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Periodic solutions for prescribed mean curvature p-Laplacian equations with a singularity of repulsive type and a time-varying delay

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Abstract

In this article, the authors study the existence of positive periodic solutions for a prescribed mean curvature p-Laplacian equation with a singularity of repulsive type and a time-varying delay

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}\right)\right)' + \beta x'(t) + g(t,x(t),x(t-\tau(t))) = p(t),$$

where $g \to -\infty$ when $x \to 0^+$. The existence of positive periodic solutions conditions is devised by using the coincidence degree theory and some analysis methods. A numerical example demonstrates the validity of the main results.

Keywords: prescribed mean curvature equation; coincidence degree theory; periodic solutions; singularity; delay

1 Introduction

The problems of periodic solution have been studied widely for some types of differential equations with a singularity (see [1–8] and the references therein). For example, in [2], Zhang studied periodic solutions for the following Liénard equation with a singularity:

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(t,x(t)\big) = 0,$$

where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, $g(t,x)$ is a T-periodic function in the first argument and can be singular at $x = 0$, *i.e.*, $g(t,x)$ can be unbounded as $x \to 0^+$.

On the basis of work of Zhang, Wang in [8] further studied periodic solutions for the Liénard equation with a singularity and a deviating argument, which is different from the literature [2],

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(t,x(t-\sigma)\big) = 0,$$

where $0 \leq \sigma < T$ is a constant, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, $g(t,x)$ is a T-periodic function in the first argument and can be singular at $x = 0$, *i.e.*, $g(t,x)$ can be unbounded as $x \to 0^+$.

Nowadays, the prescribed mean curvature equation and its modified forms, which arise from some problems associated with differential geometry and physics such as combustible gas dynamics, have been studied widely (see [9–12] and the references therein). Moreover, we note that the existence of periodic solutions for the prescribed curvature mean equations has attracted much attention from researchers. In [13], Feng considered a kind of prescribed mean curvature Liénard equation

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\bigl(u(t)\bigr)u'(t) + g\bigl(t,u\bigl(t-\tau(t)\bigr)\bigr) = e(t), \tag{1.1}$$

where $\tau, e \in C(\mathbb{R},\mathbb{R})$ are T-periodic, and $g \in C(\mathbb{R} \times \mathbb{R},\mathbb{R})$ is T-periodic in the first argument, $T > 0$ is a constant. By applying Mawhin's continuation theorem and given some sufficient conditions, the author showed that equation (1.1) has at least one periodic solution.

On the basis of work of Feng, various types of prescribed curvature mean equations have been studied (see [14–17] and the references therein). But, to the best of our knowledge, the study of positive periodic solutions for the prescribed mean curvature equation with a singularity is relatively infrequent. This is due to the fact that the mechanism on which how the solution is influenced by the singularity and the nonlinear term $(\frac{u'(t)}{\sqrt{1+(u'(t))^2}})'$ associated to prescribed mean curvature equation is far away from clear.

To address this issue, recently, Lu and Kong in [18] studied periodic solutions for a kind of prescribed mean curvature Liénard equation with a singularity and a deviating argument:

$$\left(\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\right)' + f\bigl(u(t)\bigr)u'(t) + g\bigl(t,u(t-\sigma)\bigr) = e(t), \tag{1.2}$$

where $0 \leq \sigma < T$, $g : (0,+\infty) \to \mathbb{R}$ is a continuous function and can be singular at $u = 0$. However, $\sigma = kT$, k is an integer. If $\sigma \neq kT$, it is difficult to estimate *a priori* bounds of periodic solutions by using method in [18]. Therefore, it is significant to consider time-varying delay for the prescribed mean curvature equations.

Inspired by the above facts, in this paper, we consider the following prescribed mean curvature Duffing-type equation with a singularity of repulsive type and a time-varying delay:

$$\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1+(x'(t))^2}}\right)\right)' + \beta x'(t) + g\bigl(t,x(t),x\bigl(t-\tau(t)\bigr)\bigr) = p(t), \tag{1.3}$$

where $g : [0,T] \times (0,+\infty) \times (0,+\infty) \to \mathbb{R}$ is a continuous function. g can be singular at $x = 0$, *i.e.*, g can be unbounded as $x \to 0^+$. $\tau, p \in (\mathbb{R},\mathbb{R})$ are T-periodic with $\int_0^T p(t)\,dt = 0$, β is a constant. By applying Mawhin's continuation theorem, we prove that equation (1.3) has at least one positive T-periodic solution. So, our research is meaningful and feasible.

