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Optimal controls for a class of impulsive fractional differential equations with nonlocal conditions

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Abstract

In this paper, we investigate a class of impulsive fractional differential equations with nonlocal conditions in a Banach space. Firstly, we utilize a fixed point theorem to obtain the existence of solution. Secondly, we derive the sufficient conditions for optimal controls by building approximating minimizing sequences of functions twice.

Keywords: Fractional differential equations; Nonlocal conditions; Fractional impulsive conditions; Optimal controls

1 Introduction

In recent years, along with multiple phenomena arising in physics, biophysics, engineering, and science (see [1–5] and the references therein), fractional calculus has received increasing attention. The existence of fractional differential equation has received attention of many authors (see [6–9]). The monographs of Bazhlekova [10], Guo et al. [11], Miller et al. [12], Podlubny [13], and the papers [14–16] can provide more details and references about the theory and application of fractional differential equations.

Since the end of the last century, approximate controllability of problems has been paid more and more attention to [17–22]. Impulsive differential equations and optimal conditions have been an active area of research because the impulsive differential system can fully consider the effect of abrupt changes on state. A number of papers extensively study impulsive differential equations with nonlocal differential equations or impulsive conditions only containing integer order derivatives [23, 24]. Kosmatov in [25] considered the nonlinear differential equation initial value problem:

$$\begin{cases} {}^{L}D^{\alpha}_{0^+}x(t) = f(t,x(t)), & t \in (0,1], t \neq t_k, \\ I^{1-\alpha}_{0^+}x(0) = x_0, \\ {}^{L}D^{\beta}_{0^+}x(t_k^+) - {}^{L}D^{\beta}_{0^+}x(t_k^-) = J_k(x(t_k)), & k = 1,\dots,m, \end{cases}$$

where $0 < \alpha \leq 1$, $0 < \beta < \alpha$, ${}^{L}D^{\alpha}_{0^+}$ is the Riemann–Liouville derivative of order α, $f : ([0,1] \setminus \{t_1,t_2,\dots,t_m\}) \times R \to R$ is continuous, and $f(t_k\pm,x)$ exists for all $x \in R$, $k = 1,\dots,m$, $J_k(x(t_k))$ is a continuous nonlinear map. The existence of solutions of the above nonlinear differential equation initial value problem was proved by means of Krasnoselskii's theorem in [25].

Motivated by Kosmatov's work, in this paper, we investigate a class of impulsive fractional differential equations with fractional impulsive conditions and nonlocal initial conditions. We firstly prove the existence of solution by the measure of noncompactness and fixed point theorem. Moreover, without ensuring the uniqueness of feasible pairs, we derive a set of sufficient conditions for optimal controls by building approximating minimizing sequences of functions twice. Because the approach is a novelty and the optimal controls for impulsive fractional differential equation with the initial conditions and fractional impulsive conditions are useful in practical application, so the results we obtained are more general than known results.

In this article, we consider the following impulsive fractional differential equation in an abstract space:

$$\begin{cases} {}^{L}D_{0^+}^{\alpha}x(t) = f(t,x(t)) + B(t)u(t), & t \in J' = (0,T], t \neq t_k, \\ I_{0^+}^{1-\alpha}x(0) + g(x) = x_0, & u \in U_{\mathrm{ad}}, \\ {}^{L}D_{0^+}^{\beta}x(t_k^+) - {}^{L}D_{0^+}^{\beta}x(t_k^-) = J_k(x(t_k)), & k = 1,\ldots,m, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $0 < \beta < \alpha$, $x \in X$, where X is a real Banach space, f is a nonlinear perturbation, $g(x)$ is a given X-valued function, $J_k(x(t_k))$ is a nonlinear map, U_{ad} is a control set, the control $u \in U_{\mathrm{ad}}$.

The rest of this paper is organized as follows. Section 2 involves some notations and fundamentals of fractional calculus. In Sect. 3, we derive the existence theorem of impulsive fractional differential equations by means of Darbo–Sadovskii's fixed point theorem. In Sect. 4, we establish the optimal controls for a Lagrange problem. An example is given in Sect. 5.

2 Preliminaries

In this section, we give some basic definitions and preliminary facts that will be used in the rest of the paper.

Let X be a real Banach space, $\pounds(X)$ be the class of (not necessarily bounded) linear operators in X.

The set $C([0,T];X)$ is a Banach space of all continuous functions from $[0,T]$ to X with the norm $\|u\| = \sup\{\|u(t)\|, t \in [0,T]\}$.

