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Some results on the fractional order Sturm-Liouville problems

Yuanfang Ru[1], Fanglei Wang[1*], Tianqing An[1] and Yukun An[2]

*Correspondence:
wang-fanglei@hotmail.com
[1]Department of Mathematics, College of Science, Hohai University, Nanjing, 210098, China
Full list of author information is available at the end of the article

Abstract

In this work, we introduce some new results on the Lyapunov inequality, uniqueness and multiplicity results of nontrivial solutions of the nonlinear fractional Sturm-Liouville problems

$$\begin{cases} D_{0^+}^q(p(t)u'(t)) + \Lambda(t)f(u(t)) = 0, & 1 < q \leq 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0, \end{cases}$$

where $\alpha, \beta, \gamma, \delta$ are constants satisfying $0 \neq |\beta\gamma + \alpha\gamma \int_0^1 \frac{1}{p(\tau)}\,d\tau + \alpha\delta| < +\infty$, $p(\cdot)$ is positive and continuous on $[0,1]$. In addition, some existence results are given for the problem

$$\begin{cases} D_{0^+}^q(p(t)u'(t)) + \Lambda(t)f(u(t),\lambda) = 0, & 1 < q \leq 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0, \end{cases}$$

where $\lambda \geq 0$ is a parameter. The proof is based on the fixed point theorems and the Leray-Schauder nonlinear alternative for single-valued maps.

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1 Introduction

On the one hand, since a Lyapunov-type inequality has found many applications in the study of various properties of solutions of differential equations, such as oscillation theory, disconjugacy and eigenvalues problems, there have been many extensions and generalizations as well as improvements in this field, e.g., to nonlinear second order equations, to delay differential equations, to higher order differential equations, to difference equations and to differential and difference systems. We refer the readers to [1–4] (integer order). Fractional differential equations have gained considerable popularity and importance due to their numerous applications in many fields of science and engineering including physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems. With the rapid development of the theory of fractional differential equation, there are many papers which are concerned with the Lyapunov type inequality for a certain fractional order differential equations, see [5–7] and the references therein. Recently, Ghanbari and Gholami [7] introduced the Lyapunov type inequality for a certain fractional order Sturm-Liouville problem in sense of Riemann-Liouville

$$\begin{cases} D^{\alpha}_{a^+}(p(t)u'(t)) + q(t)u(t) = 0, & 1 < \alpha \leq 2, t \in (a,b), b \neq 0, \\ u(a) = u'(a) = 0, & u(b) = 0 \end{cases}$$

like this

$$\int_a^b \int_a^b \left| \frac{q(s)}{p(\omega)} \right| ds\, d\omega > \frac{\Gamma(\alpha)}{2(b-a)^{\alpha-1}}.$$

On the other hand, many authors have studied the existence, uniqueness and multiplicity of solutions for nonlinear boundary value problems involving fractional differential equations, see [8–19]. But Lan and Lin [20] pointed out that the continuity assumptions on nonlinearities used previously are not sufficient and obtained some new results on the existence of multiple positive solutions of systems of nonlinear Caputo fractional differential equations with some of general separated boundary conditions

$$\begin{cases} -{}^cD^q z_i(t) = f_i(t, z(t)), & t \in (0,1), \\ \alpha z_i(0) - \beta z_i'(0) = 0, & \gamma z_i(1) + \delta z_i'(1) = 0, \end{cases}$$

where $z(t) = (z_1(t), \ldots, z_n(t)), f_i : [0,1] \times \mathbb{R}^n_+ \to \mathbb{R}_+$ is continuous on $[0,1] \times \mathbb{R}^n_+$, ${}^cD^q$ is the Caputo differential operator of order $q \in (1,2)$. The α, β, γ, δ are positive real numbers. The relations between the linear Caputo fractional differential equations and the corresponding linear Hammerstein integral equations are studied, which shows that suitable Lipschitz type conditions are needed when one studies the nonlinear Caputo fractional differential equations.

Motivated by these excellent works, in this paper we focus on the representation of the Lyapunov type inequality and the existence of solutions for a certain fractional order Sturm-Liouville problem

$$\begin{cases} D^q_{0^+}(p(t)u'(t)) + \Lambda(t)f(u(t)) = 0, & 1 < q \leq 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0, \end{cases} \tag{1.1}$$

where α, β, γ, δ are constants satisfying $0 \neq |\beta\gamma + \alpha\gamma \int_0^1 \frac{1}{p(\tau)}\, d\tau + \alpha\delta| < +\infty$, $p(\cdot)$ is a positive continuous function on $[0,1]$, $\Lambda(t) : [0,1] \to \mathbb{R}$ is a nontrivial Lebesgue integrable function, $f : \mathbb{R} \to \mathbb{R}$ is continuous. In addition, some existence results are given for the problem

$$\begin{cases} D^q_{0^+}(p(t)u'(t)) + \Lambda(t)f(u(t), \lambda) = 0, & 1 < q \leq 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0, \end{cases} \tag{1.2}$$

where $\lambda \geq 0$ is a parameter, $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous. For the Sturm-Liouville problems, there are many literature works on the studies of the existence and behavior of solutions to nonlinear Sturm-Liouville equations, for example, [21, 22] (integer order) and [23, 24] (fractional order).

The discussion of this manuscript is based on the fixed point theorems and the Leray-Schauder nonlinear alternative for single-valued maps. For convenience, we list the crucial lemmas as follows.

Lemma 1.1 ([25]) *Let ν be a positive measure and Ω be a measurable set with $\nu(\Omega) = 1$. Let I be an interval and suppose that u is a real function in $L(d\nu)$ with $u(t) \in I$ for all $t \in \Omega$. If f is convex on I, then*

$$f\left(\int_\Omega u(t)\,d\nu(t)\right) \geq \int_\Omega f\circ)u(t)\,d\nu(t). \tag{1.3}$$

If f is concave on I, then inequality (1.3) *holds with '$\geq$' substituted by '$\leq$'.*

Lemma 1.2 ([26]) *Let E be a Banach space, E_1 be a closed, convex subset of E, Ω be an open subset of E_1, and $0 \in \Omega$. Suppose that $T : \overline{\Omega} \to E_1$ is completely continuous. Then either*

(i) *T has a fixed point in $\overline{\Omega}$, or*

(ii) *there are $u \in \partial\Omega$ (the boundary of Ω in E_1) and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

Lemma 1.3 ([26]) *Let E be a Banach space and $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) *$\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; or*

(ii) *$\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.*

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 1.4 ([26]) *Let E be a Banach space and $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $\Omega_1 \cap K \neq \emptyset$, $\overline{\Omega_1 \cap K} \subset \Omega_2 \cap K$. Let $T : \overline{\Omega_2 \cap K} \to K$ be a completely continuous operator such that:*

(A) *$\| Tu \| \leq \| u \|$, $\forall u \in \partial(\Omega_1 \cap K)$, and*

(B) *there exists $e \in K \setminus \{0\}$ such that*

$$u \neq Tu + \mu e, \quad \text{for } u \in \partial(\Omega_2 \cap K) \text{ and } \mu > 0.$$

Then T has a fixed point in $\overline{\Omega_2 \cap K} \setminus \Omega_1 \cap K$. The same conclusion remains valid if (A) *holds on $\partial(\Omega_2 \cap K)$ and* (B) *holds on $\partial(\Omega_1 \cap K)$.*

2 Preliminaries

Definition 2.1 ([26]) *For a function u given on the interval $[a,b]$, the Riemann-Liouville derivative of fractional order q is defined as*

$$D^q_{a^+}u(t) = \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-q-1}u(s)\,ds,$$

where $n = [q] + 1$.