The rest of the paper is organized as follows. In Section 2, some necessary definitions and lemmas are introduced. The existence of periodic solutions conditions is presented in Section 3. A numerical example is illustrated to show the validity of the proposed method in Section 4.

2 Preliminary

First, we recall the following definition and lemmas.

Definition 2.1 Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A linear operator

$$L : D(L) \subset X \to Y$$

is said to be a Fredholm operator of index zero provided that

(i) $\operatorname{Im} L$ is a closed subset of Y,

(ii) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

Definition 2.2 Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, $\Omega \subset X$ be an open and bounded set;

$$L : D(L) \subset X \to Y$$

is a Fredholm operator of index zero, and we have a continuous operator

$$N : \Omega \subset X \to Y$$

being L-compact in $\bar{\Omega}$ provided that

(I) $K_p(I-Q)N(\bar{\Omega})$ is a relative compact set of X,

(II) $QN(\bar{\Omega})$ is a bounded set of Y,

where we denote $X_1 = \ker L$, $Y_2 = \operatorname{Im} L$. Then we have the decompositions $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, and we let

$$P : X \to X_1, \qquad Q : Y \to Y_1$$

be continuous linear projectors (meaning $P^2 = P$ and $Q^2 = Q$), and $K_p = L|^{-1}_{\ker P\cap D(L)}$.

Lemma 2.1 [19] *Let X and Y be two real Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \bar{\Omega} \subset X \to Y$ be L-compact on $\bar{\Omega}$. Suppose that all of the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\forall \lambda \in (0,1)$;

(2) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \ker L$;

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, *where $J : \operatorname{Im} Q \to \ker L$ is a homeomorphism map.*

Then the equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

In order to use Lemma 2.1, let us consider the following system:

$$\begin{cases} x_1'(t) = \frac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}} = \phi(x_2(t)), \\ x_2'(t) = -\beta\phi(x_2(t)) - g(t, x_1(t), x_1(t-\tau(t))) + p(t), \end{cases} \tag{2.1}$$

where $\varphi_q(s) = |s^{q-2}|s$, $\frac{1}{p} + \frac{1}{q} = 1$, $x_2(t) = \varphi_p(\frac{x_1'(t)}{\sqrt{1+(x_1'(t))^2}}) = \phi^{-1}(x_1'(t))$. Obviously, if $x(t) = (x_1(t), x_2(t))^\top$ is a solution of (2.1), then $x_1(t)$ is a solution of (1.3).

Let

$$X = Y = \{x \mid x = (x_1(t), x_2(t))^\top \in C(\mathbb{R}, \mathbb{R}^2), x(t) = x(t+T)\},$$

where the normal $\|x\| = \max\{|x_1|_0, |x_2|_0\}$, and $|x_1|_0 = \max_{t\in[0,T]} |x_1(t)|$, $|x_2|_0 = \max_{t\in[0,T]} |x_2(t)|$. It is obvious that X and Y are Banach spaces. Furthermore, for $\varphi \in C_T$, $\|\varphi\|_r = (\int_0^T |\varphi(t)|^r\,dt)^{\frac{1}{r}}$, $r > 1$.

Now we define the operator L

$$L : D(L) \subset X \to Y, \qquad Lx = x' = (x_1'(t), x_2'(t))^\top,$$

where $D(L) = \{x \mid x = (x_1(t), x_2(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2), x(t) = x(t+T)\}$.

Define a nonlinear operator $N : \bar{\Omega} \subset X \to Y$

$$Nx = \left(\frac{\varphi_q(x_2(t))}{\sqrt{1 - \varphi_q^2(x_2(t))}}, -\beta\phi(x_2(t)) - g(t, x_1(t), x_1(t - \tau(t))) + p(t) \right)^\top.$$

Then problem (2.1) can be written as $Lx = Nx$ in $\bar{\Omega}$.