The set $L^1([0,T];X)$ is a Banach space of all X-valued Bochner integrable functions defined on $[0,T]$ with the norm $\|u\|_1 = \int_0^T |u(t)|\,dt$.

We introduce $PC_\delta((0,T];X) = \{x : (0,T] \to X : x \in C(t_k,t_{k+1}),\ k = 0,\ldots,m,$ such that $x(t_k^-), x(t_k^+)$ exist, $x(t_k^-) = x(t_k)$, $k = 1,\ldots,m$, and $\lim_{t\to 0^+} t^\delta x(t)$ exists$\}$, where $\alpha + \delta - 1 \geq 0$ if endowed with the norm $\|x\|_\delta = \sup_{t\in(0,T]} t^\delta |x(t)|\,dt$, PC_δ is also a Banach space.

Definition 2.1 (see [26]) The Riemann–Liouville fractional integral is defined by

$$I_{0^+}^{\alpha}x(t) = j_\alpha(t) * u(t) = \int_0^t j_\alpha(t-s)u(s)\,ds, \quad t > 0,$$

where $*$ denotes the convolution,

$$j_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}.$$

For $x \in C([0,\infty), X)$, the Riemann–Liouville fractional derivative is defined by

$${}^{L}D^{\alpha}_{0+}x(t) = \frac{d}{dt}\big(j_{(1-\alpha)}(t) * u(t)\big).$$

Now, we introduce the Hausdorff measure of noncompactness $\sigma(\cdot)$ defined on each bounded subset Ω of the Banach space X by

$$\sigma(\Omega) = \inf\{\epsilon > 0, \Omega \text{ has a finite } \epsilon\text{-net in } X\}.$$

Some basic properties of $\sigma(\cdot)$ are given in the following lemmas.

Lemma 2.2 (see [27]) *The measure of noncompactness $\sigma(\cdot)$ satisfies the following*:

(i) $\sigma(B) = 0$ *if and only if B is relatively compact in X*;
(ii) *if $\sigma(\{x\} \cup B) = \sigma(B)$ for every $x \in X$ and every nonempty subset $B \subseteq X$*;
(iii) $\sigma(\lambda B) \le |\lambda|\sigma(B)$ *for any $\lambda \in \mathbb{R}$*;
(iv) $\sigma(B_1 + B_2) \le \sigma(B_1) + \sigma(B_2)$, *where $B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\}$*;
(v) $\sigma(B_1 \cup B_2) \le \max\{\sigma(B_1), \sigma(B_2)\}$.

For any $D \subset C([0,T], X)$, we define

$$\int_0^t D(s)\,ds = \left\{\int_0^t u(s)\,ds : u \in D\right\} \quad \text{for } t \in [0,T],$$

where $D(s) = \{u(s) \in X : u \in D\}$.

In order to prove the main result, we introduce the following fixed point theorem.

Lemma 2.3 (Darbo–Sadovskii, see [27]) *If $D \subset X$ is a convex bounded and closed set, the continuous mapping $\Lambda : D \to D$ is a σ-contraction, then Λ has at least one fixed point in D.*

The set Y is another separable reflexive Banach space where controls u take values. Let $P_f(Y) \subset Y$ be nonempty closed and convex. We assume $w : (0,T] \to P_f(Y)$ is a multivalued and measurable mapping, $w(\cdot) \subset E$, $E \subset Y$ is bounded, the admissible control set $U_{\mathrm{ad}} = S^p_w = \{u \in L^p(E) | u(t) \in w(t), \text{a.e.}\}$, $p > 1$, and $U_{\mathrm{ad}} \neq \emptyset$.

By [25], an important and new equivalent mixed type integral equation for our problem can be established.

Definition 2.4 A function $x \in PC_\delta((0,T]; X)$, whose fractional derivative ${}^{L}D^{\alpha}_{0+}x(t)$ of order $0 < \alpha < 1$ exists and is continuous on $(0,T]$ and $t \neq t_k$, is said to be a solution of problem (1.1) if it satisfies

$$\begin{aligned} x(t) = {} & \frac{(x_0 - g(x))t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\left(\sum_{0<t_k<t} t_k^{1+\beta-\alpha} J_k\big(x(t_k)\big)\right)t^{\alpha-1} \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(f\big(s,x(s)\big) + B(s)u(s)\big)\,ds. \end{aligned}$$

3 Existence of solution

In this section, we obtain sufficient conditions for the existence of solutions by utilizing the measure of noncompactness and fixed point theorem.

