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Oscillation of differential equations in the frame of nonlocal fractional derivatives generated by conformable derivatives

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Abstract

Recently, Jarad et al. in (Adv. Differ. Equ. 2017:247, 2017) defined a new class of nonlocal generalized fractional derivatives, called conformable fractional derivatives (CFDs), based on conformable derivatives. In this paper, sufficient conditions are established for the oscillation of solutions of generalized fractional differential equations of the form

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho}x(t) + f_1(t,x) = r(t) + f_2(t,x), & t > a, \\ \lim_{t\to a^+} {}_a\mathfrak{I}^{j-\alpha,\rho}x(t) = b_j \quad (j = 1,2,\ldots,m), \end{cases}$$

where $m = \lceil\alpha\rceil$, ${}_a\mathfrak{D}^{\alpha,\rho}$ is the left-fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$ in the Riemann–Liouville setting and ${}_a\mathfrak{I}^{\alpha,\rho}$ is the left-fractional conformable integral operator. The results are also obtained for CFDs in the Caputo setting. Examples are provided to demonstrate the effectiveness of the main result.

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1 Introduction

Fractional calculus is still being developed continuously and its operators are used to model complex systems where the kernel of the fractional operators reflects the nonlocality [2, 3]. The singularity of the kernel of the fractional operators has recently motivated researchers to present new types of fractional operators with nonsingular kernels and their discrete versions [4–12]. This new trend added another approach in defining fractional derivatives and integrals. In classical fractional operators, the fractionalizing process depends on iterating the weighted usual integrals or some local type derivatives in the way to get the factorial function and then replace it by the gamma function. The theory of fractional calculus with operators having nonsingular kernels depends on a limiting approach via dirac delta functions. Indeed, the fractional derivative with the nonsingular kernel is first defined so that in the limiting case $\alpha \to 0$ we get the function itself and when $\alpha \to 1$ we get the usual derivative of the function. Then the corresponding integral operators are evaluated by the help of Laplace transforms for functions whose convolution with the nonsingular kernel vanishes at the starting point a. The oscillation theory for fractional differential and difference equations was studied by some authors (see [13–21]), thus several definitions of fractional derivatives and fractional integral operators exist in the literature. In this article, we study the oscillation of fractional operators defined by the first approach mentioned above. Namely, we investigate the oscillation of a class of generalized fractional derivatives defined in [1] by iterating the local conformable derivative developed in [22]. We shall name this derivative the conformable fractional derivative (CFD). Although both the CFD with its Caputo setting and the Katugampola-type derivative studied in [23–25] coincide when $a = 0$, they are very different from each other. In fact, the kernel of CFDs depends on the end points a and b which causes many differences from the Katugampola-type one. We shall study the oscillation of conformable fractional differential equation of the form

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho}x(t) + f_1(t,x) = r(t) + f_2(t,x), \quad t > a, \\ \lim_{t\to a^+} {}_a\mathfrak{I}^{j-\alpha,\rho}x(t) = b_j \quad (j = 1,2,\ldots,m), \end{cases} \tag{1}$$

where $m = \lceil \alpha \rceil$, ${}_a\mathfrak{D}^{\alpha,\rho}$ is the left-fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$ in the Riemann–Liouville setting and ${}_a\mathfrak{I}^{\alpha,\rho}$ is the left-fractional conformable integral operator.

The objective of this paper is to study the oscillation of conformable fractional differential equations of the form (1). This will generalize the results obtained in [13, 14] when we take $a = 0$.

This paper is organized as follows. Section 2 introduces some notations and provides the definitions of conformable fractional integral and differential operators together with some basic properties and lemmas that are needed in the proofs of the main theorems. In Sect. 3, the main theorems are presented. Section 4 is devoted to the results obtained for the conformable fractional operators in the Caputo setting where we also remark the oscillation of Katugampola-type fractional operators. Examples are provided in Sect. 5 to demonstrate the effectiveness of the main theorems.

2 Notations and preliminary assertions

The left conformable derivative starting from a of a function $f : [a,\infty) \to \mathbb{R}$ of order $0 < \rho \leq 1$ is defined by

$$\big({}_aT^{\rho}f\big)(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon(t-a)^{1-\rho}) - f(t)}{\epsilon}.$$

If $(T^{\rho}f)(t)$ exists on (a,b), then $({}_aT^{\rho}f)(a) = \lim_{t\to a^+} ({}_aT^{\rho}f)(t)$.

If f is differentiable, then

$$\big({}_aT^{\rho}f\big)(t) = (t-a)^{1-\rho}f'(t). \tag{2}$$

The corresponding left conformable integral is defined as

$${}_a\mathfrak{I}^{\rho}f(x) = \int_a^x f(t)\frac{dt}{(t-a)^{1-\rho}}, \quad 0 < \rho \leq 1.$$

For the extension to the higher order $\rho > 1$, see [22].