We know

$$\ker L = \{x \mid x \in X, x' = (x_1'(t), x_2'(t))^\top = (0,0)^\top\},$$

then $x_1'(t) = 0$, $x_2'(t) = 0$, obviously $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$, thus $\ker L = \mathbb{R}^2$, and it is also easy to prove that $\operatorname{Im} L = \{y \in Y, \int_0^T y(s)\,ds = 0\}$, so L is a Fredholm operator of index zero.

Let

$$P : X \to \ker L, \qquad Px = \frac{1}{T}\int_0^T x(s)\,ds,$$

and

$$Q : Y \to \operatorname{Im} Q, \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds.$$

Let $K_p = L|_{\ker p \cap D(L)}^{-1}$, then it is easy to see that

$$(K_p y)(t) = \int_0^T G(t,s) y(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{s-T}{T}, & 0 \le t \le s; \\ \frac{s}{T}, & s \le t \le T. \end{cases}$$

It implies that $\forall \Omega \subset X$ is an open and bounded set with $\bar{\Omega} \subset X$, $K_p(I-Q)N(\bar{\Omega})$ is a relative compact set of X, $QN(\bar{\Omega})$ is a bounded set of Y, so the operator N is L-compact in $\bar{\Omega}$.

3 Main results

Theorem 3.1 *For problem* (1.3), *assume the following conditions hold*:

(g_1) (*Balance condition*) *There exist positive constants A_1 and A_2 with $A_1 < A_2$ such that if x is a positive continuous T-periodic function satisfying $\int_0^T g(t,x(t),x(t-\tau(t)))\,dt = 0$, then*

$$A_1 \le x(\varepsilon) \le A_2$$

for some $\varepsilon \in [0,T]$.

(g_2) (*Degree condition*) *$\overline{g}(x) < 0$ for all $x \in (0,A_1)$ and $\overline{g}(x) > 0$ for all $x > A_2$, where $\overline{g}(x) = \frac{1}{T}\int_0^T g(t,x(t),x(t-\tau(t)))\,dt$, $x > 0$.*

(g_3) (*Decomposition condition*) *$g(t,x(t),x(t-\tau(t))) = g_1(t,x(t-\tau(t))) + g_0(x(t))$, where g_0 is a continuous function and there exist positive constants a_i, c_i, $i = 1,2$, and b such that*

$$g\bigl(t,x(t),x\bigl(t-\tau(t)\bigr)\bigr) \le a_1x(t) + a_2x\bigl(t-\tau(t)\bigr) + b, \quad (t,x) \in [0,T] \times (0,+\infty).$$

Meanwhile, $|g_1(t,x)| \le c_1x + c_2$.

(g_4) (*Strong force condition at $x = 0$*) *$\int_0^1 g_0(x)\,dx = -\infty$.*

(g_5) *$B := (\int_0^T |p(t)|^2\,dt)^{\frac{1}{2}} + \sup_{t\in[0,T]}|p(t)| < +\infty$, $|\beta| > c_1T$, and*

$$|\beta|M_1 + T\bigl[2(a_1+a_2)M_1 + 2b + B\bigr] < 1,$$

where $M_1 = A_2 + \frac{c_1A_2T + c_2T + B\sqrt{T}}{|\beta| - c_1T}$.

Then equation (1.3) *has at least one positive T-periodic solution.*

Proof Let $\Omega_1 = \{x \in \bar{\Omega}, Lx = \lambda Nx, \forall\lambda \in (0,1)\}$. If, $\forall x \in \Omega_1$, we have

$$\begin{cases} u'(t) = \lambda\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}} = \lambda\phi(v(t)), \\ v'(t) = -\lambda\beta\phi(v(t)) - \lambda g(t,u(t),u(t-\tau(t))) + \lambda p(t), \end{cases} \tag{3.1}$$

where $v(t) = \phi^{-1}(\frac{u'(t)}{\lambda}) = \varphi_p(\frac{\frac{1}{\lambda}u'(t)}{\sqrt{1+\frac{(u'(t))^2}{\lambda^2}}})$.