Definition 2.2 ([27]) *The Riemann-Liouville fractional integral of order q for a function u is defined as*

$$I_{a^+}^q u(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} u(s)\,ds, \quad q > 0$$

provided that such integral exists.

Lemma 2.3 ([27]) *Let $q > 0$. Then*

$$I_{a^+}^q D_{a^+}^q u(t) = u(t) + \sum_{k=1}^{n} c_k t^{q-k}, \quad n = [q] + 1.$$

Lemma 2.4 *Let $h(t) \in AC[0,1]$. Then the fractional Sturm-Liouville problem*

$$\begin{cases} D_{0^+}^q (p(t)u'(t)) + h(t) = 0, & 1 < q \le 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0 \end{cases}$$

has a unique solution $u(t)$ in the form

$$u(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where

$$G(t,s) = \frac{1}{\rho\Gamma(q)} \begin{cases} [\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma \int_t^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}] - H(t,s), & 0 \le s \le t \le 1; \\ [\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma \int_s^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}], & 0 \le t \le s \le 1; \end{cases}$$

$$\rho = \beta\gamma + \alpha\gamma \int_0^1 \frac{1}{p(\tau)}\,d\tau + \alpha\delta, \qquad H(t,s) = \alpha\left[\delta + \gamma \int_t^1 \frac{d\tau}{p(\tau)}\right] \int_s^t \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau.$$

Proof From Definitions 2.1, 2.2 and Lemma 2.3, it follows that

$$u'(t) = \frac{c_1}{p(t)} - \frac{1}{\Gamma(q)p(t)} \int_0^t (t-s)^{q-1} h(s)\,ds,$$
$$u(t) = c_2 + \int_0^t \frac{c_1}{p(\tau)}\,d\tau - \int_0^t \int_0^\tau \frac{(\tau-s)^{q-1} q(s)}{\Gamma(q)p(\tau)}\,ds\,d\tau.$$

Furthermore, we have

$$u(0) = c_2, u'(0) = \frac{c_1}{p(0)},$$
$$u(1) = c_2 + \int_0^1 \frac{c_1}{p(\tau)}\,d\tau - \int_0^1 \int_0^\tau \frac{(\tau-s)^{q-1} q(s)}{\Gamma(q)p(\tau)}\,ds\,d\tau,$$
$$u'(1) = \frac{c_1}{p(1)} - \frac{1}{\Gamma(q)p(1)} \int_0^1 (1-s)^{q-1} q(s)\,ds.$$

Combining the boundary conditions, we directly get

$$c_1 = \frac{\alpha\gamma \int_0^1 \int_0^\tau \frac{(\tau-s)^{q-1}h(s)}{\Gamma(q)p(\tau)}\,ds\,d\omega + \alpha\delta \int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\,ds}{\rho},$$

$$c_2 = \frac{\beta\gamma \int_0^1 \int_0^\tau \frac{(\tau-s)^{q-1}h(s)}{\Gamma(q)p(\tau)}\,ds\,d\tau + \beta\delta \int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\,ds}{\rho}.$$

Finally, substituting c_1 and c_2, we obtain

$$\begin{aligned} u(t) &= \frac{\beta\gamma \int_0^1 \int_0^\tau \frac{(\tau-s)^{q-1}h(s)}{\Gamma(q)p(\tau)}\,ds\,d\tau + \beta\delta \int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\,ds}{\rho} \\ &\quad + \int_0^t \frac{1}{p(\omega)}\,d\tau \frac{\alpha\gamma \int_0^1 \int_0^\omega \frac{(\omega-s)^{q-1}h(s)}{\Gamma(q)p(\tau)}\,ds\,d\omega + \alpha\delta \int_0^1 \frac{(1-s)^{q-1}h(s)}{\Gamma(q)}\,ds}{\rho} \\ &\quad - \int_0^t \int_0^\tau \frac{(\tau-s)^{q-1}h(s)}{\Gamma(q)p(\tau)}\,ds\,d\tau \\ &= \frac{\beta\gamma \int_0^1 [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \beta\delta \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &\quad + \int_0^t \frac{1}{p(\tau)}\,d\tau \frac{\alpha\gamma \int_0^1 [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \alpha\delta \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &\quad - \int_0^t \left[\int_s^t \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau\right]h(s)\,ds \\ &= \int_0^1 G(t,s)h(s)\,ds. \end{aligned}$$

For $0 \leq t \leq s \leq 1$,

$$\begin{aligned} u(t) &= \frac{\beta\gamma \int_t^1 [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \beta\delta \int_t^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &\quad + \int_0^t \frac{1}{p(\tau)}\,d\tau \frac{\alpha\gamma \int_t^1 [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \alpha\delta \int_t^1 \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &= \int_t^1 \frac{1}{\rho}\left[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}\right]\left[\delta(1-s)^{q-1} + \gamma \int_s^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}\right]h(s)\,ds. \end{aligned}$$

For $0 \leq s \leq t \leq 1$,

$$\begin{aligned} u(t) &= \frac{\beta\gamma \int_0^t [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \beta\delta \int_0^t \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &\quad + \int_0^t \frac{1}{p(\tau)}\,d\tau \frac{ac \int_0^t [\int_s^1 \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau]h(s)\,ds + \alpha\delta \int_0^t \frac{(1-s)^{q-1}}{\Gamma(q)}h(s)\,ds}{\rho} \\ &\quad - \int_0^t \left[\int_s^t \frac{(\tau-s)^{q-1}}{\Gamma(q)p(\tau)}\,d\tau\right]h(s)\,ds \end{aligned}$$

$$= \frac{1}{\rho}\int_0^t \left[\beta + \alpha\int_0^t \frac{d\tau}{p(\tau)}\right]\left[\delta(1-s)^{q-1} + \gamma\int_t^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}\right]$$
$$- \alpha\left[\delta + \gamma\int_t^1 \frac{d\tau}{p(\tau)}\right]\int_s^t \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau\, h(s)\,ds. \qquad \square$$

Lemma 2.5 *Assume that* $\alpha,\beta,\gamma,\delta > 0$, *and* $p(\cdot) : [0,1] \to (0,+\infty)$. *The Green function* $G(t,s)$ *satisfies the following properties*:

