Fang and Song *Advances in Difference Equations* (2018) 2018:139
https://doi.org/10.1186/s13662-018-1580-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH
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Existence results for fractional order impulsive functional differential equations with multiple delays

Huiping Fang[1] and Mingzhu Song[2*]

*Correspondence: wyfwywyf@126.com
[2]College of Mathematics and Computer Science, Tongling University, Tongling, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the solution of impulsive fractional differential equations with multiple delays by using the nonlinear alternative of Leray–Schauder and the Banach fixed point method. Also, we prove that the equations have at least one solution or unique solution with certain conditions. In the last part, we give two examples to illustrate the usefulness of the main results.

MSC: 34A34

Keywords: Fractional order functional differential equations; Impulsive; Fixed point theorem

1 Introduction

With the development of fractional calculus and the requirement for field applications of physics, mathematics, and chemical engineering, fractional differential equations have attracted great interest in recent years (see, for example, [1–8]).

Based on the nonlinear alternative of a Leray–Schauder model, we have considered and cited previous studies on the existence of solutions of fractional differential equations for investigating the existence and uniqueness of fractional functional equations. Zhou et al. have already used Krasnoselskii's fixed point theory to study the existence and uniqueness of fractional function equations [9, 10]. Some researchers have studied fractional order impulsive differential equations and discussed the existence solution of nonlinear functional differential equations with multiple delays [11, 12]. Many authors have investigated the existence solutions of fractional function equations with impulse [13–18]. However, there are few studies on the existence of fractional order impulsive functional differential equations multi-delays. The recent development of the theory of fractional differential equations has already affected the present research.

In this paper, we should study the solutions of impulsive fractional order functional differential equations with multiple delays as follows:

$$\begin{cases} D^{\alpha}x(t) = f(t,x_t) + \sum_{i=1}^{p} x(t-r_i), & t \in J = [0,b], t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), & k = 1,2,\ldots,m, \\ x(t) = \phi(t), & t \in [-\tau,0], \end{cases} \tag{1.1}$$

where $p \in \{1,2,\dots\}$, $\tau = \max_{1\leq i\leq p}\{r_i\}$, $f : J \times \Omega \to X$ is a given function, where Ω is a phase space defined in preliminaries. $0 = t_0 < t_1 < \cdots t_m < t_{m+1} = b$, $I_k \in C(X,X)$ $(k = 1,2,\dots,m)$ are bounded functions. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the left and right limits of $x(t)$ at $t = t_k$, respectively. We assume that the histories $x_t : [-\tau,0] \to X$, $x_t(s) = x(t+s)$, $s \in [-\tau,0]$, belong to an abstract phase space Ω.

In this paper, we prove the existence and uniqueness of the solution. The idea of the paper is as follows. In the second part, we give preliminary facts and definition. In the third part, we prove the existence of solutions. In the fourth part, some examples are given to illustrate our main results.

2 Preliminaries

In this section, we give some basic definitions, notations, and results which are used throughout this paper.

Let $C(J,X)$ be the Banach space of continuous functions x from J into X with the norm $\|x\|_\infty = \sup\{x(t) : t \in J\}$, and we introduce the spaces:

$$\Omega = \bigl\{\psi : [-\tau,0] \to X \text{ such that } \psi(t) \text{ is measurable and bounded}\bigr\},$$

and define $\|\psi\|_\Omega$ by

$$\|\psi\|_\Omega = \sup_{s\in[-\tau,0]} \bigl|\psi(s)\bigr|, \quad \forall \psi \in \Omega.$$

We consider the space

$$\begin{aligned} PC = \bigl\{&x : [0,b] \to X \text{ such that } x_k \in C\bigl((t_k,t_{k+1}],X\bigr), \text{there exist } x\bigl(t_k^+\bigr) \text{ and } x\bigl(t_k^-\bigr) \\ &\text{with } x(t_k) = x\bigl(t_k^-\bigr), x_0 = \phi \in \Omega, k = 0,1,\dots,m\bigr\} \end{aligned}$$

to be a Banach space with the norm

$$\|x\|_{PC} = \max\bigl\{\|x_k\|_{J_k}, k = 0,1,\dots,m\bigr\},$$

where $x_k \in J_k = (t_k,t_{k+1}]$, $k = 0,1,\dots,m$.