Definition 2.1 ([1]) The left-fractional conformable integral operator is defined by

$$ {}_a\mathfrak{I}^{\alpha,\rho}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\frac{(x-a)^\rho-(t-a)^\rho}{\rho}\right)^{\alpha-1}\frac{f(t)\,dt}{(t-a)^{1-\rho}}, \tag{3} $$

where $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) \geq 0$.

Definition 2.2 ([1]) The left-fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) \geq 0$ in the Riemann–Liouville setting is defined by

$$ \begin{aligned} {}_a\mathfrak{D}^{\alpha,\rho}f(x) &= {}^m_a T^\rho \left({}_a\mathfrak{I}^{m-\alpha,\rho}\right)f(x) \\ &= \frac{{}^m_a T^\rho}{\Gamma(m-\alpha)}\int_a^x \left(\frac{(x-a)^\rho-(t-a)^\rho}{\rho}\right)^{m-\alpha-1}\frac{f(t)\,dt}{(t-a)^{1-\rho}}, \end{aligned} $$

where $m = \lceil \operatorname{Re}(\alpha)\rceil$, ${}^m_a T^\rho = \underbrace{{}_aT^\rho\,{}_aT^\rho\cdots{}_aT^\rho}_{m \text{ times}}$ and ${}_aT^\alpha$ is the left conformable differential operator presented in (2).

Definition 2.3 ([1]) The left-Caputo fractional conformable derivative of order $\alpha \in \mathbb{C}$, $\operatorname{Re}(\alpha) \geq 0$ is defined by

$$ \begin{aligned} {}^C_a\mathfrak{D}^{\alpha,\rho}f(x) &= {}_a\mathfrak{I}^{m-\alpha,\rho}\left({}^m_a T^\rho f(x)\right) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_a^x \left(\frac{(x-a)^\rho-(t-a)^\rho}{\rho}\right)^{m-\alpha-1}\frac{{}^m_a T^\rho f(t)}{(t-a)^{1-\rho}}\,dt. \end{aligned} \tag{4} $$

The following identity (see [1]) is essential to solving linear conformable fractional differential equations:

$$ \left({}_a\mathfrak{I}^{\alpha,\rho}(t-a)^{\rho\nu-\rho}\right)(x) = \frac{1}{\rho^\alpha}\frac{\Gamma(\nu)}{\Gamma(\alpha+\nu)}(x-a)^{\rho(\alpha+\nu-1)}, \quad \operatorname{Re}(\nu) > 0. \tag{5} $$

Lemma 2.1 (Young's inequality [26])

(i) *Let* $X, Y \geq 0$, $u > 1$, *and* $\frac{1}{u}+\frac{1}{v} = 1$, *then* $XY \leq \frac{1}{u}X^u + \frac{1}{v}Y^v$.

(ii) *Let* $X \geq 0$, $Y > 0$, $0 < u < 1$, *and* $\frac{1}{u}+\frac{1}{v} = 1$, *then* $XY \geq \frac{1}{u}X^u + \frac{1}{v}Y^v$, *where equalities hold if and only if* $Y = X^{u-1}$.

3 Oscillation of conformable fractional differential equations in the frame of Riemann

In this section we study the oscillation theory for equation (1).

Lemma 3.1 ([1]) *Let* $\operatorname{Re}(\alpha) > 0$, $m = -[-\operatorname{Re}(\alpha)]$, $f \in L(a,b)$, *and* ${}_a\mathfrak{I}^{\alpha,\rho}f \in C^m_{\rho,a}[a,b]$. *Then*

$$ {}_a\mathfrak{I}^{\alpha,\rho}\left({}_a\mathfrak{D}^{\alpha,\rho}f(x)\right) = f(x) - \sum_{j=1}^m \frac{{}_a\mathfrak{D}^{\alpha-j,\rho}f(a)}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}(x-a)^{\rho\alpha-\rho j}. $$

Using Lemma 3.1, we can write the solution representation of (1) as

$$ x(t) = \sum_{j=1}^m \frac{{}_a\mathfrak{D}^{\alpha-j,\rho}x(a)}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}(t-a)^{\rho\alpha-\rho j} + {}_a\mathfrak{I}^{\alpha,\rho}F(t,x), \tag{6} $$

where $F(t,x) = r(t) + f_2(t,x) - f_1(t,x)$ and $\rho > 0$.

