Gao *Advances in Difference Equations* (2018) 2018:121
https://doi.org/10.1186/s13662-018-1576-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Mittag–Leffler stability for a new coupled system of fractional-order differential equations on network

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Abstract

In this paper, the stability problem of a new coupled model constructed by two fractional-order differential equations for every vertex is studied. The coupled relationship is hybrid. By using the method of constructing Lyapunov functions based on graph-theoretical approach for coupled systems, sufficient conditions that the coexistence equilibrium of the coupling model is globally Mittag–Leffler stable in R^{2n} are derived. An example is given to illustrate the main results.

Keywords: Mittag–Leffler stable; Coupled model; Global stability; Caputo derivative

1 Introduction

The global-stability problem of equilibria has been investigated for coupled systems of differential equations on networks for many years [1–6]. For example, Li and Shuai developed a systematic approach that allowed one to construct global Lyapunov functions for large-scale coupled systems from building blocks of individual vertex systems by using results from graph theory. The approach was applied to several classes of coupled systems in engineering, ecology, and epidemiology. Although there exist many results about stability of coupled systems on networks (CSNs), most efforts have been devoted to CSNs whose nodes are constructed by integer-order differential equations. In fact, it is more valuable and practical to investigate a coupled system of fractional-order differential equations on the network. Recently, Li [7] investigated the global Mittag–Leffler stability of the following coupled system of fractional-order differential equations on network (CSFDEN):

$$\begin{cases} {}_{t_0}D_t^{\alpha} x_i = -\alpha_i x_i(t) + f_i(x_i(t)) + \sum_{j=1}^{n} \beta_{ij}^{x}(x_j(t) - x_i(t)), \\ x_i(t_0) = x_{it_0}, \quad i = 1, \ldots, n, \end{cases} \tag{1}$$

where D denoted Caputo fractional derivative, $\alpha \in (0,1)$. t_0 was the initial time, n ($n \geq 2$) denoted the number of vertices in the network. $(x(t))^T = (x_1(t), x_2(t), \ldots, x_n(t))^T$ denoted the state variable of the system where $x_i(t) \in R$. α_i was a positive constant. Constant β_{ij}^x represented the influence of vertex j on vertex i with $\beta_{ii}^x = 0$, $\beta_{ij}^x = -\beta_{ji}^x$, if $i \neq j$. Function f_i was Lipschitz continuous. Several sufficient conditions were obtained to ensure the Mittag–Leffler stability of CSFDEN by using graph theory and the Lyapunov method.

Furthermore, Li [8] investigated a coupled system of fractional-order differential equations on network with feedback controls (CSFDENFCs). By using the contraction mapping principle, Lyapunov method, graph theoretic approach, and inequality techniques, some sufficient conditions were derived to ensure the existence, uniqueness, and global Mittag–Leffler stability of the equilibrium point of CSFDENFCs.

As far as we know, most of researchers are interested in CSNs constructed by only one fractional-order differential equation for every vertex. To the best of author's knowledge, there are less results about CSNs constructed by two or many fractional-order differential equations for every vertex. In this paper, the coupled model (1) is generalized to the more complicated model. The vertex's dynamical character is presented by the two-dimensional system. The coupled relationship is constructed by two components of the vertex. The coupled system of fractional differential equations on network is studied. Sufficient conditions that the coexistence equilibrium of the coupling model is globally Mittag–Leffler stable in R^{2n} are derived by using the method of constructing Lyapunov functions based on graph-theoretical approach for coupled systems.

Remark 1.1 In fact, the generalization of model (1) is important and meaningful. Because a lot of ecological models can be seen as high-dimensional coupled systems. Every node is constructed by two or many differential equations in integer-order systems. For example, predator–prey models with patches and dispersal are studied by a lot of researchers [1–6].

This paper is organized as follows. Preliminary results are introduced in Sect. 2. In Sect. 3, the main results are obtained. In the sequel, an example is presented in Sect. 4. Finally, the conclusions and outlooks are drawn in Sect. 5.

2 Preliminaries

In this section, we list some definitions and theorems which will be used in the later sections.

