Xu and Wang *Advances in Difference Equations* (2018) 2018:115
https://doi.org/10.1186/s13662-018-1550-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Global existence and boundedness in a reaction–diffusion–taxis system with three species

Xue Xu[1,2*] and Yong Wang[1]

*Correspondence:
xuxue06@163.com
[1]Department of Mathematics, Harbin Institute of Technology, Harbin, P.R. China
[2]Department of Mathematics, Harbin University, Harbin, P.R. China

Abstract

The global existence and boundedness of a reaction–diffusion–taxis system with three interacting species, among which two species consist of predators *competing* for one species of prey, are investigated.

MSC: 35K57; 35K59; 35B45; 92D25

Keywords: Reaction–diffusion–taxis; Predator–prey; Global existence; Boundedness

1 Introduction

The predator–prey model [1–6] is a basic differential equation model describing the interaction *among* species, *and* considerable work has been done with regards to both Turing and non-Turing patterns where the latter often appears to be chaotic space [7, 8]. In general, the coexistence of prey population and predator population can be mostly described by the presence of positive steady states, while spatial patterns of the populations can be characterized by non-constant steady states, which have been studied for population models with random diffusions by various types of mathematical models, e.g. [9–11].

In reality, *prey are pursued and escape from predators in the spatial movement*. Such a movement is not random but directed. The object of this paper is to study the problem for three interacting species, among which two species of predators compete for one species of prey. The model is taken to be

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u - f_1(u)v_1 - f_2(u)v_2, & x\in\Omega, t>0,\\ \frac{\partial v_1}{\partial t} = d_2\Delta v_1 - \chi_1\nabla(S(v_1)\nabla u) + v_1[f_1(u)-v_1-v_2], & x\in\Omega, t>0,\\ \frac{\partial v_2}{\partial t} = d_3\Delta v_2 - \chi_2\nabla(S(v_2)\nabla u) + v_2[f_2(u)-v_1-v_2], & x\in\Omega, t>0,\\ \frac{\partial u}{\partial n} = \frac{\partial v_1}{\partial n} = \frac{\partial v_2}{\partial n} = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x)\ge 0, \qquad v_1(x,0)=v_{10}(x)\ge 0, & \\ v_2(x,0)=v_{20}(x)\ge 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where v_1, v_2 and u are the population densities of two predator species and one prey species, respectively; Ω is a bounded domain in $\mathbb{R}^N$ ($N\ge 1$) with a smooth boundary $\partial\Omega$;

n is the unit outer normal, and no flux boundary condition is imposed so the system is a closed one. Here $f_i(u)$ satisfies:

(F) $f_i(u)$ is continuously differentiable from $\mathbb{R}_+ \to \mathbb{R}_+$ and $f_i(0) = 0, f_i'(u) \geq 0$, $i = 1, 2$.

Biologically, $f_i(u)$ represents the consumption rate of the prey per predator. A typical example of f_i is the monotone Holling Type II function, $f_i(u) = m_i u/(a_i + u)$, where $m_i > 0$ is the maximal growth rate and $a_i > 0$ is the half-saturation constant. The forms of $F_1 = v_1[f_1(u) - v_1 - v_2]$ and $F_2 = v_2[f_2(u) - v_1 - v_2]$ represent the constant level of the prey u, *and* the predators v_1, v_2 have logistic growth. It is assumed that the predators v_1, v_2 are attracted by the preys u, so they move in the direction proportional to the negative gradient of prey population. That is modeled by a prey-taxis term $\chi_i \nabla(S(v_i)\nabla u)$, $i = 1, 2$, respectively, where the χ_i are the prey-taxis coefficients, and the movement *is decided also by the predator's density*, which is indicated by the function $S(v_i)$. As pointed in [12], the sensitivity function $S(u)$ satisfies the general hypotheses:

(H_1) $S : [0, \infty) \to [0, \infty)$ is continuously differentiable and $S(0) = 0$;

(H_2) There exists $C > 0$ such that $S(u) \leq Cu$ for any $u \geq 0$ and $x \in \bar{\Omega}$.