A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on $(0,\infty)$; otherwise, it is called nonoscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

We prove our results under the following assumptions:

$$xf_i(t,x) > 0 \quad (i = 1,2), x \neq 0, t \geq 0, \tag{7}$$

$$\left|f_1(t,x)\right| \geq p_1(t)|x|^{\beta} \quad \text{and} \quad \left|f_2(t,x)\right| \leq p_2(t)|x|^{\gamma}, \quad x \neq 0, t \geq 0, \tag{8}$$

$$\left|f_1(t,x)\right| \leq p_1(t)|x|^{\beta} \quad \text{and} \quad \left|f_2(t,x)\right| \geq p_2(t)|x|^{\gamma}, \quad x \neq 0, t \geq 0, \tag{9}$$

where $p_1, p_2 \in C([0,\infty),(0,\infty))$ and β, γ are positive constants.

Define

$$\Phi(t) = \Gamma(\alpha)\sum_{j=1}^{m} \frac{{}_a\mathfrak{D}^{\alpha-j,\rho}f(a)}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}(t-a)^{\rho\alpha-\rho j} \tag{10}$$

and

$$\Psi(t,T_1) = \int_a^{T_1}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{F(s,x(s))\,ds}{(s-a)^{1-\rho}}. \tag{11}$$

Theorem 3.2 *Let* $f_2 = 0$ *in* (1) *and condition* (7) *hold. If*

$$\liminf_{t\to\infty}\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} = -\infty, \tag{12}$$

and

$$\limsup_{t\to\infty}\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} = \infty, \tag{13}$$

for every sufficiently large T, *then every solution of* (1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of equation (1) with $f_2 = 0$. Suppose that $T_1 > a$ is large enough so that $x(t) > 0$ for $t \geq T_1$. Hence, (7) implies that $f_1(t,x) > 0$ for $t \geq T_1$. Using (3), we get from (6)

$$\begin{aligned}\Gamma(\alpha)x(t) &= \Gamma(\alpha)\sum_{j=1}^{m}\frac{{}_a\mathfrak{D}^{\alpha-j,\rho}f(a)}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}(t-a)^{\rho\alpha-\rho j} \\ &\quad + \int_a^{T_1}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{F(s,x(s))\,ds}{(s-a)^{1-\rho}} \\ &\quad + \int_{T_1}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{r(s)-f_1(s,x(s))\,ds}{(s-a)^{1-\rho}} \\ &\leq \Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}},\end{aligned} \tag{14}$$

where Φ and Ψ are defined in (10) and (11), respectively.

Multiplying (14) by $(\frac{t^\rho}{\rho})^{1-\alpha}$, we get

$$\begin{aligned}
0 &< \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Gamma(\alpha)x(t) \\
&\leq \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t) + \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Psi(t,T_1) \\
&\quad + \left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T_1}^t \left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}}.
\end{aligned} \tag{15}$$

Take $T_2 > T_1$. We consider two cases as follows.

Case (1): Let $0 < \alpha \leq 1$. Then $m = 1$ and $(\frac{t^\rho}{\rho})^{1-\alpha}\Phi(t) = b_1 t^{\rho-\rho\alpha}(t-a)^{\rho\alpha-\rho}$.

Since the function $h_1(t) = t^{\rho-\rho\alpha}(t-a)^{\rho\alpha-\rho}$ is decreasing for $\rho > 0$ and $\alpha < 1$, we get for $t \geq T_2$

$$\begin{aligned}
\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t)\right| &= |b_1| t^{\rho-\rho\alpha}(t-a)^{\rho\alpha-\rho} \\
&\leq |b_1| T_2^{\rho-\rho\alpha}(T_2-a)^{\rho\alpha-\rho} := c_1(T_2).
\end{aligned} \tag{16}$$

The function $h_2(t) = t^{\rho-\rho\alpha}[(t-a)^\rho - (s-a)^\rho]^{\alpha-1}$ is decreasing for $\rho > 0$ and $\alpha < 1$. Thus, we get

$$\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Psi(t,T_1)\right| \\
&\quad = \left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\left[r(s)+f_2\big(s,x(s)\big)-f_1\big(s,x(s)\big)\right]\frac{ds}{(s-a)^{1-\rho}}\right| \\
&\quad \leq \int_a^{T_1}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\left|r(s)+f_2\big(s,x(s)\big)-f_1\big(s,x(s)\big)\right|\frac{ds}{(s-a)^{1-\rho}} \\
&\quad \leq \int_a^{T_1}\left(\frac{T_2^\rho}{\rho}\right)^{1-\alpha}\left(\frac{(T_2-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1} \\
&\qquad \times \left|r(s)+f_2\big(s,x(s)\big)-f_1\big(s,x(s)\big)\right|\frac{ds}{(s-a)^{1-\rho}} \\
&\quad := c_2(T_1,T_2) \quad \text{for } t \geq T_2.
\end{aligned} \tag{17}$$

Then, from equation (15) and for $t \geq T_2$, we get

$$\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T_1}^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \geq -\left[c_1(T_2)+c_2(T_1,T_2)\right],$$

hence

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \geq -\left[c_1(T_2)+c_2(T_1,T_2)\right] > -\infty,$$

which contradicts condition (12).