Set

$$\Omega_b = \bigl\{x : [-\tau,b] \to X \setminus x \in PC(J,X) \cap \Omega\bigr\},$$

and let $\|\cdot\|_b$ be a seminorm in Ω_b defined by

$$\|x\|_b = \|\phi\|_\Omega + \sup\bigl\{\bigl|x(s)\bigr| : s \in [0,b]\bigr\}, \quad x \in \Omega_b.$$

Definition 2.1 The Riemann–Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\mu$, $\mu \geq 1$, is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad t \geq 0.$$

Definition 2.2 The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)\,ds,$$

where $n-1<\alpha<n$, $t>0$, $f\in C_{-1}^n$.

Definition 2.3 The function $x\in\Omega_b$ as follows:

$$x(t)=\begin{cases}\phi(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_s)\,ds+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^q\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}x(s)\,ds\\ \quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^q\int_{r_i}^0(t-s-r_i)^{\alpha-1}\phi(s)\,ds,\quad t\in[0,t_1];\\ x(t_1^-)+I_1(x(t_1^-))+\frac{1}{\Gamma(\alpha)}\int_{t_1}^t(t-s)^{\alpha-1}f(s,x_s)\,ds\\ \quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^q\int_{t_1-r_i}^{t-r_i}(t-s-r_i)^{\alpha-1}x(s)\,ds,\quad t\in[t_1,t_2];\\ \vdots\\ x(t_m^-)+I_m(x(t_m^-))+\frac{1}{\Gamma(\alpha)}\int_{t_m}^t(t-s)^{\alpha-1}f(s,x_s)\,ds\\ \quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^q\int_{t_m-r_i}^{t-r_i}(t-s-r_i)^{\alpha-1}x(s)\,ds,\quad t\in[t_m,b];\\ \phi(t),\quad t\in[-\tau,0],\end{cases}\tag{2.1}$$

will be called a solution of system (1.1).

For proving the existence of solution of system (1.1), we need to provide the following lemmas.

Lemma 2.1 *For $\sigma\in(0,1]$ and $0<a\le b$, we have $|a^\sigma-b^\sigma|\le(b-a)^\sigma$.*

Lemma 2.2 (Hölder's inequality) *Assume that $p,q\ge1$, and $\frac{1}{p}+\frac{1}{q}=1$. If $l\in L^p(J)$, $m\in L^q(J)$, then for $1\le p\le\infty$, $lm\in L^1(J)$ and $\|lm\|_{L^1(J)}\le\|l\|_{L^p(J)}\|m\|_{L^q(J)}$.*

For measurable functions $m:J\to R$, define the norm

$$\|m\|_{L^p(J)}=\begin{cases}(\int_J|m(t)|^p\,dt)^{\frac{1}{p}},\quad 1\le p<\infty,\\ \inf_{\mu(\bar{J})}\{\sup_{t\in J-\bar{J}}|m(t)|\},\quad p=\infty,\end{cases}$$

where $\mu(\bar{J})$ is the Lebesgue measure on $\bar{J}$. Let $L^p(J,R)$ be the Banach space of all Lebesgue measurable functions $m:J\to R$ with $\|m\|_{L^p(J)}<\infty$.

Lemma 2.3 *If X is a Banach space, $U\subset X$ is convex with $0\in U$, and $F:U\to U$ is a completely continuous operator, then either*

(i) *the set $E=\{x\in U:x=\lambda F(x),0<\lambda<1\}$ is unbounded, or*
(ii) *F has a fixed point.*

3 Main result and proofs

In this section, we give the main results on the existence of solutions of system (1.1).

Firstly, to establish our results, we add the following conditions:

(H_1) The map $f:J\times\Omega\to X$ is said to be an L^1-Caratheodory map if

(i) $t\to f(t,u)$ is measurable for each $u\in\Omega$,
(ii) $u\to f(t,u)$ is continuous for almost all $t\in J$,
(iii) for each $r>0$, there exists $h_r\in L^1(J,R^+)$ such that $\|f(t,u)\|\le h_r(t)$ for all $\|u\|_\Omega\le r$ and for almost $t\in J$.