A directed graph or digraph $G = (V,E)$ contains a set $V = \{1,2,\dots,n\}$ of vertices and a set E of arcs (i,j) leading from initial vertex i to terminal vertex j. A subgraph H of G is said to be spanning if H and G have the same vertex set. A digraph G is weighted if each arc (j,i) is assigned a positive weight. $a_{ij} > 0$ if and only if there exists an arc from vertex j to i in G.

The weight $w(H)$ of a subgraph H is the product of the weights on all its arcs. A directed path P in G is a subgraph with distinct vertices $i_1, i_2, \dots, i_m$ such that its set of arcs is $\{(i_k, i_{k+1}) : k = 1,2,\dots,m\}$. If $i_m = i_1$, we call P a directed cycle.

A connected subgraph T is a tree if it contains no cycles, directed or undirected.

A tree T is rooted at vertex i, called the root if i is not a terminal vertex of any arcs, and each of the remaining vertices is a terminal vertex of exactly one arc. A subgraph Q is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle.

Given a weighted digraph G with n vertices, the weight matrix $A = (a_{ij})_{n\times n}$ can be defined by their entry a_{ij} equals the weight of arc (j,i) if it exists, and 0 otherwise. We denote a weighted digraph as (G,A). A digraph G is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph (G,A) is strongly connected if and only if the weight matrix A is irreducible.

The Laplacian matrix of (G,A) is denoted by L. Let c_i denote the cofactor of the ith diagonal element of L. The following results are listed.

Lemma 2.1 ([6]) *Assume $n \geq 2$. Then*

$$c_i = \sum_{\mathbf{T} \in T_i} w(\mathbf{T}),$$

where T_i is the set of all spanning trees $\mathbf{T}$ of (G, A) that are rooted at vertex i, and $w(T)$ is the weight of T. In particular, if (G, A) is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.

Lemma 2.2 ([6]) *Assume $n \geq 2$. Let c_i be given in Lemma 2.1. Then the following identity holds:*

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \sum_{Q \in \mathrm{Q}} w(Q) \sum_{(s,r) \in E(C_{\mathrm{Q}})} F_{rs}(x_r, x_s).$$

Here, $F_{ij}(x_i, x_j), 1 \leq i, j \leq n$, are arbitrary functions, Q is the set of all spanning unicyclic graphs of (G, A), $w(Q)$ is the weight of Q, and C_{Q} denotes the directed cycle of Q.

If (G, A) is balanced, then

$$\sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(x_i, x_j) = \frac{1}{2} \sum_{Q \in \mathrm{Q}} w(Q) \sum_{(j,i) \in E(C_{\mathrm{Q}})} \big[F_{ij}(x_i, x_j) + F_{ji}(x_j, x_i)\big].$$

Definition 2.3 ([9]) The Caputo fractional derivative of order $\alpha \in (n-1, n)$ for a continuous function $f : R^+ \to R$ is given by

$${}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds.$$

Definition 2.4 ([7, 9]) The solution of the system

$${}_{t_0}D_t^{\alpha} x(t) = f(t, x)$$

is said to be Mittag–Leffler stable if

$$\|x(t)\| \leq \big\{m\big[x(t_0)\big] E_{\alpha}\big(-\lambda (t-t_0)^{\alpha}\big)\big\}^{b}.$$

Here, t_0 is the initial time, $\alpha \in (0,1), \lambda > 0, b > 0, m(0) = 0, m(x) \geq 0$. $m(x)$ is locally Lipschitz on $x \in B \subseteq R^n$ with Lipschitz constant m_0. $E_{\alpha}(t)$ is a Mittag–Leffler function. Moreover, the domain of the function $f(t, x)$ is $[t_0, +\infty) \times \Omega$, and the function $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x.