(i) $G(t,s) \geq 0$ *for* $0 \leq t,s \leq 1$;

(ii) *For* $0 \leq t,s \leq 1$, *there exists* $C(t) > 0$ *such that* $G(t,s)$ *satisfies the inequalities*

$$C(t)G(s,s) \leq G(t,s)$$

and

$$\min_{t\in[\theta,1-\theta]} C(t) < 1 \quad \textit{for } \theta \in \left(0,\frac{1}{2}\right).$$

(iii) *The maximum value estimate of* $G(t,s)$

$$\overline{G} = \max_{0\leq t,s\leq 1} G(t,s)$$
$$= \max\left\{\max_{s\in[0,1]} G(s,s), \max_{s\in[0,1]} G\bigl(t_0(s),s\bigr)\right\},$$

where

$$t_0(s) = s + \left[\frac{\alpha\delta(1-s)^{q-1} + \alpha\gamma\int_s^1 \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau}{\rho}\right]^{\frac{1}{q-1}}.$$

Proof (i) On the one hand, since $\alpha,\beta,\gamma,\delta > 0$, and $\beta\gamma + \alpha\gamma\int_0^1 \frac{1}{p(\tau)}\,d\tau + \alpha\delta > 0$, it is clear that $G(t,s) \geq 0$ for $0 \leq t \leq s \leq 1$. On the other hand, for $0 \leq s \leq t \leq 1$, we can verify the following inequalities:

$$\alpha\delta\int_0^t \frac{(1-s)^{q-1}}{p(\tau)}\,d\tau - \alpha\delta\int_s^t \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau \geq 0,$$
$$\frac{\alpha\gamma\int_0^t \frac{d\tau}{p(\tau)}\int_t^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}}{\alpha\gamma\int_t^1 \frac{d\tau}{p(\tau)}\int_s^t \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau} \geq \frac{\int_0^t \frac{d\tau}{p(\tau)}\int_t^1 \frac{(t-s)^{q-1}\,d\tau}{p(\tau)}}{\int_t^1 \frac{d\tau}{p(\tau)}\int_s^t \frac{(t-s)^{q-1}}{p(\tau)}\,d\tau} \geq 1.$$

Then we get $G(t,s) \geq 0$ for $0 \leq s \leq t \leq 1$.

(ii) For $0 \leq t \leq s \leq 1$,

$$\frac{\partial G(t,s)}{\partial t} = \frac{\alpha}{\rho\Gamma(q)p(t)}\left[\delta(1-s)^{q-1} + \gamma\int_s^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}\right] \geq 0. \tag{2.1}$$

Then it is easy to obtain

$$G(t,s) \leq G(s,s) \quad \text{for } 0 \leq t \leq s \leq 1.$$

For $0 \le s \le t \le 1$,

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t} &= \frac{1}{\Delta}\left\{-\beta\gamma\frac{(t-s)^{q-1}}{p(t)} + \frac{\alpha\delta(1-s)^{q-1}}{p(t)} + \alpha\gamma\frac{1}{p(t)}\int_t^1\frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau\right.\\&\quad -\alpha\gamma\int_0^t\frac{d\tau}{p(\tau)}\frac{(t-s)^{q-1}}{p(t)} - \alpha\delta\frac{(t-s)^{q-1}}{p(t)} + \alpha\gamma\frac{1}{p(t)}\int_s^t\frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau\\&\quad\left. -\alpha\gamma\int_t^1\frac{d\tau}{p(\tau)}\frac{(t-s)^{q-1}}{p(t)}\right\}\\&= \frac{1}{\rho\Gamma(q)p(t)}\left[-\rho(t-s)^{q-1} + \alpha\delta(1-s)^{q-1} + \alpha\gamma\int_s^1\frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau\right]. \qquad (2.2)\end{aligned}$$

Let

$$F(t) = -\rho(t-s)^{q-1} + \alpha\delta(1-s)^{q-1} + \alpha\gamma\int_s^1\frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau.$$

It is clear that $F'(t) = -\rho(q-1)(t-s)^{q-2} < 0$, which implies that $F(\cdot)$ is decreasing on $t \in (s,1]$. Since $F(s) > 0$ and $F(1) < 0$, there exists unique $t_0(s) \in (s,1)$ such that $F(t_0) = 0$, namely,

$$t_0(s) = s + \left[\frac{\alpha\delta(1-s)^{q-1} + \alpha\gamma\int_s^1\frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau}{\rho}\right]^{\frac{1}{q-1}}.$$

From the above discussion, we get the conclusions

$$\begin{aligned}&\frac{\partial G(t,s)}{\partial t} \ge 0, \quad \text{for } t \in [s,t_0], \text{ and } G(s,s) \le G(t,s) \le G(t_0,s),\\&\frac{\partial G(t,s)}{\partial t} \le 0, \quad \text{for } t \in [t_0,1], \text{ and } G(1,s) \le G(t,s) \le G(t_0,s).\end{aligned}$$

Furthermore, we obtain the estimate

$$G(t,s) \le G\big(t_0(s),s\big), \quad \text{for } 0 \le s \le t \le 1.$$

For $0 \le t \le s \le 1$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\beta + \alpha\int_0^t\frac{d\tau}{p(\tau)}}{\beta + \alpha\int_0^s\frac{d\tau}{p(\tau)}} \ge \frac{\beta + \alpha\int_0^t\frac{d\tau}{p(\tau)}}{\beta + \alpha\int_0^1\frac{d\tau}{p(\tau)}} = C_1(t).$$

For $0 \le s \le t \le 1$,

$$\begin{aligned}\frac{G(t,s)}{G(s,s)} &= \frac{[\beta + \alpha\int_0^t\frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma\int_t^1\frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}] - H(t,s)}{[\beta + \alpha\int_0^s\frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma\int_s^1\frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}]}\\&\ge \frac{[\beta + \alpha\int_0^t\frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma\int_t^1\frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}] - H(t,s)}{[\beta + \alpha\int_0^t\frac{d\tau}{p(\tau)}][\delta + \gamma\int_0^1\frac{d\tau}{p(\tau)}]}\end{aligned}$$