For example, in [13], the sensitivity function $q(u)$ can take the form

$$S(u) = u, \qquad S(u) = \frac{u}{1 + \varepsilon u}, \qquad S(u) = ue^{-\varepsilon u}, \tag{1.2}$$

where $\varepsilon > 0$, $m \geq 1$.

This paper *concentrates* on the global existence and boundedness of system (1.1).

2 Local existence and preliminaries

It is noticed that (1.1) has a unique non-negative local-in-time classical solution $(u(x,t), v_1(x,t), v_2(x,t))$ by using the abstract theory of quasilinear parabolic systems *in* [14]. Moreover, we can obtain the following results.

Lemma 2.1 *Assume that* (H_1), (H_2), (F) *hold and the initial data* $u_0 \in W^{1,p}(\Omega)$ *for* $p > n$ *and* $v_{i0} \in (W^{1,p}(\Omega))^2$ *for* $p > n$ $(i = 1, 2)$. *Then*

1. *There exists a positive constant* $T_{\max}$ (*the maximal existence time*) *such that the system* (1.1) *has a unique non-negative classical solution* $(u(x,t), v_1(x,t), v_2(x,t))$ *satisfying* $(u(x,t), v_1(x,t), v_2(x,t)) \in (C([0, T_{\max}); W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})))^3$ *and*

$$v_1(x,t) \geq 0, \qquad v_2(x,t) \geq 0, \qquad u_* \geq u(x,t) \geq 0, \quad x \in \Omega, 0 \leq t < T_{\max}, \tag{2.1}$$

where $u_* > 0$ *is a constant satisfying* $\int_\Omega u_0(x) = |\Omega| u_*$.

2. *If for each* $T > 0$, *there exists a constant* $M_0(T)$ *such that*

$$\left\| \left(u(x,t), v_1(x,t), v_2(x,t)\right) \right\|_\infty \leq M_0(T), \quad 0 < t < \min\{T, T_{\max}\}, \tag{2.2}$$

where $M_0(T)$ *is a constant depending on* T *and* $\|(u_0, v_{10}, v_{20})\|_{1,p}$, *then* $T_{\max} = +\infty$.

Proof From Theorem 14.6 in [14], the local existence of $(u(x,t), v_1(x,t), v_2(x,t))$ is obtained. It is noticed that every equation in (1.1) can be treated as a scalar linear equation in u and v_1, v_2, so we have $u(x,t) \geq 0$, $v_i(x,t) \geq 0$ $(i = 1, 2)$ from $u_0 \geq 0$, $v_{i0} \geq 0$ and the maximum principle for parabolic equation.

Also from (1.1) and $v_1(x,t) \geq 0$, $v_2(x,t) \geq 0$, we *have*

$$\begin{cases} \frac{\partial u}{\partial t} = d_3 \Delta u - f_1(u)v_1 - f_2(u)v_2 \leq d_3 \Delta u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega. \end{cases}$$

Using the Neumann boundary condition and the comparison principle again, we obtain $u \leq u_*$, where $u_* > 0$ is a constant satisfying $\int_\Omega u_0(x) = |\Omega| u_*$. Part 2 follows from Theorem 15.5 in [15], so we have $T_{\max} = \infty$. □

In order to obtain the global existence of non-negative solutions for system (1.1), we recall some preliminary estimates which will be used in our proof. First we review some well-known estimates for the diffusion semigroup with homogeneous Neumann boundary conditions (see [16]). For $p \in (1,\infty)$, let A denote the sectorial operator defined by

$$Au := -\Delta u \quad \text{for } u \in D(A) := \left\{ \omega \in W^{2,p}(\Omega) : \frac{\partial \omega}{\partial n} = 0 \text{ on } \partial\Omega \right\}. \tag{2.3}$$

Similarly we let $A_d u = -d\Delta u$ which satisfies the same properties as A with a scaling. Then we only collect properties of A here while the same properties for A_d will be applied in the following analysis.