Let r be a finite positive constant, and set $B_r := \{x \in X : \|x\|_\delta \le r\}$ and $W_r := \{y \in PC_\delta((0,T];X) : y(t) \in B_r, \forall t \in (0,T]\}$.

Firstly, we introduce the following hypotheses:

(H1) $f \in C((0,T] \setminus \{t_1,t_2,\dots,t_m\} \times X)$ and $f(t_k\pm,x)$ exist for all $x \in X$, $k = 1,\dots,m$.

(H2) For a finite positive constant $r > 0$, there exists a function φ_r such that

$$\|f(t,x)\| \le \varphi_r(t)$$

for a.e. $t \in (0,T]$ and $x \in B_r$, where $I_0^\alpha \varphi_r(t)$ exists and is bounded and continuous.

(H3) For each bounded set $Q \subset X$, there is a constant $q > 0$ satisfying

$$\sigma\bigl(f(t,Q)\bigr) \le q\sigma(Q),$$

where σ is the Hausdorff measure of noncompactness.

(H4) $B : (0,T] \to \pounds(Y,X)$ is an essentially bounded and compact mapping, i.e., $B \in L^\infty((0,T],\pounds(Y,X))$.

(H5) $g : PC_\delta((0,T];X) \to X$ is a compact and continuous mapping.

(H6) $J_k : X \to X$ satisfies Lipschitz condition with the Lipschitz constants $h_k > 0$:

$$\|J_k(x_1) - J_k(x_2)\| \le h_k\|x_1 - x_2\|, \quad k = 1,2,\dots,m,$$

$\forall x_1,x_2 \in PC_\delta((0,T];X)$.

Remark 3.1 From the definition U_{ad} and hypothesis (H4), we can easily prove that $Bu \in L^p((0,T];X)$ with $p > 1$ for any $u \in U_{\mathrm{ad}}$. Therefore, $Bu \in L^1((0,T];X)$ and $\|Bu\|_{L^1} < +\infty$.

Remark 3.2 Under hypothesis (H5), $\sup_{x\in W_r} \|g(x)\| < +\infty$.

Theorem 3.1 *Suppose that hypotheses* (H1)–(H6) *are satisfied, then the impulsive fractional differential equation* (1.1) *has at least one solution on* $(0,T]$ *if there is a constant* $r > 0$ *such that*

$$\begin{aligned} &T^{\alpha+\delta-1}\sup_{x\in W_r}\|g(x)\| + T^\delta \sup_{t\in(0,T]}\bigl\{I_0^\alpha|\varphi_r(t)|\bigr\} \\ &\quad + T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\biggl(\sum_{1\le k\le m} t_k^{1+\beta-\alpha}\bigl(\|J_k(0)\| + h_k r\bigr)\biggr) + \|x_0\| + \frac{T^{\alpha+\delta}\|Bu\|_{L^1}}{\alpha} \le r. \end{aligned} \tag{3.1}$$

Proof Let $x_0 \in X$ be fixed. Define the operator S on $PC_\delta((0,T];X)$ as follows:

$$\begin{aligned} Sx(t) &= \biggl[\bigl(x_0 - g(x)\bigr) + \Gamma(\alpha-\beta)\biggl(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}J_k\bigl(x(t_k)\bigr)\biggr)\biggr]t^{\alpha-1} \\ &\quad + \int_0^t (t-s)^{\alpha-1}\bigl(f\bigl(s,x(s)\bigr) + B(s)u(s)\bigr)\,ds, \quad t \in (0,T]. \end{aligned}$$

The proof process is divided into three steps.

Step I. S maps W_r into itself.

