional and applying Lyapunov–LaSalle's principle. That is novel and different from [8, 10, 14, 24].

5 Asymptotical stability analysis of equilibria

In the previous section we have found that the trivial equilibrium E_0 is unstable. In this section, we will discuss the global asymptotic stability for the equilibria E^*, E_2 and E_1, respectively. For this purpose, we first introduce the following lemma.

Lemma 2 ([35]) *Consider the following equation*:

$$\dot{\vartheta}(t) = \varrho\vartheta(t-\tau) - \varsigma\vartheta(t) - \varpi\vartheta^2(t),$$

where all parameters are positive constants, $\vartheta(t) > 0$ *for* $t \in [-\tau, 0]$, *one has*

(i) *If* $\varrho > \varsigma$, *then* $\lim_{t\to+\infty} \vartheta(t) = \frac{\varrho-\varsigma}{\varpi}$.

(ii) *If* $\varrho < \varsigma$, *then* $\lim_{t\to+\infty} \vartheta(t) = 0$.

By Lemma 2 and using an iterative technique, we can obtain the following theorems.

Theorem 7 *Under the conditions* (C2), (C3) *and* (C4), *further suppose that*

$$\begin{aligned} &2\alpha_1 > \alpha_2 k e^{-d_{11}\tau_1}, \\ &2\beta_1 > \alpha_2 e^{-d_{11}\tau_1} + \alpha_3 e^{-d_{21}\tau_2}, \\ &2\beta_2 > \alpha_3 e^{-d_{21}\tau_2} \end{aligned}$$

and

$$\alpha_1\beta_1\beta_2 > \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \alpha_2\beta_2 c_1 k e^{-d_{11}\tau_1},$$

then the interior equilibrium E^* *of system* (2) *is AS.*

Proof Under the conditions (C2), (C3) and (C4), if $2\alpha_1 > \alpha_2 k e^{-d_{11}\tau_1}$, $2\beta_1 > \alpha_2 e^{-d_{11}\tau_1} + \alpha_3 e^{-d_{21}\tau_2}$ and $2\beta_2 > \alpha_3 e^{-d_{21}\tau_2}$, by Theorem 6, one find that the interior equilibrium E^* is stable. Therefore, we need only prove that $\lim_{t\to+\infty}(x(t), y_1(t), y_2(t), z_1(t), z_2(t)) = (x^*, y_1^*, y_2^*, z_1^*, z_2^*)$.

Define

$$\begin{aligned} &U_1 = \limsup_{t\to+\infty} x(t), \qquad V_1 = \liminf_{t\to+\infty} x(t), \\ &U_2 = \limsup_{t\to+\infty} y_2(t), \qquad V_2 = \liminf_{t\to+\infty} y_2(t), \\ &U_3 = \limsup_{t\to+\infty} z_2(t), \qquad V_3 = \liminf_{t\to+\infty} z_2(t), \end{aligned}$$

in the next, we will state and prove that $U_1 = V_1 = x^*$, $U_2 = V_2 = y_2^*$, $U_3 = V_3 = z_2^*$.

From the first equation of system (2), we obtain

$$\dot{x}(t) \le x(t)\left(\alpha_1\left(1 - \frac{x(t)}{k}\right)\right).$$

By the comparison theorem, one has

$$U_1 = \limsup_{t\to+\infty} x(t) \le k \stackrel{\text{def}}{=} N_1^x. \tag{15}$$

Since $\varepsilon > 0$ is sufficiently small, then there exists a $T_{11} > 0$ such that $x(t) \le N_1^x + \varepsilon$ for $t > T_{11}$.

We obtain from the third equation of system (2), for $t > T_{11} + \tau$,

$$\dot{y}_2(t) \le \alpha_2 e^{-d_{11}\tau_1}\left(N_1^x + \varepsilon\right)y_2(t - \tau_1) - d_{12}y_2(t) - \beta_1 y_2^2(t).$$

By constructing the following auxiliary equation:

$$\dot{\mathfrak{v}}(t) = \alpha_2 e^{-d_{11}\tau_1}\left(N_1^x + \varepsilon\right)\mathfrak{v}(t - \tau_1) - d_{12}\mathfrak{v}(t) - \beta_1\mathfrak{v}^2(t).$$

Noting that condition (C4) implies that $\alpha_2 k e^{-d_{11}\tau_1} > d_{12}$, and so, by applying Lemma 2(i), we obtain that

$$\lim_{t\to+\infty} \mathfrak{v}(t) = \frac{\alpha_2 e^{-d_{11}\tau_1}(N_1^x + \varepsilon) - d_{12}}{\beta_1}.$$

Using the comparison theorem,

$$U_2 = \limsup_{t\to+\infty} y_2(t) \le \frac{\alpha_2 e^{-d_{11}\tau_1}(N_1^x + \varepsilon) - d_{12}}{\beta_1}. \tag{16}$$

Let $N_1^y = \frac{\alpha_2 e^{-d_{11}\tau_1}N_1^x - d_{12}}{\beta_1}$, since $\varepsilon > 0$ sufficiently small, thereby, $U_2 \le N_1^y$. Consequently, there exists a $T_{12} \ge T_{11} + \tau$ such that $y_2(t) \le N_1^y + \varepsilon$ for $t > T_{12}$.

From the fifth equation of system (2), we have

$$\dot{z}_2(t) \le \alpha_3 e^{-d_{21}\tau_2}\left(N_1^y + \varepsilon\right)z_2(t - \tau_2) - d_{22}z_2(t) - \beta_2 z_2^2(t) \quad \text{for } t > T_{12} + \tau.$$

Using Lemma 2(i) and comparison theorem, one can get

$$U_3 = \limsup_{t\to+\infty} z_2(t) \le \frac{\alpha_3 e^{-d_{21}\tau_2}(N_1^y + \varepsilon) - d_{22}}{\beta_2}.$$

Let $N_1^z = \frac{\alpha_3 e^{-d_{21}\tau_2}N_1^y - d_{22}}{\beta_2}$, since $\varepsilon > 0$ sufficiently small, so we obtain $U_3 \le N_1^z$. Therefore, there exists a $T_{21} \ge T_{12} + \tau$ such that $z_2(t) \le N_1^z + \varepsilon$ for $t > T_{21}$.