3 Main results

A coupled system of fractional differential equations on network is constructed as follows:

$$\begin{cases} {}_{t_0}D_t^{\alpha} x_i = -\alpha_i x_i(t) + \theta_i y_i(t) + f_i(x_i(t)) + \sum_{j=1}^{n} \beta_{ij}^{x}(y_j(t) - x_i(t)), \\ {}_{t_0}D_t^{\alpha} y_i = -\beta_i y_i(t) - \varepsilon_i x_i(t) + g_i(y_i(t)) + \sum_{j=1}^{n} \beta_{ij}^{y}(x_j(t) - y_i(t)), \\ x_i(t_0) = x_{it_0}, \qquad y_i(t_0) = y_{it_0}, \quad i = 1, \ldots, n. \end{cases} \tag{2}$$

Here, D denotes the Caputo fractional derivative, $\alpha \in (0,1)$. t_0 is the initial time, $n(n \geq 2)$ denotes the number of vertices in the network. $z(t) = (x(t), y(t))^T = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_n(t))^T$ denotes the state variable of the system where $x_i(t) \in R$ and $y_i(t) \in R$. α_i, β_i, θ_i, ε_i are all positive constants. Constant β^x_{ij} represents the influence of y_j on x_i with $\beta^x_{ii} = 0, \beta^x_{ij} = -\beta^x_{ji}$, if $i \neq j$. Constant β^y_{ij} represents the influence of x_j on y_i with $\beta^y_{ii} = 0, \beta^y_{ij} = -\beta^y_{ji}$, if $i \neq j$.

The following assumptions are given for system (2).

(H_1) Functions f_i, g_i are Lipschitz-continuous on R with Lipschitz constant $L^x_i > 0, L^y_i > 0$, respectively, i.e.,

$$\begin{aligned} \left|f_i(u) - f_i(v)\right| &\leq L^x_i |u - v|, \\ \left|g_i(u) - g_i(v)\right| &\leq L^y_i |u - v| \end{aligned}$$

for all $u, v \in R$.

(H_2) There exists a constant λ such that

$$\lambda = \min\left\{2\left(\alpha_i + \sum_{j=1}^{n} \beta^x_{ij} - L^x_i\right), 2\left(\beta_i + \sum_{j=1}^{n} \beta^y_{ij} - L^y_i\right) \,\Bigg|\, i = 1, 2, \dots, n\right\} > 0.$$

A mathematical description of a network is a directed graph consisting of vertices and directed arcs connecting them. At each vertex, the local dynamics are given by a system of differential equations called the vertex system. The directed arcs indicate interconnections and interactions among vertex systems.

Let β_{ij} represent the influence of vertex j on vertex i, with

$$\beta_{ij} = \begin{cases} \beta^x_{ij}\theta^{-1}_j, & \text{if } |\beta^x_{ij}\theta^{-1}_j| \geq |\beta^y_{ij}\varepsilon^{-1}_j|, \\ \beta^y_{ij}\varepsilon^{-1}_j, & \text{if } |\beta^x_{ij}\theta^{-1}_j| < |\beta^y_{ij}\varepsilon^{-1}_j|. \end{cases}$$

Let $A = (|\beta_{ij}|)_{n\times n}$, $A^x = (|\beta^x_{ij}|)_{n\times n}$, $A^y = (|\beta^y_{ij}|)_{n\times n}$.

A digraph (G, A) with n vertices for system (2) can be constructed as follows. Each vertex represents a patch and $(j, i) \in E(G)$ if and only if $\beta^x_{ij} \neq 0$ or $\beta^y_{ij} \neq 0$. Here, $E(G)$ denotes the set of arcs (i, j) leading from initial vertex i to terminal vertex j. At each vertex of G, the vertex dynamics are described by the following system (3):

$$\begin{cases} {}_{t_0}D^{\alpha}_t x_i = -\alpha_i x_i(t) + \theta_i y_i(t) + f_i(x_i(t)), \\ {}_{t_0}D^{\alpha}_t y_i = -\beta_i y_i(t) - \varepsilon_i x_i(t) + g_i(y_i(t)). \end{cases} \tag{3}$$

The coupling among system (2) is provided by the network. The G is strongly connected if and only if the matrix $A = (|\beta_{ij}|)_{n\times n}$ is irreducible.

In this section, the coupled system of fractional differential equations on network is studied. By using the method of constructing Lyapunov functions based on graph-theoretical approach for coupled systems, sufficient conditions that the coexistence equilibrium of the coupling model (2) is globally Mittag–Leffler stable in R^{2n} are derived.