Case (2): Let $\alpha > 1$. Then $m \geq 2$. Also, $(\frac{t-a}{t})^{\rho\alpha-\rho} < 1$ for $\alpha > 1$ and $\rho > 0$. The function $h_3(t) = (t-a)^{\rho-\rho j}$ is decreasing for $j > 1$ and $\rho > 0$. Thus, for $t \geq T_2$, we have

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Phi(t)\right| &= \left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Gamma(\alpha)\sum_{j=1}^{m}\frac{b_j(t-a)^{\rho\alpha-\rho j}}{\rho^{\alpha-j}\Gamma(\alpha-j+1)}\right| \\ &= \left|\left(\frac{t-a}{t}\right)^{\rho\alpha-\rho}\Gamma(\alpha)\sum_{j=1}^{m}\frac{b_j(t-a)^{\rho-\rho j}}{\rho^{1-j}\Gamma(\alpha-j+1)}\right| \\ &\leq \Gamma(\alpha)\sum_{j=1}^{m}\frac{|b_j|(t-a)^{\rho-\rho j}}{\rho^{1-j}\Gamma(\alpha-j+1)} \\ &\leq \Gamma(\alpha)\sum_{j=1}^{m}\frac{|b_j|(T_2-a)^{\rho-\rho j}}{\rho^{1-j}\Gamma(\alpha-j+1)} := c_3(T_2). \end{aligned} \tag{18}$$

Also, since $(\frac{t^\rho}{\rho})^{1-\alpha} < 1$ and $(\frac{(t-a)^\rho-(s-a)^\rho}{t^\rho})^{\alpha-1} < 1$ for $\alpha > 1$ and $\rho > 0$, we get

$$\begin{aligned}&\left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Psi(t,T_1)\right| \\ &= \left|\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\bigl[r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr]\frac{ds}{(s-a)^{1-\rho}}\right| \\ &\leq \int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{t^\rho}\right)^{\alpha-1}\bigl|r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr|\frac{ds}{(s-a)^{1-\rho}} \\ &\leq \int_a^{T_1}\bigl|r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr|\frac{ds}{(s-a)^{1-\rho}} := c_4(T_1). \end{aligned} \tag{19}$$

From (15), (18), and (19), we conclude that

$$\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T_1}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \geq -\bigl[c_3(T_2)+c_4(T_1)\bigr]$$

for $t \geq T_2$. Hence

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \geq -\bigl[c_3(T_2)+c_4(T_1)\bigr] > -\infty,$$

which contradicts condition (12). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually negative, similar arguments lead to a contradiction with condition (13). □

Theorem 3.3 *Let conditions* (7) *and* (8) *hold with* $\beta > \gamma$. *If*

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}} = -\infty, \tag{20}$$

and

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)-H(s)]\,ds}{(s-a)^{1-\rho}} = \infty, \tag{21}$$

for every sufficiently large T, where

$$H(s)=\frac{\beta-\gamma}{\gamma}p_1^{\frac{\gamma}{\gamma-\beta}}(s)\left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}}, \tag{22}$$

then every solution of (1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t)>0$ for $t\geq T_1>a$. Let $s\geq T_1$. Using conditions (7) and (8), we get

$$f_2(s,x)-f_1(s,x)\leq p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s).$$

Let $X=x^{\gamma}(s)$, $Y=\frac{\gamma p_2(s)}{\beta p_1(s)}$, $u=\frac{\beta}{\gamma}$, and $v=\frac{\beta}{\beta-\gamma}$, then from part (i) of Lemma 2.1 we get

$$\begin{aligned} p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s) &= \frac{\beta p_1(s)}{\gamma}\left[x^{\gamma}(s)\frac{\gamma p_2(s)}{\beta p_1(s)}-\frac{\gamma}{\beta}\left(x^{\gamma}(s)\right)^{\frac{\beta}{\gamma}}\right] \\ &= \frac{\beta p_1(s)}{\gamma}\left[XY-\frac{1}{u}X^{u}\right]\leq\frac{\beta p_1(s)}{\gamma}\frac{1}{v}Y^{v}=H(s), \end{aligned} \tag{23}$$

where H is defined by (22). Then from equation (6) we obtain

$$\begin{aligned} \Gamma(\alpha)x(t) &= \Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1} \\ &\quad\times\frac{[r(s)+f_2(s,x(s))-f_1(s,x(s))]\,ds}{(s-a)^{1-\rho}} \\ &\leq\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1} \\ &\quad\times\frac{[r(s)+p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s)]\,ds}{(s-a)^{1-\rho}} \\ &\leq\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}}. \end{aligned} \tag{24}$$