We obtain from the first equation of system (2), for $t > T_{12} + \tau$,

$$\dot{x}(t) \ge x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\left(N_1^y + \varepsilon\right)\right].$$

Using the comparison theorem,

$$V_1 = \liminf_{t\to+\infty} x(t) \geq \frac{k[\alpha_1 - c_1(N_1^y + \varepsilon)]}{\alpha_1}.$$

Let $M_1^x = \frac{k(\alpha_1 - c_1 N_1^y)}{\alpha_1}$, since $\varepsilon > 0$ is sufficiently small, then one has $V_1 \geq M_1^x$. Hence, there is a $T_{22} \geq T_{12} + \tau$ such that $x(t) \geq M_1^x - \varepsilon$ for $t > T_{22}$.

From the third equation of system (2) we obtain, for $t > \max\{T_{21}, T_{22}\}$,

$$\dot{y}_2(t) \geq \alpha_2 e^{-d_{11}\tau_1}\big(M_1^x - \varepsilon\big)y_2(t-\tau_1) - d_{12}y_2(t) - \beta_1 y_2^2(t) - c_2\big(N_1^z + \varepsilon\big)y_2(t).$$

By applying Lemma 2(i) and the standard comparison theorem, then

$$V_2 = \liminf_{t\to+\infty} y_2(t) \geq \frac{\alpha_2 e^{-d_{11}\tau_1}(M_1^x - \varepsilon) - d_{12} - c_2(N_1^z + \varepsilon)}{\beta_1}.$$

Let $M_1^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_1^x - d_{12} - c_2 N_1^z}{\beta_1}$, since $\varepsilon > 0$ sufficiently small, obviously, $V_2 \geq M_1^y$. Consequently, there exists a $T_{31} \geq \max\{T_{21}, T_{22}\}$ such that $y_2(t) \geq M_1^y - \varepsilon$ for $t > T_{31}$.

We obtain from the fifth equation of system (2), for $t > T_{31} + \tau$,

$$\dot{z}_2(t) \geq \alpha_3 e^{-d_{21}\tau_2}\big(M_1^y - \varepsilon\big)z_2(t-\tau_2) - d_{22}z_2(t) - \beta_2 z_2^2(t).$$

From this differential inequality, by applying Lemma 2(i), one can get

$$V_3 = \liminf_{t\to+\infty} z_2(t) \geq \frac{\alpha_3 e^{-d_{21}\tau_2}(M_1^y - \varepsilon) - d_{22}}{\beta_2}.$$

Let $M_1^z = \frac{\alpha_3 e^{-d_{21}\tau_2} M_1^y - d_{22}}{\beta_2}$, since $\varepsilon > 0$ is sufficiently small, then $V_3 \geq M_1^z$. Hence, there exists a $T_{32} \geq T_{31} + \tau$ such that $z_2(t) \geq M_1^z - \varepsilon$ for $t > T_{32}$.

Similar to the above discussion, we obtain from the first equation of system (2), for $t > T_{31}$,

$$\dot{x}(t) \leq x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\big(M_1^y - \varepsilon\big)\right].$$

By comparison,

$$U_1 = \limsup_{t\to+\infty} x(t) \leq \frac{k[\alpha_1 - c_1(M_1^y - \varepsilon)]}{\alpha_1}.$$

Let $N_2^x = \frac{k(\alpha_1 - c_1 M_1^y)}{\alpha_1}$, since $\varepsilon > 0$ is sufficiently small, thereby, $U_1 \leq N_2^x$. Thus, there exists a $T_{33} \geq T_{31} + \tau$ such that $x(t) \leq N_2^x + \varepsilon$ for $t > T_{33}$.

We obtain from the third equation of system (2), for $t > \max\{T_{32}, T_{33}\}$,

$$\dot{y}_2(t) \leq \alpha_2 e^{-d_{11}\tau_1}\big(N_2^x + \varepsilon\big)y_2(t-\tau_1) - d_{12}y_2(t) - \beta_1 y_2^2(t) - c_2\big(M_1^z - \varepsilon\big)y_2(t).$$

By applying Lemma 2(i) and comparison, we obtain that

$$U_2 = \limsup_{t\to+\infty} y_2(t) \leq \frac{\alpha_2 e^{-d_{11}\tau_1}(N_2^x + \varepsilon) - d_{12} - c_2(M_1^z - \varepsilon)}{\beta_1}.$$

Let $N_2^y = \frac{\alpha_2 e^{-d_{11}\tau_1} N_2^x - d_{12} - c_2 M_1^z}{\beta_1}$, since $\varepsilon > 0$ is sufficiently small, so one has $U_2 \le N_2^y$ holds. Therefore, there exists a $T_{41} \ge \max\{T_{32}, T_{33}\}$ such that $y_2(t) \le N_2^y + \varepsilon$ for $t > T_{41}$.

From the fifth equation of system (2), for $t > T_{41}$,

$$\dot{z}_2(t) \le \alpha_3 e^{-d_{21}\tau_2}\left(N_2^y + \varepsilon\right) z_2(t - \tau_2) - d_{22} z_2(t) - \beta_2 z_2^2(t).$$

Similarly, we get

$$U_3 = \limsup_{t \to +\infty} z_2(t) \le \frac{\alpha_3 e^{-d_{21}\tau_2}(N_2^y + \varepsilon) - d_{22}}{\beta_2}.$$

Let $N_2^z = \frac{\alpha_3 e^{-d_{21}\tau_2} N_2^y - d_{22}}{\beta_2}$, since $\varepsilon > 0$ is sufficiently small, then we find that $U_3 \le N_2^z$ holds. Consequently, there exists a $T_{42} \ge T_{41}$ such that $z_2(t) \le N_2^z + \varepsilon$ for $t > T_{42}$.