(H_2) $\|f(t,u)\| \le p(t)\Psi(\|u\|_\Omega)$ for almost all $t \in J$ and all $u \in \Omega$, where $p \in L^1(J,R_+)$ and $\Psi : R_+ \to (0,\infty)$ is continuous and increasing with

$$\int_{t_{k-1}}^{t_k} m(s)\,ds \le \int_{\bar{N}_{k-1}}^{\infty} \frac{ds}{\Psi(s)+s},$$

where

$$\bar{N}_0 = \upsilon(0) = \|\phi\|_\Omega + \frac{qb^\alpha}{\Gamma(\alpha+1)}\|\phi\|_\Omega, \qquad N_{k-1} = \sup_{x\in[-K_{k-2},K_{k-2}]} \bigl|I_{k-1}(x)\bigr| + M_{k-2},$$
$$\bar{N}_{k-1} = N_{k-1} + \frac{q\tau^\alpha K_{k-2}}{\Gamma(\alpha+1)}, \qquad M_{k-2} = \Gamma_{k-1}^{-1}\left(\int_{t_{k-2}}^{t_{k-1}} m(s)\,ds\right), \quad k = 2,\dots,m+2,$$

and

$$K_0 = \max\bigl(M_0, \|\phi\|_\Omega\bigr), \qquad K_k = \max(K_{k-1},K_k), \quad k = 1,\dots,m+1,$$
$$m(t) = \max\left\{\frac{b^\alpha}{\Gamma(\alpha+1)}p(t), \frac{qb^\alpha}{\Gamma(\alpha+1)}\right\},$$
$$\Gamma_{k-1}(z) = \int_{\overline{N}_{k-1}}^{z} \frac{ds}{\Psi(s)+s}, \quad z \ge \bar{N}_{k-1}, k \in \{1,2,\dots,m+2\}.$$

(H_3) The functions $I_k : X \to X$ are continuous, there exists a constant d_k such that $\|I_k(x)\| \le d_k$, $k = 1,2,\dots,m$, for all $x \in X$.

(H_4) There exist a constant $\beta \in (0,\alpha)$ and a real-valued function $l(t) \in L^{\frac{1}{\beta}}(J)$ such that $\|f(t,\phi) - f(t,\psi)\| \le l(t)\|\phi - \psi\|_\Omega$ for almost every $t \in [0,b]$ and all $\phi, \psi \in \Omega$.

(H_5) There exist positive constants c_k, $k = 1,2,\dots,m$, such that $\|I_k(x(t_k^-)) - I_k(y(t_k^-))\| \le c_k\|x-y\|$ for each $x, y \in X$.

Next, we give an existence result based on a nonlinear alternative of Leray–Schauder applied to a completely continuous operator.

Theorem 3.1 *Assume that conditions* (H_1), (H_2), *and* (H_3) *are satisfied, then system* (1.1) *has at least one solution.*

Proof The proof is carried out in the following steps.

Step 1: Consider the following problem:

$$\begin{cases} D^\alpha x(t) = f(t,x_t) + \sum_{i=1}^{p} x(t-r_i), & t \in J = [0,t_1], \\ x(t) = \phi(t), & t \in [-\tau,0]. \end{cases} \tag{3.1}$$

Define the operator $F : \Omega_{t_1} \to \Omega_{t_1}$ by

$$F(x)(t) = \begin{cases} \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x_s)\,ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}x(s)\,ds \\ \quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_{r_i}^{0}(t-s-r_i)^{\alpha-1}\phi(s)\,ds, & t \in [0,t_1], \\ \phi(t), & t \in [-\tau,0]. \end{cases} \tag{3.2}$$

Claim 1: F is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in Ω_{t_1}. Then, for $t \in [0, t_1]$, we have

$$\left\| f(s, x_{ns}, \cdot) - f(s, x_s) \right\| \leq \epsilon, \quad n \to \infty,$$

because f is continuous, for all $t \in [0, t_1]$, we have

$$\begin{aligned}
&\left| F(x_n)(t) - F(x)(t) \right| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, x_{ns}) - f(s, x_s) \right| ds \\
&\qquad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_0^{t-r_i} (t-s-r_i)^{\alpha-1} \left| x_n(s) - x(s) \right| ds.
\end{aligned}$$

Since f is L^1-Caratheodory, we obtain F is continuous.