It follows from (H1)–(H6) and (3.1), for any $x \in W_r$, that we have

$$\begin{aligned}
\left|t^{\delta} Sx(t)\right| &\leq t^{\delta}\left[\left\|\left(x_0 - g(x)\right)\right\| + \Gamma(\alpha-\beta)\left(\sum_{0<t_k<t}\left\|t_k^{1+\beta-\alpha} J_k\left(x(t_k)\right)\right\|\right)\right] t^{\alpha-1} \\
&\quad + t^{\delta}\int_0^t \left\|(t-s)^{\alpha-1}\left(f\left(s,x(s)\right) + B(s)u(s)\right)\right\| ds \\
&\leq \left[\|x_0\| + \left\|g(x)\right\| + \Gamma(\alpha-\beta)\left(\sum_{1\leq k\leq m} t_k^{1+\beta-\alpha}\left(\left\|J_k(0)\right\| + h_k r\right)\right)\right] t^{\alpha+\delta-1} \\
&\quad + t^{\delta}\int_0^t (t-s)^{\alpha-1}\left(\left\|f\left(s,x(s)\right)\right\| + \left\|B(s)u(s)\right\|\right) ds \\
&\leq \left[\|x_0\| + \left\|g(x)\right\| + \Gamma(\alpha-\beta)\left(\sum_{1\leq k\leq m} t_k^{1+\beta-\alpha}\left(\left\|J_k(0)\right\| + h_k r\right)\right)\right] T^{\alpha+\delta-1} \\
&\quad + t^{\delta}\int_0^t (t-s)^{\alpha-1}\left|\varphi_r(s)\right| ds + t^{\delta}\|Bu\|_{L^1}\int_0^t (t-s)^{\alpha-1} ds \\
&\leq \left[\|x_0\| + \left\|g(x)\right\| + \Gamma(\alpha-\beta)\left(\sum_{1\leq k\leq m} t_k^{1+\beta-\alpha}\left(\left\|J_k(0)\right\| + h_k r\right)\right)\right] T^{\alpha+\delta-1} \\
&\quad + t^{\delta} I_0^{\alpha}|\varphi_r| + t^{\delta}\|Bu\|_{L^1}\frac{t^{\alpha}}{\alpha} \\
&\leq T^{\alpha+\delta-1}\sup_{x\in W_r}\left\|g(x)\right\| + T^{\delta}\sup_{t\in(0,T]}\left\{I_0^{\alpha}\left|\varphi_r(t)\right|\right\} \\
&\quad + T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\left(\sum_{1\leq k\leq m} t_k^{1+\beta-\alpha}\left(\left\|J_k(0)\right\| + h_k r\right)\right) \\
&\quad + \|x_0\| + \frac{T^{\alpha+\delta}\|Bu\|_{L^1}}{\alpha} \leq r.
\end{aligned}$$

Step II. S is continuous in W_r.

Letting $x_1, x_2 \in W_r$, we obtain

$$\begin{aligned}
&\left|t^{\delta} Sx_1(t) - t^{\delta} Sx_2(t)\right| \\
&\quad \leq t^{\delta}\left[\left\|g(x_1) - g(x_2)\right\| + \Gamma(\alpha-\beta)\left(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\left\|J_k\left(x_1(t_k)\right) - J_k\left(x_2(t_k)\right)\right\|\right)\right] t^{\alpha-1} \\
&\qquad + t^{\delta}\int_0^T (t-s)^{\alpha-1}\left\|f\left(s,x_1(s)\right) - f\left(s,x_2(s)\right)\right\| ds \\
&\quad \leq \left[\left\|g(x_1) - g(x_2)\right\| + \Gamma(\alpha-\beta)\left(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\left\|J_k\left(x_1(t_k)\right) - J_k\left(x_2(t_k)\right)\right\|\right)\right] T^{\alpha+\delta-1} \\
&\qquad + T^{\delta}\int_0^T (t-s)^{\alpha-1}\left\|f\left(s,x_1(s)\right) - f\left(s,x_2(s)\right)\right\| ds.
\end{aligned}$$

In view of the continuity of g, J_k, f, we know that S is continuous.

Step III. S has at least one fixed point.

Set

$$I_1x(t) = \big(x_0 - g(x)\big)t^{\alpha-1},$$
$$I_2x(t) = \Gamma(\alpha-\beta)\Bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}J_k\big(x(t_k)\big)\Bigg)t^{\alpha-1},$$
$$I_3x(t) = \int_0^t (t-s)^{\alpha-1}\big(f\big(s,x(s)\big) + B(s)u(s)\big)\,ds$$

for each $x \in W_r$. In view of condition (H5), we obtain that the operator I_1 is compact in W_r, and so $\sigma(I_1W_r) = 0$. By (H1) and (H3), we can derive that $f(t,x(t))$ is a compact mapping in W_r, for the proof we can refer to Theorem 4.34 in [28]. Combining with (H4), we know that the operator I_3 is also compact in W_r, so $\sigma(I_3W_r) = 0$.