We obtain the main theorem as follows.

Theorem 3.1 *Assume that the following conditions hold*:

1. *Diagraph* (G,A) *is balanced*;
2. $A^x = (|\beta_{ij}^x|)_{n\times n}, A^y = (|\beta_{ij}^y|)_{n\times n}$ *are irreducible*;
3. *Conditions* (H_1) *and* (H_2) *hold*;
4. *The formula* $\beta_{ij}^x\theta_j^{-1} = \beta_{ij}^y\varepsilon_j^{-1}$ *holds for* $i,j = 1,2,\ldots,n$.

Then system (2) *is globally Mittag–Leffler stable*.

Proof Let $E^* = (x^*,y^*)^T = (x_1^*,x_2^*,\ldots,x_n^*,y_1^*,y_2^*,\ldots,y_n^*)^T$ be an equilibrium of (2). Assume that $e_i^x(t) = x_i(t) - x_i^*, e_i^y(t) = y_i(t) - y_i^*$ $(i = 1,2,\ldots,n)$. After calculating, we obtain that

$$\begin{aligned}
{}_{t_0}D_t^{\alpha} e_i^x(t) ={}& -\alpha_i e_i^x(t) + \theta_i e_i^y(t) + f_i\bigl(x_i^* + e_i^x(t)\bigr) - f_i\bigl(x_i^*\bigr) \\
&+ \sum_{j=1}^{n} \beta_{ij}^x\bigl(y_j^* + e_j^y(t) - x_i^* - e_i^x(t)\bigr) - \sum_{j=1}^{n} \beta_{ij}^x\bigl(y_j^* - x_i^*\bigr), \\
{}_{t_0}D_t^{\alpha} e_i^y(t) ={}& -\beta_i e_i^y(t) - \varepsilon_i e_i^x(t) + g_i\bigl(y_i^* + e_i^y(t)\bigr) - g_i\bigl(y_i^*\bigr) \\
&+ \sum_{j=1}^{n} \beta_{ij}^y\bigl(x_j^* + e_j^x(t) - y_i^* - e_i^y(t)\bigr) - \sum_{j=1}^{n} \beta_{ij}^y\bigl(x_j^* - y_i^*\bigr).
\end{aligned}$$

Let

$$e(t) = \bigl(e_1^x(t), e_1^y(t), e_2^x(t), e_2^y(t), \ldots, e_n^x(t), e_n^y(t)\bigr)$$

and

$$V_i\bigl(e_i^x(t), e_i^y(t)\bigr) = \frac{1}{2}\bigl[\varepsilon_i\bigl(e_i^x(t)\bigr)^2 + \theta_i\bigl(e_i^x(t)\bigr)^2\bigr].$$

From condition 2 of Theorem 3.1, we have the matrix A is irreducible. Furthermore, (G,A) is strongly connected. Let c_i denote the cofactor of the ith diagonal element of Laplacian matrix of (G,A). Then we have $c_i > 0$. Let

$$V\bigl(t,e(t)\bigr) = \sum_{i=1}^{n} c_i V_i\bigl(e_i^x(t), e_i^y(t)\bigr).$$

The α-derivative of V along the trajectories of system (2) is

$$\begin{aligned}
&{}_{t_0}D_t^{\alpha} V\bigl(t,e(t)\bigr) \\
&\quad= \frac{1}{2}\sum_{i=1}^{n} c_{i\,t_0}D_t^{\alpha}\bigl[\varepsilon_i\bigl(e_i^x(t)\bigr)^2 + \theta_i\bigl(e_i^y(t)\bigr)^2\bigr] \\
&\quad\le \sum_{i=1}^{n}\bigl[c_i\varepsilon_i e_i^x(t)\,{}_{t_0}D_t^{\alpha} e_i^x(t) + c_i\theta_i e_i^y(t)\,{}_{t_0}D_t^{\alpha} e_i^y(t)\bigr] \\
&\quad\le \sum_{i=1}^{n} c_i e_i^x(t) 2\Biggl(-\alpha_i - \sum_{j=1}^{n}\beta_{ij}^x + L_i^x\Biggr)\varepsilon_i e_i^x(t) + \sum_{i=1}^{n} c_i e_i^y(t) 2\Biggl(-\beta_i - \sum_{j=1}^{n}\beta_{ij}^y + L_i^y\Biggr)\theta_i e_i^y(t)
\end{aligned}$$