Claim 2: F maps bounded sets into bounded sets in Ω_{t_1}.

For any $r > 0$, there exists $l > 0$ such that, for each $x \in B_r = \{x \in \Omega_{t_1}, \|x\| \leq r\}$, we have $\|F(x)\| \leq l$.

For each $t \in [0, t_1]$, we get

$$\begin{aligned}
\left| F(x)(t) \right| &\leq \left| \phi(0) \right| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, x_s) \right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_0^{t-r_i} (t-s-r_i)^{\alpha-1} \left| x(s) \right| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{r_i}^{0} (t-s-r_i)^{\alpha-1} \left| \phi(s) \right| ds \\
&\leq \|\phi\|_\Omega + \frac{qb^\alpha}{\Gamma(\alpha+1)} \|\phi\|_\Omega + \frac{qb^\alpha r}{\Gamma(\alpha+1)} + \frac{b^\alpha}{\Gamma(\alpha+1)} \left\| h_r(t) \right\|_{L^1} := l.
\end{aligned}$$

Claim 3: F maps bounded sets into equicontinuous sets in Ω_{t_1}.

Let B_r be a bounded set of Ω_{t_1} as in Step 2. Then, for each $s_1, s_2 \in [0, t_1]$, $s_1 < s_2$, we obtain

$$\begin{aligned}
&\left\| F(x)(s_2) - F(x)(s_1) \right\| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \left\| h_r(t) \right\|_{L^1} \left(\int_{s_1}^{s_2} (s_2 - s)^{\alpha-1} ds + \int_0^{s_1} \left[(s_2-s)^{\alpha-1} - (s_1-s)^{\alpha-1} \right] ds \right) \\
&\qquad + \frac{r}{\Gamma(\alpha)} \sum_{i=1}^{q} \left(\int_{s_1-r_i}^{s_2-r_i} (s_2 - s - r_i)^{\alpha-1} ds \right. \\
&\qquad \left. + \int_0^{s_1-r_i} \left[(s_2-s-r_i)^{\alpha-1} - (s_1-s-r_i)^{\alpha-1} \right] ds \right) \\
&\quad \leq \left[\frac{3\|h_r(t)\|_{L^1}}{\Gamma(\alpha+1)} + \frac{3qr}{\Gamma(\alpha+1)} \right] (s_2 - s_1)^\alpha.
\end{aligned}$$

If $s_2 \to s_1$, then $\|F(x)(s_2) - F(x)(s_1)\| \to 0$. According to a consequence of Steps 1–3, and together with the Arzela–Ascoli theorem, we can deduce that F is continuous and completely continuous.

Claim 4: (A priori bounds).

Let x be a possible solution of the equation $x = \lambda P(x)$ with $\lambda \in (0,1)$. Then, for each $t \in [0,t_1]$, we have

$$\begin{aligned}\left|x(t)\right| &\le \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x_s)\,ds + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}x(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_{r_i}^{0}(t-s-r_i)^{\alpha-1}\phi(s)\,ds.\end{aligned}$$

By (H_3), we have

$$\left|x(t)\right| \le \|\phi\|_\Omega + \frac{qb^\alpha}{\Gamma(\alpha+1)}\|\phi\|_\Omega + \frac{b^\alpha}{\Gamma(\alpha+1)}\int_0^t p(s)\Psi\left(\|x_s\|_\Omega\right)ds + \frac{qb^\alpha}{\Gamma(\alpha+1)}\int_0^t \left|x(s)\right|ds.$$

Let us define $\omega(t)$ as

$$\omega(t) = \sup\left\{\left|x(s)\right| : -\tau \le s \le t\right\}, \quad 0 \le t \le t_1.$$

Then we have

$$\omega(t) \le \|\phi\|_\Omega + \frac{qb^\alpha}{\Gamma(\alpha+1)}\|\phi\|_\Omega + \int_0^t m(s)\left(\Psi\left(\omega(s)\right) + \omega(s)\right)ds.$$