Lemma 2.2 *Assume that $m \in \{0,1\}$, $p \in [1,\infty]$ and $q \in (1,\infty)$. Then there exists some positive constant C_2, such that*

$$\|u\|_{m,p} \leq C_2 \left\| (A+1)^\theta u \right\|_q \tag{2.4}$$

for any $u \in D((A+1)^\theta)$, where $\theta \in (0,1)$ satisfies

$$m - \frac{n}{p} < 2\theta - \frac{n}{q}.$$

If in addition $q \geq p$, then there exist $C_3 > 0$ and $\gamma > 0$ such that, for any $u \in L^p(\Omega)$,

$$\left\| (A+1)^\theta e^{-t(A+1)} u \right\|_q \leq C_3 t^{-\theta - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})} e^{-\gamma t} \|u\|_p \tag{2.5}$$

where the associated diffusion semigroup $\{e^{-t(A+1)}\}_{t\geq 0}$ maps $L^p(\Omega)$ into $D((A+1)^\theta)$. Moreover, for any $p \in (1,\infty)$ and $\varepsilon > 0$, there exist $C_4 > 0$ and $\mu > 0$ such that

$$\left\| (A+1)^\theta e^{-tA} \nabla \cdot u \right\|_p \leq C_4 t^{-\theta - \frac{1}{2} - \varepsilon} e^{-\mu t} \|u\|_p \tag{2.6}$$

which is valid for all $\mathbb{R}^n$-valued $u \in L^p(\Omega)$.

3 Global existence and boundedness of solutions

In this section we prove the global existence and boundedness of solutions for system (1.1). The main step toward the result is to establish a uniform bound of $v_1(x,t)$, $v_2(x,t)$ in $L^k(\Omega)$ for any $k \geq 2$. First we show that the solution $v_1(x,t)$ *and* $v_2(x,t)$ *are* bounded in $L^1(\Omega)$.

Lemma 3.1 *Assume that* (H_1), (H_2) *and* (F) *hold. Then there exists a constant* $C_0 > 0$ *such that the predator component of* (1.1) *satisfies the following estimate*:

$$\int_\Omega v_1(x,t) + v_2(x,t)\,dx < C_0 \quad \textit{for all } t \in (0, T_{\max}). \tag{3.1}$$

Proof Let $\int_\Omega v_i(x,t)\,dx = Q_i(t)$ $(i = 1,2)$, $\int_\Omega u(x,t)\,dx = Q_3(t)$. Then we have

$$\begin{aligned}
\frac{\partial Q_1}{\partial t} &= \int_\Omega d_2\Delta v_1 - \chi_1\nabla\big(S(v_1)\nabla u\big) + v_1\big(f_1(u) - v_1 - v_2\big)\,dx \\
&= d_2\int_\Omega \frac{\partial v_1}{\partial \nu}\,dS - \chi_1\int_\Omega S(v_1)\frac{\partial u}{\partial \nu}\,dS + \int_\Omega v_1\big(f_1(u) - v_1 - v_2\big)\,dx, \\
\frac{\partial Q_2}{\partial t} &= \int_\Omega d_3\Delta v_2 - \chi_2\nabla\big(S(v_2)\nabla u\big) + v_2\big(f_2(u) - v_1 - v_2\big)\,dx \\
&= d_3\int_\Omega \frac{\partial v_2}{\partial \nu}\,dS - \chi_2\int_\Omega S(v_2)\frac{\partial u}{\partial \nu}\,dS + \int_\Omega v_2\big(f_2(u) - v_1 - v_2\big)\,dx, \\
\frac{\partial Q_3}{\partial t} &= \int_\Omega d_1\Delta u - f_1(u)v_1 - f_2(u)v_2\,dx \\
&= d_1\int_\Omega \frac{\partial u}{\partial \nu}\,dS + \int_\Omega -f_1(u)v_1 - f_2(u)v_2\,dx.
\end{aligned} \tag{3.2}$$

From the Neumann boundary conditions and the uniform boundedness of u in Lemma 2.1, we have

$$\frac{\partial(Q_1 + Q_2 + Q_3)}{\partial t} = \int_\Omega -(v_1 + v_2)^2\,dx \le 0. \tag{3.3}$$

Consequently,

$$Q_1 + Q_2 < Q_1 + Q_2 + Q_3 \le \int_\Omega (u_0 + v_{10} + v_{20})\,dx := C_0. \qquad \square$$

Now we carry out the L^p bound of v_1, v_2 for $p \ge 2$. The following Gagliardo–Nirenberg inequality plays a key role in our proof (see [17] for detail).