$$
\begin{aligned}
&= \frac{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta(1-s)^{q-1} + \gamma \int_t^1 \frac{(t-s)^{q-1}d\tau}{p(\tau)}] - \alpha[\delta + \gamma \int_t^1 \frac{d\tau}{p(\tau)}] \int_s^t \frac{(t-s)^{q-1}}{p(\tau)}\,d\tau}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\geq \frac{\beta\delta(1-t)^{q-1} + \beta\gamma \int_t^1 \frac{(t-s)^{q-1}d\tau}{p(\tau)} + \alpha\delta(1-s)^{q-1} \int_0^t \frac{d\tau}{p(\tau)} + \alpha\gamma \int_0^t \frac{d\tau}{p(\tau)} \int_t^1 \frac{(t-s)^{q-1}d\tau}{p(\tau)}}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\quad - \frac{\alpha\delta(1-s)^{q-1} \int_s^t \frac{d\tau}{p(\tau)} + \alpha\gamma \int_t^1 \frac{d\tau}{p(\tau)} \int_s^t \frac{(t-s)^{q-1}}{p(\tau)}\,d\tau}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&= \frac{\beta\delta(1-t)^{q-1} + \gamma(t-s)^{q-1}[\beta \int_t^1 \frac{d\tau}{p(\tau)} + \alpha \int_0^t \frac{d\tau}{p(\tau)} \int_t^1 \frac{d\tau}{p(\tau)} - \alpha \int_t^1 \frac{d\tau}{p(\tau)} \int_s^t \frac{d\tau}{p(\tau)}]}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\quad + \frac{\alpha\delta(1-s)^{q-1}[\int_0^t \frac{d\tau}{p(\tau)} - \int_s^t \frac{d\tau}{p(\tau)}]}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\geq \frac{\beta\delta(1-t)^{q-1} + \gamma(t-s)^{q-1}[\beta \int_t^1 \frac{d\tau}{p(\tau)} + \alpha \int_0^t \frac{d\tau}{p(\tau)} \int_t^1 \frac{d\tau}{p(\tau)} - \alpha \int_t^1 \frac{d\tau}{p(\tau)} \int_0^t \frac{d\tau}{p(\tau)}]}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\quad + \frac{\alpha\delta(1-s)^{q-1} \int_0^s \frac{d\tau}{p(\tau)}}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} \\
&\geq \frac{\beta\delta(1-t)^{q-1}}{[\beta + \alpha \int_0^t \frac{d\tau}{p(\tau)}][\delta + \gamma \int_0^1 \frac{d\tau}{p(\tau)}]} = C_2(t).
\end{aligned}
$$

Choosing $C(t) = \min\{C_1(t), C_2(t)\}$, we get $C(t)G(s,s) \leq G(t,s)$. □

3 Existence results I

Theorem 3.1 (Lyapunov type inequality) *Assume that* $\alpha, \beta, \gamma, \delta > 0$, $p(\cdot) : [0,1] \to (0,+\infty)$, *and let* $\Lambda(t) : [0,1] \to R$ *be a nontrivial Lebesgue integrable function. Then, for any nontrivial solution of the fractional Sturm-Liouville problem*

$$
\begin{cases}
D_{0^+}^q (p(t)u'(t)) + \Lambda(t)u(t) = 0, & 1 < q \leq 2, t \in (0,1), \\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0,
\end{cases}
$$

the following so-called Lyapunov type inequality will be satisfied:

$$
\int_0^1 |\Lambda(s)|\,ds > \frac{1}{\overline{G}},
$$

where $\overline{G}$ *is defined in* (*iii*) *of Lemma* 2.5.

Proof From Lemma 2.4 and the triangular inequality, we get

$$
|u(t)| = \left| \int_0^1 G(t,s)\Lambda(s)u(s)\,ds \right| \leq \int_0^1 G(t,s)|\Lambda(s)u(s)|\,ds.
$$

Let E denote the Banach space $C[0,1]$ with the norm defined by $\|u\| = \max_{t\in[0,1]} |u(t)|$. Via some simple computations, we can obtain

$$\begin{aligned}
\|u(t)\| &\le \max_{t\in[0,1]} \int_0^1 G(t,s)\left|\Lambda(s)u(s)\right| ds \\
&\le \|u(t)\| \max_{t\in[0,1]} \int_0^1 G(t,s)\left|\Lambda(s)\right| ds \\
&\le \|u(t)\| \int_0^1 \Bigl[\max_{t\in[0,1]} G(t,s)\Bigr]\left|\Lambda(s)\right| ds,
\end{aligned}$$

namely,

$$\int_0^1 \left|\Lambda(s)\right| ds > \frac{1}{\overline{G}}. \qquad \square$$

Theorem 3.2 (Generalized Lyapunov type inequality) *Assume that $\alpha, \beta, \gamma, \delta > 0$, $p(\cdot) : [0,1] \to (0,+\infty)$, and let $\Lambda(t) : [0,1] \to \mathbb{R}$ be a nontrivial Lebesgue integrable function, $f(u)$ is a positive function on $\mathbb{R}$. Then, for any nontrivial solution of the fractional Sturm-Liouville problem* (1.1), *the following so-called Lyapunov type inequality will be satisfied*:

$$\int_0^1 \left|\Lambda(s)\right| ds > \frac{u^*}{\overline{G}\max_{u\in[u_*,u^*]} f(u)},$$

where

$$u_* = \min_{t\in[0,1]} u(t), \qquad u^* = \max_{t\in[0,1]} u(t).$$

Proof From the similar proof of Theorem 3.1, we get

$$\left|u(t)\right| \le \int_0^1 G(t,s)\left|\Lambda(s)\right| f\bigl(u(s)\bigr)\, ds.$$

Since f is continuous and concave, then using Jensen"s inequality (1.3), we obtain

$$\begin{aligned}
\|u(t)\| &\le \max_{t\in[0,1]} \int_0^1 G(t,s)\left|\Lambda(s)\right| f\bigl(u(s)\bigr)\, ds \\
&\le \int_0^1 \Bigl[\max_{t\in[0,1]} G(t,s)\Bigr]\left|\Lambda(s)\right| f\bigl(u(s)\bigr)\, ds \\
&\le \overline{G}\left|\Lambda(t)\right|_{L^1} \int_0^1 \frac{|\Lambda(s)|}{|\Lambda(t)|_{L^1}} f\bigl(u(s)\bigr)\, ds \\
&\le \overline{G} \max_{u\in[u_*,u^*]} f(u)\left|\Lambda(t)\right|_{L^1},
\end{aligned}$$

namely,

$$\int_0^1 \left|\Lambda(s)\right| ds > \frac{u^*}{\overline{G}\max_{u\in[u_*,u^*]} f(u)}. \qquad \square$$

For convenience, we give some notations:

$$\varpi = \max_{t\in[0,1]} \left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds \right];$$
$$\varsigma = \min_{t\in[\theta,1-\theta]} C(t)\cdot \int_0^1 G(s,s)\Lambda(s)\,ds.$$

Theorem 3.3 *Let $\Lambda(t):[0,1]\to\mathbb{R}_+$ be a nontrivial Lebesgue integrable function and $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the Lipschitz condition*

$$\bigl|f(x)-f(y)\bigr| \le L|x-y|, \quad \forall x,y\in\mathbb{R}, L>0.$$

Then problem (1.1) *has a unique solution if $L\varpi < 1$.*

Proof By Lemma 2.4, the solution of problem (1.1) is equivalent to a fixed point of the operator $T:E\to E$ defined by $T(u(t)) = \int_0^1 G(t,s)\Lambda(s)f(u(s))\,ds$.