$$
\begin{aligned}
&+ c_i\varepsilon_i\theta_i e_i^y(t)e_i^x(t) - c_i\theta_i\varepsilon_i e_i^x(t)e_i^y(t) + \sum_{i=1}^n c_i\varepsilon_i\beta_{ij}^x\theta_j^{-1}\theta_j e_i^x e_j^y + \sum_{i=1}^n c_i\theta_i\beta_{ij}^y\varepsilon_j^{-1}\varepsilon_j e_i^y e_j^x \\
&= \sum_{i=1}^n c_i e_i^x(t)2\left(-\alpha_i - \sum_{j=1}^n \beta_{ij}^x + L_i^x\right)\varepsilon_i e_i^x(t) + \sum_{i=1}^n c_i e_i^y(t)2\left(-\beta_i - \sum_{j=1}^n \beta_{ij}^y + L_i^y\right)\theta_i e_i^y(t) \\
&+ c_i\varepsilon_i\theta_i e_i^y(t)e_i^x(t) - c_i\theta_i\varepsilon_i e_i^x(t)e_i^y(t) + \sum_{i=1}^n c_i a_{ij} F_{ij}^x(t,x,y) + \sum_{i=1}^n c_i a_{ij} F_{ij}^y(t,x,y).
\end{aligned}
$$

Here, $a_{ij} = |\beta_{ij}| = |\beta_{ij}^x\theta_j^{-1}| = |\beta_{ij}^y\varepsilon_j^{-1}|$ and $F_{ij}^x(t,x,y) = \operatorname{sgn}(\beta_{ij})\varepsilon_i\theta_j e_i^x e_j^y$, $F_{ij}^y(t,x,y) = \operatorname{sgn}(\beta_{ij})\varepsilon_j\theta_i e_i^y e_j^x$. From (G,A)'s balanced and strongly connected character, it follows that

$$
\begin{aligned}
&\sum_{i=1}^n c_i a_{ij} F_{ij}^x(t,x,y) = \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[F_{ij}^x(t,x,y) + F_{ji}^x(t,x,y)\right], \\
&\sum_{i=1}^n c_i a_{ij} F_{ij}^y(t,x,y) = \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[F_{ij}^y(t,x,y) + F_{ji}^y(t,x,y)\right].
\end{aligned}
$$

Furthermore, we obtain that

$$
\begin{aligned}
&\sum_{i=1}^n c_i a_{ij}\left[F_{ij}^x(t,x,y) + F_{ij}^y(t,x,y)\right] \\
&\quad= \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[F_{ij}^x(t,x,y) + F_{ji}^y(t,x,y)\right] \\
&\qquad+ \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[F_{ij}^y(t,x,y) + F_{ji}^x(t,x,y)\right] \\
&\quad= \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[\operatorname{sgn}(\beta_{ij})\theta_i\varepsilon_j e_i^x e_j^y + \operatorname{sgn}(\beta_{ji})\theta_i\varepsilon_j e_i^x e_j^y\right] \\
&\qquad+ \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[\operatorname{sgn}(\beta_{ij})\varepsilon_j\theta_i e_i^y e_j^x + \operatorname{sgn}(\beta_{ij})\varepsilon_j\theta_i e_i^y e_j^x\right] \\
&\quad= \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[\operatorname{sgn}(\beta_{ij})\theta_i\varepsilon_j e_i^x e_j^y - \operatorname{sgn}(\beta_{ij})\theta_i\varepsilon_j e_i^x e_j^y\right] \\
&\qquad+ \frac{1}{2}\sum_{Q\in \mathrm{Q}} w(Q) \sum_{(j,i)\in E(C_{\mathrm{Q}})} \left[\operatorname{sgn}(\beta_{ij})\varepsilon_j\theta_i e_i^y e_j^x - \operatorname{sgn}(\beta_{ij})\varepsilon_j\theta_i e_i^y e_j^x\right] \\
&\quad= 0 + 0 \\
&\quad= 0.
\end{aligned}
$$