Let us take the right-hand side of the above inequality as $\upsilon(t)$. Then we have

$$c = \upsilon(0) = \|\phi\|_\Omega + \frac{qb^\alpha}{\Gamma(\alpha+1)}\|\phi\|_\Omega, \qquad \omega(t) \le \upsilon(t), \quad t \in [0,t_1]$$

and

$$\upsilon'(t) \le m(t)\left(\Psi\left(\omega(t)\right) + \omega(t)\right), \quad t \in [0,t_1].$$

Using the nondecreasing character of Ψ, we get

$$\upsilon'(t) \le m(t)\left(\Psi\left(\upsilon(t)\right) + \upsilon(t)\right), \quad t \in [0,t_1].$$

Then, for each $t \in [0,t_1]$, we have

$$\Gamma_1\left(\upsilon(t)\right) = \int_{\upsilon(0)}^{\upsilon(t)} \frac{ds}{\Psi(s)+s} \le \int_0^{t_1} m(s)\,ds < \int_{\upsilon(0)}^{\infty}\frac{ds}{\Psi(s)+s}.$$

This implies that $\upsilon(t) < \infty$. So, there is a constant K_0 such that $\upsilon(t) \le \Gamma_1^{-1}(\int_0^{t_1} m(s)\,ds) = M_0$, $t \in [0,t_1]$. So, $\|x_t\|_\Omega \le \omega(t) < \upsilon(t) < M_0$, $t \in [0,t_1]$, then$\|x\|_\infty \le \max\{\|\phi\|_\Omega, M_0\} = K_0$.

Step 2: Consider the following problem:

$$\begin{cases} D^\alpha x(t) = f(t,x_t) + \sum_{i=1}^{p} x(t-r_i), & t \in [t_1,t_2], \\ x(t_1^+) = x(t_1^-) + I_1(x(t_1^-)), \\ x(t) = x_{m-1}(t), & t \in [-\tau,t_1]. \end{cases} \tag{3.3}$$

Define the operator $F_1 : \Omega_{t_2} \to \Omega_{t_2}$ by

$$F_1(x)(t) = \begin{cases} x(t_1^-) + I_1(x(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} f(s, x_s)\, ds \\ \quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{t_1-r_i}^{t-r_i} (t-s-r_i)^{\alpha-1} x(s)\, ds, \quad t \in [t_1, t_2], \\ x(t_1^+) = x(t_1^-) + I_1(x(t_1^-)). \end{cases} \tag{3.4}$$

As in Step 1, we can show that F_1 is continuous and completely continuous if x is a possible solution of the equation $x = \lambda F_1 x$ for some $\lambda \in (0,1)$.

Note that

$$\left|x\left(t_1^+\right)\right| \le \sup_{r \in [-K_0, K_0]} \left|I_1(r)\right| + K_0 := N_1$$

and

$$\begin{aligned} \left|x(t)\right| &\le N_1 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{t_1-r_i}^{t_1} (t-s-r_i)^{\alpha-1} \left|x(s)\right| ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \left|f(s,x_s)\right| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{q} \int_{t_1}^{t-r_i} (t-s-r_i)^{\alpha-1} \left|x(s)\right| ds \\ &\le N_1 + \frac{qK_0}{\Gamma(\alpha+1)} \left|(t-t_1)^\alpha - (t-t_1+r_i)^\alpha\right| + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \left|f(s,x_s)\right| ds \\ &\quad + \frac{q}{\Gamma(\alpha+1)} |t-t_1-r_i|^\alpha \int_{t_1}^{t} \left|x(s)\right| ds. \end{aligned}$$

By (H_2), we have

$$\left|x(t)\right| \le N_1 + \frac{q\tau^\alpha K_0}{\Gamma(\alpha+1)} + \frac{b^\alpha}{\Gamma(\alpha+1)} \int_{t_1}^{t} p(s)\Psi\left(\|x_s\|_\Omega\right) ds + \frac{qb^\alpha}{\Gamma(\alpha+1)} \int_{t_1}^{t} \left|x(s)\right| ds.$$

Let us define $\omega(t)$ as

$$\omega(t) = \sup\left\{\left|x(s)\right| : -\tau \le s \le t\right\}, \quad 0 \le t \le t_2.$$

Then we have

$$\omega(t) \le N_1 + \int_{t_1}^{t} m(s)\left(\Psi\left(\omega(s)\right) + \omega(s)\right) ds.$$