Lemma 3.2 *Let* $u \in L^p(\Omega)$ *and* $D^k u \in L^q(\Omega)$ *where* $p, q \in [1,\infty]$. *Then, for the derivatives* $D^i u$, $i \in [0,k)$, *there exists a constant* $C_5 > 0$ *such that*

$$\|D^i u\|_h \le C_5\big(\|D^k u\|_q^\lambda \|u\|_p^{1-\lambda} + \|u\|_m\big), \tag{3.4}$$

where

$$\frac{1}{h} - \frac{i}{n} = \lambda\left(\frac{1}{q} - \frac{k}{n}\right) + (1-\lambda)\frac{1}{p}, \quad m > 0,$$

and λ *satisfies*

$$\frac{i}{k} \le \lambda \le 1.$$

Moreover, we recall the following elementary inequality (*see* [18]).

Lemma 3.3 *Assume that $y,z \in \mathbb{R}$, $y,z \geq 0$ and $r > 0$, then we have*

$$(y+z)^r \leq 2^r\bigl(y^r + z^r\bigr). \tag{3.5}$$

Theorem 3.4 *Assume that (F), (H_1) and (H_2) hold. Then, for any $k \geq 2$, there exists a positive constant $C_1 > 0$ such that*

$$\bigl\|v_1(\cdot,t)\bigr\|_k \leq C_1, \qquad \bigl\|v_2(\cdot,t)\bigr\|_k \leq C_1, \quad \textit{for all } t \in (0, T_{\max}). \tag{3.6}$$

Proof First we show that, for any $\tau \in (0, T_{\max})$, there exists a constant $H(\tau) > 0$ such that

$$\bigl\|u(\cdot,t)\bigr\|_{1,\infty} \leq H(\tau), \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.7}$$

Let $\tau \in (0, T_{\max})$ be given such that $\tau < 1$, and choose $q > n$ and $\theta \in (\frac{1}{2}(1+\frac{n}{q}), 1)$. The first equation of (1.1) can be rewritten as

$$\frac{\partial u}{\partial t} = d_1 \Delta u - u + \varphi(u, v_1, v_2), \tag{3.8}$$

where $\varphi(u,v_1,v_2) = (u - f_1(u)v_1 - f_2(u)v_2)$. Then from the variation of constants formula for (3.8), we have

$$u(\cdot,t) = e^{-t(A_{d_1}+1)}u_0 + \int_0^t e^{-(t-s)(A_{d_1}+1)}\varphi\bigl(u(\cdot,t), v_1(\cdot,t), v_2(\cdot,t)\bigr)\,ds.$$

From (2.4) and (2.5) we have

$$\begin{aligned}
\bigl\|u(\cdot,t)\bigr\|_{1,\infty} &\leq C_2 \bigl\|(A_{d_1}+1)^\theta u(\cdot,t)\bigr\|_q \\
&\leq C_3 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \bigl\|u - f_1(u)v_1 - f_2(u)v_2\bigr\|_q\,ds + C_3 t^{-\theta} e^{-\gamma t}\|u_0\|_q \\
&\leq C_3 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \bigl\|u(\cdot,t)\bigr\|_\infty\,ds + C_3 t^{-\theta} e^{-\gamma t}\|u_0\|_q \\
&\leq C_3 t^{-\theta} + C_3 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)}\,ds \leq C_3 t^{-\theta} + C_3 \int_0^\infty \sigma^{-\theta} e^{-\gamma\sigma}\,d\sigma \\
&\leq C_3\bigl(\tau^{-\theta}+1\bigr) := H(\tau) \quad \text{for all } t \in (\tau, T_{\max}),
\end{aligned} \tag{3.9}$$