We obtain from the first equation of system (2)

$$\dot{x}(t) \ge x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\left(N_2^y + \varepsilon\right)\right] \quad \text{for } t > T_{41} + \tau.$$

Using a comparison argument,

$$V_1 = \liminf_{t \to +\infty} x(t) \ge \frac{k[\alpha_1 - c_1(N_2^y + \varepsilon)]}{\alpha_1}.$$

Let $M_2^x = \frac{k(\alpha_1 - c_1 N_2^y)}{\alpha_1}$, since $\varepsilon > 0$ is sufficiently small, then obviously $V_1 \ge M_2^x$ holds. Hence, there is a $T_{43} \ge T_{41} + \tau$ such that $x(t) \ge M_2^x - \varepsilon$ for $t > T_{43}$.

We obtain from the third equation of system (2), for $t > \max\{T_{42}, T_{43}\}$,

$$\dot{y}_2(t) \ge \alpha_2 e^{-d_{11}\tau_1}\left(M_2^x - \varepsilon\right) y_2(t - \tau_1) - d_{12} y_2(t) - \beta_1 y_2^2(t) - c_2\left(N_2^z + \varepsilon\right) y_2(t).$$

By applying Lemma 2(i), one can get

$$V_2 = \liminf_{t \to +\infty} y_2(t) \ge \frac{\alpha_2 e^{-d_{11}\tau_1}(M_2^x - \varepsilon) - d_{12} - c_2(N_2^z + \varepsilon)}{\beta_1}.$$

Let $M_2^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_2^x - d_{12} - c_2 N_2^z}{\beta_1}$, since $\varepsilon > 0$ is sufficiently small, then one has $V_2 \ge M_2^y$ holds. Thus, there exists a $T_{51} \ge \max\{T_{42}, T_{43}\} + \tau$ such that $y_2(t) \ge M_2^y - \varepsilon$ for $t > T_{51}$.

From the fifth equation of system (2), for $t > T_{51}$,

$$\dot{z}_2(t) \ge \alpha_3 e^{-d_{21}\tau_2}\left(M_2^y - \varepsilon\right) z_2(t - \tau_2) - d_{22} z_2(t) - \beta_2 z_2^2(t).$$

By applying Lemma 2(i) and the standard comparison theorem, one obtains

$$V_3 = \liminf_{t \to +\infty} z_2(t) \ge \frac{\alpha_3 e^{-d_{21}\tau_2}(M_2^y - \varepsilon) - d_{22}}{\beta_2}.$$

Let $M_2^z = \frac{\alpha_3 e^{-d_{21}\tau_2} M_2^y - d_{22}}{\beta_2}$, since $\varepsilon > 0$ is sufficiently small, thereby, $V_3 \ge M_2^z$ holds. Therefore, there exists a $T_{52} \ge T_{51}$ such that $z_2(t) \ge M_2^z - \varepsilon$ for $t > T_{52}$.

So far, we have completed the first step of the iterative scheme. Repeating the above argument and using mathematical induction, we obtain six sequences N_n^x, N_n^y, N_n^z, M_n^x, M_n^y, M_n^z, $n = 1, 2, \ldots$, such that, for $n \geq 2$,

$$\begin{aligned} N_n^x &= \frac{k(\alpha_1 - c_1 M_{n-1}^y)}{\alpha_1}, \qquad N_n^y = \frac{\alpha_2 e^{-d_{11}\tau_1} N_n^x - d_{12} - c_2 M_{n-1}^z}{\beta_1}, \\ M_n^x &= \frac{k(\alpha_1 - c_1 N_n^y)}{\alpha_1}, \qquad M_n^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_n^x - d_{12} - c_2 N_n^z}{\beta_1}, \\ N_n^z &= \frac{\alpha_3 e^{-d_{21}\tau_2} N_n^y - d_{22}}{\beta_2}, \qquad M_n^z = \frac{\alpha_3 e^{-d_{21}\tau_2} M_n^y - d_{22}}{\beta_2}. \end{aligned} \tag{17}$$

By analyzing, one can get

$$M_n^x \leq V_1 \leq U_1 \leq N_n^x, \qquad M_n^y \leq V_2 \leq U_2 \leq N_n^y, \qquad M_n^z \leq V_3 \leq U_3 \leq N_n^z. \tag{18}$$

From (17), we obtain that

$$N_{n+1}^y = \beta_1^{-1}\left(1 - \beta_1^{-1}\Phi\right)(\Phi + \beta_1) y_2^* + \left(\beta_1^{-1}\Phi\right)^2 N_n^y, \tag{19}$$

where

$$\Phi = \alpha_1^{-1}\alpha_2 c_1 k e^{-d_{11}\tau_1} + \alpha_3 \beta_2^{-1} c_2 e^{-d_{21}\tau_2}.$$

As $N_n^y \geq y_2^*$ and $\alpha_1\beta_1\beta_2 > \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \alpha_2\beta_2 c_1 k e^{-d_{11}\tau_1}$, one can obtain from (19) that

$$\begin{aligned} N_{n+1}^y - N_n^y &= \beta_1^{-1}\left(1 - \beta_1^{-1}\Phi\right)(\Phi + \beta_1) y_2^* + \left[\left(\beta_1^{-1}\Phi\right)^2 - 1\right] N_n^y \\ &\leq \left(1 - \beta_1^{-1}\Phi\right)\left(1 + \beta_1^{-1}\Phi\right) y_2^* + \left[\left(\beta_1^{-1}\Phi\right)^2 - 1\right] y_2^* \\ &= 0. \end{aligned}$$

Consequently, the sequence N_n^y is monotonically decreasing and

$$\lim_{n\to+\infty} N_n^y = \frac{\alpha_1(\alpha_2\beta_2 k e^{-d_{11}\tau_1} - \beta_2 d_{12} + c_2 d_{22})}{\alpha_1\beta_1\beta_2 + \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \alpha_2\beta_2 c_1 k e^{-d_{11}\tau_1}} = y_2^*. \tag{20}$$