In the sequel, we have

$$
{}_{t_0}D_t^\alpha V\bigl(t,e(t)\bigr) \le -\lambda V\bigl(t,e(t)\bigr).
$$

Let

$$
{}_{t_0}D_t^\alpha V\bigl(t,e(t)\bigr) + M(t) = -\lambda V\bigl(t,e(t)\bigr).
$$

Using the Laplace transform for the equation above, we obtain that

$$s^{\alpha}w(s) - w(0)s^{\alpha-1} + M(s) = -\beta w(s),$$

where $w(s)$, $M(s)$ are the Laplace transform of $V(t,e(t))$ and $M(t)$, respectively. Using the inverse Laplace transform for the formula above, we have

$$V\bigl(t,e(t)\bigr) \le V\bigl(0,e(0)\bigr)E_{\alpha}\bigl(-\beta t^{\alpha}\bigr).$$

By the definition of $V(t,e(t))$, we obtain that system (2) is globally Mittag–Leffler stable. Then the proof is completed. □

By Theorem 3.1, we obtain the following corollary naturally.

Corollary 3.2 *Consider the model*

$$\begin{cases} {}_{t_0}D_t^{\alpha}x_i = -\alpha_i x_i(t) + \theta_i y_i(t) + f_i(x_i(t)) + \sum_{j=1}^{n}\beta_{ij}^{x}(y_j(t) - x_i(t)), \\ {}_{t_0}D_t^{\alpha}y_i = -\beta_i y_i(t) - \varepsilon_i x_i(t) + g_i(y_i(t)) + \sum_{j=1}^{n}\beta_{ij}^{x}(x_j(t) - y_i(t)), \\ x_i(t_0) = x_{it_0}, \qquad y_i(t_0) = y_{it_0}, \quad i = 1,\ldots,n. \end{cases} \tag{4}$$

Assume that (G,A) is balanced and $A = A^x = (|\beta_{ij}^{x}|)_{n\times n}$ is irreducible, $\theta_i = \varepsilon_i$ for any $i = 1,2,\ldots,n$, conditions (H_1) and (H_2) hold. Then system (4) *is globally Mittag–Leffler stable.*

4 An example

In this section, an example is presented to illustrate Theorem 3.1. Consider the following system of fractional equations on network:

$$\begin{cases} {}_{t_0}D_t^{\alpha}x_1(t) = -\alpha_1 x_1(t) + \theta_1 y_1(t) + f_1(x_1(t)) + \sum_{j=1}^{n}\beta_{1j}^{x}(y_j(t) - x_1(t)), \\ {}_{t_0}D_t^{\alpha}y_1(t) = -\beta_1 y_1(t) - \varepsilon_1 x_1(t) + g_1(y_1(t)) + \sum_{j=1}^{n}\beta_{1j}^{y}(x_j(t) - y_1(t)), \\ {}_{t_0}D_t^{\alpha}x_2(t) = -\alpha_2 x_2(t) + \theta_2 y_2(t) + f_2(x_2(t)) + \sum_{j=1}^{n}\beta_{2j}^{x}(y_j(t) - x_2(t)), \\ {}_{t_0}D_t^{\alpha}y_2(t) = -\beta_2 y_2(t) - \varepsilon_2 x_2(t) + g_2(y_2(t)) + \sum_{j=1}^{n}\beta_{2j}^{y}(x_j(t) - y_2(t)), \end{cases} \tag{5}$$

where, $\alpha = 0.5$, $\alpha_1 = \alpha_2 = 5$, $\beta_1 = \beta_2 = 9$, $2\theta_1 = 2\theta_2 = \varepsilon_1 = \varepsilon_2 = 1$, $f_1(x_1(t)) = \sin(x_1(t))$, $f_2(x_2(t)) = \sin(x_2(t))$, $g_1(y_1(t)) = \sin(2y_1(t))$, $g_2(y_2(t)) = \sin(2y_2(t))$, and $\beta_{11}^{x} = \beta_{22}^{x} = \beta_{11}^{y} = \beta_{22}^{y} = 0$, $\beta_{12}^{x} = -\beta_{21}^{x} = 3$, $\beta_{12}^{y} = -\beta_{21}^{y} = 6$. Therefore, we have

$$A = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 6 & -6 \\ -6 & 6 \end{pmatrix}.$$