Let $\sup_{t\in[0,1]}|f(0)| = \nu$. Now we show that $T:B_r\subset B_r$, where $B_r = \{u\in C[0,1]: \|u\| < r\}$ with $r > \frac{\nu\varpi}{1-L\varpi}$. For $u\in B_r$, one has $|f(u)| = |f(u)-f(0)+f(0)| \le L|u|+\nu \le Lr+\nu$. Furthermore, we have

$$\begin{aligned}
\bigl\|T(u)(t)\bigr\| &= \left\| \int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \right\| \\
&= \int_0^t G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\
&\le \left\| \int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \right\| \\
&\le (Lr+\nu)\max_{t\in[0,1]} \left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds \right] \\
&= (Lr+\nu)\varpi \le r,
\end{aligned}$$

which yields $T:B_r\subset B_r$.

For any $x,y\in E$, we have

$$\begin{aligned}
\bigl\|T(x)-T(y)\bigr\| &= \left\| \int_0^1 G(t,s)\Lambda(s)f\bigl(x(s)\bigr)\,ds - \int_0^1 G(t,s)\Lambda(s)f\bigl(y(s)\bigr)\,ds \right\| \\
&\le \sup_{t\in[0,1]} \left\{ \int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\bigl|f\bigl(x(s)\bigr)-f\bigl(y(s)\bigr)\bigr|\,ds \right. \\
&\quad \left. + \int_t^1 G(s,s)\Lambda(s)\bigl|f\bigl(x(s)\bigr)-f\bigl(y(s)\bigr)\bigr|\,ds \right\} \\
&\le L\max_{t\in[0,1]} \left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds \right] \|x-y\| \\
&= L\varpi\,\|x-y\|.
\end{aligned}$$

Since $L\varpi < 1$, from the Banach's contraction mapping principle it follows that there exists a unique fixed point for the operator T which corresponds to the unique solution for problem (1.1). This completes the proof. □

Theorem 3.4 *Let* $\Lambda(t) : [0,1] \to R^+$ *be a nontrivial Lebesgue integrable function and* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying the following*:

(F0) *There exists a positive constant* K *such that* $|f(u)| \leq K$ *for* $u \in \mathbb{R}$.

Then problem (1.1) *has at least one solution.*

Proof First, since the function $p : [0,1] \to (0,+\infty)$ is continuous, we get $p_* = \min_{t\in[0,1]} p(t) > 0$. Further, from (2.1) and (2.2), we get the following estimates respectively:

for $0 \leq t \leq s \leq 1$,

$$\begin{aligned} 0 < \frac{\partial G(t,s)}{\partial t} &= \frac{\alpha}{\rho\Gamma(q)p(t)}\left[\delta(1-s)^{q-1} + \gamma\int_s^1 \frac{(\tau-s)^{q-1}\,d\tau}{p(\tau)}\right] \\ &\leq \frac{\alpha}{\rho\Gamma(q)p_*}\left[\delta + \gamma\int_0^1 \frac{d\tau}{p(\tau)}\right]; \end{aligned}$$

for $0 \leq s \leq t \leq 1$,

$$\begin{aligned} \left|\frac{\partial G(t,s)}{\partial t}\right| &= \left|\frac{1}{\rho\Gamma(q)p(t)}\left[-\rho(t-s)^{q-1} + \alpha\delta(1-s)^{q-1} + \alpha\gamma\int_s^1 \frac{(\tau-s)^{q-1}}{p(\tau)}\,d\tau\right]\right| \\ &\leq \frac{1}{\rho\Gamma(q)p_*}\left[\rho + \alpha\delta + \alpha\gamma\int_0^1 \frac{d\tau}{p(\tau)}\right]; \end{aligned}$$

which implies that $|\frac{\partial G(t,s)}{\partial t}|$ is bounded for $0 \leq s,t \leq 1$, namely, there exists $S > 0$ such that $|\frac{\partial G(t,s)}{\partial t}| \leq S$. Combining with $|f(t,u)| \leq K$ for $t \in [0,1]$, $t \in R$, we obtain

$$\left|(Tu)'(t)\right| = \left|\int_0^1 \frac{\partial G(t,s)}{\partial t}\Lambda(s)f\big(u(s)\big)\,ds\right| \leq SK\big\|\Lambda(t)\big\|_{L^1}.$$

Hence, for any $t_1, t_2 \in [0,1]$, we have

$$\left|(Tu)(t_2) - (Tu)(t_1)\right| = \left|\int_{t_1}^{t_2} (Tu)'(t)\,dt\right| \leq SK\big\|\Lambda(t)\big\|_{L^1}|t_2 - t_1|.$$

This means that T is equicontinuous on [0,1]. Thus, by the Arzelà-Ascoli theorem, the operator T is completely continuous.

Finally, let $B_r = \{u \in E : \|u\| < r\}$ with $r = K\varpi + 1$. If u is a solution for the given problem, then, for $\lambda \in (0,1)$, we obtain

$$\begin{aligned} \|u\| &= \lambda\big\|Tu(t)\big\| = \lambda\left\|\int_0^1 G(t,s)\Lambda(s)f\big(u(s)\big)\,ds\right\| \\ &= \lambda\left\|\int_0^t G(t,s)\Lambda(s)f\big(u(s)\big)\,ds + \int_t^1 G(t,s)\Lambda(s)f\big(u(s)\big)\,ds\right\| \\ &< \max_{t\in[0,1]}\int_0^t G\big(t_0(s),s\big)\Lambda(s)\big|f\big(u(s)\big)\big|\,ds + \int_t^1 G(s,s)\big|\Lambda(s)f\big(u(s)\big)\big|\,ds \\ &\leq K\max_{t\in[0,1]}\left[\int_0^t G\big(t_0(s),s\big)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds\right] \\ &\leq K\varpi, \end{aligned}$$

which yields a contradiction. Therefore, by Lemma 1.2, the operator T has a fixed point in E. □

Theorem 3.5 *Let* $\Lambda(t) : [0,1] \to R^+$ *be a nontrivial Lebesgue integrable function and* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function satisfying* (F0). *In addition, the following assumption holds*:

(F1) *There exists a positive constant* r_1 *such that*

$$f(u) \geq \varsigma^{-1} r_1 \quad \text{for } u \in [0, r_1].$$

Then problem (1.1) *has at least one solution.*

Proof Define a cone P of the Banach space E as $P = \{u \in E : u \geq 0\}$. From the proof of Theorem 3.4, we know that $T : P \to P$ is completely continuous. Set $P_{r_i} = \{u \in P : \|u\| < r_i\}$.