Therefore, from (17) and (20) we see that the sequences N_n^x and N_n^z are decreasing and the sequences M_n^x, M_n^y and M_n^z are increasing, furthermore,

$$\begin{aligned} &\lim_{n\to+\infty} N_n^x = x^*, \qquad \lim_{n\to+\infty} N_n^z = z_2^*, \\ &\lim_{n\to+\infty} M_n^x = x^*, \qquad \lim_{n\to+\infty} M_n^y = y_2^*, \qquad \lim_{n\to+\infty} M_n^z = z_2^*. \end{aligned} \tag{21}$$

Hence,

$$\lim_{t\to+\infty} x(t) = x^*, \qquad \lim_{t\to+\infty} y_2(t) = y_2^*, \qquad \lim_{t\to+\infty} z_2(t) = z_2^*.$$

We obtain from (6)

$$y_1(t) = \frac{\alpha_2 \int_{t-\tau_1}^{t} e^{d_{11}s} x(s) y_2(s)\, ds}{e^{d_{11}t}}. \tag{22}$$

According to L'Hospital's rule, one can get

$$\begin{aligned}\lim_{t\to+\infty} y_1(t) &= \lim_{t\to+\infty} \frac{\alpha_2[e^{d_{11}t}x(t)y_2(t) - e^{d_{11}(t-\tau_1)}x(t-\tau_1)y_2(t-\tau_1)]}{d_{11}e^{d_{11}t}} \\ &= \frac{\alpha_2}{d_{11}} \lim_{t\to+\infty}\bigl[x(t)y_2(t) - e^{-d_{11}\tau_1}x(t-\tau_1)y_2(t-\tau_1)\bigr] \\ &= \frac{\alpha_2}{d_{11}}\bigl(1-e^{-d_{11}\tau_1}\bigr)x^*y_2^* = y_1^*. \end{aligned} \tag{23}$$

We obtain from (7) that

$$z_1(t) = \frac{\alpha_3 \int_{t-\tau_2}^t e^{d_{21}s}z_2(s)y_2(s)\,ds}{e^{d_{21}t}}. \tag{24}$$

According to L'Hospital's rule, one has

$$\begin{aligned}\lim_{t\to+\infty} z_1(t) &= \lim_{t\to+\infty} \frac{\alpha_3[e^{d_{21}t}y_2(t)z_2(t) - e^{d_{21}(t-\tau_2)}y_2(t-\tau_2)z_2(t-\tau_2)]}{d_{21}e^{d_{21}t}} \\ &= \frac{\alpha_3}{d_{21}} \lim_{t\to+\infty}\bigl[y_2(t)z_2(t) - e^{-d_{21}\tau_2}y_2(t-\tau_1)z_2(t-\tau_1)\bigr] \\ &= \frac{\alpha_3}{d_{21}}\bigl(1-e^{-d_{21}\tau_2}\bigr)y_2^*z_2^* = z_1^*. \end{aligned} \tag{25}$$

This completes the proof. □

In the next, we will discuss the global stability of the boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$ of system (2) when

$$\alpha_1\alpha_3 e^{-d_{21}\tau_2}\bigl(\alpha_2 k e^{-d_{11}\tau_1} - d_{12}\bigr) - d_{22}\bigl(\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1}\bigr) < 0.$$

Theorem 8 *The delays have great impacts on the dynamics for system* (2). *More precisely, let* $m_1 = \frac{1}{d_{11}}\ln\frac{\alpha_2 c_1 k}{\alpha_1\beta_1}$, $m_2 = \frac{1}{d_{11}}\ln\frac{\alpha_2 k}{d_{12}}$ *and* $m_4 = \max\{\frac{1}{d_{21}}\ln\frac{\alpha_3(\alpha_1\beta_1 - c_1 d_{12})}{\beta_1 c_1 d_{22}}, \frac{1}{d_{21}}\ln\frac{\alpha_3 y_2^0}{d_{22}}\}$, *if* $\tau_1 \in (m_1, m_2)$ *and* $\tau_2 \in (m_4, +\infty)$, *then the boundary equilibrium* E_2 *of system* (2) *is AS.*

Proof By $\tau_1 \in (m_1, m_2)$, one finds that (C1) and $\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1} > 0$ hold. Thus, the boundary equilibrium E_2 exists. At the same time, by $\tau_2 \in (m_4, +\infty)$, it is obvious that $\alpha_3 e^{-d_{21}\tau_2}(\alpha_2 k e^{-d_{11}\tau_1} - d_{12}) - \beta_1 d_{22} < 0$ and $\alpha_3 y_2^0 e^{-d_{21}\tau_2} < d_{22}$.

Using Theorem 5, we have found that the boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$ is LAS. Therefore, it is sufficient to show that $\lim_{t\to+\infty}(x(t), y_1(t), y_2(t), z_1(t), z_2(t)) = (x^0, y_1^0, y_2^0, 0, 0)$.

Since $\alpha_2 e^{-d_{11}\tau_1}(k+\varepsilon) > d_{12}$, the same arguments as those in the proof of Theorem 7 show that (15), (16) hold, i.e.,

$$\begin{aligned} U_1 &= \limsup_{t\to+\infty} x(t) \le k \stackrel{\text{def}}{=} N_1^x, \\ U_2 &= \limsup_{t\to+\infty} y_2(t) \le \frac{\alpha_2 e^{-d_{11}\tau_1}N_1^x - d_{12}}{\beta_1} \stackrel{\text{def}}{=} N_1^y. \end{aligned}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_{12} \ge T_{11} + \tau$ such that $y_2(t) \le N_1^y + \varepsilon$ for $t > T_{12}$.

We obtain from the fifth equation of system (2), for $t > T_{12} + \tau$,

$$\dot{z}_2(t) \le \alpha_3 e^{-d_{21}\tau_2}\left(N_1^y + \varepsilon\right) z_2(t-\tau_2) - d_{22} z_2(t) - \beta_2 z_2^2(t).$$

By applying Lemma 2(ii) and the standard comparison theorem, one has

$$\lim_{t\to+\infty} z_2(t) = 0.$$

Thus, for $\varepsilon > 0$ sufficiently small, there exists a $T_{21} \ge T_{12} + \tau$ such that $0 < z_2(t) < \varepsilon$ for $t > T_{21}$.