Integrating the second equation of (3.1) from 0 to T, we have

$$\int_0^T g\bigl(t,u(t),u\bigl(t-\tau(t)\bigr)\bigr)\,dt = 0. \tag{3.2}$$

Combining with (g_1), we can see that there exist positive constants A_1, A_2, and $\varepsilon \in [0,T]$ such that

$$A_1 \le u(\varepsilon) \le A_2. \tag{3.3}$$

Therefore, we have

$$\begin{aligned} |u|_0 &= \max_{t\in[0,T]} |u(t)| \le \max_{t\in[0,T]} \left| u(\varepsilon) + \int_\varepsilon^t u'(s)\,ds \right| \\ &\le A_2 + \int_0^T |u'(s)|\,ds \\ &\le A_2 + \sqrt{T}\|u'\|_2. \end{aligned} \tag{3.4}$$

Multiplying the second equation of (3.1) by $u'(t)$ and integrating on the interval $[0, T]$, we can get

$$\begin{aligned} 0 &= \int_0^T v'(t)u'(t)\,dt \\ &= -\int_0^T \beta\bigl(u'(t)\bigr)^2\,dt - \lambda\int_0^T g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr)u'(t)\,dt + \lambda\int_0^T p(t)u'(t)\,dt. \end{aligned}$$

It follows from (g_3) that

$$\begin{aligned} \int_0^T \beta\bigl(u'(t)\bigr)^2\,dt &= -\int_0^T g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr)u'(t)\,dt + \lambda\int_0^T p(t)u'(t)\,dt \\ &= -\int_0^T \bigl[g_1\bigl(t, u\bigl(t-\tau(t)\bigr)\bigr) + g_0\bigl(u(t)\bigr)\bigr]u'(t)\,dt + \lambda\int_0^T p(t)u'(t)\,dt \\ &= -\int_0^T g_1\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr)u'(t)\,dt + \lambda\int_0^T p(t)u'(t)\,dt, \end{aligned}$$

i.e.,

$$\begin{aligned} |\beta|\int_0^T |u'(t)|^2\,dt &\le \int_0^T \bigl|g_1\bigl(t, u\bigl(t-\tau(t)\bigr)\bigr)\bigr||u'(t)|\,dt + \int_0^T |p(t)||u'(t)|\,dt \\ &\le \int_0^T \bigl(c_1\bigl|u\bigl(t-\tau(t)\bigr)\bigr| + c_2\bigr)|u'(t)|\,dt + \int_0^T |p(t)||u'(t)|\,dt \\ &\le c_1|u|_0\sqrt{T}\|u'\|_2 + c_2\sqrt{T}\|u'\|_2 + B\|u'\|_2, \end{aligned}$$

which together with (3.4) gives

$$\begin{aligned} |\beta|\|u'\|_2^2 &\le c_1|u|_0\sqrt{T}\|u'\|_2 + c_2\sqrt{T}\|u'\|_2 + B\|u'\|_2 \\ &\le c_1\bigl[A_2 + \sqrt{T}\|u'\|_2\bigr]\sqrt{T}\|u'\|_2 + c_2\sqrt{T}\|u'\|_2 + B\|u'\|_2 \\ &= c_1T\|u'\|_2^2 + (c_1A_2\sqrt{T} + c_2\sqrt{T} + B)\|u'\|_2. \end{aligned} \tag{3.5}$$

It follows from $|\beta| > aT$ that

$$\|u'\|_2 \le \frac{c_1A_2\sqrt{T} + c_2\sqrt{T} + B}{|\beta| - c_1T}. \tag{3.6}$$

Substituting (3.6) into (3.4), we obtain

$$|u|_0 \le A_2 + \frac{c_1A_2T + c_2T + B\sqrt{T}}{|\beta| - c_1T} := M_1. \tag{3.7}$$

Furthermore, from the second equation of (3.1), we can get

$$\int_0^T \left|v'(t)\right| dt \leq \int_0^T |\beta|\left|u'(t)\right| dt + \lambda \int_0^T \left|g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr)\right| dt \\ + \lambda \int_0^T \left|p(t)\right| dt. \tag{3.8}$$