For $u \in \partial P_{r_1}$, one has $0 \leq u \leq r_1$. For $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} T\bigl(u(t)\bigr) &= \int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\ &\geq \int_0^1 C(t)G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\ &\geq \min_{t\in[\theta,1-\theta]} C(t) \cdot \int_0^1 G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\ &\geq \min_{t\in[\theta,1-\theta]} C(t) \cdot \int_0^1 G(s,s)\Lambda(s)\,ds \cdot r_1 \\ &> r_1 = \|u\|. \end{aligned}$$

Choosing $r_2 > K\varpi$. Then, for $u \in \partial P_{r_2}$, we have

$$\begin{aligned} \bigl\|T\bigl(u(t)\bigr)\bigr\| &= \left\| \int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \right\| \\ &= \left\| \int_0^t G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \right\| \\ &\leq \max_{t\in[0,1]} \left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds \right] K \\ &< r_2 = \|u\|. \end{aligned}$$

Then, by Lemma 1.3, problem (1.1) has at least one positive solution $u(t)$ belonging to E such that $r_1 \leq \|u\| \leq r_2$. □

Theorem 3.6 *Let* $\Lambda(t) : [0,1] \to \mathbb{R}_+$ *be a nontrivial Lebesgue integrable function,* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function and satisfy the following assumptions*:

(F2) *There exists a nondecreasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\bigl|f(u)\bigr| \leq \varphi\bigl(\|u\|\bigr), \quad \forall u \in \mathbb{R};$$

(F3) *There exists a constant* $R > 0$ *such that* $\frac{R}{\varpi\varphi(R)} > 1$.

Then problem (1.1) *has at least one solution.*

Proof From the proof of Theorem 3.4, we know that T is completely continuous. Now we show that (ii) of Lemma 1.2 does not hold. If u is a solution of (1.1), then, for $\lambda \in (0,1)$, we obtain

$$\begin{aligned}
\|u\| &= \lambda \| T(u(t)) \| = \lambda \left\| \int_0^1 G(t,s)\Lambda(s)f(u(s))\,ds \right\| \\
&= \lambda \left\| \int_0^t G(t,s)\Lambda(s)f(u(s))\,ds + \int_t^1 G(t,s)\Lambda(s)f(u(s))\,ds \right\| \\
&< \max_{t\in[0,1]} \left[\int_0^t G(t_0(s),s)\Lambda(s)|f(u(s))|\,ds + \int_t^1 G(s,s)\Lambda(s)|f(u(s))|\,ds \right] \\
&\le \max_{t\in[0,1]} \left[\int_0^t G(t_0(s),s)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds \right] \varphi(\|u\|) \\
&\le \varpi \varphi(\|u\|).
\end{aligned}$$

Let $B_R = \{u \in E : \|u\| < R\}$. From the above inequality and (F3), it yields a contradiction. Therefore, by Lemma 1.2, the operator T has a fixed point in B_R. □

Theorem 3.7 *Let* $\Lambda(t) : [0,1] \to \mathbb{R}_+$ *be a nontrivial Lebesgue integrable function and* $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function. Suppose that* (F2) *and* (F3) *hold. In addition, the following assumption holds*:

(F4) *There exists a positive constant* r *with* $r < R$ *and a function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying*

$$\begin{aligned}
&f(u) \ge \psi(\|u\|), \quad \text{for } u \in [0, \varsigma r], \\
&\psi(\varsigma r) \ge r.
\end{aligned}$$

If $\varsigma < 1$, *then* (1.1) *has at least one positive solution* $u(t)$.

Proof Let $B_r = \{u \in E : \|u\| < r\}$.

Part (I). For any $u \in \partial(B_R \cap P)$, from (F3) and (F4) it follows that

$$\begin{aligned}
\| T(u(t)) \| &= \left\| \int_0^1 G(t,s)\Lambda(s)f(u(s))\,ds \right\| \\
&= \left\| \int_0^t G(t,s)\Lambda(s)f(u(s))\,ds + \int_t^1 G(t,s)\Lambda(s)f(u(s))\,ds \right\| \\
&< \max_{t\in[0,1]} \int_0^t G(t_0(s),s)\Lambda(s)f(u(s))\,ds + \int_t^1 G(s,s)\Lambda(s)f(u(s))\,ds \\
&\le \max_{t\in[0,1]} \left[\int_0^t G(t_0(s),s)\Lambda(s)\,ds + \int_t^1 G(s,s\Lambda(s))\,ds \right] \varphi(\|u\|) \\
&= \varpi \|\varphi(R) \\
&\le R = \|u\|,
\end{aligned}$$

which implies that (A) of Lemma 1.4 holds.

Now we prove that $u \neq T(u) + \mu$ for $u \in \partial(B_{\varsigma r} \cap P)$ and $\mu > 0$. On the contrary, if there exists $u_0 \in \partial(B_{\varsigma r} \cap P)$ and $\mu_0 > 0$ such that $u_0 = T(u_0) + \mu_0$, then, for $t \in [\theta, 1-\theta]$, one has

$\min_{t\in[\theta,1-\theta]} C(t) > 0$. Furthermore, from (F5) it follows that

$$\begin{aligned}
u_0(t) &= T\big(u_0(t)\big) + \mu_0 \\
&= \int_0^1 G(t,s)\Lambda(s)f\big(u_0(s)\big)\,ds + \mu_0 \\
&\geq \int_0^1 C(t)G(s,s)\Lambda(s)f\big(u_0(s)\big)\,ds + \mu_0 \\
&\geq \min_{t\in[\theta,1-\theta]} C(t)\int_0^1 G(s,s)\Lambda(s)f\big(u_0(s)\big)\,ds + \mu_0 \\
&\geq \min_{t\in[\theta,1-\theta]} C(t)\int_0^1 G(s,s)\Lambda(s)\psi(\varsigma r)\,ds + \mu_0 \\
&= \varsigma r + \mu_0.
\end{aligned}$$

Furthermore, we get

$$\varsigma r > \min_{t\in[\theta,1-\theta]} u_0(t) \geq \varsigma r + \mu_0 > \varsigma r,$$

which yields a contradiction. So (B) of Lemma 1.4 holds.

Therefore, Lemma 1.3 guarantees that T has at least one fixed point. □

Theorem 3.8 *Let* $\Lambda(t) : [0,1] \to \mathbb{R}_+$ *be a nontrivial Lebesgue integrable function and* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function satisfying* (F0). *In addition, the following assumptions hold*:

(F5) $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$;

(F6) *There exists* $\overline{R} > 0$ *such that* $\min_{u\in[\vartheta\overline{R},\overline{R}]} f(u) > \sigma\overline{R}$, *where*

$$\begin{aligned}
&0 < \vartheta = \eta\Big[\min_{t\in[\theta,1-\theta]} C(t)\Big] < 1, \\
&0 < \eta = \left[\max_{0\leq s\leq 1} \frac{G(t_0(s),s)}{G(s,s)}\right]^{-1} \leq 1, \\
&\sigma = \left[\min_{t\in[\theta,1-\theta]} C(t)\int_\theta^{1-\theta} G(s,s)\Lambda(s)\,ds\right]^{-1}.
\end{aligned}$$

Then problem (1.1) *has at least two solutions*.