We obtain from the first equation of system (2), for $t > T_{12} + \tau$,

$$\dot{x}(t) \ge x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\left(N_1^y + \varepsilon\right)\right].$$

By the comparison theorem,

$$V_1 = \liminf_{t\to+\infty} x(t) \ge \frac{k[\alpha_1 - c_1(N_1^y + \varepsilon)]}{\alpha_1}.$$

Let $M_1^x = \frac{k(\alpha_1 - c_1 N_1^y)}{\alpha_1}$, since $\varepsilon > 0$ is sufficiently small, obviously, $V_1 \ge M_1^x$ holds. Therefore, there exists a $T_{22} \ge T_{12} + \tau$ such that $x(t) \ge M_1^x - \varepsilon$ for $t > T_{22}$.

We obtain from the third equation of system (2), for $t > \max\{T_{21}, T_{22}\}$,

$$\dot{y}_2(t) \ge \alpha_2 e^{-d_{11}\tau_1}\left(M_1^x - \varepsilon\right) y_2(t-\tau_1) - d_{12} y_2(t) - \beta_1 y_2^2(t) - c_2 \varepsilon y_2(t).$$

By applying Lemma 2(i) and the standard comparison theorem, one has

$$V_2 = \liminf_{t\to+\infty} y_2(t) \ge \frac{\alpha_2 e^{-d_{11}\tau_1}(M_1^x - \varepsilon) - d_{12} - c_2\varepsilon}{\beta_1}.$$

Let $M_1^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_1^x - d_{12}}{\beta_1}$, since $\varepsilon > 0$ sufficiently small, so we get $V_2 \ge M_1^y$. Consequently, there exists a $T_{31} \ge \max\{T_{21}, T_{22}\}$ such that $y_2(t) \ge M_1^y - \varepsilon$ for $t > T_{31}$.

Similar to the above discussion, we obtain from the first equation of system (2), for $t > T_{31} + \tau$,

$$\dot{x}(t) \le x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\left(M_1^y - \varepsilon\right)\right].$$

By the comparison theorem,

$$U_1 = \limsup_{t\to+\infty} x(t) \le \frac{k[\alpha_1 - c_1(M_1^y - \varepsilon)]}{\alpha_1}.$$

Let $N_2^x = \frac{k(\alpha_1 - c_1 M_1^y)}{\alpha_1}$, for $\varepsilon > 0$ sufficiently small, one has $U_1 \ge N_2^x$. Hence, there exists a $T_{32} \ge T_{31} + \tau$ such that $x(t) \le N_2^x + \varepsilon$ for $t > T_{33}$.

We obtain from the third equation of system (2), for $t > \max\{T_{32}, T_{21}\}$,

$$\dot{y}_2(t) \le \alpha_2 e^{-d_{11}\tau_1}\left(N_2^x + \varepsilon\right) y_2(t-\tau_1) - d_{12} y_2(t) - \beta_1 y_2^2(t).$$

By applying Lemma 2(i) and comparison, one can get

$$U_2 = \limsup_{t\to+\infty} y_2(t) \le \frac{\alpha_2 e^{-d_{11}\tau_1}(N_2^x + \varepsilon) - d_{12}}{\beta_1}.$$

Let $N_2^y = \frac{\alpha_2 e^{-d_{11}\tau_1} N_2^x - d_{12}}{\beta_1}$, since $\varepsilon > 0$ is sufficiently small, thereby, $U_2 \le N_2^y$. Accordingly, there exists a $T_{41} \ge \max\{T_{32}, T_{21}\}$ such that $y_2(t) \le N_2^y + \varepsilon$ for $t > T_{41}$.

From the first equation of system (2), for $t > T_{41} + \tau$,

$$\dot{x}(t) \ge x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\left(N_2^y + \varepsilon\right)\right].$$

By the comparison theorem,

$$V_1 = \liminf_{t\to+\infty} x(t) \ge \frac{k[\alpha_1 - c_1(N_2^y + \varepsilon)]}{\alpha_1}.$$

Let $M_2^x = \frac{k(\alpha_1 - c_1 N_2^y)}{\alpha_1}$, since $\varepsilon > 0$ is sufficiently small, then obviously $V_1 \ge M_2^x$. Therefore, there exists a $T_{42} \ge T_{41} + \tau$ such that $x(t) \ge M_2^x - \varepsilon$ for $t > T_{42}$.

We obtain from the third equation of system (2), for $t > \max\{T_{42}, T_{21}\}$,

$$\dot{y}_2(t) \ge \alpha_2 e^{-d_{11}\tau_1}\left(M_2^x - \varepsilon\right) y_2(t - \tau_1) - d_{12} y_2(t) - \beta_1 y_2^2(t) - c_2 \varepsilon y_2(t).$$

By applying Lemma 2(i), one has

$$V_2 = \liminf_{t\to+\infty} y_2(t) \ge \frac{\alpha_2 e^{-d_{11}\tau_1}(M_2^x - \varepsilon) - d_{12} - c_2\varepsilon}{\beta_1}.$$

Let $M_2^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_2^x - d_{12}}{\beta_1}$, for $\varepsilon > 0$ sufficiently small, so we can get $V_2 \ge M_2^y$. Consequently, there exists a $T_{51} \ge \max\{T_{42}, T_{21}\} + \tau$ such that $y_2(t) \ge M_2^y - \varepsilon$ for $t > T_{51}$.