Write

$$I_+ = \bigl\{t \in [0,T] : g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr) \geq 0\bigr\};$$
$$I_- = \bigl\{t \in [0,T] : g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr) \leq 0\bigr\}.$$

Then we can get from (3.2) and (g_3)

$$\begin{aligned} &\int_0^T \left|g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr)\right| dt \\ &\quad = \int_{I_+} g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr) dt - \int_{I_-} g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr) dt \\ &\quad = 2\int_{I_+} g\bigl(t, u(t), u\bigl(t-\tau(t)\bigr)\bigr) dt \\ &\quad \leq 2a_1 \int_0^T u(t)\, dt + 2a_2 \int_0^T u\bigl(t-\tau(t)\bigr) dt + 2\int_0^T b\, dt \\ &\quad \leq 2(a_1 + a_2)T|u|_0 + 2bT. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8) and combining with (3.6) and (3.7), we obtain

$$\begin{aligned} \int_0^T \left|v'(t)\right| dt &\leq |\beta|\sqrt{T}\left\|u'\right\|_2 + \lambda\bigl[2(a_1 + a_2)T|u|_0 + 2bT\bigr] + \lambda BT \\ &\leq |\beta|M_1 + T\bigl[2(a_1 + a_2)M_1 + 2b + B\bigr]. \end{aligned} \tag{3.10}$$

Integrating the first equation of (3.1) on the interval $[0,T]$, we have

$$\int_0^T \frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}\, dt = 0.$$

Then we can see that there exists $\eta \in [0,T]$ such that $v(\eta) = 0$. It implies that

$$\left|v(t)\right| = \left|\int_\eta^t v'(s)\, ds + v(\eta)\right| \leq \int_0^T \left|v'(s)\right| ds,$$

which combining with (3.10) gives

$$\begin{aligned} \left|v(t)\right| &\leq \int_0^T \left|v'(s)\right| ds \\ &\leq |\beta|M_1 + T\bigl[2(a_1 + a_2)M_1 + 2b + B\bigr] \\ &:= \rho. \end{aligned}$$

Since $|\beta|M_1 + T(2(a_1 + a_2)M_1 + 2b + B) < 1$, we have

$$|v|_0 = \max_{t\in[0,T]} \bigl|v(t)\bigr| \leq \rho < 1. \tag{3.11}$$

From (3.1), we can also have

$$\bigl|u'\bigr|_0 \leq \lambda \cdot \max_{t\in[0,T]} \frac{|v(t)|^{q-1}}{\sqrt{1 - v^{2(q-1)}(t)}} \leq \frac{\lambda\rho^{q-1}}{1 - \rho^{2(q-1)}}. \tag{3.12}$$

On the other hand, from the second equation of (3.1) and by (g_3), we can see that

$$v'(t) = -\beta u'(t) - \lambda g\bigl(t, u(t), u\bigl(t - \tau(t)\bigr)\bigr) + \lambda p(t).$$

Take $\omega \in [0, T]$, then

$$\begin{aligned} v'(\omega) &= -\beta u'(\omega) - \lambda g\bigl(\omega, u(\omega), u\bigl(\omega - \tau(\omega)\bigr)\bigr) + \lambda p(\omega) \\ &= -\beta u'(\omega) - \lambda\bigl[g_1\bigl(\omega, u\bigl(\omega - \tau(\omega)\bigr)\bigr) + g_0\bigl(u(\omega)\bigr)\bigr] + \lambda p(\omega). \end{aligned} \tag{3.13}$$

Multiplying both sides of equation (3.13) by $u'(\omega)$, we have

$$\begin{aligned} v'(\omega)u'(\omega) = {}& -\beta u'(\omega)u'(\omega) - \lambda\bigl[g_1\bigl(\omega, u\bigl(\omega - \tau(\omega)\bigr)\bigr) + g_0\bigl(u(\omega)\bigr)\bigr]u'(\omega) \\ & + \lambda p(\omega)u'(\omega). \end{aligned} \tag{3.14}$$