where C_3 denotes a generic constant that may vary from line to line. For any $k \geq 2$, from (1.1), (3.9), (H_2) and Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega v_1^k &= k\int_\Omega v_1^{k-1}(v_1)_t \\
&\leq k\int_\Omega v_1^{k-1} d_2 \Delta v_1 + k\int_\Omega v_1^{k-1}\chi_1 \nabla\cdot\bigl(S(v_1)\nabla u\bigr) + k\int_\Omega v_1^{k-1}v_1\bigl(f_1(u) - v_1 - v_2\bigr) \\
&= -k(k-1)d_2\int_\Omega v_1^{k-2}|\nabla v_1|^2 - k(k-1)\chi_1\int_\Omega v_1^{k-2}S(v_1)\nabla u\cdot\nabla v_1 \\
&\quad + k\int_\Omega v_1^{k-1}v_1\bigl(f_1(u) - v_1 - v_2\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{-4(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + Ik(k-1)\int_\Omega v_1^{k-2}S(v_1)|\nabla v_1| + k\int_\Omega v_1^{k-1}v_1 f_1(u) \\
&\le \frac{-4(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + CIk(k-1)\int_\Omega u_1^{k-1}|\nabla v_1| + Ek\int_\Omega v_1^k \\
&= \frac{-4(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + CIk(k-1)\frac{2}{k}\int_\Omega v_1^{\frac{k}{2}}\big|\nabla v_1^{\frac{k}{2}}\big| + Ek\int_\Omega v_1^k \\
&\le \frac{-4(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 \\
&\quad + CI(k-1)\bigg(\frac{2}{CIk}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + \frac{CIk}{2}\int_\Omega v_1^k\bigg) + Ek\int_\Omega v_1^k \\
&\le \frac{-2(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + \bigg(\frac{C^2I^2k(k-1)}{2} + Ek\bigg)\int_\Omega v_1^k, \qquad (3.10)
\end{aligned}
$$

where $I = \chi_1 H(\tau)$ is a positive constant. Then we have

$$
\frac{d}{dt}\int_\Omega v_1^k \le \frac{-2(k-1)d_2}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + \bigg(\frac{C^2I^2k(k-1)}{2} + Ekc\bigg)\int_\Omega v_1^k. \qquad (3.11)
$$

From Lemma 3.2 and Lemma 3.3, we find that

$$
\begin{aligned}
\int_\Omega v_1^k &= \big\|v_1^{\frac{k}{2}}\big\|_2^2 \\
&\le C_5\Big(\big\|\nabla v_1^{\frac{k}{2}}\big\|_2^{\lambda}\big\|v_1^{\frac{k}{2}}\big\|_{\frac{2}{k}}^{1-\lambda} + \big\|v_1^{\frac{k}{2}}\big\|_{\frac{2}{k}}\Big)^2 \\
&\le C_5\Big(\big\|\nabla v_1^{\frac{k}{2}}\big\|_2^{\lambda}\|1+v_1\|_1^{\frac{k}{2}(1-\lambda)} + \|1+v_1\|_1^{\frac{k}{2}}\Big)^2 \\
&\le C_5\Big(\big\|\nabla v_1^{\frac{k}{2}}\big\|_2^{\lambda}\big(|\Omega|+C_1\big)^{\frac{k}{2}(1-\lambda)} + \big(|\Omega|+C_1\big)^{\frac{k}{2}}\Big)^2 \\
&\le C_6\Big(\big\|\nabla v_1^{\frac{k}{2}}\big\|_2^{2\lambda} + 1\Big), \qquad (3.12)
\end{aligned}
$$

where

$$
\lambda = \frac{kn-n}{2+kn-n} \in (0,1), \qquad (3.13)
$$

for any $k \ge 2$. Since (3.13) implies that $2\lambda < 2$, then from (3.12) we obtain

$$
\int_\Omega v_1^k \le C_7\Big(\big\|\nabla v_1^{\frac{k}{2}}\big\|_2^2 + 1\Big). \qquad (3.14)
$$

By using Young's inequality and (3.14), we obtain

$$
\bigg(\frac{C^2I^2k(k-1)d_1}{2} + Ekc + 1\bigg)\int_\Omega v_1^k \le \frac{2(k-1)d_1}{k}\int_\Omega \big|\nabla v_1^{\frac{k}{2}}\big|^2 + C_8, \qquad (3.15)
$$

for some $C_8 > 0$. Combining (3.11) and (3.15), we have

$$
\frac{d}{dt}\int_\Omega v_1^k + \int_\Omega v_1^k \le C_8. \qquad (3.16)
$$

Integrating (3.16), we arrive at

$$\int_\Omega v_1^k \le \max\left\{\int_\Omega v_{10}^k, C_8\right\} := R_1. \tag{3.17}$$

Similarly, we can get

$$\int_\Omega v_2^k \le \max\left\{\int_\Omega v_{20}^k, C_9\right\} := R_2, \tag{3.18}$$

which are the desired results. □

Next we establish the L^∞ bound for $v_1(x,t)$, $v_2(x,t)$. The following Sobolev inequality will be used in forthcoming proofs.