So far, we have completed the first step of the iterative scheme. Repeating the above argument and using mathematical induction, we obtain four sequences N_n^x, N_n^y, M_n^x, M_n^y, $n = 1, 2, \ldots$, such that, for $n \ge 2$,

$$\begin{aligned} N_n^x &= \frac{k(\alpha_1 - c_1 M_{n-1}^y)}{\alpha_1}, \qquad N_n^y = \frac{\alpha_2 e^{-d_{11}\tau_1} N_n^x - d_{12}}{\beta_1}, \\ M_n^x &= \frac{k(\alpha_1 - c_1 N_n^y)}{\alpha_1}, \qquad M_n^y = \frac{\alpha_2 e^{-d_{11}\tau_1} M_n^x - d_{12}}{\beta_1}. \end{aligned} \tag{26}$$

By analyzing, we can get

$$M_n^x \le V_1 \le U_1 \le N_n^x, \qquad M_n^y \le V_2 \le U_2 \le N_n^y. \tag{27}$$

From (26), one has

$$N_{n+1}^y = \frac{(\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1})(\alpha_2 k e^{-d_{11}\tau_1} - d_{12})}{\alpha_1 \beta_1^2} + \left(\frac{\alpha_2 c_1 k e^{-d_{11}\tau_1}}{\alpha_1\beta_1}\right)^2 N_n^y. \tag{28}$$

As $N_n^y \geq y_2^0$, we can obtain from (28)

$$\begin{aligned} N_{n+1}^y - N_n^y &= \frac{(\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1})(\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1})}{(\alpha_1\beta_1^2)^2} y_2^0 \\ &\quad + \left[\left(\frac{\alpha_2 c_1 k e^{-d_{11}\tau_1}}{\alpha_1\beta_1}\right)^2 - 1\right] N_n^y \\ &\leq \frac{(\alpha_1\beta_1 - \alpha_2 c_1 k e^{-d_{11}\tau_1})(\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1})}{(\alpha_1\beta_1^2)^2} y_2^0 \\ &\quad + \frac{(\alpha_2 c_1 k e^{-d_{11}\tau_1} - \alpha_1\beta_1)(\alpha_2 c_1 k e^{-d_{11}\tau_1} + \alpha_1\beta_1)}{(\alpha_1\beta_1^2)^2} y_2^0 \\ &= 0. \end{aligned}$$

Therefore, the sequence N_n^y is monotonically decreasing and

$$\lim_{n\to+\infty} N_n^y = \frac{\alpha_1(\alpha_2 k e^{-d_{11}\tau_1} - d_{12})}{\alpha_1\beta_1 + \alpha_2 c_1 k e^{-d_{11}\tau_1}} = y_2^0. \tag{29}$$

Then from (26) and (29) we see that the sequence N_n^x is decreasing and the sequences M_n^x and M_n^y are increasing, furthermore,

$$\lim_{n\to+\infty} N_n^x = x^0, \qquad \lim_{n\to+\infty} M_n^x = x^0, \qquad \lim_{n\to+\infty} M_n^y = y_2^0. \tag{30}$$

Hence, we obtain

$$\lim_{t\to+\infty} x(t) = x^0, \qquad \lim_{t\to+\infty} y_2(t) = y_2^0, \qquad \lim_{t\to+\infty} z_2(t) = 0.$$

Similar to the proof of (22)–(25), by a direct computation, we obtain

$$\begin{aligned} &\lim_{t\to+\infty} y_1(t) = \frac{\alpha_2}{d_{11}}\left(1 - e^{-d_{11}\tau_1}\right) x^0 y_2^0 = y_1^0, \\ &\lim_{t\to+\infty} z_1(t) = 0. \end{aligned}$$

This completes the proof. □

In the next, we shall study the global stability of the axial equilibrium $E_1(k,0,0,0,0)$ of system (2) when $k\alpha_2 e^{-d_{11}\tau_1} < d_{12}$.

Theorem 9 *The delay due to the maturity of the predator has great impacts on the dynamics for system* (2). *More precisely, if* $\tau_1 \in (m_2,+\infty)$, *then the axial equilibrium* E_1 *of system* (2) *is AS. In this case, all predators will go to extinction.*

Proof By $\tau_1 \in (m_2,+\infty)$, one finds that $\alpha_2 k e^{-d_{11}\tau_1} < d_{12}$ holds. Using Theorem 4, we find that the axial equilibrium $E_1(k,0,0,0,0)$ is LAS. Hence, it suffices to prove that $\lim_{t\to+\infty}(x(t),y_1(t),y_2(t),z_1(t),z_2(t)) = (k,0,0,0,0)$.

The same arguments as those in the proof of Theorem 7 show that (15) holds, i.e.

$$\limsup_{t\to+\infty} x(t) \leq k. \tag{31}$$

Hence, for $\varepsilon > 0$ sufficiently small, satisfying $\alpha_2 e^{-d_{11}\tau_1}(k+\varepsilon) < d_{12}$, there is a $T_1 > 0$ such that $x(t) \leq k + \varepsilon$ for $t > T_1$.

We obtain from the third equation of system (2), for $t > T_1 + \tau$,

$$\dot{y}_2(t) \leq \alpha_2 e^{-d_{11}\tau_1}(k+\varepsilon)y_2(t-\tau_1) - d_{12}y_2(t) - \beta_1 y_2^2(t).$$

By applying Lemma 2(ii) and comparison, one can get

$$\lim_{t\to+\infty} y_2(t) = 0.$$

Consequently, for any $\varepsilon > 0$ sufficiently small, there exists a $T_2 > T_1 + \tau$ such that $0 < y_2(t) < \varepsilon$ for $t > T_2$.

From the first equation of system (2), for $t > T_2$,

$$\dot{x}(t) \geq x(t)\left[\alpha_1\left(1 - \frac{x(t)}{k}\right) - c_1\varepsilon\right].$$

Using the comparison theorem,

$$\liminf_{t\to+\infty} x(t) \geq \frac{k(\alpha_1 - c_1\varepsilon)}{\alpha_1}.$$

This inequality holds for $\varepsilon > 0$ sufficiently small, one has

$$\liminf_{t\to+\infty} x(t) \geq k. \tag{32}$$

By (31) and (32), we obtain

$$\lim_{t\to+\infty} x(t) = k.$$

We obtain from the first equation of system (2), for $t > T_2$,

$$\dot{z}_2(t) \leq \alpha_3 e^{-d_{21}\tau_2}\varepsilon z_2(t-\tau_2) - d_{22}z_2(t) - \beta_2 z_2^2(t).$$

By applying Lemma 2(ii) and comparison, one can get

$$\lim_{t\to+\infty} z_2(t) = 0.$$

Similar to the proof of (22)–(25), we obtain $\lim_{t\to+\infty} y_1(t) = 0$, $\lim_{t\to+\infty} z_1(t) = 0$.