Let $\varepsilon \in [0, T]$ be as in (3.3). For any $\omega \in [\varepsilon, T]$, integrating equation (3.14) on the interval $[\varepsilon, T]$, we have

$$\begin{aligned} \lambda \int_{u(\varepsilon)}^{u(\omega)} g_0(u)\,du &= \lambda \int_{\varepsilon}^{\omega} g_0\bigl(u(\omega)\bigr)u'(\omega)\,d\omega \\ &= -\int_{\varepsilon}^{\omega} v'(\omega)u'(\omega)\,d\omega - \int_{\varepsilon}^{\omega} \beta\bigl(u'(\omega)\bigr)^2\,d\omega \\ &\quad - \lambda \int_{\varepsilon}^{\omega} g_1\bigl(\omega, u\bigl(\omega - \tau(\omega)\bigr)\bigr)u'(\omega)\,d\omega + \lambda \int_{\varepsilon}^{\omega} p(\omega)u'(\omega)\,d\omega. \end{aligned} \tag{3.15}$$

By (3.12) and (3.15), we get

$$\begin{aligned} &\lambda\biggl|\int_{u(\varepsilon)}^{u(\omega)} g_0(u)\,du\biggr| \\ &\quad = \lambda\biggl|\int_{\varepsilon}^{\omega} g_0\bigl(u(\omega)\bigr)u'(\omega)\,d\omega\biggr| \\ &\quad \leq \int_{\varepsilon}^{\omega} \bigl|v'(\omega)u'(\omega)\bigr|\,d\omega + \int_{\varepsilon}^{\omega} \bigl|\beta\bigl(u'(\omega)\bigr)^2\bigr|\,d\omega \\ &\qquad + \lambda \int_{\varepsilon}^{\omega} \bigl|g_1\bigl(\omega, u\bigl(\omega - \tau(\omega)\bigr)\bigr)u'(\omega)\bigr|\,d\omega + \lambda \int_{\varepsilon}^{\omega} \bigl|p(\omega)u'(\omega)\bigr|\,d\omega \\ &\quad \leq \bigl|u'\bigr|_0 \int_{\varepsilon}^{\omega} \bigl|v'(t)\bigr|\,dt + |\beta|T\bigl|u'\bigr|_0^2 + \lambda G_{M_1}T\bigl|u'\bigr|_0 + \lambda T|p|_0\bigl|u'\bigr|_0 \\ &\quad \leq \bigl|u'\bigr|_0\bigl[|\beta|M_1 + 2T(a_1 + a_2)M_1 + 2bT + BT\bigr] + |\beta|T\bigl|u'\bigr|_0 \end{aligned}$$

$$\begin{aligned}&+\lambda TG_{M_1}+\lambda T|p|_0]\\ &\leq \frac{\lambda\rho^{q-1}}{1-\rho^{2(q-1)}}\Bigg[|\beta|M_1+2T(a_1+a_2)M_1+2bT+BT\\ &\quad+|\beta|T\frac{\lambda\rho^{q-1}}{1-\rho^{2(q-1)}}+\lambda TG_{M_1}+\lambda T|p|_0\Bigg],\end{aligned}\tag{3.16}$$

where $G_{M_1}=\max_{|u|\leq M_1}g_1(t,u)$. It follows from (3.16) that

$$\left|\int_{u(\varepsilon)}^{u(\omega)}g_0(u)\,du\right|<+\infty.$$

According to (g_4), we can see that there exists a constant $M_2>0$ such that, for $\omega\in[\varepsilon,T]$,

$$u(\omega)\geq M_2.\tag{3.17}$$

For the case $\omega\in[0,\varepsilon]$, we can handle it similarly. Thus, we have

$$u(t)\geq M_2,\quad \forall t\in[0,T].\tag{3.18}$$

Let us define

$$0<D_1=\min\{A_1,M_2\}\quad\text{and}\quad D_2=\max\{A_2,M_1\}.$$

Then by (3.3), (3.7), and (3.18) we can obtain

$$D_1\leq u(t)\leq D_2.\tag{3.19}$$

Set

$$\Omega=\left\{x=(u,v)^\top\in X:\frac{D_1}{2}<u(t)<D_2+1,|v|_0<\rho_1<\frac{\rho+1}{2}\right\}.$$