Proof From Lemma 2.5, we can derive the following inequalities:

$$\begin{aligned}
\big\|T\big(u(t)\big)\big\| &= \left\|\int_0^1 G(t,s)\Lambda(s)f\big(u(s)\big)\,ds\right\| \\
&= \left\|\int_0^t G(t,s)\Lambda(s)f\big(u(s)\big)\,ds + \int_t^1 G(t,s)\Lambda(s)f\big(u(s)\big)\,ds\right\| \\
&\leq \max_{t\in[0,1]}\left[\int_0^t G\big(t_0(s),s\big)\Lambda(s)f\big(u(s)\big)\,ds + \int_t^1 G(s,s)\Lambda(s)f\big(u(s)\big)\,ds\right]
\end{aligned}$$

$$\begin{aligned}
&= \max_{t\in[0,1]}\left[\int_0^t \frac{G(t_0(s),s)}{G(s,s)}G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right] \\
&\leq \max_{0\leq s\leq 1}\frac{G(t_0(s),s)}{G(s,s)}\left[\int_0^1 G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right]
\end{aligned}$$

and

$$\begin{aligned}
T\bigl(u(t)\bigr) &= \int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\
&\geq \int_0^1 C(t)G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds.
\end{aligned}$$

Combining the two inequalities, we have

$$T\bigl(u(t)\bigr) \geq C(t)\eta\bigl\|T\bigl(u(t)\bigr)\bigr\|.$$

Define a subcone $\widehat{P}$ of the Banach space E as $\widehat{P} = \{u \in E : u \geq C(t)\eta\|u(t)\|\}$. From the standard process, we know that $T : \widehat{P} \to \widehat{P}$ is completely continuous. Set $\widehat{P}_r = \{u \in \widehat{P} : \|u\| < r\}$.

Since $\lim_{u\to 0^+}\frac{f(u)}{u} = 0$, there exist $\epsilon > 0$ and $r > 0$ such that $f(u) < \epsilon u$, for $0 \leq u \leq r$, where ϵ satisfies $\epsilon\varpi < 1$. For $u \in \partial\widehat{P}_r$, we have

$$\begin{aligned}
\bigl\|T\bigl(u(t)\bigr)\bigr\| &= \left\|\int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right\| \\
&= \left\|\int_0^t G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right\| \\
&\leq \epsilon\max_{t\in[0,1]}\left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds\right]\|u\| \\
&< \|u\|.
\end{aligned}$$

In a similar way, we choose $R > K\varpi$. Then, for $u \in \partial\widehat{P}_R$, we have

$$\begin{aligned}
\bigl\|T\bigl(u(t)\bigr)\bigr\| &= \left\|\int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right\| \\
&= \left\|\int_0^t G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right\| \\
&\leq \max_{t\in[0,1]}\left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds\right]K \\
&< R = \|u\|.
\end{aligned}$$

For any $u \in \partial P_{\overline{R}}$, choosing $t^* \in (\theta, 1-\theta)$, it is easy to verify that $u(t^*) \in [\vartheta\overline{R}, \overline{R}]$. Furthermore, we have

$$\begin{aligned}
T\bigl(u(t^*)\bigr) &= \int_0^1 G\bigl(t^*,s\bigr)\Lambda(s)f\bigl(u(s)\bigr)\,ds \\
&\geq C\bigl(t^*\bigr)\int_\theta^{1-\theta} G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\geq C(t^*)\int_\theta^{1-\theta} G(s,s)\Lambda(s)\min_{u\in[\vartheta\overline{R},\overline{R}]} f\big(u(s)\big)\,ds\\
&\geq \Big[\min_{t\in[\theta,1-\theta]} C(t)\Big]\int_\theta^{1-\theta} G(s,s)\Lambda(s)\sigma\overline{R}\,ds\\
&= \overline{R} = \|u\|.
\end{aligned}
$$

Then by Lemma 1.3, problem (1.1) has at least two positive solutions $r \leq \|u_1(t)\| \leq \overline{R}$ and $\overline{R} \leq \|u_2(t)\| \leq R$. □

Example 1 Let us consider the problem

$$
\begin{cases}
D_{0^+}^q(p(t)u'(t)) + \Lambda(t)\arctan u = 0, & 1 < q \leq 2, t \in (0,1),\\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0.
\end{cases}
$$

Since $|f(u)| = |\arctan u| < \pi$, this problem has a solution by Theorem 3.4. If $\Lambda(t)$ satisfies

$$
\varpi = \max_{t\in[0,1]}\left[\int_0^t G\big(t_0(s),s\big)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds\right] < 1.
$$

It is easy to get that

$$
f'(u) = (\arctan u)' = \frac{1}{1+u^2} \leq 1 = L.
$$

Therefore, this problem has a unique solution by Theorem 3.3.

Example 2 Let us consider the problem

$$
\begin{cases}
D_{0^+}^q(p(t)u'(t)) + \Lambda(t)e^{-u^{100}} = 0, & 1 < q \leq 2, t \in (0,1),\\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0.
\end{cases}
$$

Since $f(u) = e^{-u^{100}} \leq 1$, we can choose $r_1 = \varpi + 1$. Then it is clear that

$$
f(u) \leq 1 < \varpi^{-1}r_1 \quad \text{for } u \in [0, r_1],
$$

which implies that (F1) holds. Finally, for any $r > 0$, we have $f(u) \geq e^{-r^{100}}$ for $u \in [0,r]$. Since $\lim_{r\to 0^+}\frac{e^{-r^{100}}}{\varsigma^{-1}r} = +\infty$, there exists $r_2 < r_1$ such that $f(u) \geq \varsigma^{-1}r_2$ for $u \in [0, r_2]$, which implies that (F1) holds. Therefore, this problem has a unique solution by Theorem 3.5.

Example 3 Let us consider the problem

$$
\begin{cases}
D_{0^+}^q(p(t)u'(t)) + \Lambda(t)e^{-u^2}(\arctan u + \sin u + 2) = 0, & 1 < q \leq 2, t \in (0,1),\\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0.
\end{cases}
$$

It is clear that $|f(u)| = |e^{-u^2}(\arctan u^{\frac{1}{5}} + \sin u^{\frac{1}{3}} + 2)| \leq \|u\|^{\frac{1}{5}} + \|u\|^{\frac{1}{3}} + 2 = \varphi(\|u\|)$, $\forall u \in R$. Then (F2) holds. Furthermore, for sufficiently large $R > 0$, the inequality $\frac{R}{\varpi\varphi(R)} > 1$ obviously holds, namely, (F3) holds. Then this problem has at least one solution by Theorem 3.6.

For $u \in R^+$, since $f(u) = e^{-u^2}(\arctan u^{\frac{1}{5}} + \sin u^{\frac{1}{3}} + 2) \geq e^{-u^2} \geq e^{-\|u\|^2} = \psi(\|u\|)$, we have $f(u) \geq \psi(\|u\|)$ for $u \in [0, \varsigma r]$, for any $r > 0$. Via some simple computations, we get $\lim_{r\to 0^+} \frac{\psi(\varsigma r)}{r} = +\infty$. Then there exists sufficiently small $r > 0$ such that $\psi(\varsigma r) \geq r$. From the above discussions, we have that (F4) holds. Therefore, this problem has at least one positive solution $u(t)$ for $\varsigma < 1$ by Theorem 3.7.