The proof is complete. □

Remark 4 It is obvious that $\alpha_1\beta_1\beta_2 > \alpha_1\alpha_3 c_2 e^{-d_{21}\tau_2} + \alpha_2\beta_2 c_1 k e^{-d_{11}\tau_1}$ implies $\alpha_1\beta_1 > \alpha_2 c_1 k e^{-d_{11}\tau_1}$. And then, by calculating, the condition (C4) can reduce to $\tau_2 < m_4$. Therefore, by Theorem 7, if the interior equilibrium $E^*(x^*, y_1^*, y_2^*, z_1^*, z_2^*)$ of system (2) is GAS, then the τ_2 must satisfy $\tau_2 < m_4$.

Remark 5 From Theorem 8, when $\tau_1 \in (m_1, m_2)$ and $\tau_2 \in (m_4, +\infty)$, then the boundary equilibrium $E_2(x^0, y_1^0, y_2^0, 0, 0)$ of system (2) is AS, i.e., the prey species and the predator species will coexist, the top-predator species will go extinct. Comparing with Remark 4, one can find that longer delay τ_2 will lead the top-predator species to extinction.

Remark 6 According to Theorem 9, when $\tau_1 \in (m_2, +\infty)$, then the axial equilibrium $E_1(k, 0, 0, 0, 0)$ of system (2) is AS, i.e., all predators will go extinct. Comparing with the Remark 5, it is obvious that longer delay τ_1 will lead the predators to extinction.

6 Numerical simulation

In this section, one example is presented to demonstrate the correctness and effectiveness of the obtained results.

Example 1 Consider the following system with two different time delays:

$$\begin{cases} \dot{x}(t) = x(t)[15(1 - \frac{x(t)}{4}) - 5y_2(t)], \\ \dot{y}_1(t) = 6x(t)y_2(t) - \ln 2y_1(t) - 6e^{-\ln 2\tau_1}x(t-\tau_1)y_2(t-\tau_1), \\ \dot{y}_2(t) = 6e^{-\ln 2\tau_1}x(t-\tau_1)y_2(t-\tau_1) - y_2(t) - 5y_2^2(t) - \frac{1}{4}y_2(t)z_2(t), \\ \dot{z}_1(t) = 6y_2(t)z_2(t) - \ln 2z_1(t) - 6e^{-\ln 2\tau_2}y_2(t-\tau_2)z_2(t-\tau_2), \\ \dot{z}_2(t) = 6e^{-\ln 2\tau_2}y_2(t-\tau_2)z_2(t-\tau_2) - z_2(t) - 2z_2^2(t), \end{cases} \tag{33}$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are constant time delay.

Case I. Let $\tau_1 = 2$ and $\tau_2 = 1$, then

$$\begin{aligned} &(\mathrm{C2}) \quad \alpha_1\alpha_3c_2e^{-d_{21}\tau_2} + \beta_2c_1d_{12} - c_1c_2d_{22} = 20 > 0; \\ &(\mathrm{C3}) \quad \alpha_2\beta_2ke^{-d_{11}\tau_1} - \beta_2d_{12} + c_2d_{22} = 10.25 > 0; \\ &(\mathrm{C4}) \quad \alpha_1\alpha_3e^{-d_{21}\tau_2}\big(\alpha_2ke^{-d_{11}\tau_1} - d_{12}\big) - d_{22}\big(\alpha_1\beta_1 + \alpha_2c_1ke^{-d_{11}\tau_1}\big) = 120 > 0; \\ &\qquad\quad 2\alpha_1 - \alpha_2ke^{-d_{11}\tau_1} = 24 > 0; \\ &\qquad\quad 2\beta_1 > \alpha_2e^{-d_{11}\tau_1} + \alpha_3e^{-d_{21}\tau_2} = 5.5 > 0; \\ &\qquad\quad 2\beta_2 > \alpha_3e^{-d_{21}\tau_2} = 1 > 0, \quad \text{and} \\ &\qquad\quad \alpha_1\beta_1\beta_2 - \alpha_1\alpha_3c_2e^{-d_{21}\tau_2} - \alpha_2\beta_2c_1ke^{-d_{11}\tau_1} = 78.75 > 0. \end{aligned}$$

Thus, the conditions of Theorem 7 hold and the interior equilibrium $E^*(\frac{526}{177}, \frac{291{,}141}{31{,}329\ln 2}, \frac{123}{177}, \frac{11{,}808}{10{,}443\ln 2}, \frac{32}{59})$ of system (33) is AS. The numerical simulation is shown in Fig. 1.

Case II. Let $\tau_1 = 2$ and $\tau_2 = 10$, then

$$\begin{aligned} &(\mathrm{C1}) \quad \alpha_2ke^{-d_{11}\tau_1} - d_{12} = 5 > 0; \\ &\qquad\quad \alpha_3y_2^0e^{-d_{21}\tau_2} - d_{22} = \frac{30}{7} \times \frac{1}{2^{10}} - 1 < 0; \\ &\qquad\quad \alpha_3e^{-d_{21}\tau_2}\big(\alpha_2ke^{-d_{11}\tau_1} - d_{12}\big) - \beta_1d_{22} = 5\left(\frac{6}{2^{10}} - 1\right) < 0, \quad \text{and} \\ &\qquad\quad \alpha_1\beta_1 - \alpha_2c_1ke^{-d_{11}\tau_1} = 45 > 0. \end{aligned}$$