The rest of the proof is the same as that of Theorem 3.2 and hence is omitted. □

Theorem 3.4 *Let $\alpha\geq 1$ and suppose that* (7) *and* (9) *hold with $\beta<\gamma$. If*

$$\limsup_{t\to\infty}\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_{T}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}}=\infty \tag{25}$$

and

$$\liminf_{t\to\infty}\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha}\int_{T}^{t}\left(\frac{(t-a)^{\rho}-(s-a)^{\rho}}{\rho}\right)^{\alpha-1}\frac{[r(s)-H(s)]\,ds}{(s-a)^{1-\rho}}=-\infty \tag{26}$$

for every sufficiently large T, where H is defined by (22), *then every bounded solution of* (1) *is oscillatory.*

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1). Then there exist constants M_1 and M_2 such that

$$M_1 \le x(t) \le M_2 \quad \text{for } t \ge a. \tag{27}$$

Assume that x is a bounded eventually positive solution of (1). Then there exists $T_1 > a$ such that $x(t) > 0$ for $t \ge T_1 > a$. Using conditions (7) and (9), we get $f_2(s,x) - f_1(s,x) \ge p_2(s)x^{\gamma}(s) - p_1(s)x^{\beta}(s)$. Using (ii) of Lemma 2.1 and similar to the proof of (23), we find

$$p_2(s)x^{\gamma}(s) - p_1(s)x^{\beta}(s) \ge H(s) \quad \text{for } s \ge T_1.$$

From (6) and similar to (24), we obtain

$$\Gamma(\alpha)x(t) = \Phi(t) + \Psi(t,T_1) + \int_{T_1}^{t} \left(\frac{(t-a)^{\rho} - (s-a)^{\rho}}{\rho}\right)^{\alpha-1} \frac{[r(s) + H(s)]\,ds}{(s-a)^{1-\rho}}.$$

Multiplying by $(\frac{t^{\rho}}{\rho})^{1-\alpha}$, we get

$$\begin{aligned}\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \Gamma(\alpha)x(t) \ge{}& \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \Phi(t) + \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \Psi(t,T_1) \\ &+ \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \int_{T_1}^{t} \left(\frac{(t-a)^{\rho} - (s-a)^{\rho}}{\rho}\right)^{\alpha-1} \frac{[r(s) + H(s)]\,ds}{(s-a)^{1-\rho}}.\end{aligned} \tag{28}$$

Take $T_2 > T_1$. We consider two cases as follows.

Case (1): Let $\alpha = 1$. Then (16) and (17) are still correct. Hence, from (28) and using (27), we find that

$$\begin{aligned}M_2\Gamma(\alpha) \ge{}& \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \Gamma(\alpha)x(t) \ge -c_1(T_2) - c_2(T_1,T_2) \\ &+ \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \int_{T_1}^{t} \left(\frac{(t-a)^{\rho} - (s-a)^{\rho}}{\rho}\right)^{\alpha-1} \frac{[r(s) + H(s)]\,ds}{(s-a)^{1-\rho}}\end{aligned}$$

for $t \ge T_2$. Thus, we get

$$\begin{aligned}&\limsup_{t\to\infty} \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \int_{T_1}^{t} \left(\frac{(t-a)^{\rho} - (s-a)^{\rho}}{\rho}\right)^{\alpha-1} \frac{[r(s) + H(s)]\,ds}{(s-a)^{1-\rho}} \\ &\quad\le c_1(T_2) + c_2(T_1,T_2) + M_2\Gamma(\alpha) < \infty,\end{aligned}$$

which contradicts condition (25).

Case (2): Let $\alpha > 1$. Then (18) and (19) are still true. Hence, from (28) and using (27), we find that

$$\begin{aligned}M_2\Gamma(\alpha)\left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \ge{}& -c_3(T_2) - c_4(T_1) \\ &+ \left(\frac{t^{\rho}}{\rho}\right)^{1-\alpha} \int_{T_1}^{t} \left(\frac{(t-a)^{\rho} - (s-a)^{\rho}}{\rho}\right)^{\alpha-1} \frac{[r(s) + H(s)]\,ds}{(s-a)^{1-\rho}}\end{aligned}$$

for $t \geq T_2$. Since $\lim_{t\to\infty}(\frac{t^\rho}{\rho})^{1-\alpha} = 0$ for $\alpha > 1$, we conclude that