Then, under condition (H6), we notice that $\{J_k\}_{k=1}^m$ are Lipschitz continuous. For $x_1, x_2 \in W_r$, we have

$$\begin{aligned}
&\big|t^\delta I_2x_1(t) - t^\delta I_2x_2(t)\big| \\
&\quad\le t^\delta\Gamma(\alpha-\beta)\Bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\big\|J_k\big(x_1(t_k)\big) - J_k\big(x_2(t_k)\big)\big\|\Bigg)t^{\alpha-1} \\
&\quad\le \Gamma(\alpha-\beta)\Bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\big\|J_k\big(x_1(t_k)\big) - J_k\big(x_2(t_k)\big)\big\|\Bigg)T^{\alpha+\delta-1} \\
&\quad\le T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}h_k\|x_1 - x_2\|.
\end{aligned}$$

Therefore,

$$\sigma(I_2W_r) \le T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}h_k\sigma(W_r),$$
$$\sigma(SW_r) \le \sigma(I_1W_r) + \sigma(I_2W_r) + \sigma(I_3W_r) \le T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\Bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}h_k\Bigg)\sigma(W_r).$$

By (3.1) we get

$$T^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}h_k < 1.$$

By Lemma 2.3 we immediately deduce that the operator S has a fixed point in $x \in W_r$, i.e., (1.1) has at least one solution. This completes the proof. □

Without the Lipschitz continuity of f, we cannot obtain the uniqueness of solution of the impulsive fractional differential equation (1.1). Hence, we let $\mathrm{Sol}(u)$ be the space of all solutions of system (1.1) in W_r for each $u \in U_{\mathrm{ad}}$.

4 Optimal controls for problem

Let $x^u \in W_r$ denote the solution of system (1.1) associated with the control $u \in U_{\mathrm{ad}}$.

We discuss the following limited Lagrange problem (P):

$$J\big(x^u,u\big)=\int_0^T \psi\big(t,x^u(t),u(t)\big)\,dt.$$

For all $u\in U_{\mathrm{ad}}$, we can find $x^0\in W_r\subseteq PC_\delta((0,T];X)$ and $u^0\in U_{\mathrm{ad}}$ such that

$$J\big(x^0,u^0\big)\le J\big(x^u,u\big),$$

where $x^0\in W_r$ denotes the solution of system (1.1) associated with the control $u^0\in U_{\mathrm{ad}}$.

We introduce the following hypotheses:

The function $\psi:(0,T]\times X\times Y$ satisfies the following:

($\Psi1$) The function $\psi:(0,T]\times X\times Y\to R\cup\infty$ is Borel measurable.

($\Psi2$) ψ is sequentially lower continuous on $X\times Y$ for a.e. $t\in(0,T]$.

($\Psi3$) ψ is convex on Y for a.e. $t\in(0,T]$ and all $x\in X$.

($\Psi4$) There exist two constants $c\ge 0, d>0$ and $\phi\in L^1((0,T]:R)$ such that

$$\psi\big(t,x^u(t),u(t)\big)\ge\phi(t)+c\|x\|+d\|u\|_Y^p.$$

Remark 4.1 If, for all $x(\cdot)\in W_r$, a pair $(x(\cdot),u(\cdot))$ satisfies system (1.1), we say that the pair is feasible. If the pair (x^u,u) is feasible, then $x^u\subset W_r$.

Theorem 4.1 *Suppose that hypotheses* ($\Psi1$)–($\Psi4$) *are satisfied, then problem* (P) *has at least one optimal feasible pair if it satisfies the condition of Theorem* 3.1.

Proof Step I. For each $u\in U_{\mathrm{ad}}$, set

$$J(u)=\inf_{x^u\in\mathrm{Sol}(u)}J\big(x^u,u\big).$$

If $\mathrm{Sol}(u)$ has infinite elements, obviously we do not have to prove in the case of

$$J(u)=\inf_{x^u\in\mathrm{Sol}(u)}J\big(x^u,u\big)=+\infty.$$

If $\mathrm{Sol}(u)$ has finite elements, there is some $\hat{x}^u\in\mathrm{Sol}(u)$ such that $J(\hat{x}^u,u)=\inf_{x^u\in\mathrm{Sol}(u)}J(x^u,u)=J(u)$.

We suppose that $J(u)=\inf_{x^u\in\mathrm{Sol}(u)}J(x^u,u)<+\infty$. One obtains $J(u)>-\infty$ by conditions ($\Psi1$)–($\Psi4$).

In view of the definition of the infimum, we can find a sequence $\{x_n^u\}_{n=1}^\infty\subseteq\mathrm{Sol}(u)$, which satisfies $J(x_n^u,u)\to J(u)$ as $\to\infty$.

Since $\{(x_n^u,u)\}_{n=1}^\infty$ is a sequence of feasible pairs, one has

$$\begin{aligned}x_n^u(t)&=\big(x_0-g\big(x_n^u\big)\big)t^{\alpha-1}+\Gamma(\alpha-\beta)\biggl(\sum_{0<t_k<t}t_k^{1+\beta-\alpha}J_k\big(x_n^u(t_k)\big)\biggr)t^{\alpha-1}\\&\quad+\int_0^t(t-s)^{\alpha-1}\big(f\big(s,x_n^u(s)\big)+B(s)u(s)\big)\,ds,\end{aligned}\tag{4.1}$$

where $t\in(0,T]$.