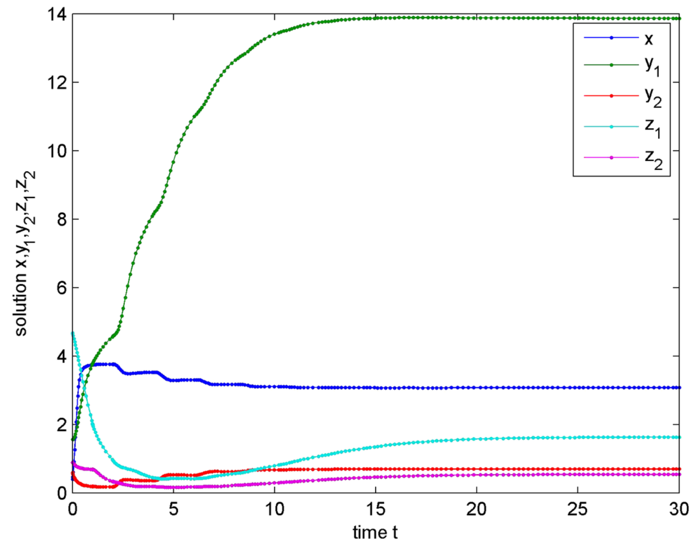

Figure 1 The temporal solution found by numerical integration of system (33) with $\tau_1 = 2$, $\tau_2 = 1$ and $(\phi(\theta), \varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (0.4, \frac{27}{25\ln 2}, 0.6, \frac{81}{50\ln 2}, 0.9)$

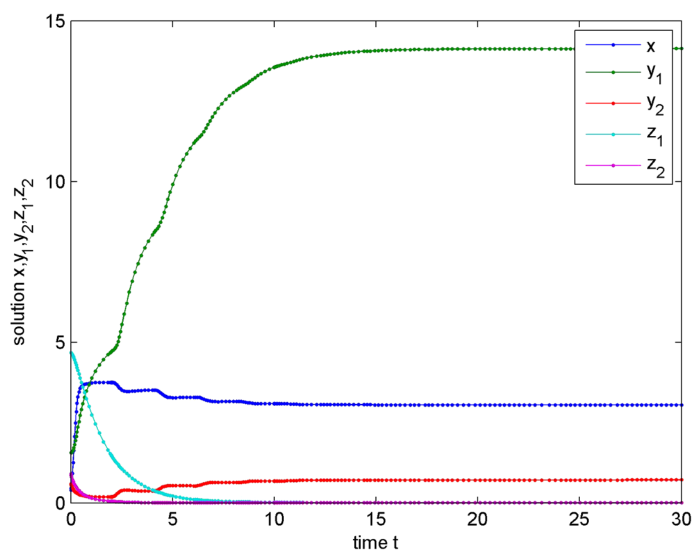

Figure 2 The temporal solution found by numerical integration of system (33) with $\tau_1 = 2$, $\tau_2 = 10$ and $(\phi(\theta), \varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (0.4, \frac{27}{25\ln 2}, 0.6, \frac{81}{25\ln 2} \times (1 - \frac{1}{2^{10}}), 0.9)$

According to Theorem 8, we can show that the boundary equilibrium $E_2(\frac{64}{21}, \frac{480}{49\ln 2}, \frac{5}{7}, 0, 0)$ of system (33) is AS. The numerical simulation illustrates our result (see Fig. 2).

Case III. Let $\tau_1 = 15$ and $\tau_2 = 1$, then

$$\alpha_2 k e^{-d_{11}\tau_1} - d_{12} = \frac{3}{2^{12}} - 1 < 0.$$

Therefore, the condition of Theorem 9 holds and the axial equilibrium $E_1(4, 0, 0, 0, 0)$ of system (33) is AS. The numerical simulations also confirm this phenomenon (see Fig. 3).

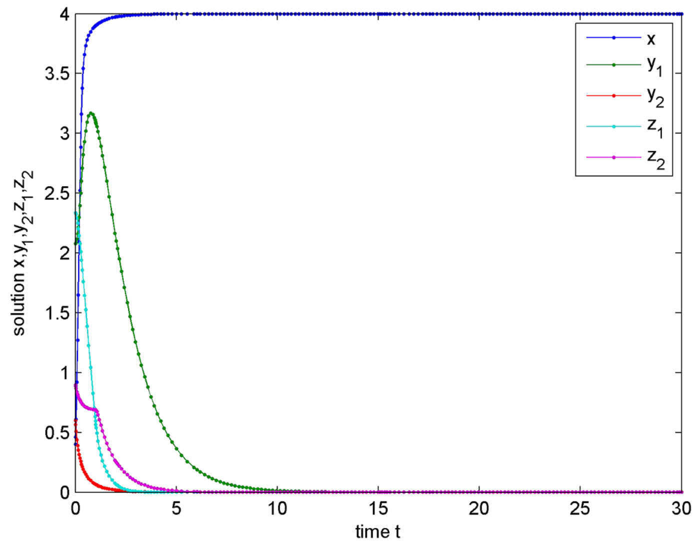

Figure 3 The temporal solution found by numerical integration of system (33) with $\tau_1 = 15$, $\tau_2 = 1$ and $(\phi(\theta), \varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta)) = (0.4, \frac{36}{25\ln 2} \times (1 - \frac{1}{2^{15}}), 0.6, \frac{81}{50\ln 2}, 0.9)$

7 Discussion

In this paper, by taking full consideration of maturity and stage structure of the predators, a new delayed three-species food-chain model with stage structure for predators is proposed and investigated. The positivity and boundedness of solutions of the model have been verified. By analyzing system (2), the existence and stability of four nonnegative equilibria of system are proved. And (C1) determines the existence of the boundary equilibrium E_2; (C2)–(C4) determine the existence of the boundary equilibrium E^*; the trivial equilibrium E_0 and the axial equilibrium E_1 exist irrespective of any parameters.

Some interesting findings show that the delays have great impacts on dynamical behaviors for the system: if the delay τ_2 is too large, that will account for the top-predator species going to extinction; if the delay τ_1 is too large, that will account for the predators to extinction. More precisely, according to Theorems 8 and 9, if $\tau_1 \in (m_1, m_2)$ and $\tau_2 \in (m_4, +\infty)$, then the prey species and the predator species will coexist, the top-predator species will go extinct; if $\tau_1 \in (m_2, +\infty)$, then all the predators will go extinct.

The obtained results in this paper may provide some new insights for predicting the dynamical behaviors of the food-chain system and protecting the ecological balance in a real ecosystem. By the way, we consider an autonomous system and the coefficient parameters of our model are restricted to constant. However, it would be very challenging whether one can derive sufficient conditions for the dynamical behaviors of the three-species food-chain model with time-varying coefficients. This will be our future study.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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