Then the condition (1) of Lemma 2.1 is satisfied. Suppose that there exists $x\in\partial\Omega\cap\ker L$ such that $QNx=\frac{1}{T}\int_0^T Nx(s)\,ds=(0,0)^\top$, *i.e.*,

$$\begin{cases}\frac{1}{T}\int_0^T\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}\,dt=0,\\ \frac{1}{T}\int_0^T[-\beta\frac{\varphi_q(v(t))}{\sqrt{1-\varphi_q^2(v(t))}}-g(t,u(t),u(t-\tau(t)))+p(t)]\,dt=0.\end{cases}\tag{3.20}$$

Since $\ker L=\mathbb{R}^2$, and $u\in\mathbb{R}$, $v\in\mathbb{R}$ are constant, combining with the first equation of (3.20), we obtain

$$v=0<\rho_1.$$

From the second equation of (3.20), we have

$$\frac{1}{T}\int_0^T g\big(t,u(t),u\big(t-\tau(t)\big)\big)\,dt=0.$$

From (g_1) we can see that

$$\frac{D_1}{2} < D_1 < A_1 \le u(t) \le A_2 < D_2 < D_2 + 1,$$

which contradicts the assumption $x \in \partial\Omega$. So, for all $x \in \ker L \cap \partial\Omega$, we have $QNx \neq 0$. Then the condition (2) of Lemma 2.1 is satisfied.

In the following, we prove that the condition (3) of Lemma 2.1 is also satisfied.

Let

$$z = Kx = K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \frac{A_1+A_2}{2} \\ v \end{pmatrix},$$

then we have

$$x = z + \begin{pmatrix} \frac{A_1+A_2}{2} \\ v \end{pmatrix}.$$

Define $J : \operatorname{Im} Q \to \ker L$ to be a linear isomorphism with

$$x = z + \begin{pmatrix} v \\ -u \end{pmatrix},$$

and define

$$H(\mu, x) = \mu Kx + (1-\mu)JQNx, \quad \forall (x,\mu) \in \Omega \times [0,1].$$

Then

$$H(\mu,x) = \begin{pmatrix} \mu u - \frac{\mu(A_1+A_2)}{2} \\ \mu v \end{pmatrix} + \frac{1-\mu}{T}\begin{pmatrix} \int_0^T [\frac{cv^{q-1}}{\sqrt{1-v^{2(q-1)}}} + g(t,u(t),u(t-\tau(t)))]\,dt \\ \int_0^T \frac{v^{q-1}}{\sqrt{1-v^{2(q-1)}}}\,dt \end{pmatrix}. \quad (3.21)$$

Now we claim that $H(\mu,x)$ is a homotopic mapping. Assume, by way of contradiction, that there exist $\mu_0 \in [0,1]$ and $x_0 = \binom{u_0}{v_0} \in \partial\Omega$ such that $H(\mu_0,x_0) = 0$.

Substituting μ_0 and x_0 into (3.21), we have

$$H(\mu,x) = \begin{pmatrix} \mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\frac{cv_0^{q-1}}{\sqrt{1-v_0^{2(q-1)}}} + (1-\mu_0)\overline{g}(u_0) \\ \mu_0 v_0 + (1-\mu_0)\frac{v_0^{q-1}}{\sqrt{1-v_0^{2(q-1)}}} \end{pmatrix}. \quad (3.22)$$

Since $H(\mu_0,x_0) = 0$, we can see that

$$\mu_0 v_0 + (1-\mu_0)\frac{v_0^{q-1}}{\sqrt{1-v_0^{2(q-1)}}} = 0.$$

Combining with $\mu_0 \in [0,1]$, we obtain $v_0 = 0$. Thus $u_0 = A_1$ or A_2.