Lemma 3.5 *Let*

$$2^* = \begin{cases} \infty, & n \le 2, \\ \frac{n}{n-2}, & n > 2. \end{cases}$$

Then, for any $1 < \alpha \le 2^*$ *and* $k > 0$, *there exists a positive constant* M_0 *such that*

$$\left(\int_\Omega u^{(k+1)\alpha}\,dx\right)^{\frac{1}{\alpha}} \le M_0 \int_\Omega \left(\left|\nabla\left(u^{\frac{k+1}{2}}\right)\right|^2 + u^{k+1}\right)dx. \tag{3.19}$$

We make a key progress on the boundedness estimates of v_1, v_2.

Theorem 3.6 *Let* $(u(x,t), v_1(x,t), v_2(x,t))$ *be the solution of* (1.1). *Assume that* (F), (H_1) *and* (H_2) *hold, then there exists a positive constant* M *such that*

$$\left\|v_1(\cdot,t)\right\|_\infty \le M, \qquad \left\|v_2(\cdot,t)\right\|_\infty \le M \quad \text{for all } t \in (0, T_{\max}). \tag{3.20}$$

Proof We use semigroup arguments (*see* [13, 16]) to get the L^∞-bound of v_1, v_2. As pointed out in Theorem 3.4, by using the variation of constants formula, we have

$$\begin{aligned} v_1(\cdot,t) &= e^{-t(A_{d_2}+1)}v_{10} - \chi_1 \int_0^t e^{-(t-s)(A_{d_2}+1)}\nabla\cdot\left(S\left(v_1(\cdot,t)\right)\nabla u(\cdot,t)\right)ds \\ &\quad + \int_0^t e^{-(t-s)(A_{d_2}+1)}\psi\left(v_1(\cdot,t), v_2(\cdot,t), u(\cdot,t)\right)ds \\ &:= E_1 + E_2 + E_3, \end{aligned} \tag{3.21}$$

where $\psi(u(\cdot,t), v_1(\cdot,t), v_2(\cdot,t)) = v_1(f_1(u) - v_1 - v_2)$. Then we estimate the L^∞-bound for each of E_1, E_2 and E_3 separately. We also choose $\tau < 1$ as done in Theorem 3.4.

For E_1, we find that

$$\left\|E_1(\cdot,t)\right\|_\infty \le C_3\tau^{\vartheta}e^{-\epsilon t}\|v_{10}\|_q \le \|v_{10}\|_\infty \quad \text{for all } t \in (\tau, T_{\max}), \tag{3.22}$$

where $\vartheta \in (\frac{n}{2q}, 1)$, $q > n$ and $\epsilon > 0$.

For E_2, set $m = 0$, $q > n$ and $p = \infty$ in Lemma 2.2, so we can choose $\theta \in (\frac{n}{2q}, \frac{1}{2})$. In this case, we have $\varepsilon \in (0, \frac{1}{2} - \theta)$. Then there exist positive constants C_{10} and μ such that

$$\begin{aligned}
\|E_2(\cdot,t)\|_\infty &\le C_2\|(A_{d_2}+1)^\theta E_2(\cdot,t)\|_q \\
&= \chi_1 C_2\int_0^t \|(A_{d_2}+1)^\theta e^{-(t-s)(A_{d_2}+1)}\nabla\cdot\big(S(v_1(\cdot,t))\nabla u(\cdot,t)\big)\|_q\,ds \\
&\le \chi C_4\int_0^t e^{-(t-s)}\|(A_{d_2}+1)^\theta e^{-(t-s)A_{d_2}}\nabla\cdot\big(S(v_1(\cdot,t))\nabla u(\cdot,t)\big)\|_q\,ds \\
&\le C_{10}\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}e^{-(\mu+1)(t-s)}\|S(v_1(\cdot,t))\nabla u(\cdot,t)\|_q\,ds
\end{aligned} \tag{3.23}$$

for all $t \in (0, T_{\max})$. From (3.9), we have

$$\|\nabla u(\cdot,t)\|_\infty \le H(\tau) \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.24}$$