Step II. We can find some $\hat{x}^u\in\mathrm{Sol}(u)$ that satisfy $J(\hat{x}^u,u)=\inf_{x^u\in\mathrm{Sol}(u)}J(x^u,u)=J(u)$.

Claim 1 *$\{x_n^u\}_{n=1}^{\infty}$ is precompact in $PC_\delta((0,T];X)$ for all $u \in U_{\mathrm{ad}}$.*

For proving the result, we set

$$x_n^u = I_1 x_n^u + I_2 x_n^u + I_3 x_n^u.$$

Referring to the proof of Theorem 3.1, we get that $\{I_1 x_n^u\}_{n=1}^{\infty} \subset PC_\delta((0,T];X)$ and $\{I_3 x_n^u\}_{n=1}^{\infty} \subset PC_\delta((0,T];X)$ are both precompact. Furthermore, it follows from assumption (H6) that the operator I_2 is Lipschitz continuous in $PC_\delta((0,T];X)$, where Lipschitz constant $(T+1)^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha} h_k$. Hence, by the given condition and the Hausdorff measure of noncompactness, one has

$$\begin{aligned}\rho\bigl(\{x_n^u\}_{n=1}^{\infty}\bigr) &\le \rho\bigl(I_1\{x_n^u\}_{n=1}^{\infty}\bigr) + \rho\bigl(I_2\{x_n^u\}_{n=1}^{\infty}\bigr) + \rho\bigl(I_3\{x_n^u\}_{n=1}^{\infty}\bigr) \\ &\le (T+1)^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha} h_k \rho\bigl(\{x_n^u\}_{n=1}^{\infty}\bigr).\end{aligned}$$

Therefore we can conclude that $\rho(\{x_n^u\}_{n=1}^{\infty}) = 0$ from the above inequality, so the set $\{x_n^u\}_{n=1}^{\infty}$ is precompact in $PC_\delta((0,T];X)$ for $u \in U_{\mathrm{ad}}$.

Claim 2 $\hat{x}^u \in \mathrm{Sol}(u)$.

In general, we assume that $x_n^u \to \hat{x}^u$, as$n \to \infty$, in $PC_\delta((0,T];X)$ for $u \in U_{\mathrm{ad}}$. In terms of the dominated convergence theorem and the continuity of g, J_i, and f, when we take the limit $n \to \infty$ on both sides for (4.1), we obtain

$$\begin{aligned}\hat{x}^u(t) &= \bigl(x_0 - g(\hat{x}^u)\bigr)t^{\alpha-1} + \Gamma(\alpha-\beta)\biggl(\sum_{0<t_k<t} t_k^{1+\beta-\alpha} J_k\bigl(\hat{x}_n^u(t_k)\bigr)\biggr)t^{\alpha-1} \\ &\quad + \int_0^t (t-s)^{\alpha-1}\bigl(f\bigl(s,\hat{x}_n^u(s)\bigr) + B(s)u(s)\bigr)\,ds,\end{aligned}$$

where $0 \le t \le T$, which shows that $\hat{x}^u \in \mathrm{Sol}(u)$.

Step III. $J(\hat{x}^u,u) = \inf_{x^u\in \mathrm{Sol}(u)} J(x^u,u) = J(u)$ for all $u \in U_{\mathrm{ad}}$.

According to the Balder theorem and hypotheses $(\Psi 1)$–$(\Psi 4)$, and using the definition of a feasible pair, we obtain

$$J(u) = \lim_{n\to\infty}\int_0^T \psi\bigl(t,x_n^u(t),u(t)\bigr)\,dt \ge \int_0^T \psi\bigl(t,\hat{x}^u(t),u(t)\bigr)\,dt = J\bigl(\hat{x}^u,u\bigr) \ge J(u).$$

Hence, $J(\hat{x}^u,u) = J(u)$. This implies that $J(u)$ at $\hat{x}^u \in PC_\delta((0,T];X)$ to the minimum for all $u \in U_{\mathrm{ad}}$.

Step IV. There exists $u_0 \in U_{\mathrm{ad}}$ such that $J(u_0) \le J(u)$ for $\forall u \in U_{\mathrm{ad}}$.