Then we obtain that $c_1 = c_2 = 6$. Obviously, (G,A) is strongly connected and balanced. It is easy to obtain that conditions H_1, H_2 hold. According to Theorem 3.1, system (5) has an equilibrium point $(0,0,0,0)$ which is globally Mittag–Leffler stable. The solution and Lyapunov function $V(t,e(t))$ for system (5) with initial value $x_0 = (0,0,0,0.1)$ are shown in Figs. 1 and 2.

Model (5) can be regarded as the ecology model with two patches and dispersal. Some kind of rate of change for species can be denoted by fractional order differential for the

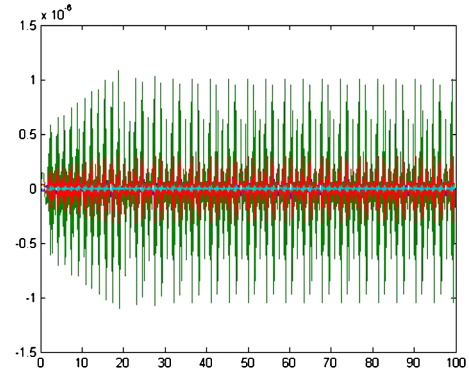

Figure 1 The solution

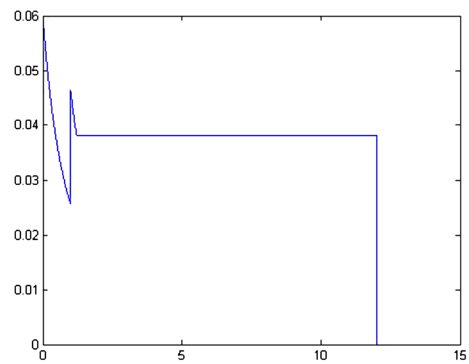

Figure 2 The Lyapunov function

species. This kind of rate of change of species x_i is related to y_j through the dispersal. Moreover, the rate of change of species y_i is related to x_j through the dispersal. The model with patches and dispersal is important in application [1–6]. In the sequel, model (5) is universal and useful in reality.

5 Conclusions and utlooks

In this paper, the new coupled model constructed by two fractional-order differential equations for every vertex is studied. The coupled relationship is constructed by two components of the vertex. By using the method of constructing Lyapunov functions based on graph-theoretical approach for coupled systems, sufficient conditions that the coexistence equilibrium of the coupling model is globally Mittag–Leffler stable in R^{2n} are derived. Finally, an example is given to illustrate the main results.

Theorem 3.1 is the main result of this paper. This result is different from the previous studies. Firstly, Theorem 3.1 is a generalization of the main result of Li [7]. The model in this paper is more complicated for every vertex and the conditions of Theorem 3.1 are different from the result in Li [7]. Secondly, Theorem 3.1 is different from Shuai's results [6] with the integer order differential. The fractional order differential is the main character for this paper. Moreover, the hybrid coupled relation is new and different from the previous papers [1–7]. There are also many difficulties in proving Theorem 3.1. One is the proof of $\sum_{i=1}^{n} c_i a_{ij}[F_{ij}^{x}(t,x,y) + F_{ij}^{y}(t,x,y)] = 0$; the other is the construction of the Lyapunov function for the coupled system.

Further studies on this subject are being carried out by the presenting author in the two aspects: one is to study the model with time delay; the other is to discuss the method to design control terms.

Acknowledgements
This research is supported by the Natural Science Foundation of HeiLongJiang Province (No. QC2009C99), Science and Technology Planning Project of Daqing City under Grant No. zd-2017-49.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 2 November 2017 Accepted: 21 March 2018 Published online: 03 April 2018

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