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_{T_1}^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}} \leq c_3(T_2)+c_4(T_1) < \infty,$$

which contradicts condition (25). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually bounded negative, similar arguments lead to a contradiction with condition (26). □

4 Oscillation of conformable fractional differential equations in the frame of Caputo

In this section, we study the oscillation of conformable fractional differential equations in the Caputo setting of the form

$$\begin{cases} {}_a^C\mathfrak{D}^{\alpha,\rho}x(t)+f_1(t,x)=r(t)+f_2(t,x), & t>a\\ {}_a^kT^\rho x(a)=b_k \quad (k=0,1,\dots,m-1), \end{cases} \tag{29}$$

where $m = \lceil\alpha\rceil$ and ${}_a^C\mathfrak{D}^{\alpha,\rho}$ is defined by (4).

Lemma 4.1 [1] *Let* $f \in C^m_{\rho,a}[a,b]$, $\alpha \in \mathbb{C}$. *Then*

$${}_a\mathfrak{I}^{\alpha,\rho}\left({}_a^C\mathfrak{D}^{\alpha,\rho}f(x)\right)=f(x)-\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho f(a)(x-a)^{\rho k}}{\rho^k k!}.$$

Using Lemma 4.1, the solution representation of (29) can be written as

$$x(t)=\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho x(a)(t-a)^{\rho k}}{\rho^k k!}+{}_a\mathfrak{I}^{\alpha,\rho}F(t,x), \tag{30}$$

where $F(t,x) = r(t) + f_2(t,x) - f_1(t,x)$ and $\rho > 0$.

Define

$$\chi(t)=\Gamma(\alpha)\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho x(a)(t-a)^{\rho k}}{\rho^k k!}. \tag{31}$$

Theorem 4.2 *Let* $f_2 = 0$ *in* (29) *and condition* (7) *hold. If*

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}}=-\infty \tag{32}$$

and

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}}=\infty \tag{33}$$

for every sufficiently large T, *then every solution of* (29) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of equation (29) with $f_2 = 0$. Suppose that $T_1 > a$ is large enough so that $x(t) > 0$ for $t \geq T_1$. Hence (7) implies that $f_1(t,x) > 0$ for $t \geq T_1$. Using (3), we get from (30)

$$\begin{aligned}
\Gamma(\alpha)x(t) &= \Gamma(\alpha)\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho x(a)(t-a)^{\rho k}}{\rho^k k!} \\
&\quad + \int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{F(s,x(s))\,ds}{(s-a)^{1-\rho}} \\
&\quad + \int_{T_1}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)-f_1(s,x(s))\,ds}{(s-a)^{1-\rho}} \\
&\leq \chi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}},
\end{aligned} \tag{34}$$

where χ and Ψ are defined in (31) and (11), respectively.

Multiplying (34) by $(\frac{t^\rho}{\rho})^{1-m}$, we get

$$\begin{aligned}
0 &< \left(\frac{t^\rho}{\rho}\right)^{1-m}\Gamma(\alpha)x(t) \\
&\leq \left(\frac{t^\rho}{\rho}\right)^{1-m}\chi(t)+\left(\frac{t^\rho}{\rho}\right)^{1-m}\Psi(t,T_1) \\
&\quad + \left(\frac{t^\rho}{\rho}\right)^{1-m}\int_{T_1}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}}.
\end{aligned} \tag{35}$$

Take $T_2 > T_1$. We consider two cases as follows.

Case (1): Let $0 < \alpha \leq 1$. Then $m = 1$ and $(\frac{t^\rho}{\rho})^{1-m}\chi(t) = \Gamma(\alpha)b_0$.

The function $h_4(t) = (\frac{(t-a)^\rho-(s-a)^\rho}{\rho})^{\alpha-1}$ is decreasing for $\rho > 0$, $t > T_2 > s$, and $\alpha < 1$. Thus, we get

$$\begin{aligned}
&\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Psi(t,T_1)\right| \\
&\quad = \left|\int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\bigl[r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr]\frac{ds}{(s-a)^{1-\rho}}\right| \\
&\quad \leq \int_a^{T_1}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\bigl|r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr|\frac{ds}{(s-a)^{1-\rho}} \\
&\quad \leq \int_a^{T_1}\left(\frac{(T_2-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\bigl|r(s)+f_2\bigl(s,x(s)\bigr)-f_1\bigl(s,x(s)\bigr)\bigr|\frac{ds}{(s-a)^{1-\rho}} \\
&\quad := c_5(T_1,T_2).
\end{aligned}$$

Then, from equation (35) and for $t \geq T_2$, we get

$$\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_{T_1}^{t}\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \geq -\bigl[\Gamma(\alpha)b_0+c_5(T_1,T_2)\bigr],$$

hence

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}}$$
$$\geq -\left[\Gamma(\alpha)b_0+c_5(T_1,T_2)\right]>-\infty,$$

which contradicts condition (32).