Let us take the right-hand side of the above inequality as $\upsilon(t)$. Then we have

$$c = N_1 + \frac{q\tau^\alpha K_0}{\Gamma(\alpha+1)}, \qquad \omega(t) \le \upsilon(t), \quad t \in [t_1, t_2]$$

and

$$\upsilon'(t) \le m(t)\left(\Psi\left(\omega(t)\right) + \omega(t)\right), \quad t \in [t_1, t_2].$$

Using the nondecreasing character of Ψ, we get

$$\upsilon'(t) \leq m(t)\big(\Psi\big(\upsilon(t)\big) + \upsilon(t)\big), \quad t \in [t_1, t_2].$$

Then, for each $t \in [t_1, t_2]$, we have

$$\Gamma_2\big(\upsilon(t)\big) = \int_{\upsilon(t_1)}^{\upsilon(t)} \frac{ds}{\Psi(s) + s} \leq \int_{t_1}^{t_2} m(s)\,ds < \int_{\upsilon(t_1)}^{\infty} \frac{ds}{\Psi(s) + s}.$$

This implies that $\upsilon(t) < \infty$. So, there is a constant K_1 such that $\upsilon(t) \leq \Gamma_2^{-1}(\int_{t_1}^{t_2} m(s)\,ds) = M_1$, $t \in [t_1, t_2]$. So, $\|x_t\|_\Omega \leq \omega(t) < \upsilon(t) < M_1$, $t \in [t_1, t_2]$, then $\|x\|_\infty \leq \max\{K_0, M_1\} = K_1$.

Step 3: We continue this process and take into account that $x_m := x|_{[t_m, b]}$ is a solution to the problem

$$\begin{cases} D^\alpha x(t) = f(t, x_t) + \sum_{i=1}^p x(t - r_i), & t \in [t_m, b], \\ x(t_m^+) - x(t_m^-) = I_m(x_{m-1}(t_m^-)), \\ x(t) = x_{m-1}(t), & t \in [-\tau, t_m]. \end{cases} \tag{3.5}$$

The solution of system (1.1) is then defined by

$$x(t) = \begin{cases} x_0(t), & t \in [-\tau, t_1], \\ x_1(t), & t \in (t_1, t_2], \\ \vdots \\ x_m(t), & t \in (t_m, b]. \end{cases}$$

□

Next, we should use the Banach contraction principle to prove that F has a fixed point.

Theorem 3.2 *Assume that conditions* (H_4) *and* (H_5) *hold, then system* (1.1) *has a unique solution on* J*, provided that the following inequality holds*:

$$\frac{1}{\Gamma(\alpha)}\left[\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} hb^{\alpha-\beta} + \frac{b^\alpha}{\alpha}\right] + \sum_{k=1}^m c_k < 1, \quad h = \left(\int_0^b l(s)^{\frac{1}{\beta}}\,ds\right)^\beta.$$

Proof Let F be the function defined by (3.1), then $F : \Omega_b \to \Omega_b$ is well defined according to Theorem 3.1.

For every $x, y \in \Omega_b$ and $t \in [-\tau, 0]$,

$$\big\|F\big(x(t)\big) - F\big(y(t)\big)\big\| = \big\|\phi(t) - \psi(t)\big\| = 0.$$

For each $t \in [0, t_1]$, from conditions (H_4) and (H_5), we have

$$\begin{aligned} &\big\|F\big(x(t)\big) - F\big(y(t)\big)\big\| \\ &\quad\leq \frac{1}{\Gamma(\alpha)}\bigg\|\int_0^t (t-s)^{\alpha-1}\big(f(s, x_s) - f(s, y_s)\big)\,ds \\ &\qquad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^q \int_0^{t-r_i} (t-s-r_i)^{\alpha-1}\big\|x(s) - y(s)\big\|\,ds\bigg\| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{l}{\Gamma(\alpha)}\left\| \int_0^t (t-s)^{\alpha-1}\,ds\|x_s-y_s\|_\Omega \right.\\
&\quad \left. + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}\,ds\big\|x(s)-y(s)\big\| \right\|\\
&\leq \left[\frac{1}{\Gamma(\alpha)}\left(\int_0^t \big((t-s)^{\alpha-1}\big)^{\frac{1}{1-\beta}}\,ds\right)^{1-\beta}\left(\int_0^b l(s)^{\frac{1}{\beta}}\,ds\right)^{\beta} + \frac{b^\alpha}{\Gamma(\alpha+1)}\right]\big\|x(s)-y(s)\big\|\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} h t_1^{\alpha-\beta} + \frac{t_1^\alpha}{\alpha}\right]\big\|x(s)-y(s)\big\|.
\end{aligned}
$$