Suppose that $\inf_{u\in U_{\mathrm{ad}}} J(u) < +\infty$. Referring to the proof of Step I, we deduce that $\inf_{u\in U_{\mathrm{ad}}} J(u) > -\infty$, and there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq U_{\mathrm{ad}}$ satisfying $J(u_n) \to \inf_{u\in U_{\mathrm{ad}}} J(u)$ as $n \to \infty$. We utilize that $L^p((0,T];Y)$ is a reflexive Banach space and $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^p((0,T];Y)$. So we can assume that $\{u_n\}_{n=1}^{\infty}$ converges weakly to some $u_0 \in L^p((0,T];Y)$ as $n \to \infty$.

The set U_{ad} is convex and closed, so by Mazur's lemma we can deduce that $u_0 \in U_{\mathrm{ad}}$. For $n \geq 1$, $\hat{x}^{u_n}$ is the solution for (1.1) associated with u_n, where $J(u_n)$ attains its minimum. Hence the pair $(\hat{x}^{u_n}, u_n)$ is feasible, it satisfies the following:

$$\begin{aligned}\hat{x}^{u_n}(t) = {} & \bigl(x_0 - g(\hat{x}^{u_n})\bigr)t^{\alpha-1} + \Gamma(\alpha-\beta)\biggl(\sum_{0<t_k<t} t_k^{1+\beta-\alpha} J_k\bigl(\hat{x}^{u_n}(t_k)\bigr)\biggr)t^{\alpha-1} \\ & + \int_0^t (t-s)^{\alpha-1}\bigl(f\bigl(s,\hat{x}^{u_n}(s)\bigr) + B(s)u(s)\bigr)\,ds,\end{aligned} \tag{4.2}$$

where $0 \leq t \leq T$.

Set

$$\begin{aligned}\acute{I}_3\hat{x}^{u_n}(t) &= \int_0^t (t-s)^{\alpha-1} f\bigl(s,\hat{x}^{u_n}(s)\bigr)\,ds, \\ Zu_n(t) &= \int_0^t (t-s)^{\alpha-1} B(s)u_n(s)\,ds.\end{aligned}$$

We have

$$\hat{x}^{u_n}(t) = I_1\hat{x}^{u_n}(t) + I_2\hat{x}^{u_n}(t) + \acute{I}_3\hat{x}^{u_n}(t) + Zu_n(t),$$

where $0 \leq t \leq T$.

We know that $\{I_1\hat{x}^{u_n}\}_{n=1}^{\infty}$ and $Zu_n(t)$ are both precompact in $PC_\delta((0,T];X)$ and I_2 satisfies the Lipschitz continuity in $PC_\delta((0,T];X)$ with the Lipschitz constant

$$(T+1)^{\alpha+\delta-1}\Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha} h_k < 1.$$

Moreover, in view of given assumptions, we can see that $\{\acute{I}_3\hat{x}^{u_n}\}_{n=1}^{\infty} \subseteq PC_\delta((0,T];X)$ is precompact.

Referring to the proof of Theorem 3.1, we imply that $\rho(\{\hat{x}^{u_n}\}_{n=1}^{\infty}) = 0$, i.e., $\{\hat{x}^{u_n}\}_{n=1}^{\infty} \subseteq PC_\delta((0,T];X)$ is precompact. Thus we can find a subsequence which is connected with $\hat{x}^{u_0} \in PC_\delta((0,T];X)$ and $\{\hat{x}^{u_n}\}_{n=1}^{\infty}$ satisfying that $\hat{x}^{u_n} \to \hat{x}^{u_0}$, as $n \to \infty$ in $PC_\delta((0,T];X)$. When we take the limit $n \to \infty$ on both sides for (4.2), we get

$$\begin{aligned}\hat{x}^{u_0}(t) = {} & \bigl(x_0 - g(\hat{x}^{u_0})\bigr)t^{\alpha-1} + \Gamma(\alpha-\beta)\biggl(\sum_{0<t_k<t} t_k^{1+\beta-\alpha} J_k\bigl(\hat{x}^{u_0}(t_k)\bigr)\biggr)t^{\alpha-1} \\ & + \int_0^t (t-s)^{\alpha-1}\bigl(f\bigl(s,\hat{x}^{u_0}(s)\bigr) + B(s)u(s)\bigr)\,ds,\end{aligned}$$

where $0 \leq t \leq T$. Therefore the pair $(\hat{x}^u, u_0)$ is feasible.