Case (2): Let $\alpha>1$. Then $m\geq 2$. Also, $(\frac{t-a}{t})^{\rho m-\rho}<1$ for $m\geq 2$ and $\rho>0$. The function $h_3(t)=(t-a)^{\rho(k-m+1)}$ is decreasing for $k<m-1$ and $\rho>0$. Thus, for $t\geq T_2$, we have

$$\begin{aligned}\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\chi(t)\right| &= \left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Gamma(\alpha)\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho x(a)(t-a)^{\rho k}}{\rho^k k!}\right| \\ &= \left|\left(\frac{t-a}{t}\right)^{\rho m-\rho}\Gamma(\alpha)\sum_{k=0}^{m-1}\frac{{}_a^kT^\rho x(a)(t-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!}\right| \\ &\leq \Gamma(\alpha)\sum_{k=0}^{m-1}\frac{|{}_a^kT^\rho x(a)|(t-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!} \\ &\leq \Gamma(\alpha)\sum_{k=0}^{m-1}\frac{|{}_a^kT^\rho x(a)|(T_2-a)^{\rho(k-m+1)}}{\rho^{k-m+1}k!}:=c_6(T_2). \end{aligned} \tag{36}$$

Also, since $(\frac{t^\rho}{\rho})^{1-m}<1$ and $(\frac{(t-a)^\rho-(s-a)^\rho}{t^\rho})^{\alpha-1}<1$ for $\alpha>1$ and $\rho>0$, and similar to (19) we get

$$\left|\left(\frac{t^\rho}{\rho}\right)^{1-m}\Psi(t,T_1)\right|\leq c_4(T_1).$$

Then, from (35) and (36), we get a contradiction with condition (32). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually negative, similar arguments lead to a contradiction with condition (33). □

We state the following two theorems without proof.

Theorem 4.3 *Let conditions* (7) *and* (8) *hold with* $\beta>\gamma$. *If*

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}}=-\infty$$

and

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)-H(s)]\,ds}{(s-a)^{1-\rho}}=\infty$$

for every sufficiently large T, *where* H *is defined by* (22), *then every solution of* (29) *is oscillatory.*

Theorem 4.4 *Let* $\alpha\geq 1$ *and suppose that* (7) *and* (9) *hold with* $\beta<\gamma$. *If*

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}}=\infty$$

and

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)-H(s)]\,ds}{(s-a)^{1-\rho}}=-\infty$$

for every sufficiently large T, where H is defined by (22), *then every bounded solution of* (29) *is oscillatory.*

5 Examples

In this section, we construct numerical examples to illustrate the effectiveness of our theoretical results.

Example 5.1 Consider the Riemann conformable fractional differential equation

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho}x(t)+x^5(t)\ln(t+e) \\ \quad =\frac{2\rho^\alpha(t-a)^{\rho(2-\alpha)}}{\Gamma(3-\alpha)}+[(t-a)^{10\rho}-(t-a)^{\frac{2}{3}\rho}]\ln(t+e)+x^{\frac{1}{3}}(t)\ln(t+e), \\ \lim_{t\to a^+}{}_a\mathfrak{I}^{1-\alpha,\rho}x(t)=0,\quad 0<\alpha<1,\rho>0, \end{cases}\tag{37}$$

where $m=1, f_1(t,x)=x^5(t)\ln(t+e)$, $r(t)=\frac{2\rho^\alpha(t-a)^{\rho(2-\alpha)}}{\Gamma(3-\alpha)}+[(t-a)^{10\rho}-(t-a)^{\frac{2}{3}\rho}]\ln(t+e)$, and $f_2(t,x)=x^{\frac{1}{3}}(t)\ln(t+e)$. It is easy to verify that conditions (7) and (8) are satisfied for $\beta=5$, $\gamma=\frac{1}{3}$ and $p_1(t)=p_2(t)=\ln(t+e)$. However, we show in the following that condition (20) does not hold. For every sufficiently large $T\geq 1$ and all $t\geq T$, we have $r(t)>0$. Calculating $H(s)$ as defined by (22), we find that $H(s)=14(15)^{-\frac{15}{14}}\ln(s+e)\geq 0.77$. Then, using (5) with $\nu=1$, we get

$$\begin{aligned}
&\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{[r(s)+H(s)]\,ds}{(s-a)^{1-\rho}}\\
&\quad\geq\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{H(s)\,ds}{(s-a)^{1-\rho}}\\
&\quad\geq\liminf_{t\to\infty}0.77\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{(s-T)^0\,ds}{(s-a)^{1-\rho}}\\
&\quad=\liminf_{t\to\infty}0.77\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\Gamma(\alpha)\big({}_a\mathfrak{I}^{\alpha,\rho}(s-T)^0\big)(t)\\
&\quad=\liminf_{t\to\infty}\frac{0.77t^\rho}{\rho\alpha}\left(\frac{t-T}{t}\right)^{\rho\alpha}=\infty.
\end{aligned}$$