Similarly, for each $t \in [t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
&\big\|F\big(x(t)\big) - F\big(y(t)\big)\big\|\\
&\quad \leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}\big(f(s,x_s)-f(s,y_s)\big)\,ds\right\|\\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}\big\|x(s)-y(s)\big\|\,ds\\
&\qquad + \sum_{k=1}^{m}\big\|I_k\big(x\big(t_k^-\big)\big) - I_k\big(y\big(t_k^-\big)\big)\big\|\\
&\quad \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} l(s)\,ds\|x_s-y_s\|_\Omega\\
&\qquad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{q}\int_0^{t-r_i}(t-s-r_i)^{\alpha-1}\,ds\big\|x(s)-y(s)\big\| + \sum_{k=1}^{m} c_k\|x-y\|\\
&\quad \leq \left[\frac{1}{\Gamma(\alpha)}\left(\int_0^t \big((t-s)^{\alpha-1}\big)^{\frac{1}{1-\beta}}\,ds\right)^{1-\beta}\left(\int_0^b l(s)^{\frac{1}{\beta}}\,ds\right)^{\beta}\right.\\
&\qquad \left. + \frac{b^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m} c_k\right]\big\|x(s)-y(s)\big\|\\
&\quad \leq \frac{1}{\Gamma(\alpha)}\left[\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta} h b^{\alpha-\beta} + \frac{b^\alpha}{\alpha} + \sum_{k=1}^{m} c_k\right]\big\|x(s)-y(s)\big\|.
\end{aligned}
$$

Therefore, F is a contraction operator. Hence, F has a unique fixed point by the Banach contraction principle, that is, system (1.1) has a unique solution. □

(H_4') There exists a positive constant l such that $\|f(t,\phi)-f(t,\psi)\| \leq l\|\phi-\psi\|_\Omega$ for almost every $t \in [0,b]$ and all $\phi, \psi \in \Omega$.

Corollary 3.1 *Assume that conditions* (H_4') *and* (H_5) *hold, then system* (1.1) *has a unique solution on J provided that the following inequality* $\frac{(l+1)b^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m} c_k < 1$ *holds.*

4 Examples

In this section we provide some examples to illustrate the usefulness of our main results.

Example 4.1 We consider the first fractional impulsive system as follows:

$$\begin{cases} D^{\alpha}x(t) = \frac{1}{1+e^t}(\int_{-1}^{0} x(t+\theta)\,d\theta)^2 + x(t-1), & t \in J = [0,1], t \neq t_1, \\ \Delta x|_{t=t_1} = I_1(x(t_1^-)), \\ x(t) = \phi(t), & t \in [-1,0], \end{cases} \tag{4.1}$$

where $f(t,\psi) = \frac{1}{1+e^t}(\int_{-1}^{0} \psi(\theta)\,d\theta)^2$, $\alpha = \frac{1}{2}$, $t_1 \in (0,1)$. Set $p(t) = \frac{1}{1+e^t}$, $\Psi(x) = x^2 + \frac{1}{4}$, then $f(t,y) \leq \frac{1}{1+e^t}\Psi(\|y\|_{\Omega})$, $y \in \Omega$, $t \in [0,1]$, $\int_0^1 m(t)\,dt = \frac{1}{\sqrt{\pi}} \leq \int_0^{\infty} \frac{1}{u^2+u+\frac{1}{4}}\,du = 2$. So, all the assumptions in Theorem 3.1 are satisfied, our results can be applied to problem (4.1).