If $u_0 = A_1$, it follows from (g_2) that $g(u_0) < 0$, then substituting $v_0 = 0$ into (3.22), we have

$$\begin{aligned}
&\mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\frac{\beta v_0^{q-1}}{\sqrt{1 - v_0^{2(q-1)}}} + (1-\mu_0)\overline{g}(u_0) \\
&\quad = \mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\overline{g}(u_0) \\
&\quad < \mu_0\left(u_0 - \frac{A_1 + A_2}{2}\right) < 0. \qquad (3.23)
\end{aligned}$$

If $u_0 = A_2$, it follows from (g_2) that $g(u_0) > 0$, then substituting $v_0 = 0$ into (3.22), we have

$$\begin{aligned}
&\mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\frac{\beta v_0^{q-1}}{\sqrt{1 - v_0^{2(q-1)}}} + (1-\mu_0)\overline{g}(u_0) \\
&\quad = \mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\overline{g}(u_0) \\
&\quad > \mu_0\left(u_0 - \frac{A_1 + A_2}{2}\right) > 0. \qquad (3.24)
\end{aligned}$$

Combining with (3.23) and (3.24), we can see that $H(\mu_0, x_0) \neq 0$, which contradicts the assumption. Therefore $H(\mu, x)$ is a homotopic mapping and $x^\top H(\mu, x) \neq 0$, $\forall (x, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1]$. Then

$$\begin{aligned}
\deg(JQN, \Omega \cap \ker L, 0) &= \deg\bigl(H(0, x), \Omega \cap \ker L, 0\bigr) \\
&= \deg\bigl(H(1, x), \Omega \cap \ker L, 0\bigr) \\
&= \deg(Kx, \Omega \cap \ker L, 0) \\
&= \sum_{x \in K^{-1}(0)} \operatorname{sgn}\bigl|K'(x)\bigr| \\
&= 1 \neq 0.
\end{aligned}$$

Thus, the condition (3) of Lemma 2.1 is also satisfied. Therefore, by applying Lemma 2.1, we can conclude that equation (1.3) has at least one positive T-periodic solution. □

4 Numerical example

In this section, we provide an example to illustrate results from the previous sections.

Example 4.1 As an application, we consider the following example:

$$\begin{aligned}
&\left(\varphi_p\left(\frac{x'(t)}{\sqrt{1 + (x'(t))^2}}\right)\right)' + \frac{1}{50}u'(t) + \frac{1}{24}(2 + \sin 100t)u\bigl(t - \sin^2 50t\bigr) \\
&\quad - \frac{1}{u(t)} = \frac{1}{72}\sin 100t. \qquad (4.1)
\end{aligned}$$

Conclusion *Problem* (4.1) *has at least one positive* $\frac{\pi}{50}$*-periodic solution.*

Proof Corresponding to Theorem 3.1 and (1.3), we have

$$
\begin{aligned}
&p(t) = \frac{1}{72}\sin 100t, \\
&g\bigl(t, u(t), u\bigl(t - \sin^2 50t\bigr)\bigr) = \frac{1}{24}(2 + \sin 100t)u\bigl(t - \sin^2 50t\bigr) - \frac{1}{u(t)},
\end{aligned}
$$

then we can have and choose

$$
\begin{aligned}
&T = \frac{\pi}{50}, \qquad a_1 = c_2 = 0.1, \qquad a_2 = c_1 = \frac{1}{8}, \qquad b = \frac{1}{101}, \qquad \beta = \frac{1}{50}, \\
&A_1 = 1, \qquad A_2 = 5,
\end{aligned}
$$

and $B := (\int_0^T |p(t)|^2\,dt)^{\frac{1}{2}} + \sup_{t\in[0,T]} |p(t)| < \frac{1}{36} < +\infty$. Then we can see that (g_1)-(g_4) hold. Moreover, $|\beta| > c_1 T$ and

$$
|\beta|M_1 + T\bigl[2(a_1 + a_2)M_1 + 2b + B\bigr] \approx 0.324 < 1.
$$

Hence, by applying Theorem 3.1, we can see that equation (4.1) has at least one positive T-periodic solution. □

Remark 4.1 Since all the results in [1–19] and the references therein cannot be applicable to equation (4.1) for solving positive periodic solutions, Theorem 3.1 in this paper is essentially new.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have equally contributed to obtaining the new results in this article and also read and approved the final manuscript.

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