However, using (5) with $\nu=3$, one can easily verify that $x(t)=(t-a)^{2\rho}$ is a nonoscillatory solution of (37). The initial condition is also satisfied because

$${}_a\mathfrak{I}^{1-\alpha,\rho}(t-a)^{2\rho}=\frac{2\rho^{\alpha-1}(t-a)^{\rho(3-\alpha)}}{\Gamma(4-\alpha)}.$$

Example 5.2 Consider the conformable fractional differential equation

$$\begin{cases} {}_a\mathfrak{D}^{\alpha,\rho}x(t)+x^3(t)=\sin t, \\ \lim_{t\to a^+}{}_a\mathfrak{I}^{1-\alpha,\rho}x(t)=0,\quad 0<\alpha<1, \end{cases}\tag{38}$$

where $f_1(t,x) = x^3(t)$, $r(t) = \sin t$, and $f_2(t,x) = 0$. Then condition (7) holds. Furthermore, one can easily check that

$$\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{\sin s\,ds}{(s-a)^{1-\rho}} = -\infty$$

and

$$\limsup_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-\alpha}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{\sin s\,ds}{(s-a)^{1-\rho}} = \infty.$$

This shows that conditions (12) and (13) of Theorem 3.2 hold. Hence, every solution of (38) is oscillatory.

Example 5.3 Consider the Caputo conformable fractional differential equation

$$\begin{cases} {}_a^C\mathfrak{D}^{\alpha,\rho}x(t) + e^t x^3(t) = \frac{\rho^\alpha(t-a)^{\rho(1-\alpha)}}{\Gamma(2-\alpha)} + (t-a)^{3\rho}e^t, \\ x(a) = 0, \quad 0 < \alpha < 1, \rho > 0, \end{cases} \tag{39}$$

where $m = 1$, $f_1(t,x) = e^t x^3(t)$, $r(t) = \frac{\rho^\alpha(t-a)^{\rho(1-\alpha)}}{\Gamma(2-\alpha)}$, and $f_2(t,x) = 0$. Then condition (7) is satisfied. However, condition (32) does not hold since

$$\begin{aligned} &\liminf_{t\to\infty}\left(\frac{t^\rho}{\rho}\right)^{1-m}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{r(s)\,ds}{(s-a)^{1-\rho}} \\ &\quad\geq \liminf_{t\to\infty}\int_T^t\left(\frac{(t-a)^\rho-(s-a)^\rho}{\rho}\right)^{\alpha-1}\frac{(s-T)^0\,ds}{(s-a)^{1-\rho}} \\ &\quad= \liminf_{t\to\infty}\frac{1}{\alpha\rho^\alpha}(t-T)^{\rho\alpha} = \infty. \end{aligned}$$

Using (2), (5) with $\nu = 2$ and the fact that

$${}_a^C\mathfrak{D}^{\alpha,\rho}(t-a)^{\rho(\nu-1)} = {}_a\mathfrak{I}^{1-\alpha,\rho}\,{}_aT^\rho(t-a)^{\rho(\nu-1)},$$

one can easily check that $x(t) = (t-a)^\rho$ is a nonoscillatory solution of (39).

Remark 5.1 The oscillation of fractional differential equations in the frame of Katugampola-type fractional derivatives studied in [23–25] can be investigated in a similar way as we have done in this article for CFDs and their Caputo settings. The reader can verify sufficient conditions and the proofs by observing the kernel which is free from the starting point a.

6 Conclusion

In this article, the oscillation theory for conformable fractional differential equations was studied. Sufficient conditions for the oscillation of solutions of Riemann conformable fractional differential equations of the form (1) were given in three theorems in Sect. 3. As $\rho \to 1$ in these theorems, we get the results obtained in [13] and [14] when $a = 0$. The main approach is based on applying Young's inequality which will help us in obtaining sharper conditions. The oscillation for the Caputo conformable fractional differential equations has been investigated as well. Numerical examples have been presented to demonstrate the effectiveness of the obtained results. We shall discuss the case when $\rho \to 0$ in the future work. Namely, we shall discuss the oscillation of Hadamard-type fractional differential equations with kernels both depending or not depending on the starting point.

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