Example 4.2 We consider the second fractional impulsive system as follows:

$$\begin{cases} D^{\alpha}x(t) = \frac{1}{10+e^t}(\int_{-1}^{0} x(t+\theta)\,d\theta)^2 + x(t-1), & t \in J = [0,1], t \neq t_1, \\ \Delta x|_{t=t_1} = I_1(x(t_1^-)), \\ x(t) = \phi(t), & t \in [-1,0], \end{cases} \tag{4.2}$$

where $f(t,\psi) = \frac{1}{10+e^t}(\int_{-1}^{0} \psi(\theta)\,d\theta)^2$, $\alpha = \frac{1}{2}$, $I_1(x(t_1^-)) = \frac{1}{3}$, $t_1 \in (0,1)$, then $\|f(t,x) - f(t,y)\| \leq \frac{1}{11}\|x-y\|$, $\forall x,y \in \Omega_b$, $(\frac{1}{11}+1)\frac{1}{\Gamma(\frac{1}{2}+1)} + 0 = \frac{24}{11\sqrt{\pi}} < 1$. So, all the assumptions in Corollary 3.1 are satisfied, our results can be applied to problem (4.2).

5 Conclusions

In this paper, we use the nonlinear alternative of Leray–Schauder and the Banach fixed point theorem to prove the existence and uniqueness of solution for the fractional order impulsive functional differential equations with multiple delays.

Acknowledgements
The research of H. Fang is partially supported by the key project of Provincial Excellent Talents in University of Anhui Province (gxyqZD2016300, gxyqZD2018077), the Natural Science Foundation of Huangshan University (2017xkjq001). The research of M. Song is partially supported by the Natural Science Foundation of Anhui Province (1708085MA04).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Huangshan University, Huangshan, P.R. China. [2]College of Mathematics and Computer Science, Tongling University, Tongling, P.R. China.

Publisher's Note

Received: 18 January 2018 Accepted: 1 April 2018 Published online: 17 April 2018

References

1. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
3. Kosmatov, N.: Integral equations and initial value problems for nonlinear differential equations of fractional order. Nonlinear Anal. **70**, 2521–2529 (2009)
4. Zhang, S.Q.: Existence of solution for a boundary value problem of fractional order. Acta Math. Sci. **26**, 220–228 (2006)
5. Lv, Z.W., Liang, J., Xiao, T.J.: Solutions to fractional differential equations with nonlocal initial condition in Banach spaces. Adv. Differ. Equ. **2010**, Article ID 340349 (2010)
6. Goodrich, C.S.: Existence of a positive solution to systems of differential equations of fractional order. Comput. Math. Appl. **62**, 1251–1268 (2011)

7. Yang, X., Wei, Z.L., Dong, W.: Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 85–92 (2012)
8. Tang, X.S.: Existence of solutions of four-point boundary value problems for fractional differential equations at resonance. J. Appl. Math. Comput. **51**, 145–160 (2016)
9. Zhou, Y.: Existence and uniqueness of fractional functional differential equations with unbounded delay. Int. J. Dyn. Syst. Differ. Equ. **1**, 239–244 (2008)
10. Zhou, Y., Jiao, F., Li, J.: Existence and uniqueness for fractional neutral differential equations with infinite delay. Nonlinear Anal. **71**, 3249–3256 (2009)
11. Ouahab, A.: Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay. J. Math. Anal. Appl. **323**, 456–472 (2006)
12. Darwish, M.A., Ntouyas, S.K.: Functional differential equations of fractional order with state-dependent delay. Dyn. Syst. Appl. **18**, 539–550 (2009)
13. Fečkan, M., Zhou, Y., Wang, J.R.: On the concept and existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 3050–3060 (2012)
14. Wang, J.R., Fečkan, M., Zhou, Y.: On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. **8**, 345–361 (2011)
15. Shu, X.B., Lai, Y.Z., Chen, Y.M.: The existence of mild solutions for impulsive fractional partial differential equations. Nonlinear Anal. TMA **74**, 2003–2011 (2011)
16. Chen, A., Chen, Y.: Existence of solutions to anti-periodic boundary value problem for nonlinear fractional differential equations with impulses. Adv. Differ. Equ. **2011**, Article ID 915689 (2011)
17. Jiang, H.P.: Existence results for fractional order functional differential equations with impulse. Comput. Math. Appl. **64**, 3477–3483 (2012)
18. Shu, X.B., Shi, Y.J.: A study on the mild solution of impulsive fractional evolution equations. Appl. Math. Comput. **15**, 465–476 (2016)