Hence, there exists $C_{11} > 0$ such that

$$\|S(v_1(\cdot,t))\nabla u(\cdot,t)\|_q \le C_{11} \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.25}$$

Therefore, we obtain, for all $t \in (\tau, T_{\max})$,

$$\begin{aligned}
\|E_2(\cdot,t)\|_\infty &\le C_{11}C_{12}\int_0^t (t-s)^{-\theta-\frac{1}{2}-\varepsilon}e^{-(\mu+1)(t-s)}\,ds \\
&\le C_{11}C_{12}\int_0^\infty \sigma^{-\theta-\frac{1}{2}-\varepsilon}e^{-(\mu+1)\sigma}\,d\sigma \\
&\le C_{13}\Gamma\left(\frac{1}{2}-\theta-\varepsilon\right),
\end{aligned} \tag{3.26}$$

where $\Gamma(x)$ is the Gamma function. Since $\frac{1}{2} - \theta - \varepsilon > 0$, then $\Gamma(\frac{1}{2} - \theta - \varepsilon)$ is positive and real-valued.

Finally, for E_3, by using (2.4) and (2.5), we have

$$\begin{aligned}
\|E_3(\cdot,t)\|_{1,p} &\le C_2\|(A_{d_2}+1)^\theta E_3(\cdot,t)\|_q \\
&\le C_3\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\|v_1(f_1(u)-v_1-v_2)\|_q\,ds \\
&\le C_3\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\|v_1 f_1(u)\|_q\,ds \\
&\le C_3\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\big(\|v_1(\cdot,t)\|_q + E\|u(\cdot,t)\|_q\big)\,ds \\
&\le C_3\int_0^t (t-s)^{-\theta}e^{-\gamma(t-s)}\,ds \\
&\le C_3\int_0^\infty \sigma^{-\theta}e^{-\gamma\sigma}\,d\sigma \le C_3\Gamma(1-\theta) \quad \text{for all } t \in (\tau, T_{\max}),
\end{aligned} \tag{3.27}$$

where C_3 denotes a generic constant that may vary from line to line, and $\Gamma(1-\theta) > 0$ for $1-\theta > 0$. For $p > n$, from the Sobolev embedding theorem, we have

$$\|E_3(\cdot,t)\|_\infty \le C_{14}\Gamma(1-\theta) \quad \text{for all } t \in (\tau, T_{\max}). \tag{3.28}$$

Therefore, by (3.22), (3.26) and (3.28), we see that $\|v_1(\cdot,t)\|_\infty$ is bounded for $t \in (0, T_{\max})$.

Similarly, we see that $\|v_2(\cdot,t)\|_\infty$ is bounded for $t \in (0, T_{\max})$.

Lemma 2.1 part 2 implies that $T_{\max} = \infty$ and therefore $(u(x,t), v_1(x,t), v_2(x,t))$ is bounded for all $(x,t) \in \Omega \times (0,\infty)$. □

Theorem 3.7 *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \ge 1)$ with smooth boundary $\partial\Omega$. Suppose that $d_1, d_2, d_3 > 0$, $\chi_1 \ge 0$, $\chi_2 \ge 0$, (F), (H_1) and (H_2) hold. For any $(u_0, v_{10}, v_{20}) \in [W^{1,p}(\Omega)]^3$ where $p > n$, satisfying $u_0(x) \ge 0$, $v_{10}(x) \ge 0$, $v_{20}(x) \ge 0$ for $x \in \Omega$, the system* (1.1) *possesses a unique global classical solution $(u(x,t), v_1(x,t), v_2(x,t))$ satisfying $(u, v_1, v_2) \in (C([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^3$, and $(u(x,t), v_1(x,t), v_2(x,t))$ is uniformly bounded in $\Omega \times (0,\infty)$, i.e. there is a constant $M(u_0, v_{10}, v_{20}) > 0$ such that $\|u(\cdot,t)\|_\infty + \|v_1(\cdot,t)\|_\infty + \|v_2(\cdot,t)\|_\infty \le M(u_0, v_{10}, v_{20})$ for all $t \in [0,\infty)$.*