In view of the Balder theorem and assumptions $(\Psi 1)$–$(\Psi 4)$, we have

$$\begin{aligned}\inf_{u\in U_{\mathrm{ad}}} J(u) &= \lim_{n\to\infty}\int_0^T \psi\bigl(t,\hat{x}^{u_n}(t),u_n(t)\bigr)\,dt \geq \int_0^T \psi\bigl(t,\hat{x}^{u_0}(t),u_0(t)\bigr)\,dt \\ &= J\bigl(\hat{x}^{u_0},u_0\bigr) \geq \inf_{u\in U_{\mathrm{ad}}} J(u).\end{aligned}$$

Thus,

$$J(\hat{x}^{u_0}, u_0) = J(u_0) = \inf_{x^{u_0} \in \mathrm{Sol}(u_0)} J(x^{u_0}, u_0).$$

Furthermore,

$$J(u_0) = \inf_{u \in U_{\mathrm{ad}}} J(u),$$

i.e., J at $u_0 \in U_{\mathrm{ad}}$ to the minimum. □

5 An example

Example 5.1 Consider the following impulsive fractional differential equations:

$$\begin{cases} {}^L D_{0^+}^{\alpha} x(t) = \frac{e^{-t}|x(t)|}{(1+e^t)(1+|x(t)|)} + u(t), \quad t \in J' = (0,1], t \neq t_1, \\ I_{0^+}^{1-\alpha} x(0) = g(x), \\ {}^L D_{0^+}^{\beta} x(t_1^+) - {}^L D_{0^+}^{\beta} x(t_1^-) = \frac{|x(t_1)|}{3+|x(t_1)|}. \end{cases} \tag{5.1}$$

We take

$$f : (0,1] \times R \to R \quad \text{and} \quad f(t,x) = \frac{e^{-t}|x|}{(1+e^t)(1+|x|)},$$

$$g : PC_\delta\bigl((0,T];X\bigr) \to X \quad \text{and} \quad g(x) \text{ is a continuous and compact map,}$$

$$J_1 : X \to X \quad \text{and} \quad J_1(x) = \frac{|x|}{3+|x|}.$$

Let $x_1, x_2 \in B_r$ and $t \in (0,1]$, we have

$$\begin{aligned} \bigl|f(t,x_1) - f(t,x_2)\bigr| &= \frac{e^{-t}}{(1+e^t)} \left| \frac{|x_1|}{1+|x_1|} - \frac{|x_2|}{1+|x_2|} \right| = \frac{e^{-t}||x_1| - |x_2||}{(1+e^t)(1+|x_1|)(1+|x_2|)} \\ &\le \frac{e^{-t}}{1+e^t}|x_1 - x_2| \le \frac{e^{-t}}{2}|x_1 - x_2|. \end{aligned}$$

Obviously, for each $x_1, x_2 \in B_r$ and $t \in (0,1]$,

$$\bigl|f(t,x)\bigr| = \frac{e^{-t}}{1+e^t}\frac{|x|}{1+|x|} \le \frac{e^{-t}}{1+e^t} \le \frac{e^{-t}}{2}.$$

$$\bigl|J_1(x_1) - J_1(x_2)\bigr| \le \frac{1}{3}|x_1 - x_2|.$$

So hypotheses (H1)–(H6) are satisfied.

We define the cost function $J(x^u, u) = \int_0^1 (\|x^u(t)\|_X^2 + \|u(t)\|_Y^2)\,dt$, where (x^u, u) is a feasible pair. If the inequality

$$\begin{aligned} &T^{\alpha+\delta-1} \sup_{x \in W_r} \bigl\|g(x)\bigr\| + T^{\delta} \sup_{t \in (0,1]} \left\{ I_0^{\alpha} \left| \frac{e^{-t}}{2} \right| \right\} + \frac{r}{3} T^{\alpha+\delta-1} \Gamma(\alpha - \beta) t_1^{1+\beta-\alpha} \\ &\quad + \|x_0\| + \frac{T^{\alpha+\delta}\|Bu\|_{L^1}}{\alpha} \le r \end{aligned}$$

holds, so all the conditions in Theorem 3.1 are satisfied, our results can be applied to problem (5.1).

6 Conclusions

In this paper, we have considered the existence of solution and optimal controls for a class of impulsive fractional differential equations with fractional impulsive conditions and nonlocal conditions. By constructing operator S, using the measure of noncompactness method and fixed point theorem, we derived some sufficient assumptions to guarantee the existence of solution. By constructing approximating minimizing sequences of functions twice, we also obtained the optimal controls of impulsive fractional differential equations. Our results are more general than known results. Based on the results of this paper, we will investigate the impulsive fractional semilinear differential equations in the next paper.

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The authors declare that they have no competing interest.

Authors' contributions
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