Example 4 Let us consider the problem

$$\begin{cases} D_{0^+}^q(p(t)u'(t)) + \Lambda(t)\frac{2\sigma}{(2\vartheta)^2e^{-2\vartheta}}u^2e^{-u} = 0, & 1 < q \leq 2, t \in (0,1), \\ \alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0. \end{cases}$$

Since $f(u) = \frac{2\sigma+1}{(2\vartheta)^2e^{-2\vartheta}}u^2e^{-u}$, via some simple computations, we can verify that (F0) and (F5) hold. In addition, since $f'(u) = \frac{2\sigma+1}{(2\vartheta)^2e^{-2\vartheta}}e^{-u}(2u - u^2) = \frac{2\sigma+1}{(2\vartheta)^2e^{-2\vartheta}}e^{-u}u(2-u)$, it is clear that $f'(u) > 0$ for $u \in (0,2)$; $f'(u) < 0$ for $u \in (2,+\infty)$. Let $\overline{R} = 2$, then for any $u \in [2\vartheta, 2]$, we have $\min_{u\in[2\vartheta,2]} f(u) = \frac{2\sigma+1}{(2\vartheta)^2e^{-2\vartheta}}(2\vartheta)^2e^{-2\vartheta} > 2\sigma$. Therefore, this problem has at least two positive solutions $u(t)$ by Theorem 3.8.

4 Existence results II

Theorem 4.1 *Let $\Lambda(t) : [0,1] \to \mathbb{R}_+$ be a nontrivial Lebesgue integrable function and $f : \mathbb{R} \times [0,+\infty) \to \mathbb{R}$ be a continuous function satisfying the following*:

(H) *There exists a positive constant K such that $|f(u,\lambda)| \leq K$ for $u \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$.*

Then problem (1.2) *has at least one solution.*

This result can be directly derived from the proof of Theorem 3.4.

Now define a cone P of the Banach space E as $P = \{u \in E : u \geq 0\}$. Let $P_{r_i} = \{u \in P : \|u\| < r_i\}$. Define T by

$$T\big(u(t)\big) = \int_0^1 G(t,s)\Lambda(s)f\big(u(s),\lambda\big)\,ds.$$

From the proof of Theorem 3.4, we know that $T : P \to P$ is completely continuous.

Theorem 4.2 *Let $\Lambda(t) : [0,1] \to \mathbb{R}_+$ be a nontrivial Lebesgue integrable function and f be a nonnegative continuous function satisfying* (H). *If $f(0,0) > 0$, then there exists $\lambda^* > 0$ such that problem* (1.2) *has at least one solution for $0 \leq \lambda < \lambda^*$.*

Proof Since $f(u,\lambda)$ is continuous and $f(0,0) > 0$, for any given $\epsilon > 0$ (sufficiently small), there exists $\delta > 0$ such that $f(u,\lambda) > f(0,0) - \epsilon$ if $0 \leq u < \delta$, $0 \leq \lambda < \delta$. Choosing $r_1 < \min\{\delta, \varsigma(f(0,0) - \epsilon)\}$ and $\lambda^* = \delta$. Then, for any $u \in \partial P_{r_1}$ and $t \in [\theta, 1-\theta]$, we have

$$\begin{aligned} T\big(u(t)\big) &= \int_0^1 G(t,s)\Lambda(s)f\big(u(s),\lambda\big)\,ds \\ &\geq \int_0^1 C(t)G(s,s)\Lambda(s)f\big(u(s),\lambda\big)\,ds \\ &\geq \min_{t\in[\theta,1-\theta]} C(t) \cdot \int_0^1 G(s,s)\Lambda(s)f\big(u(s),\lambda\big)\,ds \end{aligned}$$

$$
\begin{aligned}
&\geq \min_{t\in[\theta,1-\theta]} C(t)\cdot\int_0^1 G(s,s)\Lambda(s)\,ds\cdot\bigl(f(0,0)-\epsilon\bigr)\\
&> r_1 = \|u\|.
\end{aligned}
$$

Choosing $r_2 > K\varpi$. Then, for $u \in \partial P_{r_2}$, we have

$$
\begin{aligned}
\bigl\|T\bigl(u(t)\bigr)\bigr\| &= \left\|\int_0^1 G(t,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\right\|\\
&\leq \max_{t\in[0,1]}\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)f\bigl(u(s)\bigr)\,ds + \int_t^1 G(s,s)\Lambda(s)f\bigl(u(s)\bigr)\,ds\\
&\leq \max_{t\in[0,1]}\left[\int_0^t G\bigl(t_0(s),s\bigr)\Lambda(s)\,ds + \int_t^1 G(s,s)\Lambda(s)\,ds\right]K\\
&< r_2 = \|u\|.
\end{aligned}
$$

Then, by Lemma 1.3, problem (1.2) has at least one positive solution $u(t)$ belonging to E such that $r_2 \leq \|u\| \leq r_1$. □

Corollary 4.3 *Let* $\Lambda(t):[0,1]\to\mathbb{R}_+$ *be a nontrivial Lebesgue integrable function and* f *be a nonnegative continuous function satisfying* (H). *If* $\lim_{u\to 0^+} f(u,\lambda) = f(0,0) > 0$, *then problem* (1.2) *has at least one solution for any* $\lambda \geq 0$.

Example 5 Let us consider the problem

$$
\begin{cases}
D_{0^+}^q(p(t)u'(t)) + \Lambda(t)(\arctan u^2 + e^{-\lambda}) = 0, & 1 < q \leq 2, t \in (0,1),\\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0.
\end{cases}
$$

It is clear that (H) holds and $f(0,0) > 0$. Then there exists $\lambda^* > 0$ such that this problem has at least one solution for $0 \leq \lambda < \lambda^*$.

Example 6 Let us consider the problem

$$
\begin{cases}
D_{0^+}^q(p(t)u'(t)) + \Lambda(t)e^{-\lambda u^{100}} = 0, & 1 < q \leq 2, t \in (0,1),\\
\alpha u(0) - \beta p(0)u'(0) = 0, & \gamma u(1) + \delta p(1)u'(1) = 0.
\end{cases}
$$

It is clear that (H) holds and $\lim_{u\to 0^+} f(u,\lambda) = f(0,0) > 0$. Then this problem has at least one solution for any $\lambda > 0$.

5 Conclusion

In this manuscript, the authors prove some new existence results as well as uniqueness and multiplicity results on fractional boundary value problems.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, College of Science, Hohai University, Nanjing, 210098, China. [2]Department of Mathematics, College of Science, Nanjing University of Aeronautics and Astronautics, Nanjing, 210098, China.

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