Proof Combining the results established in Lemma 2.1 and Theorem 3.6, we obtain the desired conclusions. □

4 Conclusions

This paper *focuses* on the global existence and boundedness of system (1.1) under more general conditions. The fact that two *predators* compete *for* one prey species makes it harder to study the global dynamics for this model, *and* our analysis would *also apply* to other three component systems for the L^∞ estimates.

Acknowledgements

The authors would like to thank the anonymous referees for their good comments. This work is supported by Natural Science Foundation of of Heilongjiang Province of China (No.A 2017007) and the Ph.D. Programs Foundation of Harbin University of China (No. HUDF 2016-001).

Competing interests

We claim that none of the authors have any competing interests in the manuscript.

Authors' contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

Publisher's Note

Received: 28 November 2017 Accepted: 8 March 2018 Published online: 27 March 2018

References

1. Kazarinov, N., van den Driessche, P.: A model predator–prey systems with functional response. Math. Biosci. **39**, 125–134 (1978)
2. Lotka, A.J.: Elements of Physical Biology. Williams & Wilkins, Baltimore (1925)
3. Medvinsky, A.B., Petrovskii, S.V., Tikhonova, I.A., Malchow, H., Li, B.L.: Spatiotemporal complexity of plankton and fish dynamics. SIAM Rev. **44**(3), 311–370 (2002)
4. Murray, J.D.: Mathematical Biology. I. An Introduction. 3rd edn. Interdisciplinary Applied Mathematics, vol. 17. Springer, New York (2002)
5. Owen, M.R., Lewis, M.A.: How predation can slow, stop or reverse a prey invasion. Bull. Math. Biol. **63**, 655–684 (2001)
6. Rosenzweig, L.M.: Paradox of enrichment: destabilization of exploitation ecosystems in ecological time. Science **171**(3969), 385–387 (1971)
7. Meixner, M., De Wit, A., Bose, S., Schöll, E.: Generic spatiotemporal dynamics near codimension-two Turing–Hopf bifurcations. Phys. Rev. E **55**, 6690–6697 (1997)

8. Ricard, M.R., Mischler, S.: Turing instabilities at Hopf bifurcation. J. Nonlinear Sci. **19**, 467–496 (2009)
9. Ko, W., Ryu, K.: Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a prey refuge. J. Differ. Equ. **231**, 534–550 (2006)
10. Wang, J.F., Shi, J.P., Wei, J.J.: Dynamics and pattern formation in a diffusive predator–prey system with strong Allee effect in prey. J. Differ. Equ. **251**, 1276–1304 (2011)
11. Wang, J.F., Shi, J.P., Wei, J.J.: Global bifurcation analysis and pattern formation inhomogeneous diffusive predator–prey systems. J. Differ. Equ. **260**, 3495–3523 (2016)
12. Wu, S.N., Shi, J.P., Wu, B.Y.: Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis. J. Differ. Equ. **260**(7), 5847–5874 (2016)
13. Winkler, M.: Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity. Math. Nachr. **283**, 1664–1673 (2010)
14. Amann, H.: Dynamic theory of quasilinear parabolic equations II, Reaction–diffusion systems. Differ. Integral Equ. **3**, 13–75 (1990)
15. Amann, H.: Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In: Function Spaces, Differential Operators and Nonlinear Analysis (Friedrichroda, 1992), Teubner-Texte Math., vol. 133, pp. 9-126 Teubner, Stuttgart (1993)
16. Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**, 52–107 (2005)
17. Nirenberg, L.: On elliptic partial differential equations. Ann. Sc. Norm. Super. Pisa **13**(3), 115–162 (1959)
18. Winkler, M., Djie, K.C.: Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect. Nonlinear Anal. **72**, 1044–1064 (2010)